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## Foliation dynamics and leaf invariants

STEVEN HURDER<sup>(1)</sup>

### §1. Statement of results

Let  $\mathcal{F}$  be a codimension- $n$  foliation of a smooth manifold  $M$  without boundary.  $M$  may be either compact or open, and assume  $\mathcal{F}$  is transversally  $C^2$ . The purpose of this note is to examine the relation between the linear holonomy of the leaves of  $\mathcal{F}$  and the growth rates of the leaves.

**THEOREM 1.** *Let  $\mathcal{F}$  and  $M$  be as above. Given a leaf  $L \subset M$  of  $\mathcal{F}$ , suppose its linear holonomy group  $\Gamma_L \subset GL(n, \mathbb{R})$  is not amenable. Then  $\mathcal{F}$  has a leaf  $L'$  which contains  $L$  in its closure, and for all Riemannian metrics on  $M$ ,  $L'$  has exponential growth.*

Amenability is taken in the sense of topological groups, where  $\Gamma_L$  is endowed with the topology from  $GL(n, \mathbb{R})$ .

We actually prove a slightly more general result, from which Theorem 1 follows by standard methods.

**THEOREM 2.** *Let  $\mathcal{G}$  be a pseudogroup of local diffeomorphisms of  $\mathbb{R}^n$ , all of whose elements are defined at and fix the origin  $0 \in \mathbb{R}^n$ , and are  $C^2$  in a neighborhood of  $0$ . Let  $\Gamma$  denote the linear group of Jacobians at  $0$  of the elements of  $\mathcal{G}$ . If  $\Gamma$  is not amenable, then the action of  $\mathcal{G}$  on  $\mathbb{R}^n$  has an orbit with exponential growth and which contains  $0$  in its closure.*

The normal bundle to  $\mathcal{F}$  is denoted by  $Q$ . The restriction of  $Q$  to a leaf  $L$  is well-known to be a flat  $\mathbb{R}^n$ -vector bundle, to which there are associated characteristic classes [12] obtained from the relative Lie algebra cohomology of  $(\mathfrak{gl}_n, O_n)$ . They are given by a map

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \rightarrow H^*(L).$$

The leaf classes of  $L$  consist of the image of  $\chi_L$ .

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**THEOREM 3.** *Let  $\mathcal{F}$  be a foliation of  $M$  as above. Suppose there exists  $y \in H^m(\mathfrak{gl}_n, O_n)$  with  $m > 1$  and  $\chi_L(y) \neq 0$ . Then the linear holonomy group  $\Gamma_L$  of  $L$  is not amenable.*

**COROLLARY 4.** *Let  $\mathcal{F}$  and  $M$  be as above. Suppose that all leaves of  $\mathcal{F}$  have non-exponential growth. Then for every leaf  $L$  of  $\mathcal{F}$ , the linear holonomy group  $\Gamma_L$  is amenable, and all leaf classes of  $L$  in degrees greater than one are zero.*

The hypothesis  $m > 1$  is necessary. For example, a flow on  $M$  with a linearly attracting closed orbit  $L$  has  $\chi_L(y_1) \neq 0$ , where  $y_1$  is the standard generator of  $H^1(\mathfrak{gl}_n, O_n)$ . All orbits of the flow have at most linear growth, hence non-exponential, and the holonomy group of  $L$  is  $\mathbb{Z}$ , which is amenable.

Corollary 4 can be viewed as a generalization to all of the characteristic classes for flat bundles of a result due to Hirsch and Thurston. The Main Theorem of [7] implies that the Euler class of the restriction  $Q|_L \rightarrow L$  is zero if the foliated normal sphere bundle to  $L$  has an invariant transverse measure. This will be the case, for example, when  $\mathcal{F}$  has a leaf  $L'$  of non-exponential growth with  $L$  contained in the closure of  $L'$ .

Theorem 1 is complementary to a result of Zimmer (Theorem 5.5 of [20]; see also Corollary 4.3 of [10]): If  $\mathcal{F}$  is amenable, then there exists a measurable framing  $s$  of  $Q \rightarrow M$  such that for almost every leaf  $L$ , there is a closed amenable subgroup  $G_L \subset GL(n, \mathbb{R})$  for which the linear holonomy along  $L$ , with respect to  $s$ , takes values in  $G_L$ . For example,  $\mathcal{F}$  will be amenable if almost every leaf has subexponential growth.

