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Foliation dynamics and leaf invariants

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§1. Statement of results

Let $\mathcal{F}$ be a codimension-n foliation of a smooth manifold M without boundary. M may be either compact or open, and assume $\mathcal{F}$ is transversally C^2. The purpose of this note is to examine the relation between the linear holonomy of the leaves of $\mathcal{F}$ and the growth rates of the leaves.

THEOREM 1. *Let $\mathcal{F}$ and M be as above. Given a leaf $L \subset M$ of $\mathcal{F}$, suppose its linear holonomy group $\Gamma_L \subset GL(n, \mathbb{R})$ is not amenable. Then $\mathcal{F}$ has a leaf L' which contains L in its closure, and for all Riemannian metrics on M, L' has exponential growth.*

Amenability is taken in the sense of topological groups, where Γ_L is endowed with the topology from $GL(n, \mathbb{R})$.

We actually prove a slightly more general result, from which Theorem 1 follows by standard methods.

THEOREM 2. *Let $\mathcal{G}$ be a pseudogroup of local diffeomorphisms of $\mathbb{R}^n$, all of whose elements are defined at and fix the origin $0 \in \mathbb{R}^n$, and are C^2 in a neighborhood of 0. Let Γ denote the linear group of Jacobians at 0 of the elements of $\mathcal{G}$. If Γ is not amenable, then the action of $\mathcal{G}$ on $\mathbb{R}^n$ has an orbit with exponential growth and which contains 0 in its closure.*

The normal bundle to $\mathcal{F}$ is denoted by Q. The restriction of Q to a leaf L is well-known to be a flat $\mathbb{R}^n$-vector bundle, to which there are associated characteristic classes [12] obtained from the relative Lie algebra cohomology of $(\mathfrak{gl}_n, O_n)$. They are given by a map

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \to H^*(L).$$

The leaf classes of L consist of the image of χ_L.

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THEOREM 3. *Let $\mathcal{F}$ be a foliation of M as above. Suppose there exists* $y \in H^m(\mathrm{gl}_n, O_n)$ *with* $m > 1$ *and* $\chi_L(y) \neq 0$. *Then the linear holonomy group* Γ_L *of L is not amenable.*

COROLLARY 4. *Let $\mathcal{F}$ and M be as above. Suppose that all leaves of $\mathcal{F}$ have non-exponential growth. Then for every leaf L of $\mathcal{F}$, the linear holonomy group* Γ_L *is amenable, and all leaf classes of L in degrees greater than one are zero.*

The hypothesis $m > 1$ is necessary. For example, a flow on M with a linearly attracting closed orbit L has $\chi_L(y_1) \neq 0$, where y_1 is the standard generator of $H^1(\mathrm{gl}_n, O_n)$. All orbits of the flow have at most linear growth, hence non-exponential, and the holonomy group of L is Z, which is amenable.

Corollary 4 can be viewed as a generalization to all of the characteristic classes for flat bundles of a result due to Hirsch and Thurston. The Main Theorem of [7] implies that the Euler class of the restriction $Q \mid L \to L$ is zero if the foliated normal sphere bundle to L has an invariant transverse measure. This will be the case, for example, when $\mathcal{F}$ has a leaf L' of non-exponential growth with L contained in the closure of L'.

Theorem 1 is complementary to a result of Zimmer (Theorem 5.5 of [20]; see also Corollary 4.3 of [10]): If $\mathcal{F}$ is amenable, then there exists a measurable framing s of $Q \to M$ such that for *almost every* leaf L, there is a closed amenable subgroup $G_L \subset GL(n, \mathbb{R})$ for which the linear holonomy along L, with respect to s, takes values in G_L. For example, $\mathcal{F}$ will be amenable if almost every leaf has subexponential growth.

Note that the set of leaves of $\mathcal{F}$ with non-trivial linear holonomy has measure zero (Lemma 7.2 of [10]), so Zimmer's theorem does not imply our Theorem 1. With the stronger hypothesis that every leaf of $\mathcal{F}$ has nonexponential growth, Theorem 1 implies that for every leaf L, there exists a framing s_L of $Q \mid L \to L$ for which the linear holonomy along L, with respect to s_L, takes values in an amenable subgroup G_L. It is an open problem to find sufficient conditions on the dynamics of $\mathcal{F}$ that imply $Q \to M$ has a measurable framing s, with respect to which every leaf has amenable linear holonomy.

This work arose out of the study [9], and was motivated by an attempt to generalize to all codimensions the results of Duminy [2] relating the Godbillon-Vey class in codimension-one with leaf dynamics. For a further discussion, see [10].

We now give an idea of the proofs. Theorem 3 is based on the observation that the well-known explicit Lie algebra forms, representing the generators of $H^*(\mathrm{gl}_n, O_n)$, are exact when restricted to the Lie algebra of a maximal amenable subgroup of $GL(n, \mathbb{R})$. This is proven in §3. The heart of this paper is the proof of

Theorem 2. It is useful to compare Theorem 2 with Tits' Theorem [19]: a non-amenable linear group Γ contains a free non-abelian subgroup on two generators. From this it is easy to see that the linear action of Γ on $\mathbb{R}^n$ has orbits of exponential growth. Two problems arise when one tries to use this to show the pseudogroup $\mathcal{G}$ has orbits of exponential growth. First, control must be maintained over the domains of the appropriate holonomy maps from $\mathcal{G}$. This is achieved by finding an element $\gamma_0^{-1} \in \mathcal{G}$ with non-trivial contracting stable manifold, and then applying our elements from $\mathcal{G}$ to some power of γ_0^{-1}. The second, more delicate problem is to control how well the orbit under $\mathcal{G}$ of a given point is "shadowed" by the corresponding orbits under Γ. This latter problem occupies §5, and is where the C^2-assumption on $\mathcal{G}$ is needed. It is doubtful that Theorem 2 holds if we are just given that $\mathcal{G}$ is C^1. Finally, we remark that the proof of Theorem 2 is reminiscent of the proof given in [5] of a special case of Tits' Theorem.

