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The cyclic homology of the group rings

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Introduction

Let Γ be a discrete group; denote by $\langle\Gamma\rangle$ the set of conjugacy classes of elements of Γ, and by $\langle\Gamma\rangle'$ respectively $\langle\Gamma\rangle''$ the subsets of $\langle\Gamma\rangle$ represented by elements $x\in\Gamma$, of finite respectively infinite order. For each $x\in\Gamma$ let $\Gamma_x=\{g\in\Gamma \mid gx=xg\}$ be the centralizer of x, $\{x\}$ the subgroup generated by x and N_x the quotient group $\Gamma_x/\{x\}$. If x' and x'' belong to the same conjugacy class $\hat{x}$, $\Gamma_{x'}$ and $\Gamma_{x''}$ as well as $N_{x'}$ and $N_{x''}$ are isomorphic and we can write $\Gamma_{\hat{x}}$ and $N_{\hat{x}}$ for their isomorphism class.

For R a commutative ring with unit we denote by $R[G]$ the R-group ring of G; it is an R-algebra with unit. We will denote by $HH_*(R[G])$ respectively $HC_*(R[G])$ its Hochschild respectively cyclic homology (see [C] or [LQ]). There is a long exact sequence (called Gysin–Connes sequence) which connects them,

$$\cdots \to HH_*(R[G] \xrightarrow{I} HC_*(R[G]) \xrightarrow{S} HC_{*-2}(R[G]) \xrightarrow{B} HH_{*-1}(R[G]) \to \cdots$$

(see [C] [LQ] or [B]).

THEOREM I′. *If k is a field of characteristic zero then:*

1) $HH_*(k[G])=\bigoplus_{\hat{x}\in\langle G\rangle} H_*(BG_{\hat{x}};k)$

2) $HC_*(k[G])=\bigoplus_{\hat{x}\in\langle G\rangle'} H_*(BN_{\hat{x}};k)\otimes HC_*(k)^{(1)}+\bigoplus_{\hat{x}\in\langle G\rangle''} H_*(BN_{\hat{x}};k)$.

Moreover the Gysin–Connes exact sequence decomposes as a direct sum of exact sequences parametrized by $\hat{x}\in\langle G\rangle$. If x is of finite order resp. of infinite order the corresponding exact sequence is the homology Gysin sequence of the fibration $BN_x\to BN_x\times BS^1 \xrightarrow{pr_2} BS^1$ resp. $BG_x\to BN_x\to BS^1$.

Here $H_*(X;R)$ denotes the homology of the space X with coefficients in R, and BG the classifying space of the group G. The general case of an arbitrary

[1] It is well known that $HC_*(k)=H_*(BS^1;k)$ see [LQ].

commutative unitary ring R requires more definitions so the result about $HC_*(R[G])$ given by Theorem I is stated at the end of Section I. As a consequence, we can describe $HC_*(R[G * H])$ and $HC_*(R[G\times H])$ (the last one under additional hypotheses) in terms of $R[G]$ and $R[H]$ as follows:

PROPOSITION II. $HC_*(R[G*H]) = HC_*(R[G]) + HC_*(R[H]) + \bigoplus_{\hat\alpha\in U} R_{\hat\alpha}$ $U = \{\hat\alpha\in\langle G*H\rangle \mid \hat\alpha\cap e_G*H=\emptyset,\ \hat\alpha\cap G*e_H=\emptyset\}$ *with* $R_{\hat\alpha}=R$ *regarded as a graded* R*-module concentrated in the degree zero.*

DEFINITION. We say that the group G has the property "h" if for any element of infinite order the fibration

$$B\{x\} = S^1 \to BG_x \to BN_x$$

is rationally trivial (or equivalently the Gysin homomorphism $H_*(BN_x; Q)\to H_{*-2}(BN_x; Q)$ is trivial). Clearly all abelian groups have property "h."

PROPOSITION III. *If G has property "h" and k is a field of characteristic zero then*

$$HC_*(k[G\times H]) = \bigoplus_{\hat x\in\langle G\rangle'} H_*(BN_{\hat x}; k)\otimes HC_*(k[H])$$
$$+ \bigoplus_{\hat x\in\langle G\rangle''} H_*(BN_{\hat x}; k)\otimes HH_*(k[H]).$$

COROLLARY IV.

$$HC_*(k[H\times Z]) = HC_*(k[H]) + HC_{*-1}(k[H]) + \bigoplus_{\alpha\in Z\setminus\{0\}} (HH_*(k[H]))_\alpha;$$

the last sum is the direct sum of copies of $HH_*(k[H])$ indexed by $\alpha\in Z\setminus\{0\}$.

Besides the cyclic holology, Connes has defined the periodic cyclic homology $PHC_*(k[G]) = \varprojlim(\cdots\to HC_{*+2n}(k[G])\to HC_{*+2n-2}(k[G])\to\cdots)$, for $*=0$, or 1. The results above implies the analogous formulae for periodic cyclic homology.