Note that the set of leaves of  $\mathcal{F}$  with non-trivial linear holonomy has measure zero (Lemma 7.2 of [10]), so Zimmer's theorem does not imply our Theorem 1. With the stronger hypothesis that every leaf of  $\mathcal{F}$  has nonexponential growth, Theorem 1 implies that for every leaf  $L$ , there exists a framing  $s_L$  of  $Q|_L \rightarrow L$  for which the linear holonomy along  $L$ , with respect to  $s_L$ , takes values in an amenable subgroup  $G_L$ . It is an open problem to find sufficient conditions on the dynamics of  $\mathcal{F}$  that imply  $Q \rightarrow M$  has a measurable framing  $s$ , with respect to which every leaf has amenable linear holonomy.

This work arose out of the study [9], and was motivated by an attempt to generalize to all codimensions the results of Duminy [2] relating the Godbillon-Vey class in codimension-one with leaf dynamics. For a further discussion, see [10].

We now give an idea of the proofs. Theorem 3 is based on the observation that the well-known explicit Lie algebra forms, representing the generators of  $H^*(\mathfrak{gl}_n, O_n)$ , are exact when restricted to the Lie algebra of a maximal amenable subgroup of  $GL(n, \mathbb{R})$ . This is proven in §3. The heart of this paper is the proof of

Theorem 2. It is useful to compare Theorem 2 with Tits' Theorem [19]: a non-amenable linear group  $\Gamma$  contains a free non-abelian subgroup on two generators. From this it is easy to see that the linear action of  $\Gamma$  on  $\mathbb{R}^n$  has orbits of exponential growth. Two problems arise when one tries to use this to show the pseudogroup  $\mathcal{G}$  has orbits of exponential growth. First, control must be maintained over the domains of the appropriate holonomy maps from  $\mathcal{G}$ . This is achieved by finding an element  $\gamma_0^{-1} \in \mathcal{G}$  with non-trivial contracting stable manifold, and then applying our elements from  $\mathcal{G}$  to some power of  $\gamma_0^{-1}$ . The second, more delicate problem is to control how well the orbit under  $\mathcal{G}$  of a given point is "shadowed" by the corresponding orbits under  $\Gamma$ . This latter problem occupies §5, and is where the  $C^2$ -assumption on  $\mathcal{G}$  is needed. It is doubtful that Theorem 2 holds if we are just given that  $\mathcal{G}$  is  $C^1$ . Finally, we remark that the proof of Theorem 2 is reminiscent of the proof given in [5] of a special case of Tits' Theorem.

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**§2. Growth types and leaf classes**

Let  $\mathcal{F}$  denote a fixed codimension  $n$ , transversally  $C^2$  foliation on a manifold  $M$ ,  $L$  a fixed leaf of  $\mathcal{F}$ , and  $h$  a Riemannian metric on  $M$ . Given a basepoint  $x \in L$ , let  $B(x, r) \subset L$  denote the ball of radius  $r$  in the submanifold metric on  $L$ . The metric  $h$  induces a volume element on  $L$ , and  $\text{vol}\{B(x, r)\}$  will denote the total volume of  $B(x, r)$ . The growth function of  $L$  is  $G(x, h, r) = \text{vol}\{B(x, r)\}$ .

With respect to the choice of  $x$  and  $h$ , the growth type of  $L$  is said to be:

subexponential if  $\limsup_{r \rightarrow \infty} \frac{1}{r} \log G(x, h, r) = 0$

nonexponential if  $c_L \equiv \liminf_{r \rightarrow \infty} \frac{1}{r} \log G(x, h, r) = 0$

exponential if  $c_L > 0$ .

If  $M$  is compact, then the growth type of  $L$  is independent of the choices of  $x$  and  $h$ , [6], [16], and thus is an invariant of the way  $L$  is embedded in  $M$ .

The growth rate of a finitely generated group  $H$  is defined in a similar way (cf. [14]). Let  $\{g_1, \dots, g_s\}$  be a reflexive generating set for  $H$ ; reflexive means that some  $g_i$  is the identity element, and for each  $i$ ,  $g_i^{-1} = g_j$  for some  $j$ . The word metric on  $H$  is then defined by

$$|g| \leq p \text{ if } g = g_{i_1} \cdots g_{i_p} \text{ for some integers } 1 \leq i_1, \dots, i_p \leq s.$$

Set  $H_p = \{g \in H \text{ with } |g| \leq p\}$ . Let  $\#S$  denote the cardinality of a set  $S$ . We say  $H$  has subexponential growth if

$$c_H = \limsup_{p \rightarrow \infty} \frac{1}{p} \log \# H_p = \liminf_{p \rightarrow \infty} \frac{1}{p} \log \# H_p$$

is zero, and exponential growth if  $c_H > 0$ .