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§2. Growth types and leaf classes

Let $\mathcal{F}$ denote a fixed codimension n, transversally C^2 foliation on a manifold M, L a fixed leaf of $\mathcal{F}$, and h a Riemannian metric on M. Given a basepoint $x \in L$, let $B(x, r) \subset L$ denote the ball of radius r in the submanifold metric on L. The metric h induces a volume element on L, and $\mathrm{vol}\{B(x, r)\}$ will denote the total volume of $B(x, r)$. The growth function of L is $G(x, h, r) = \mathrm{vol}\{B(x, r)\}$.

With respect to the choice of x and h, the growth type of L is said to be:

subexponential if $\limsup_{r\to\infty} \frac{1}{r} \log G(x, h, r) = 0$

nonexponential if $c_L \equiv \liminf_{r\to\infty} \frac{1}{r} \log G(x, h, r) = 0$

exponential if $c_L > 0$.

If M is compact, then the growth type of L is independent of the choices of x and h, [6], [16], and thus is an invariant of the way L is embedded in M.

The growth rate of a finitely generated group H is defined in a similar way (cf. [14]). Let $\{g_1, \ldots, g_s\}$ be a reflexive generating set for H; reflexive means that some g_i is the identity element, and for each i, $g_i^{-1} = g_j$ for some j. The word metric on H is then defined by

$$|g| \leq p \text{ if } g = g_{i_1} \cdots g_{i_p} \quad \text{for some integers } 1 \leq i_1, \ldots, i_p \leq s.$$

Set $H_p = \{g \in H \text{ with } |g| \leq p\}$. Let $\#S$ denote the cardinality of a set S. We say H has subexponential growth if

$$c_H \equiv \limsup_{p \to \infty} \frac{1}{p} \log \# H_p = \liminf_{p \to \infty} \frac{1}{p} \log \# H_p$$

is zero, and exponential growth if $c_H > 0$.

For a countable pseudogroup $\mathcal{G}$ of local diffeomorphisms of $\mathbb{R}^n$, all of which are defined at and fix $0 \in \mathbb{R}^n$, we define the orbit growth type of $\mathcal{G}$ as in Plante [16]. First, assume $\mathcal{G}$ is finitely generated with reflexive generating set $\{\gamma_1, \ldots, \gamma_s\}$. For $\gamma \in \mathcal{G}$ with y in the domain of γ, we say $|\gamma y|_y \leq p$ if there are integers $1 \leq i_1, \ldots, i_p \leq s$ with γ_{i_k} defined at $\gamma_{i_{k-1}} \circ \cdots \circ \gamma_{i_1}(y)$ and $\gamma(y) = \gamma_{i_p} \circ \cdots \circ \gamma_{i_1}(y)$. Then set

$$\text{Orbit}\,(y, \mathcal{G}, p) = \{\gamma y \text{ such that } \gamma \in \mathcal{G} \text{ with } |\gamma y|_y \leq p\}$$

$$c(y, \mathcal{G}) = \liminf_{p \to \infty} \frac{1}{p} \log \# \text{Orbit}\,(y, \mathcal{G}, p).$$

We say $\mathcal{G}$ has exponential orbit growth at y if $c(y, \mathcal{G}) > 0$ and nonexponential otherwise. For a non-finitely generated groupoid $\mathcal{G}$, we say it has exponential orbit growth at y if this is true for some finitely generated subpseudogroup $\mathcal{G}_0 \subset \mathcal{G}$.

Given a regular foliation chart $\phi: U \to \mathbb{R}^m$ with $\phi(x) = 0$ (cf. §4 of [16]), a closed path ξ in L based at x determines a holonomy map $\gamma_\xi : (V, 0) \to (W, 0)$ for some open neighborhoods V and W of $0 \in \mathbb{R}^n$, [3], [6], [16]. Given a finitely generated subgroup $H \subset \pi_1(L, x)$, choose closed paths $\{\xi_1, \ldots, \xi_d\}$ representing a generating set of H, let $\mathcal{G}$ denote the pseudogroup generated by the elements $\{\gamma_{\xi_1}, \ldots, \gamma_{\xi_d}\}$. We extend the generating set to a reflexive set $\{\gamma_{\xi_1}, \ldots, \gamma_{\xi_d}\}$. We extend the generating set to a reflexive set $\{\gamma_{\xi_1}, \ldots, \gamma_{\xi_s}\}$, and let V be an open neighborhood of $0 \in \mathbb{R}^n$ on which all of the γ_{ξ_i} are defined. The following result is then implicit in §4 of [16]; see also Chapter IX of [6]:

PROPOSITION 2.1. *Let $y \in V$ and suppose $\mathcal{G}$ has exponential orbit growth at y. Then for all Riemannian metrics on M, the leaf L' of $\mathcal{F}$ through y has exponential growth.*

It is clear that Theorem 1 follows from Proposition 2.1 and Theorem 2.

Given a foliation chart $\phi: U \to \mathbb{R}^m$ centered at x, the linear holonomy map of L is given by $dh: \pi_1(L, x) \to GL(n, \mathbb{R})$, where for $\mathfrak{a} \in \pi_1(L, x)$ choose a closed path ξ in L representing $\mathfrak{a}$, let γ_ξ denote the holonomy map associated to ξ, then set $dh(\mathfrak{a}) = J_0\gamma_\xi$, the Jacobian matrix at 0. The image $\Gamma = \Gamma_L$ of dh is the linear holonomy group of L with respect to the chart (U, ϕ). For a different choice of foliation chart centered at x, the map dh is changed by conjugating with some element of $GL(n, \mathbb{R})$. Thus the conjugacy class of Γ in $GL(n, \mathbb{R})$ is an invariant of the germ of $\mathcal{F}$ along L.