For each $\hat x\in\langle G\rangle''$ represented by $x\in G$ let

$$T_*(\hat x; R) = \lim(\cdots\to H_{*+2n}(BN_x; R)\xrightarrow{S} H_{*+2n-2}(BN_x; R)\to\cdots)$$

with S the Gysin homomorphism of the fibration $B\{x\} = S^1\to BG_x\to BN_x$ and let

$$K_*(BG;R)=\begin{cases}\bigoplus_{n\geq 0} H_{2n}(BG;R) & \text{if } *=0\\ \bigoplus_{n\geq 0} H_{2n+1}(BG;R) & \text{if } *=1\end{cases}$$

THEOREM I'_{p}

$$PHC_*(k[G])=\bigoplus_{\hat{x}\in\langle G\rangle'} K_*(BN_{\hat{x}};k)+\bigoplus_{\hat{x}\in\langle G\rangle''} T_*(\hat{x};k).$$

PROPOSITION II_{p}. *$PHC_*(R[G*H])=PHC_*(R[G])+PHC_*(R[H])$, where $G*H$ is the free product of G and H.*

COROLLARY IV. $PHC_*(k[G\times Z])=PHC_*(k[G])+PHC_{*-1}(k[G])$.

It should be noticed that homotopy theoretic computations permit to identify the right side of the equality 2) in Theorem I to the equivariant homology $H_*^{S^1}(BG_*^{S^1};R)$. T. Goodwillie [G], Burghelea–Fiedorowicz [BF], and others have proved that for a space X, and S^1-equivariant homology of X^{S^1} is isomorphic to the "cyclic homology" of the space X. The proof given here is, however, the "right one" for the case of discrete groups, since it explains the decomposition in terms of conjugacy classes and the different behaviour of finite order elements compared to the infinite order elements. A. Connes has noticed an interesting analogy with the Selberg–Trace formula, but this will not be discussed here.

The paper is organized as follows: In Section I we present the proof of Theorem I. The proof of Proposition 1.8 and 1.6′, important steps in the proof of Theorem I are deferred to Section II; this because we use the fibration associated with the cyclic set, a geometric tool not really essential for Theorem I. Algebraic proofs for Propositions 1.6′ and 1.8 are possible (but apparently longer).

In Section III we derive all the other statements contained in this Introduction and in Section III we will present few comments on the torsion free groups. The author thanks the referee for useful suggestions in improving the exposition; in particular the concept of cyclic groupoid as used here was suggested by him.

Section I

The purpose of this section is to calculate the cyclic homology of the group ring of the group Γ and prove Theorem I stated at the end of this section. In order to do this we need the theory of cyclic sets introduced in [C] for which we define Hochschild homology, cyclic homology and Gysin Connes exact sequence. For a group Γ one defines a cyclic set $\tilde{\mathscr{C}}(\Gamma)$, whose Gysin Connes sequence is the same as the Gysin Connes sequence of the group ring $R[\Gamma]$ (Proposition 1.4). We show that this cyclic set splits naturally as a disjoint union of cyclic sets parameterized by the conjugacy classes of elements of Γ, $\tilde{\mathscr{C}}(\Gamma)=\bigcup_{\hat{x}\in\langle\Gamma\rangle}\tilde{\mathscr{C}}(\Gamma)^{\hat{x}}$. Some observations (1.1, 1.2, 1.5) about nerves of groupoids allows us to define inside of each cyclic set $\tilde{\mathscr{C}}^{\hat{x}}(\Gamma)$ a cyclic set of special type $\tilde{\mathscr{L}}(\Gamma_x; x)$ having the same Gysin Connes exact sequence as $\tilde{\mathscr{C}}^{\hat{x}}(\Gamma)$. This implies that the cyclic sets $\bigcup_{\hat{x}\in\langle\Gamma\rangle}\tilde{\mathscr{L}}(\Gamma_x, x)$ and $\tilde{\mathscr{C}}(\Gamma)$ have identical Gysin Connes sequences; with the inclusion $\bigcup_{\hat{x}\in\langle\Gamma\rangle}\tilde{\mathscr{L}}(\Gamma_x, x)\to\tilde{\mathscr{C}}(\Gamma)$ inducing an isomorphism in both Hochschild and cyclic homology. Propositions 1.6, 1.6′ and 1.8 give the description of each of these Gysin Connes sequences.

Recall that a simplicial set $X=(X_n, d_n^i, s_n^i)$ consists of the sets X_n, and the maps $d_n^i: X_n\to X_{n-1}$, $s_n^i: X_n\to X_{n+1}$, $i=0,1,\ldots,n$, which satisfy the usual commutation relation. A cyclic set $\tilde{X}=(X, t_*)$ consists of a simplicial set X equipped with a cyclic structure t_* i.e., a sequence of maps $t_n: X_n\to X_n$ so that $(t_n)^{n+1}=id$, $d_n^i t_n=t_{n-1}d_n^{i-1}$, $s_n^i t_n=t_{n+1}s_n^{i-1}$ for $1\le i\le n$. The morphisms of cyclic sets are morphisms of simplicial sets which commute with the cyclic structure.