For a countable pseudogroup  $\mathcal{G}$  of local diffeomorphisms of  $\mathbb{R}^n$ , all of which are defined at and fix  $0 \in \mathbb{R}^n$ , we define the orbit growth type of  $\mathcal{G}$  as in Plante [16]. First, assume  $\mathcal{G}$  is finitely generated with reflexive generating set  $\{\gamma_1, \dots, \gamma_s\}$ . For  $\gamma \in \mathcal{G}$  with  $y$  in the domain of  $\gamma$ , we say  $|\gamma y|_y \leq p$  if there are integers  $1 \leq i_1, \dots, i_p \leq s$  with  $\gamma_{i_k}$  defined at  $\gamma_{i_{k-1}} \circ \dots \circ \gamma_{i_1}(y)$  and  $\gamma(y) = \gamma_{i_p} \circ \dots \circ \gamma_{i_1}(y)$ . Then set

$$\text{Orbit}(y, \mathcal{G}, p) = \{\gamma y \text{ such that } \gamma \in \mathcal{G} \text{ with } |\gamma y|_y \leq p\}$$

$$c(y, \mathcal{G}) = \liminf_{p \rightarrow \infty} \frac{1}{p} \log \# \text{Orbit}(y, \mathcal{G}, p).$$

We say  $\mathcal{G}$  has exponential orbit growth at  $y$  if  $c(y, \mathcal{G}) > 0$  and nonexponential otherwise. For a non-finitely generated groupoid  $\mathcal{G}$ , we say it has exponential orbit growth at  $y$  if this is true for some finitely generated subpseudogroup  $\mathcal{G}_0 \subset \mathcal{G}$ .

Given a regular foliation chart  $\phi: U \rightarrow \mathbb{R}^m$  with  $\phi(x) = 0$  (cf. §4 of [16]), a closed path  $\xi$  in  $L$  based at  $x$  determines a holonomy map  $\gamma_\xi: (V, 0) \rightarrow (W, 0)$  for some open neighborhoods  $V$  and  $W$  of  $0 \in \mathbb{R}^n$ , [3], [6], [16]. Given a finitely generated subgroup  $H \subset \pi_1(L, x)$ , choose closed paths  $\{\xi_1, \dots, \xi_d\}$  representing a generating set of  $H$ , let  $\mathcal{G}$  denote the pseudogroup generated by the elements  $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$ . We extend the generating set to a reflexive set  $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$ . We extend the generating set to a reflexive set  $\{\gamma_{\xi_1}, \dots, \gamma_{\xi_d}\}$ , and let  $V$  be an open neighborhood of  $0 \in \mathbb{R}^n$  on which all of the  $\gamma_{\xi_i}$  are defined. The following result is then implicit in §4 of [16]; see also Chapter IX of [6]:

**PROPOSITION 2.1.** *Let  $y \in V$  and suppose  $\mathcal{G}$  has exponential orbit growth at  $y$ . Then for all Riemannian metrics on  $M$ , the leaf  $L'$  of  $\mathcal{F}$  through  $y$  has exponential growth.*

It is clear that Theorem 1 follows from Proposition 2.1 and Theorem 2.

Given a foliation chart  $\phi: U \rightarrow \mathbb{R}^m$  centered at  $x$ , the linear holonomy map of  $L$  is given by  $dh: \pi_1(L, x) \rightarrow GL(n, \mathbb{R})$ , where for  $\alpha \in \pi_1(L, x)$  choose a closed path  $\xi$  in  $L$  representing  $\alpha$ , let  $\gamma_\xi$  denote the holonomy map associated to  $\xi$ , then set  $dh(\alpha) = J_0\gamma_\xi$ , the Jacobian matrix at 0. The image  $\Gamma = \Gamma_L$  of  $dh$  is the linear holonomy group of  $L$  with respect to the chart  $(U, \phi)$ . For a different choice of foliation chart centered at  $x$ , the map  $dh$  is changed by conjugating with some element of  $GL(n, \mathbb{R})$ . Thus the conjugacy class of  $\Gamma$  in  $GL(n, \mathbb{R})$  is an invariant of the germ of  $\mathcal{F}$  along  $L$ .