The leaf classes of L are obtained by considering the pullback via dh of the continuous cohomology of $GL(n, \mathbb{R})$. Recall from Haefliger [4] or Stasheff [18] that the continuous cohomology $H_c^*(G)$ of a topological group G is the cohomology of the cochain complex of real valued group cochains on the discrete group G^δ which are continuous with respect to the topology on G. The basic result is:

THEOREM 2.2 (van Est [4]). *Let G be a Lie group, and let $K \subset G$ be a maximal compact subgroup with G/K contractible. Then there is a natural isomorphism*

$$H^*(\mathfrak{g}, K) \cong H_c^*(G)$$

where $\mathfrak{g}$ is the Lie algebra of G and $H^*(\mathfrak{g}, K)$ is the relative Lie algebra cohomology.

For $G = GL(n, \mathbb{R})$, it is well known that

$$H_c^*(GL(n, \mathbb{R})) \cong H^*(\mathfrak{gl}_n, O_n) \cong \Lambda(y_1, y_3, \ldots, y_{n'}) \tag{2.3}$$

where y_i is a closed O_n-basic form on $\mathfrak{gl}_n$ of degree $2i-1$, and n' is the largest odd integer less than $(n+1)$, (cf. Chapter 5 of [13].) Given an index $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \cdots < i_r \leq n'$ we write $y_I = y_{i_1} \wedge \cdots \wedge y_{i_r}$. The proof of Theorem 3 will depend upon the identification in (2.3) of $H_c^*(GL(n, \mathbb{R}))$, and the naturality in the conclusion of van Est's theorem.

Define the characteristic map χ_L as the composition

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \cong H_c^*(GL(n, \mathbb{R})) \xrightarrow{dh^*} H^*(\pi_1(L, x)) \longrightarrow H^*(L)$$

where we use that $\pi_1(L, x)$ is discrete so that

$$H_c^*(\pi_1(L, x)) \cong H^*(B\pi_1(L, x)) \to H^*(L),$$

where the second map is induced from the natural map $L \to B\pi_1(L, x)$. For a more detailed discussion of the leaf classes, see Kamber–Tondeur [12], Chapter 6 of [13] or Shulman–Tischler [17].

§3. Structure of the linear holonomy group

In this section we analyze how the structure of a countable subgroup $\Gamma \subset GL(n, \mathbb{R})$ is related to the map $H_c^*(GL(N, \mathbb{R})) \to H^*(\Gamma)$. Theorem 3 will follow from this, and we also establish some preliminary results needed for the proof of Theorem 2.

Consider $GL(n, \mathbb{R})$ as the real points of $GL(n, \mathbb{C})$ and let G denote the algebraic closure of Γ in $GL(n, \mathbb{C})$. The identity component G_0 of G has finite index, and passing to the subgroup $\Gamma \cap G_0$ does not affect the statements or conclusions of Theorems 2 and 3. Thus, we can assume G is connected.

Let $G^1 = [G, G]$ be the commutator subgroup of G, and set $G^{k+1} = [G^k, G^k]$. Similarly define $\Gamma^{k+1} = [\Gamma^k, \Gamma^k]$.

LEMMA 3.1. *G^k is closed and connected for all k.*

Proof. See §17.2 of [8], for example. □

We denote the algebraic closure of a group $H \subset GL(n, \mathbb{C})$ by $\bar{H}$.

LEMMA 3.2. *The algebraic closure $\overline{\Gamma^k} = G^k$.*

Proof. The inclusion $\overline{\Gamma^k} \subset G^k$ is immediate, so it suffices to show $G^k \subseteq \overline{\Gamma^k}$. By definition $\bar{\Gamma} = G$, and we proceed by induction: assume $\overline{\Gamma^l} = G^l$ for $l < k$. Consider the commutator map

$$c: GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \to GL(n, \mathbb{C})$$

with $c(g, h) = [g, h]$. This is algebraic, so $H = c^{-1}(\overline{\Gamma^k})$ is algebraically closed. Clearly, $\Gamma^{k-1} \times \Gamma^{k-1} \subseteq H$ so $\overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$. Now $\overline{\Gamma^{k-1} \times \Gamma^{k-1}}$ is a group containing $\overline{\Gamma^{k-1}} \times e$ and $e \times \overline{\Gamma^{k-1}}$, so by induction $G^{k-1} \times G^{k-1} \subset \overline{\Gamma^{k-1}} \times \overline{\Gamma^{k-1}} \subset \overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$. Since G^k is generated as a group by the image $c(G^{k-1} \times G^{k-1})$, we are done. □

As each G^k is connected, there exists a least integer N such that $G^k = G^{k+1}$ for all $k \geq N$. The key to the proof of Theorem 2 is to understand the properties of Γ^N, which we now study.

DEFINITION 3.3 [20]. A topological group H is *amenable* if every continuous affine action of H on a compact convex separable set has a fixed point.

A connected amenable Lie group is a compact extension of a solvable group. For $H \subset GL(n, \mathbb{C})$ amenable, Moore proves in [15] that H is conjugate to a subgroup of one of 2^n standard maximal amenable algebraic subgroups.

DEFINITION 3.4. A subgroup $H \subset GL(n, \mathbb{C})$ is *distal* if for each $g \in H$, all eigenvalues of g have unit length.

PROPOSITION 3.4 (Conze–Guivarc'h [1]). *A distal subgroup of $GL(n, \mathbb{C})$ is amenable.*

For the linear group Γ we now observe:

LEMMA 3.6. *If Γ^k is distal for any $k > 0$, then G is amenable.*

Proof. Suppose that Γ^k is distal. Then Γ^k is amenable, so by Moore [15] its algebraic closure G^k is also amenable. This implies G is amenable, for G is obtained from G^k by a finite number of abelian extensions. □

COROLLARY 3.7. *If Γ is not amenable, then G^N is not trivial, and for all $k > 0$ the group Γ^k is not distal.*

This corollary is the starting point for the proof of Theorem 2 in the next section. We now prove Theorem 3. First, note that the inclusion induced map $H_c^*(GL(n, \mathbb{C})) \to H_c^*(GL(n, \mathbb{R}))$ is onto, since $H^*(\mathfrak{gl}_n\mathbb{C}, U_n) \to H^*(\mathfrak{gl}_n, O_n)$ is onto (e.g., see Chapter 7 of [13]). By the remarks of §2, Theorem 3 then follows from:

PROPOSITION 3.8. *Let $i : \Gamma \to GL(n, \mathbb{C})$ be the inclusion, and suppose that Γ is amenable. Then*

$$i^* : H_c^m(GL(n, \mathbb{C})) \to H^m(\Gamma)$$

is zero for all $m > 1$.