For a cyclic set $\tilde{X}=(X_n, d_n^i, s_n^i, t_n)$ let $T_*(X:R)$ be the chain complex associated with the simplicial set (X_n, d_n^i, s_n^i) i.e. $T_n(\tilde{X}; R)$ is the free R-module generated by X_n and $d_n: T_n(\tilde{X}; R)\to T_{n-1}(\tilde{X}; R)$ is given by $d_n=\sum_{0\le i\le n}(-1)^i d_n^i$. The homology $H_*(T_*(\tilde{X}; R))$, which is by definition the homology of (X_n, d_n^i, s_n^i) or equivalently, of the geometric realization of (X_n, d_n^i, s_n^i) will be called the Hochschild homology of $\tilde{X}$ with coefficients in R and denoted by $HH_*(\tilde{X}; R)$.

Following [C] or [LQ] one can also associate to $\tilde{X}$ the bicomplex $(E_{p,q}(\tilde{X}; R), d_{p,q}^I, d_{p,q}^{II})$ with $E_{p,q}(\tilde{X}; R)=T_p(X; R)$, $d_{p,q}^I=d_p$(resp. $\sum_{1\le i\le p}(-1)^i d_p^i$) if q is even (resp. odd), $d_{p,q}^{II}=1+\tau_p+\cdots+\tau_p^p$(resp. $1-\tau_p$, resp. 0) if $q\neq 0$ and even (resp. q odd, resp. $q=0$). Here $\tau_p=(-1)^p t_p$.

The homology of the total complex $(E_*(\tilde{X}; R), D)$ with $D=d^I+d^{II}$ is then the cyclic homology of $\tilde{X}$ with coefficients in R. As noticed in [B] this homology can also be calculated by using the complex denoted $(C_*(\tilde{X}; R), {}_\beta d)$, which is described as follows: $C_n(\tilde{X}; R)=\bigoplus_{i\ge 0}T_{n-2i}(\tilde{X}; R)$ with ${}_\beta d$ being given by ${}_\beta d(x_n, x_{n-2}, x_{n-4},\ldots)=(dx_n+\beta x_{n-2}, dx_{n-2}+\beta x_{n-4},\ldots)$. Here $\beta_n: T_n(\tilde{X}; R)\to T_{n+1}(\tilde{X}; R)$ is given by the formula $\beta_n=(-1)^n(1-\tau_{n+1})\tilde{S}_n^n(1+\tau_n+\tau_n^2+\cdots+\tau_n^n)$ with $\tilde{S}_n^n$ being the R-linear extension of the degeneracy map s_n^n. One then has a short exact sequence

$$0\to T_*(\tilde{X}; R)\xrightarrow{I} C_*(\tilde{X}; R)\xrightarrow{S}\sum^2 C_*(\tilde{X}; R)\to 0 \qquad (*)$$

with I being induced by the inclusion, for each n, of $T_n(\tilde{X}; R)$ in

$\bigoplus_{i\geq 0} T_{n-2i}(\tilde{X}; R) = C_n(\tilde{X}; R)$ and S by the projection of $\bigoplus_{i\geq 0} T_{n-2i}(\tilde{X}; R)$ onto $\bigoplus_{i\geq 1} T_{n-2i}(\tilde{X}; R)$. This short exact sequence induces a long exact sequence called the Gysin Connes sequence:

$$\cdots \to HH_*(\tilde{X}; R) \xrightarrow{I} HC_*(\tilde{X}; R) \xrightarrow{S} HC_{*-2}(\tilde{X}; R) \xrightarrow{B} HH_{*-1}(\tilde{X}; R) \to \cdots$$

OBSERVATION 1.1. *Let $f: \tilde{X} \to \tilde{Y}$ be a morphism of cyclic sets. Then if $HH_*(f)$ is an isomorphism, $HC_*(f)$ is also an isomorphism.*

It suffices to note that f induces a morphism between the Gysin Connes exact sequences associated with $\tilde{X}$ and $\tilde{Y}$. A simple inspection of the resulting diagram shows by induction that $HC_n(f)$ is an isomorphism.

Let us recall that a groupoid $\mathscr{C}$ is a small category all of whose morphisms are isomorphisms. The nerve (or the classifying space) of the groupoid $\mathscr{C}$ is the simplicial set Nerve $\mathscr{C}$ whose set of n-simplexes consist of strings $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots A_n \xrightarrow{\alpha_n} A_{n+1}$ with faces and degeneracies defined in the obvious way.