The leaf classes of  $L$  are obtained by considering the pullback via  $dh$  of the continuous cohomology of  $GL(n, \mathbb{R})$ . Recall from Haefliger [4] or Stasheff [18] that the continuous cohomology  $H_c^*(G)$  of a topological group  $G$  is the cohomology of the cochain complex of real valued group cochains on the discrete group  $G^\delta$  which are continuous with respect to the topology on  $G$ . The basic result is:

**THEOREM 2.2** (van Est [4]). *Let  $G$  be a Lie group, and let  $K \subset G$  be a maximal compact subgroup with  $G/K$  contractible. Then there is a natural isomorphism*

$$H^*(\mathfrak{g}, K) \cong H_c^*(G)$$

where  $\mathfrak{g}$  is the Lie algebra of  $G$  and  $H^*(\mathfrak{g}, K)$  is the relative Lie algebra cohomology.

For  $G = GL(n, \mathbb{R})$ , it is well known that

$$H_c^*(GL(n, \mathbb{R})) \cong H^*(\mathfrak{gl}_n, O_n) \cong \Lambda(y_1, y_3, \dots, y_{n'}) \tag{2.3}$$

where  $y_i$  is a closed  $O_n$ -basic form on  $\mathfrak{gl}_n$  of degree  $2i - 1$ , and  $n'$  is the largest odd integer less than  $(n + 1)$ , (cf. Chapter 5 of [13].) Given an index  $I = (i_1, \dots, i_r)$  with  $1 \leq i_1 < \dots < i_r \leq n'$  we write  $y_I = y_{i_1} \wedge \dots \wedge y_{i_r}$ . The proof of Theorem 3 will depend upon the identification in (2.3) of  $H_c^*(GL(n, \mathbb{R}))$ , and the naturality in the conclusion of van Est's theorem.

Define the characteristic map  $\chi_L$  as the composition

$$\chi_L : H^*(\mathfrak{gl}_n, O_n) \cong H_c^*(GL(n, \mathbb{R})) \xrightarrow{dh^*} H^*(\pi_1(L, x)) \longrightarrow H^*(L)$$

where we use that  $\pi_1(L, x)$  is discrete so that

$$H_c^*(\pi_1(L, x)) \cong H^*(B\pi_1(L, x)) \rightarrow H^*(L),$$

where the second map is induced from the natural map  $L \rightarrow B\pi_1(L, x)$ . For a more detailed discussion of the leaf classes, see Kamber–Tondeur [12], Chapter 6 of [13] or Shulman–Tischler [17].

### §3. Structure of the linear holonomy group

In this section we analyze how the structure of a countable subgroup  $\Gamma \subset GL(n, \mathbb{R})$  is related to the map  $H_c^*(GL(N, \mathbb{R})) \rightarrow H^*(\Gamma)$ . Theorem 3 will follow from this, and we also establish some preliminary results needed for the proof of Theorem 2.

Consider  $GL(n, \mathbb{R})$  as the real points of  $GL(n, \mathbb{C})$  and let  $G$  denote the algebraic closure of  $\Gamma$  in  $GL(n, \mathbb{C})$ . The identity component  $G_0$  of  $G$  has finite index, and passing to the subgroup  $\Gamma \cap G_0$  does not affect the statements or conclusions of Theorems 2 and 3. Thus, we can assume  $G$  is connected.

Let  $G^1 = [G, G]$  be the commutator subgroup of  $G$ , and set  $G^{k+1} = [G^k, G^k]$ . Similarly define  $\Gamma^{k+1} = [\Gamma^k, \Gamma^k]$ .

LEMMA 3.1.  $G^k$  is closed and connected for all  $k$ .

*Proof.* See §17.2 of [8], for example.  $\square$

We denote the algebraic closure of a group  $H \subset GL(n, \mathbb{C})$  by  $\bar{H}$ .

LEMMA 3.2. The algebraic closure  $\bar{\Gamma^k} = G^k$ .