Proof. Let $\Lambda \subset GL(n, \mathbb{C})$ be a maximal amenable subgroup containing Γ. From [15] we know there is a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{C}^n$ and integers $\{n_1, \ldots, n_d\}$ with $n_1 + \cdots + n_d = n$ such that with respect to this basis, Λ has the form:

$$\begin{bmatrix} R^+U_{n_1} & * & \cdot & \cdot & * & * \\ 0 & * & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & R^+U_{n_d} \end{bmatrix}$$

Here, $R^+U_{n_i}$ denotes the positive reals product with the unitary group of dimension n_i. Let $U_n \subset GL(n, \mathbb{C})$ be the unitary subgroup with respect the basis $\{v_1, \ldots, v_n\}$.

The map i^* factors through the map $H_c^*(GL(n, \mathbb{C})) \to H_c^*(\Lambda)$, so it will suffice to show this latter map is trivial in degrees greater than one. Let λ be the Lie algebra of Λ, and let $U \subset \Lambda$ be a maximal compact subgroup with $U = \Lambda \cap U_n = U_{n_1} \times \cdots \times U_{n_d}$. By the van Est Theorem, it suffices to show that for Lie algebra cohomology,

$$j^* : H^m(\mathfrak{gl}_n\mathbb{C}, U_n) \to H^m(\lambda, U)$$

is zero when $m > 1$.

Let $\tilde{\mathfrak{t}}$ be the solvable radical of λ, let $\mathfrak{n}$ be the nilradical and $\tilde{\mathfrak{d}}$ the subspace of the complex diagonal matrices with $\tilde{\mathfrak{t}} = \mathfrak{n} \oplus \tilde{\mathfrak{d}}$. The intersection $\tilde{\mathfrak{t}} \cap \mathfrak{u}$ consists of purely imaginary diagonal matrices, so we consider $\mathfrak{t} = \tilde{\mathfrak{t}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$ as those matrices in $\mathfrak{t}$ with real diagonal entries. Similarly define $\mathfrak{d} = \tilde{\mathfrak{d}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$ so that $\mathfrak{t} = \mathfrak{n} \oplus \mathfrak{d}$. As $\mathfrak{t}$ is normal in λ, if follows from the definition of relative Lie algebra cohomology that

$$H^*(\lambda, U) \cong H^*(\mathfrak{t}^U) \cong H^*(\mathfrak{t})^U,$$

where superscript U means the $Ad(U)$-invariant subspace. The adjoint action of U on $\mathfrak{t}$, λ and $\mathfrak{gl}_n\mathbb{C}$ are all compatible, so we get:

$$\begin{tikzcd}
H^m(\mathfrak{gl}_n\mathbb{C}, U_n) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^m(\lambda, U) \arrow[d, "\cong"] \\
H^m(\mathfrak{gl}_n\mathbb{C})^{U_n} \arrow[r, "r^*"] & H^m(\mathfrak{t})^U
\end{tikzcd}$$

We will show $r^* = 0$ for $m > 1$.

Recall from (p. 116 of [13]) that the generator $y_i \in H^{2i-1}(\mathfrak{gl}_n\mathbb{C})$ is represented by the *ad* $GL(n, \mathbb{C})$-invariant form on $\mathfrak{gl}_n\mathbb{C}$,

$$y_i = k_i \operatorname{tr}(\Theta \wedge \underbrace{[\Theta, \Theta] \wedge \cdots \wedge [\Theta, \Theta]}_{(i-1)\text{-factors}})$$

where Θ is the Maurer–Cartan form and k_i is a scalar. The algebra $\mathfrak{n}$ is an ideal in $\mathfrak{t}$ as an *associative* algebra, and $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}$ so for $i > 1$ the form $r^*(y_i)$ on $\mathfrak{t}$ is obtained by taking the traces of elements of $\mathfrak{n}$, which all have trace zero. Thus, $r^*(y_i) = 0$. As the $\{y_i\}$ generate the algebra $H^*(\mathfrak{gl}_n\mathbb{C})$, we are done. $\square$

As a corollary of the above proof, we have the general fact about Lie algebra cohomology which is useful in other contexts as well.

PROPOSITION 3.9. *Let G be an amenable subgroup of $GL(n,\mathbb{R})$ with Lie algebra $\mathfrak{g}$ and maximal compact subgroup $K=G\cap O_n$. Then $H^m(\mathfrak{gl}_n, O_n)\to H^m(\mathfrak{g}, K)$ is the zero map for all $m>1$. In particular, the restriction of the forms y_i to $\mathfrak{g}$ are exact for all $i>1$ and odd.*

§4. Action of Γ on an attracting subspace

Let $\Gamma\subset GL(n,\mathbb{R})$ be a non-amenable countable subgroup, $G\subset GL(n,\mathbb{C})$ its connected algebraic closure and N the integer defined in §3 for which $G^N=G^{N+1}$. By Corollary 3.7 the group Γ^{N+1} is not distal, so there exists $f\in\Gamma^{N+1}$ with an eigenvalue of modulus greater than one. Let $\mu_1,\dots,\mu_s$ be the eigenvalues of f and set

$$\mu=\max\{|\mu_1|,|\mu_1^{-1}|,\dots,|\mu_s|,|\mu_s^{-1}|\}.$$

By reordering the μ_i and replacing f with f^{-1} if necessary, one can assume

$$\mu=|\mu_1|=\cdots=|\mu_r|>\lambda=|\mu_{r+1}|\geqslant\cdots\geqslant|\mu_s|.$$