OBSERVATION 1.2. *If $\mathscr{C}$ is a groupoid such that for any $A, B \in ob\mathscr{C}$* Hom $(A, B) \neq \varnothing$, *and $\mathscr{C}_A$ denotes the full subgroupoid with only one object A, then the inclusion $\mathscr{C}_A \to \mathscr{C}$ induces a simplicial inclusion* Nerve $\mathscr{C}_A \to$ Nerve $\mathscr{C}$ *which is a homotopy equivalence (hence it induces an isomorphism on homology).*

A cyclic groupoid $(\mathscr{C}, \varepsilon)$ is a groupoid equipped with a "cyclic structure" ε which associates to each object A a morphism $\varepsilon_A \in \mathrm{Hom}(A, A)$ such that $f \cdot \varepsilon_A = \varepsilon_B \cdot f$ for $f \in \mathrm{Hom}(A, B)$. For any groupoid $\mathscr{C}$ let $(\mathscr{C}, id)$ be the cyclic groupoid obtained by taking the trivial cyclic structure $\varepsilon_A = id_A$. A cyclic groupoid gives rise to a cyclic set $\mathrm{Nerve}(\mathscr{C}, \varepsilon) = (\mathrm{Nerve}\,\mathscr{C}, t_*)$ with $t_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1^{-1} \circ \alpha_2^{-1} \cdots \alpha_n^{-1} \circ \varepsilon_{A_n}, \alpha_1, \ldots, \alpha_{n-1})$, $\alpha_i \in \mathrm{Hom}(A_i, A_{i+1})$.

OBSERVATION 1.3. $HC_*(\mathrm{Nerve}(\mathscr{C}, id); R) = \bigoplus_{k\geq 0} HH_{*-2k}(\mathrm{Nerve}(\mathscr{C}, id); R)$ and the Gysin–Connes exact sequence reduces to the short exact sequences

$$O \to HH_*(\mathrm{Nerve}(\mathscr{C}, id); R) \to \bigoplus_{k\geq 0} HH_{*-2k}(\mathrm{Nerve}(\mathscr{C}, id); R) \to \bigoplus_{k\geq 1} HH_{*-2k}(\mathrm{Nerve}(\mathscr{C}, id); R) \to 0$$

By Observation 1.1 it suffices to verify the statement for groupoids with one element, hence for groups. This was already done by Karoubi in "Homologie cyclique des groupes et algèbres – C. R. Acad. Sci t. 297 p. 381-4" section II.

EXAMPLES. Let G be a discrete group and $x \in \mathrm{Center}\, G$.

1) One also denotes by G the group G regarded as a groupoid with only one object $*$ and Hom $(*, *) = G$, and one denotes by (G, x) the cyclic groupoid whose cyclic structure is given by x. Henceforth we will write $\tilde{\mathscr{L}}(G, x)$ instead of Nerve (G, x).

2) $\mathscr{C}(G)$ denotes the groupoid whose objects are the elements of G and $\mathrm{Hom}\,(g', g'') = \{\alpha \in G / \alpha^{-1} g' \alpha = g''\}$. In what follows $(\mathscr{C}(G), \bar{\varepsilon})$ will denote the cyclic groupoid $\mathscr{C}(G)$ with cyclic structure $\bar{\varepsilon}_g = g$, and $\tilde{\mathscr{C}}(G)$ will denote its associated cyclic set.

3) If $\hat{g}$ denotes the conjugacy class of g in G, let $\mathscr{C}(G)^{\hat{g}}$ be the full subgroupoid of $\mathscr{C}(G)$ whose objects are the elements $g \in G$ in the conjugacy class $\hat{g}$; then we have the cyclic subgroupoid $(\mathscr{C}(G)^{\hat{g}}, \bar{\varepsilon})$ of $(\mathscr{C}(G), \bar{\varepsilon})$ and, therefore, the cyclic subset $\tilde{\mathscr{C}}(G)^{\hat{g}} \subset \tilde{\mathscr{C}}(G)$. It is easy to see that $\tilde{\mathscr{C}}(G)$ decomposes as a disjoint union $\bigcup_{\hat{g} \in \langle G \rangle} \tilde{\mathscr{C}}(G)^{\hat{g}}$.

PROPOSITION 1.4. *The Gysin Connes sequence for $R[G]$ can be naturally identified with the Gysin Connes sequence of the cyclic set $\tilde{\mathscr{C}}(G)$ with coefficients in R; in particular the Hochschild homology $HH_*(R[G])$ (resp. cyclic homology $HC_*(R[G])$ are isomorphic to $HH_*(\tilde{\mathscr{C}}(G); R)$ (resp. $HC_*(\tilde{\mathscr{C}}(G); R)$).*

Proof. We define a natural isomorphism of chain complexes $\theta_G : T_*(\tilde{\mathscr{C}}(G); R) \to T_*(R[G])$ which commute with the corresponding homomorphisms β_*; here $T_*(R[G])$ is the chain complex (see [L, Q] or [B$_2$]) which calculates the Hochschild homology of $R[G]$. θ_G sends the generator $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $T_*(\tilde{\mathscr{C}}(G); R)$ to the element $(\alpha_n^{-1} \alpha_{n-1}^{-1} \cdots \alpha_1^{-1} g \otimes \alpha_1 \otimes \cdots \otimes \alpha_n)$ of $T_n(R[G]) = \underbrace{R[G] \otimes \cdots \otimes R[G]}_{n+1}$ with g being the "source" of the morphism $\alpha \in \mathscr{C}(G)$. Clearly θ_G induces an isomorphism $\tilde{\theta}_G : C_*(\tilde{\mathscr{C}}(G); R) \to C_*(R[G])$ making the following diagram commutative and, therefore, proving our statement.