*Proof.* The inclusion  $\bar{\Gamma^k} \subset G^k$  is immediate, so it suffices to show  $G^k \subseteq \bar{\Gamma^k}$ . By definition  $\bar{\Gamma} = G$ , and we proceed by induction: assume  $\bar{\Gamma^l} = G^l$  for  $l < k$ . Consider the commutator map

$$c: GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$$

with  $c(g, h) = [g, h]$ . This is algebraic, so  $H = c^{-1}(\bar{\Gamma^k})$  is algebraically closed. Clearly,  $\Gamma^{k-1} \times \Gamma^{k-1} \subseteq H$  so  $\overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$ . Now  $\overline{\Gamma^{k-1} \times \Gamma^{k-1}}$  is a group containing  $\overline{\Gamma^{k-1}} \times e$  and  $e \times \overline{\Gamma^{k-1}}$ , so by induction  $G^{k-1} \times G^{k-1} \subset \overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset \overline{\Gamma^{k-1} \times \Gamma^{k-1}} \subset H$ . Since  $G^k$  is generated as a group by the image  $c(G^{k-1} \times G^{k-1})$ , we are done.  $\square$

As each  $G^k$  is connected, there exists a least integer  $N$  such that  $G^k = G^{k+1}$  for all  $k \geq N$ . The key to the proof of Theorem 2 is to understand the properties of  $\Gamma^N$ , which we now study.

DEFINITION 3.3 [20]. A topological group  $H$  is *amenable* if every continuous affine action of  $H$  on a compact convex separable set has a fixed point.

A connected amenable Lie group is a compact extension of a solvable group. For  $H \subset GL(n, \mathbb{C})$  amenable, Moore proves in [15] that  $H$  is conjugate to a subgroup of one of  $2^n$  standard maximal amenable algebraic subgroups.

**DEFINITION 3.4.** A subgroup  $H \subset GL(n, \mathbb{C})$  is *distal* if for each  $g \in H$ , all eigenvalues of  $g$  have unit length.

**PROPOSITION 3.4** (Conze–Guivarc’h [1]). *A distal subgroup of  $GL(n, \mathbb{C})$  is amenable.*

For the linear group  $\Gamma$  we now observe:

**LEMMA 3.6.** *If  $\Gamma^k$  is distal for any  $k > 0$ , then  $G$  is amenable.*

*Proof.* Suppose that  $\Gamma^k$  is distal. Then  $\Gamma^k$  is amenable, so by Moore [15] its algebraic closure  $G^k$  is also amenable. This implies  $G$  is amenable, for  $G$  is obtained from  $G^k$  by a finite number of abelian extensions.  $\square$

**COROLLARY 3.7.** *If  $\Gamma$  is not amenable, then  $G^N$  is not trivial, and for all  $k > 0$  the group  $\Gamma^k$  is not distal.*

This corollary is the starting point for the proof of Theorem 2 in the next section. We now prove Theorem 3. First, note that the inclusion induced map  $H_c^*(GL(n, \mathbb{C})) \rightarrow H_c^*(GL(n, \mathbb{R}))$  is onto, since  $H^*(\mathfrak{gl}_n \mathbb{C}, U_n) \rightarrow H^*(\mathfrak{gl}_n, O_n)$  is onto (e.g., see Chapter 7 of [13]). By the remarks of §2, Theorem 3 then follows from:

**PROPOSITION 3.8.** *Let  $i : \Gamma \rightarrow GL(n, \mathbb{C})$  be the inclusion, and suppose that  $\Gamma$  is amenable. Then*

$$i^* : H_c^m(GL(n, \mathbb{C})) \rightarrow H^m(\Gamma)$$

is zero for all  $m > 1$ .

*Proof.* Let  $\Lambda \subset GL(n, \mathbb{C})$  be a maximal amenable subgroup containing  $\Gamma$ . From [15] we know there is a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{C}^n$  and integers  $\{n_1, \dots, n_d\}$  with  $n_1 + \dots + n_d = n$  such that with respect to this basis,  $\Lambda$  has the form:

$$\begin{bmatrix} R^+U_{n_1} & * & \cdot & \cdot & * & * \\ 0 & * & \cdot & \cdot & * & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & R^+U_{n_d} \end{bmatrix}$$

Here,  $R^+U_n$  denotes the positive reals product with the unitary group of dimension  $n_i$ . Let  $U_n \subset GL(n, \mathbb{C})$  be the unitary subgroup with respect the basis  $\{v_1, \dots, v_n\}$ .