Let $\{v(i,j)\mid 1\leqslant i\leqslant s;\ 1\leqslant j\leqslant r(i)\}$ be a basis of $\mathbb{C}^n$ in which f has Jordan form:

$$\begin{aligned} fv(i,1)&=\mu_i\cdot v(i,1)\\ fv(i,j)&=\mu_i[v(i,j)+v(i,j-1)] \quad \text{for} \quad 1\leqslant j\leqslant r(i). \end{aligned} \tag{4.1}$$

We also require that $v(i,1)=\overline{v(j,1)}$ if $\mu_i=\overline{\mu_j}$, where $\overline{}$ denotes complex conjugate. Set

$$V(i)=\operatorname{Span}\{v(i,j)\mid 1\leqslant j\leqslant r(i)\}.$$

Note that $V(i)$ is stable under f, and there is a superdiagonal nilpotent matrix $N(i)$ so that

$$f\mid V(i)=\mu_i[Id+N(i)].$$

Let $V_{\mathbb{C}}=\bigoplus_{i=1}^r V(i)$ and $W_{\mathbb{C}}=\bigoplus_{i=r+1}^s V(i)$ so that $\mathbb{C}^n=V_{\mathbb{C}}\oplus W_{\mathbb{C}}$. Since f is real, both $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are the complexifications of the real subspaces $V=V_{\mathbb{C}}\cap\mathbb{R}^n$ and $W=W_{\mathbb{C}}\cap\mathbb{R}^n$.

Endow $\mathbb{C}^n$ with the Hermitian metric for which the vectors $\{v(i, j)\}$ are orthonormal. Let $|v|$ denote the length of $v \in \mathbb{C}^n$, and for $A \in GL(n, \mathbb{C})$ we set

$$|A| = \sup_{|v|=1} |Av|.$$

Define $\pi : \mathbb{R}^n - \{0\} \to S^{n-1}$ by $\pi(v) = v/|v|$. For a subspace $Z \subset \mathbb{R}^n$, let Z^1 denote the set of unit vectors in Z.

Note that (4.1) implies for all $k > 0$ and $1 \leq i \leq s$,

$$[f \mid V(i)]^k = \mu_i^k\left[Id + \binom{k}{1}N(i) + \cdots + \binom{k}{n}N(i)^n\right] \tag{4.2}$$

where $N(i)^j = 0$ for $j \geq r(i)$. Let $q(k) = \sum_{j=0}^{n-1} \binom{k}{j}$, a polynomial of degree $(n-1)$ in k. Then (4.2) and our choice of metric yields:

LEMMA 4.3

a) *For $v \in V$,*

$$|v|\, \mu^k \leq |f^k(v)| \leq \mu^k q(k)\, |v|.$$

b) *For $w \in W$,*

$$|w|\, |\mu_s|^k \leq |f^k(w)| \leq \lambda^k q(k)\, |w|. \quad \square$$

Define the arctangent function $a : \mathbb{R}^n - W \to R^+$ between V and W by the rule:

For $y \in \mathbb{R}^n$ with $y = v + w$, $v \in V$, $w \in W$, $0 \neq v$,

$$a(y) = \frac{|w|}{|v|}.$$

LEMMA 4.4. *For all $y \in \mathbb{R}^n - W$ and $k > 0$,*

$$\left(\frac{|\mu_s|}{\mu}\right)^k \frac{1}{q(k)} a(y) \leq a(f^k(y)) \leq \left(\frac{\lambda}{\mu}\right)^k q(k) a(y).$$

Proof. For $y = v + w$, $f^k(y) = v_k + w_k$ where $w_k = f^k w \in W$ and $v_k = f^k v \in V$. Thus, $a(f^k(y)) = |w_k|/|v_k|$ and Lemma 4.3 yields the estimate. $\square$

The last result needed for the constructions in §5 asserts that Γ^N contains enough elements to map all of the strong expanding manifold V of f into the domain $\mathbb{R}^n - (V \cup W)$. We remark that if it were possible to find a single $g \in \Gamma$ for which $gV \cap V = \{0\}$ and $gV \cap W = \{0\}$, then a much simplified proof of Theorem 2 would be possible along the lines of [5]. As it is, we make do with the following:

PROPOSITION 4.5. *Let $v \in V$ be a non-zero vector. Then there exists $g \in \Gamma^N$ such that*

$$gv \notin V \text{ and } v \cdot gv \neq 0 \tag{4.6}$$

and hence $gv \notin W$.

Proof. Suppose to the contrary that for all $g \in \Gamma^N$, either $gv \in V$ or $v \cdot gv = 0$. These are algebraic conditions on Γ, so by Lemma 3.2 they also hold for all $g \in G^N$. Now G^N is irreducible as it is a connected algebraic group, so either $gv \in V$ for all $g \in G^N$, or $v \cdot gv = 0$ for all $g \in G^N$. Clearly we must have the first case, so $G^N \cdot v \subset V$. Let $\tilde{V}$ denote the span of $G^N v$. Then $\tilde{V}$ is a subspace of V stable under Γ^N, hence $f \mid \tilde{V}$ is in the commutator group of $\Gamma^N \mid \tilde{V}$. But the determinant of $f \mid \tilde{V}$ is $\mu^{\dim \tilde{V}} > 1$, which contradicts $f \mid \tilde{V}$ being a product of commutators. □

The condition (4.6) is open for $v \in V$, so given any $v \in V^1$ and $g_v \in \Gamma^N$ satisfying (4.6), there is a $\delta(v) > 0$ so that for the *closed* $2\delta(v)$-ball $B(v, 2\delta(v))$ in $\mathbb{R}^n$ centered at v, we have (4.6) is satisfied for g_v and all $y \in B(v, 2\delta(v))$. Since V^1 is compact, we can choose a finite set $\{g_1, \ldots, g_d\} \subset \Gamma^N$ and radii $\{\delta_1, \ldots, \delta_d\}$ so that the balls $B_i = B(v_i, \delta_i) \cap V^1$ cover V^1, and (4.6) is satisfied for each g_i with $y \in B(v_i, 2\delta_i)$. Note this implies that for $1 \leq i \leq d$, the arctangent a is defined and bounded away from zero on the set $g_i B(v_i, 2\delta_i)$.