$$\begin{tikzcd}
0 \arrow[r] & T_*(\tilde{\mathscr{C}}(G); R) \arrow[r] \arrow[d, "\theta_G"] & C_*(\tilde{\mathscr{C}}(G); R) \arrow[r] \arrow[d, "\tilde{\theta}_G"] & \sum^2 C_*(\tilde{\mathscr{C}}(G); R) \arrow[r] \arrow[d, "\Sigma^2 \tilde{\theta}_G"] & 0 \\
0 \arrow[r] & T_*(R[G]) \arrow[r] & C_*(R[G]) \arrow[r] & \sum^2 C_*(R[G]) \arrow[r] & 0
\end{tikzcd}$$

Let us fix in each conjugacy class $\hat{g}$ a representative $g \in G$. One defines $\tilde{\iota}^g : \tilde{\mathscr{L}}(G_g; g) \to \tilde{\mathscr{C}}(G)^g$ as the map of cyclic sets induced by the functor $\iota^{\hat{g}} : G_g \to G$ defined by $\iota^{\hat{g}}(*) = g$ and $\iota^{\hat{g}}(\alpha) = \alpha$ $\alpha \in \mathrm{Hom}\,(*, *)$, and one considers $\tilde{\iota} : \bigcup_{\hat{g} \in \langle G \rangle} \tilde{\mathscr{L}}(G_g, g) \to \tilde{\mathscr{C}}(G)$. Using Observations 1.1 and 1.2 we conclude.

OBSERVATION 1.5. *The Gysin Connes exact sequence of the cyclic set $\tilde{\mathscr{C}}(G)$ is the direct sum of the Gysin Connes exact sequences of $\tilde{\mathscr{L}}(G_g, g)$ indexed by $\hat{g} \in \langle G \rangle$ (for any coefficients R).*

PROPOSITION 1.6. *If $g \in \text{Center}\, G$ is of finite order and k is a field of characteristic zero then the Gysin Connes exact sequence of $\tilde{\mathscr{L}}(G, g)$ (with coefficients in k) is isomorphic to the homology Gysin sequence (with coefficients in k) of the trivial fibration*

$$B(G/\{g\}) \to B(G/\{g\}) \times BS^1 \to BS^1.$$

Proof. If $G = e$ this is Observation 1.3. If $g \neq e$ the map of cyclic spaces $\tilde{\mathscr{L}}(G, g) \to \tilde{\mathscr{L}}(G/\{g\}, e)$ induces by Observation 1.1 an isomorphism between the Gysin Connes exact sequences.

PROPOSITION 1.7. *$HC_*(\tilde{\mathscr{L}}(Z, 1); R) = R$ (resp. 0) if $* = 0$ (resp. $\neq 0$).*

Proof. First, notice that $HH_*(\tilde{\mathscr{L}}(Z, 1); R) = R$ (resp. 0) if $* = 0, 1$ (resp. $\neq 0, 1$, since $HH_*(\tilde{\mathscr{L}}(Z, 1); R)$ is equal to $H_*(BZ; R)$. The result will follow from Gysin Connes exact sequence once one concludes that $HC_1(\tilde{\mathscr{L}}(Z, 1);\ R) = 0$. To verify this we will check that any generator $[m]$ of $C_1(\tilde{\mathscr{L}}(Z, 1); R) = T_1(\tilde{\mathscr{L}}(Z, 1); R)$, is a boundary, where $T_1(\mathscr{L}(Z, 1); R)$ is the free R-module on generators $\{[m]/m \in Z\}$. Indeed for each $[m]$ there exists $(x_2, x_0) \in T_2(\cdots) \oplus T_0(\cdots) = C_2(\cdots)$ such that ${}_{\beta}d(x_2, x_0) = [m]$, where $T_2(\cdots)$ (resp. $T_0(\cdots)$) is the free R-module generated by the symbols $[m, n]$, $m, n \in Z$ (resp. *); since β_0: $T_0(\cdots) \to T_1(\cdots)$ is given by $\beta(*) = [0] + [1]$ one can take $x_2 = -[m-1, 1] - [m-2, 1] - \cdots [0, 1] - (m-1)[0, 0]$ and $x_0 = m(*)$ if $m \geq 0$.