Finally, replacing f with a positive multiple if necessary, we can assume that $\mu > 3$, and for all $1 \leq i \leq d$ both $\mu > |g_i|$ and $\mu > |g_i^{-1}|$. By our choice of metric on $\mathbb{C}^n$ and (4.1), we also have both $|f| < 2\mu$ and $|f^{-1}| < 2\mu$.

§5. Exponential growth on the expanding manifold.

Let $\mathcal{G}$ be the groupoid given in Theorem 2 and Γ the linear group of Jacobians at 0. Assume that Γ is not amenable. Let $f \in \Gamma^{N+1}$ and $\{g_1, \ldots, g_d\} \subset \Gamma^N$ be chosen as in §4. Choose $\gamma \in \mathcal{G}$ with $J_0\gamma = f$, and for each $1 \leq i \leq d$ choose $\gamma_i \in \mathcal{G}$ with $J_0\gamma_i = g_i$. For notational convenience, set $\gamma_0 = \gamma$. Let $D \subset \mathbb{R}^n$ be an open neighborhood of 0 on which all of the γ_i are defined. Let $\mathcal{G}_0$ denote the subgroupoid of $\mathcal{G}$ generated by the set $\{\gamma_0, \ldots, \gamma_d\}$. We will show $\mathcal{G}_0$ has a continuum of orbits with exponential growth.

By the stable manifold theorem (cf. [11]) applied to γ^{-1}, there is a connected submanifold $S \subset D$ with $0 \in S$, the tangent space T_0S at 0 is equal V, and γ^{-1} is uniformly contracting on S. In particular, $\gamma^{-1}S \subset S$. By a change of coordinates on $\mathbb{R}^n$, we can assume S is an open neighborhood of 0 in V.

Before entering into the details of the proof of Theorem 2, a brief overview of the argument may help the reader. We first define an open cone $C \subset S$ whose points satisfy $\lim_{k\to\infty} \pi(\gamma^{-k}y) \in V^1$ and $|\gamma^{-k}y| < \mu^{-k/2}$. For an appropriate constant e_0, we set $y_p = \gamma^{-pe_0}y$ for a given $y \in C$. For each $p > 0$ we construct a subset $\mathcal{R}_p \subset \mathcal{G}_0$ consisting of 2^p words of length $\leq m_0 \cdot p$, such that the linear parts of the words in $\mathcal{R}_p$ move y_p to 2^p distinct points. We furthermore obtain an exponentially decreasing lower bound on the distance between these 2^p points. Using Taylor's theorem for C^2-maps, and for e_0 sufficiently large so that y_p is sufficiently small, we conclude that $\mathcal{R}_p \cdot y_p$ consists of 2^p distinct points. The last remark is that in constructing $\mathcal{R}_p$, we use a version of the "ping-pong" lemma of [5]. In our version, the orbits are repeatedly returned to the attractor V by applying high powers of f, and are then scattered back into $\mathbb{R}^n - (V \cup W)$ by the elements of $\{g_1, \ldots, g_d\}$. Thus, all of the orbits we build concentrate on the subspace V, and one does not have the bilateral symmetry inherent in the method of Tits. Instead of 2 players, one can think of this as an instructor with many students.

Recall that for a C^2-diffeomorphism ϕ with $\phi(0) = 0$, Taylor's Theorem gives an estimate on the spherical error between ϕ and $J_0\phi$, and the estimate is linear in y:

For all $\epsilon > 0$ sufficiently small, there exists $k(\phi, \varepsilon) > 0$ so that

$$\frac{|\phi y - J_0\phi y|}{|y|} < k(\phi, \epsilon) \cdot |y| \quad \text{for all } |y| < \epsilon. \tag{5.1}$$

As an immediate consequence we have:

LEMMA 5.2. *Let $\mathcal{R} = \{\phi_1, \ldots, \phi_p\}$ be a set of local C^2-diffeomorphisms of an open neighborhood U of $0 \in \mathbb{R}^n$ into $\mathbb{R}^n$ with $\phi_i(0) = 0$ for all i. Let $\epsilon > 0$ be sufficiently small so that there exists constants $k(\phi_i, \epsilon)$ for which (5.1) holds. Then for $K = \max_{i \leq i \leq p} k(\phi_i, \epsilon)$ and $y \in U$ with $|y| < \epsilon$, suppose that*

$$|J_0\phi_i y - J_0\phi_j y| > 2 \cdot K \cdot |y|^2 \quad \textit{for all} \quad i \neq j.$$

Then the set $\mathcal{R} \cdot y = \{\phi_i y \mid 1 \leq i \leq p\}$ consist of p distinct points. □

LEMMA 5.3. *There exists $\delta > 0$ and an integer $b > 0$ such that $|\gamma^{-b}y| < \mu^{-b/2}|y|$ for all $y \in S$ with $|y| < \delta$.*

Proof. By Lemma 4.3 there exists an integer $b>0$ for which $|f^{-b} \mid V|<\mu^{-3b/4}$. Choose $\delta>0$ sufficiently small so that

$$\delta \cdot k(\epsilon, \gamma^{-b})<\{\mu^{-b/2}-\mu^{-3b/4}\}$$

where ϵ is such that (5.1) holds for γ^{-b}, and $\delta<\epsilon$. Then

$$\begin{aligned}|\gamma^{-b}y| &\leq |\gamma^{-b}y-f^{-b}y|+|f^{-b}y|\\ &\leq |y|^2 \cdot k(\epsilon, \gamma^{-b})+\mu^{-3b/4}|y|\\ &\leq \mu^{-b/2}|y|. \quad \square\end{aligned}$$

For b, δ as in (5.3) we replace f, γ and μ with f^b, γ^b and μ^b, so we can assume:

$$|\gamma^{-p}y|<\mu^{-p/2}|y| \quad \text{for all} \quad p>0,\ y\in S,\ |y|<\delta \tag{5.4}$$

Choose $\epsilon>0$ to satisfy $\epsilon<\delta$, $\epsilon<\mu^{-1}$ and there exists a constant K_0 so that for all $\phi\in\{\gamma, \gamma^{-1}, \gamma_1, \ldots, \gamma_d\}$, condition (5.1) holds for all $|y|<\epsilon$ and $k(\phi, \epsilon)=K_0$. Then set

$$C=\{y\in S \mid 0<|y|<\epsilon\}$$

These remarks are then summarized by

COROLLARY 5.5. *$\gamma^{-1}C\subset C$, and for all $p>0$ and $y\in C$,*

$$|\gamma^{-p}y|<\mu^{-p/2}\cdot\epsilon. \quad \square$$

Set $K=\max\{K_0, 2\mu\}$ and $\epsilon_p=\min\{\epsilon, K^{-p}\}$. For a word $\phi=\phi_1\circ\cdots\circ\phi_p$ of length $\leq p$ with each $\phi_i\in\{\gamma_0, \ldots, \gamma_d\}$, we estimate the constant $k(\phi, \epsilon_p)$ required for (5.1):

LEMMA 5.6. *For ϕ, K and ϵ_p as above*

$$|\phi y-J_0\phi y|<K^{2p}|y|^2 \quad \textit{for} \quad |y|<\epsilon_p \tag{5.7}$$

Thus, $K(\phi, \epsilon_p)\leq K^{2p}$.

Proof. For $p=1$, (5.7) follows from the definition of K. Assume (5.7) holds for

ϕ of length $(p-1)$, and set $\tilde{\phi}_2=\phi_2\circ\cdots\circ\phi_p$. Then

$$\begin{aligned}|\phi y-J_0\phi y| &= |\phi_1\circ\tilde{\phi}_2 y-J_0\phi_1\circ J_0\tilde{\phi}_2 y|\\ &\leqslant |y|^2\cdot\{|J_0\phi_1|\cdot K^{2p-2}+|J_0\tilde{\phi}_2|^2\cdot K\\ &\quad +2\,|J_0\tilde{\phi}_2|\,|y|\,K^{2p-1}+|y|^2\,K^{4p-3}\}.\end{aligned}$$

From $|J_0\phi_1|\leqslant K$, $|J_0\tilde{\phi}_2|<K^{p-1}$ and $|y|<K^{-p}$ we conclude

$$\begin{aligned}|\phi y-J_0\phi y| &< |y|^2\{K^{2p-1}+K^{2p-1}+2K^{2p-2}+K^{2p-3}\}\\ &= |y|^2\cdot K^{2p-1}\left\{2+\frac{2}{K^2}+\frac{1}{K^3}\right\}\\ &\leqslant |y|^2\cdot K^{2p}\end{aligned}$$

since $K>\mu>3$. □

LEMMA 5.8. *For* $g\in\{f, f^{-1}, g_1,\ldots, g_d\}$ *and all* $u_1, u_2\in\mathbb{R}^n$, $|gu_1|>\dfrac{1}{2\mu}|u_1|$ *and* $|gu_1-gu_2|>\dfrac{1}{2\mu}|u_1-u_2|$.

Proof. $|g|<2\mu$, so $|gw|<2\mu\cdot|w|$ and hence for $w=g^{-1}u_1$ or $w=g^{-1}(u_1-u_2)$ we get the estimate. □

Recall that $\{B_i=B(v_i,\delta_i)\mid 1\leqslant i\leqslant d\}$ is the covering of V^1 by closed balls in V defined at the end of §4. By compactness of the sets $g_iB_i(v_i, 2\delta_i)$ and the continuity of the arctangent function a on them, there exists constants $0<c_1<c_2$ for which $c_1<a(g_iy)<c_2$ for all $1\leqslant i\leqslant d$ and $y\in B(v_i, 2\delta_i)$.

Set $X=\{x\in\mathbb{R}^n\mid |x|=1$ and $c_1\leqslant a(x)\leqslant c_2\}$.

For $\delta>0$, set

$$\begin{aligned}A(\delta) &= \{x\in\mathbb{R}^n\mid |x|=1\quad\text{and}\quad a(x)<\delta\}\\ A_i(\delta) &= \{x\in A(\delta)\mid x=v+w,\ v\in B_i,\ w\in W\}\end{aligned}$$

Note the sets $\{A_1(\delta),\ldots, A_d(\delta)\}$ cover $A(\delta)$. Choose $\delta_0>0$ sufficiently small so that for all $1\leqslant i\leqslant d$, $g_iA_i(2\delta_0)\subset X$. Lemma 4.4 implies there exists an integer e for which $f^p(X)\subset A(\delta_0)$ for all $p\geqslant e$. Set $m_0=2d\cdot e+1$, and define

$$c_1=\underset{\substack{y,z\in X\\ 1\leqslant i<j\leqslant 2d}}{\text{infimum}}\ |\pi f^{i\cdot e}y-\pi f^{j\cdot e}z|.$$