PROPOSITION 1.8. *If $g \in \text{Center}\, G$ is an element of infinite order the Gysin Connes exact sequence of $\tilde{\mathscr{L}}(G, g)$ with coefficients in R is the same as the homology Gysin sequence (with coefficients in R) of the fibration $BG \to B(G/\{g\}) \to BS^1$. In particular $HC_*(\tilde{\mathscr{L}}(G, g); R) = H_*(BG/\{g\}; R)$.*

If $g \in \text{Center}\, G$ is an element of order n let $\alpha \in H^2(BG/\{g\}; Z_n)$ be the characteristic map of the fibration $B\{g\} \to BG \to BG/\{g\}$, and let $\tau_n \in H^2(BS^1; Z_n) \simeq \text{Hom}\,(Z, Z_n)$ be the cohomology class corresponding to the projection $Z \to Z_n$. Let $(BG/\{g\} \times BS^1) \overset{g}{\times} K(Z_n, 1)$ be the total space of the fibration over $BG/\{g\} \times BS^1$ with fibre $K(Z_n, 1)$ defined by the class $\sigma + \tau_n \in H^2(BG/\{g\} \times BS^1; Z_n)$.

PROPOSITION 1.6′. *If* $g \in \text{Center } G$ *is of order* n *then*

$$HC_*(\tilde{\mathscr{L}}(G, g); R) = H_*((BG/\{g\} \times BS^1) \overset{g}{\times} K(Z_n, 1); R).$$

The proofs of Proposition 1.8 and 1.6′ will be given in the next section.

Proposition 1.4, Observation 1.5, Proposition (1.6′ (Proposition 1.6 if R is field of characteristic zero) and Proposition 1.8 imply:

THEOREM I. 1) $HH_*(R[G]) = \bigoplus_{\hat{x} \in \langle G \#'} H_*(BG_x; R)$

2) $HC_*(R[G]) = \bigoplus_{\hat{x} \in \langle G \rangle'} H_*((BN_x \times BS^1) \times K(Z_n, 1); R) + \bigoplus_{\hat{x} \in \langle G \rangle''} H_*(BN_x; R)$ *with* x *a chosen representative in* $\hat{x}$. *Moreover, the Gysin Connes exact sequence is a direct sum of exact sequences parameterized by* $\hat{x} \in \langle G \rangle$; *the exact sequence corresponding to* $\hat{x}$ *is the homology Gysin sequence (with coefficients in* R*) of the fibration* $BG_x \to BN_x \to BS^1$ *if* x *has infinite order and of the trivial fibration* $BN_x \to BN_x \times BS^1 \to BS^1$ *if* x *has finite order and* R *has characteristic zero.*

Clearly Theorem I implies theorem I′.

Section II

It is shown in [B–F] that to any cyclic set $\tilde{X} = (X, t_*)$ one can associate in a natural way a fibration $\|X\| \xrightarrow{i_{\tilde{X}}} |||\tilde{X}||| \xrightarrow{p_{\tilde{X}}} BS^1$ whose fibre $\|X\|$ is the geometric realization of the underlying simplicial set X. It is also shown there that the homology Gysin sequence of this fibration is the same as the Gysin Connes sequence of the cylic set $\tilde{X}$.

Proof of Proposition 1.8. Let $\tilde{\mathscr{L}}(Z, 1) \xrightarrow{i} \tilde{\mathscr{L}}(G, g) \xrightarrow{\pi} \tilde{\mathscr{L}}(G/\{g\}, e)$ be the morphism of cyclic sets induced by $i: Z \to G$, $i(1) = g$ and the projection $G \to G/\{g\}$; let $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3$ denote the spaces $\tilde{\mathscr{L}}(Z, 1)$, $\tilde{\mathscr{L}}(G, g)$ and $\tilde{\mathscr{L}}(G/\{g\}; e)$ respectively. One can construct the following homotopy commulative diagram, whose vertical lines are fibrations up to homotopy

$$\begin{tikzcd}
B(G/\{g\}) \arrow[r, "i_{\tilde{X}_3}"] & B(G/\{g\}) \times BS^1 \arrow[r, "\lambda"] & BS^1 \\
B(G) \arrow[u, "\|\pi\|"] \arrow[r, "i_{\tilde{X}_2}"] & |||\tilde{\mathscr{L}}(G, g)||| \arrow[u, "|||\pi|||"] \arrow[r] & ES^1 \arrow[u] \\
B\{g\} = S^1 \arrow[u, "\varepsilon'"] \arrow[r, "\simeq"] & S^1 \arrow[u, "\varepsilon"] \arrow[r, "\simeq"] & S^1 \arrow[u] \\
 & (2) & (1)
\end{tikzcd}$$

We can choose BS^1 to be a commutative topological group so that λ is given by $\lambda(u, v) = \lambda_1(u) + \lambda_2(v)$, $u \in BG/\{g\}$ $v \in BS^1$ with λ_1 (resp. λ_2) the restriction of λ to $BG/\{g\} \times *$ resp. $* \times BS^1$. Note that $\varepsilon_* : \pi_1(S^1) = Z \to \pi_1(\|\|\tilde{\mathscr{L}}(G, g)\|\|)$ is trivial as $i_{X_2} \cdot \varepsilon'$ factors through $\|\|\tilde{\mathscr{L}}(\{g\}, g)\|\|$ (which is contractible by Proposition 1.7). Hence, λ_2 is a homotopy equivalence. We claim that the fibration (2) is isomorphic to one induced from the universal fibration (1) by the projection $B(G/\{x\}) \times BS^1 \xrightarrow{pr_2} BS^1$. To see this, one defines a homotopy equivalence $\theta : B(G/\{g\}) \times BS^1 \to B(G/\{g\}) \times BS^1$ by $\theta(u, v) = (u, -\deg(\lambda_2) \cdot \lambda_1(u) + v)$ and notices that $\lambda \cdot \theta = pr_2$. This implies Proposition 1.8.

Proof of Proposition 1.6′. We have to show that $\tilde{\mathscr{L}}(G, g)$ is homotopy equivalent to $(BN \times BS^1) \overset{g}{\times} K(Z_n; 1)$. It suffices to verify that the restriction of the fibration

$$K(Z_n; 1) = B\{g\} \to \|\|\tilde{\mathscr{L}}(G, g)\|\| \xrightarrow{\pi} \|\|\tilde{\mathscr{L}}(G/\{g\}, e)\|\| = BG/\{g\} \times BS^1$$

to $BG/\{g\} \times *$ (resp. $* \times BS^1$) has σ (resp. τ_n) as its characteristic class. This follows immediately from the commutativity at the diagrams

$$\begin{array}{ccccc}
B\{g\} & \longrightarrow & \|\|\tilde{\mathscr{L}}(G, g)\|\| & \xrightarrow{\pi} & BG/\{g\} \times BS^1 \\
\uparrow & & \uparrow{\scriptstyle i_{\tilde{X}}} & & \uparrow{\scriptstyle i_{\tilde{Y}}} \\
B\{g\} & \longrightarrow & BG & \longrightarrow & BG/\{g\}
\end{array}$$

where $\tilde{X} = \tilde{\mathscr{L}}(G, g)$, $\tilde{Y} = \tilde{\mathscr{L}}(G/\{g\}, e)$, and

$$\begin{array}{ccccc}
B\{g\} & \longrightarrow & \|\|\tilde{\mathscr{L}}(G, g)\|\| & \xrightarrow{\|\|\pi\|\|} & BG/\{g\} \times BS^1 \\
\uparrow & & \uparrow{\scriptstyle in} & & \uparrow \\
K(Z_n, 1) & \longrightarrow & \|\|\tilde{\mathscr{L}}(\{g\}, g)\|\| & \longrightarrow & B\{g\}/\{g\} \times BS^1 \\
\uparrow & & \uparrow & & \uparrow \\
K(Z, 1) & \longrightarrow & \|\|\tilde{\mathscr{L}}(Z, 1)\|\| & \longrightarrow & B(Z/\{1\}) \times BS^1
\end{array}$$

where in: $\tilde{\mathscr{L}}(\{g\}, g) \to \tilde{\mathscr{L}}(G, g)$ is induced by the inclusion $\{g\} \subset G$.

Section III

The purpose of this section is to complete the proof of the remaining statements from Introduction.

Proof of Proposition II. Let $P = G * H$. The reader can verify that $P_x \neq \{x\}$ only if $\hat{x}$ contains a representative of either form $g_1 * e_H$ or $e_G * g_2$ in which case $P_{\hat{x}}$ is isomorphic to either $G_{\hat{g}_1}$ or $H_{\hat{g}_2}$. With this observation in mind one applies Theorem I. Q.E.D.

Let $x \in \text{Center } G$, $y \in \text{Center } H$ hence $(x, y) \in \text{Center } G \times H$, and let $\{x\}$, $\{y\}$, $\{(x, y)\}$ denote the subgroup of G resp. H, resp. $G \times H$ generated by x resp. y, resp. $(x \times y)$. Proposition III follows by applying Observation 1.5, the homology (actually homotopy) equivalence $|||\tilde{\mathscr{L}}(\Gamma_x, x)||| \sim |||\mathscr{C}^{\hat{x}}(\Gamma)|||$ and Observation 3.1 below.

OBSERVATION 3.1.[(1)] a) *If both x and y are of finite order then*

$$H_*(|||\tilde{\mathscr{L}}(G \times H, (x, y))|||; k) = H_*(BG \times BH \times BS^1; k).$$

b) *If x is of infinite order and y of finite order then*

$$H_*(|||\tilde{\mathscr{L}}(G \times H, (x, y)|||; k) = H_*(B(G/\{x\}; k) \otimes H_*(BH; k).$$

c) *If x and y are of infinite order and the fibration $S^1 = B\{y\} \to BH \to B(H\{y\})$ has trivial Chern class $\sigma_H \in H^2(B(H/\{y\}); R)$ then*

$$H_*(|||\tilde{\mathscr{L}}(G \times H, (x, y))|||; R) = H_*(B(G) \times B(H/\{y\}); R).$$