Choose $e_0 > 1$ so that for all $p \geqslant 1$,

$$\mu^{(2m_0 - e_0/2)p} < \frac{c_1}{2K^{2p} \cdot \epsilon \cdot 2^{2 \cdot p \cdot m_0}} \tag{5.9}$$

and

$$\mu^{e_0} > K^4. \tag{5.10}$$

For all non-zero $y \in C$ we now show the groupoid $\mathcal{R}_0$ has exponential orbit growth on y. Fix a choice of $0 \neq y \in C$. For $p > 0$ set $y_p = \gamma^{-p \cdot e_0} y$. By Lemma 5.5 and (5.10) we have $|y_p| < K^{-2p}\epsilon < \epsilon_p$, and then (5.9) yields

$$\begin{aligned} 2\,|y_p|^2\, K^{2p} &\leqslant 2K^{2p} \cdot |y_p| \cdot \mu^{-p \cdot e_0/2} \cdot \epsilon \\ &\leqslant \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \end{aligned}$$

We can now define the set $\mathcal{R}_p$, which consists of 2^p words of length $\leqslant p \cdot m_0$ in $\mathcal{R}_0$. The set $\mathcal{R}_p$ will be chosen so that for all $\phi \neq \psi \in \mathcal{R}_p$,

$$|J_0 \phi y_p - J_0 \psi y_p| > \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \tag{5.12}$$

By Lemma 5.2 and (5.11), the set $\mathcal{R}_p y_p = \{\phi y_p \mid \phi \in \mathcal{R}_p\}$ consists of 2^p distinct points. Thus, $\mathcal{R}_p \cdot \gamma^{-pe_0}$ consists of words of length $\leqslant (m_0 + e_0)p$, and applied to y yields 2^p distinct orbits. Since $y_p \to 0$, this will finish the proof of Theorem 2.

Fix p, choose i_0 with $\pi y_p \in B(i_0)$, and consider the $2d$ points

$$F_1 = \{\pi f^{e \cdot k} g_{i_0} y_p \mid 1 \leqslant k \leqslant 2d\} \subset A(\delta_0).$$

There exists an integer i_1 with $1 \leqslant i_1 \leqslant d$ for which $Q_1 = F_1 \cap A_{i_1}$ contains at least 2 points.

Now proceed inductively, and suppose i_{q-1}, F_{q-1} and Q_{q-1} have been chosen with $Q_{q-1} = F_{q-1} \cap A_{i_{q-1}}$ and $\# Q_{q-1} \geqslant 2^{q-1}$. The set

$$F_q = \{\pi f^{e \cdot k} g_{i_{q-1}} Q_{q-1} \mid 1 \leqslant k \leqslant 2d\} \subset A(\delta_0)$$

consists of at least $2d \cdot 2^{q-1}$ points, since

$$g_{i_{q-1}} A_{i_{q-1}} \subset X \quad \text{and} \quad f^{e \cdot k}(X) \cap f^{e \cdot j}(X) = \emptyset \quad \text{for} \quad j \neq k.$$

Therefore, there exists i_q with $Q_q = F_q \cap A_{i_q}$ containing at least 2^q points. This completes the inductive step.

Let F_p be the set obtained in this inductive fashion; let $\mathcal{R}_p$ be the set of words in $\{\gamma_0, \ldots, \gamma_d\}$ corresponding to the words in $\{f, g_1, \ldots, g_d\}$ which are applied to y_p to obtain the points in F_p. A typical element of $\mathcal{R}_p$ has the form

$$\phi = \gamma^{e \cdot k_p} \circ \gamma_{i_{p-1}} \circ \gamma^{e \cdot k_{p-1}} \circ \gamma_{i_{p-2}} \circ \cdots \circ \gamma_{i_0}$$

for some integers $1 \le k_1, \ldots, k_p \le 2d$. The length of ϕ is at most $p \cdot m_0$ with respect to the set $\{\gamma_0, \ldots, \gamma_p\}$, and $\mathcal{R}_p y_p$ consists of at least $2d \cdot 2^{p-1} \ge 2^p$ points, once we have established the estimate (5.12).

Let $\phi \ne \psi \in \mathcal{R}_p$ and let $g = J_0\phi$, $h = J_0\psi$ be their linear parts. There are integers $1 \le k_1, \ldots, k_p \le 2d$ and $1 \le j_1, \ldots, j_p \le 2d$ for which

$$g = f^{e \cdot j_p} g_{i_{p-1}} \cdots f^{e \cdot j_1} g_{i_0}$$
$$h = f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_1} g_{i_0}$$

Let q be the largest integer such that $j_{q-1} \ne k_{q-1}$. Set

$$\xi = f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_q} g_{i_{q-1}}$$
$$g' = \xi^{-1} g,\ h' = \xi^{-1} h$$

Apply Lemma 5.8 at most $q \cdot m_0$ times to obtain

$$\begin{aligned} |gy_p - hy_p| &= |\xi(g'y_p - h'y_p)| \\ &\ge (2\mu)^{-qm_0} |g'y_p - h'y_p|. \end{aligned}$$

Next, g' and h' have length $\le pm_0$, so Lemma 5.8 again yields

$$\min\{|g'y_p|, |h'y_p|\} \ge (2\mu)^{-pm_0} |y_p|.$$

Hence,

$$|g'y_p - h'y_p| \ge (2\mu)^{-pm_0} |y_p| \cdot |\pi g'y_p - \pi h'y_p| \ge (2\mu)^{-pm_0} \cdot |y_p| \cdot c_1$$

and so

$$|gy_p - hy_p| \ge (2\mu)^{-2pm_0} |y_p| \cdot c_1. \quad \square$$

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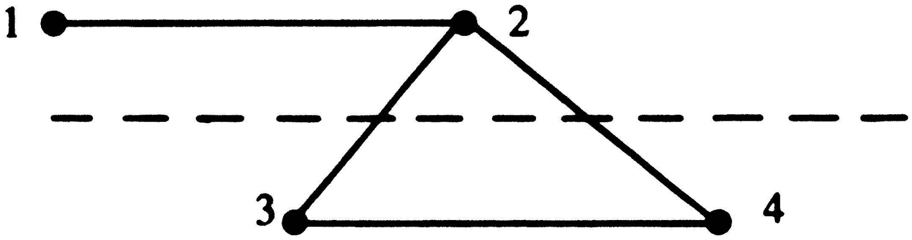

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