Proof. a) is in fact Proposition 1.6. If either x or y are of infinite order $|||\tilde{\mathscr{L}}(G \times H; (x, y))|||$ has the same homology as $B(G \times H/\{(x, y)\})$ by Proposition 1.8. Notice that we have the fibration

$$B\left(\frac{\{x\} \times \{y\}}{\{(x, y)\}}\right) \to B(G \times H/\{(x, y)\}) \to B(G/\{x\}) \times B(H/\{y\}).$$

If x is of infinite order and y of finite order, hence the group $\{x\} \times \{y\}/\{(x, y)\}$ is finite, the fibration implies

$$\begin{aligned} H_*(B(G \times H/\{(x, y)\}); k) &= H_*(B(G/\{x\}) \times B(H/\{y\}); k) \\ &= H_*(B(G/\{x\}) \times BH; k) \end{aligned}$$

which is b). If both x and y are of infinite order, it is not hard to check that the

[1] Here k denotes a field of characteristic zero and R a commutative unitary ring.

fibration

$$S^1 = B\frac{\{x\}\times\{y\}}{\{(x, y)\}} \to B(G\times H/\{(x, y)\}) \to B(G/\{x\})\times B(H\{y\})$$

is a fibration of Chern class $\sigma_G+\sigma_H$ where $\sigma_G \in H^2(B(G/\{x\}; R)$ respectively $\sigma_H \in H^2(B(H/\{y\}); R)$ are the Chern classes of the fibration $B\{x\}\to BG\to B(G/\{x\})$ resp. $B\{y\}\to BH\to B(H/\{y\})$. If $\sigma_H=0$ then clearly $H_*(B(G\times H/\{(x, y)\}); R)=H_*(B(G)\times B(H/\{y\}); R)$.

Theorem I'_p, Proposition II_p and Corollary IV_p follows from Theorem I′, Proposition II and Corollary IV.

Section IV

If G is a torsion free group, i.e. the only element of finite order is zero, the formula given by Theorem I becomes:

$$HC_*(R[G]) = H_*(BG; R)\otimes H_*(BS^1; R) + \bigoplus_{\hat{x}\in\langle G\rangle\setminus\hat{e}} H_*(BN_{\hat{x}}; R).$$

Theorem I also implies

$$PHC_*(R[G]) = K_*(BG; R) + \bigoplus_{\hat{x}\in\langle G\rangle\setminus\hat{e}} T_*(\hat{x}; R).$$

If $N_{\hat{x}}$ is of finite homological dimension then $T(\hat{x}; R)$ is zero for any element of infinite order. In general this is not the case. For instance let Γ be a discrete group for which there exists $\varphi: K(\Gamma, 1)\to CP^\infty$ which is a homology equivalence. Such Γ always exists by a theorem of Kan–Thurston see [BDH]; one can even assume that Γ is torsion free. Let $S^1 \xrightarrow{i} E \to K(\Gamma, 1)$ be the pullback of the universal S^1-bundle by φ; clearly $E = K(G, 1)$ and if $x = i_*(1)\in G$, $i_*: Z = \pi_1(S^1)\to\pi_1(E)$ we have $\tilde{H}_*(BG; R)=0$ for any R and $T_*(\hat{x}; R)=R$ (resp. 0) if $*$ is even (resp. odd). In spite of this it will not be very surprising if the following conjecture is true:

CONJECTURE. *If $K(\Gamma, 1)$ has the homotopy type of a finite CW-complex, then $T(\hat{x}; R)=0$ for any $\hat{x}\in\langle\Gamma\rangle''$.*

This is obviously true for those Γ which are fundamental groups of compact riemannian manifolds of nonpositive sectional curvature.

REFERENCES

[B] D. BURGHELEA. *Cyclic homology and the algebraic K-theory of spaces I.* (Preprint 1983-Preprint IHES 1984).

[B–F] D. BURGHELEA and Z. FIEDOROWICZ. *Cyclic homology and the algebraic K-theory of spaces II.* (to appear).

[BDH] C. BAUMSLAG, E. DYER and A. HELLER. *The topology of discrete groups.* J. Pure App. Algebra *16* (1980), 1–47.

[C] A. CONNES. *Cohomologie cyclique et foncteurs* Ext^n: C.R. Acad. Paris 296 série I, 1983 p. 953. See also M. Karoubi, *Homologie cyclique et K-théorie algébrique II.* C.R. Acad. Paris. 297 Série I. pp. 513.

[G] T. GOODWILLIE. *Cyclic homology, derivations and the free loop space.* Preprint 1984.

[LQ] J. L. LODAY and D. QUILLEN. *Cyclic homology and the Lie algebra of matrices.* To appear in Commentarii Math. Helv.

[Q] D. QUILLEN. *Higher algebraic K-theory I.* Springer Lecture Notes in Math. Vol. 341 (1972), 85–147.

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