The map  $i^*$  factors through the map  $H_c^*(GL(n, \mathbb{C})) \rightarrow H_c^*(\Lambda)$ , so it will suffice to show this latter map is trivial in degrees greater than one. Let  $\lambda$  be the Lie algebra of  $\Lambda$ , and let  $U \subset \Lambda$  be a maximal compact subgroup with  $U = \Lambda \cap U_n = U_{n_1} \times \dots \times U_{n_a}$ . By the van Est Theorem, it suffices to show that for Lie algebra cohomology,

$$j^* : H^m(\mathfrak{gl}_n \mathbb{C}, U_n) \rightarrow H^m(\lambda, U)$$

is zero when  $m > 1$ .

Let  $\tilde{\mathfrak{t}}$  be the solvable radical of  $\lambda$ , let  $\mathfrak{n}$  be the nilradical and  $\tilde{\mathfrak{d}}$  the subspace of the complex diagonal matrices with  $\tilde{\mathfrak{t}} = \mathfrak{n} \oplus \tilde{\mathfrak{d}}$ . The intersection  $\tilde{\mathfrak{t}} \cap \mathfrak{u}$  consists of purely imaginary diagonal matrices, so we consider  $\mathfrak{t} = \tilde{\mathfrak{t}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$  as those matrices in  $\mathfrak{t}$  with real diagonal entries. Similarly define  $\mathfrak{d} = \tilde{\mathfrak{d}}/(\tilde{\mathfrak{t}} \cap \mathfrak{u})$  so that  $\mathfrak{t} = \mathfrak{n} \oplus \mathfrak{d}$ . As  $\mathfrak{t}$  is normal in  $\lambda$ , it follows from the definition of relative Lie algebra cohomology that

$$H^*(\lambda, U) \cong H^*(\mathfrak{t}^U) \cong H^*(\mathfrak{t})^U,$$

where superscript  $U$  means the  $Ad(U)$ -invariant subspace. The adjoint action of  $U$  on  $\mathfrak{t}$ ,  $\lambda$  and  $\mathfrak{gl}_n \mathbb{C}$  are all compatible, so we get:

$$\begin{array}{ccc} H^m(\mathfrak{gl}_n \mathbb{C}, U_n) & \xrightarrow{j^*} & H^m(\lambda, U) \\ \downarrow \cong & & \downarrow \cong \\ H^m(\mathfrak{gl}_n \mathbb{C})^{U_n} & \xrightarrow{r^*} & H^m(\mathfrak{t})^U \end{array}$$

We will show  $r^* \equiv 0$  for  $m > 1$ .

Recall from (p. 116 of [13]) that the generator  $y_i \in H^{2i-1}(\mathfrak{gl}_n \mathbb{C})$  is represented by the  $ad GL(n, \mathbb{C})$ -invariant form on  $\mathfrak{gl}_n \mathbb{C}$ ,

$$y_i = k_i \operatorname{tr}(\underbrace{\Theta \wedge [\Theta, \Theta] \wedge \dots \wedge [\Theta, \Theta]}_{(i-1)\text{-factors}})$$

where  $\Theta$  is the Maurer–Cartan form and  $k_i$  is a scalar. The algebra  $\mathfrak{n}$  is an ideal in  $\mathfrak{t}$  as an associative algebra, and  $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{n}$  so for  $i > 1$  the form  $r^*(y_i)$  on  $\mathfrak{t}$  is obtained by taking the traces of elements of  $\mathfrak{n}$ , which all have trace zero. Thus,  $r^*(y_i) = 0$ . As the  $\{y_i\}$  generate the algebra  $H^*(\mathfrak{gl}_n \mathbb{C})$ , we are done.  $\square$

As a corollary of the above proof, we have the general fact about Lie algebra cohomology which is useful in other contexts as well.

**PROPOSITION 3.9.** *Let  $G$  be an amenable subgroup of  $GL(n, \mathbb{R})$  with Lie algebra  $\mathfrak{g}$  and maximal compact subgroup  $K = G \cap O_n$ . Then  $H^m(\mathfrak{g}, O_n) \rightarrow H^m(\mathfrak{g}, K)$  is the zero map for all  $m > 1$ . In particular, the restriction of the forms  $y_i$  to  $\mathfrak{g}$  are exact for all  $i > 1$  and odd.*

**§4. Action of  $\Gamma$  on an attracting subspace**

Let  $\Gamma \subset GL(n, \mathbb{R})$  be a non-amenable countable subgroup,  $G \subset GL(n, \mathbb{C})$  its connected algebraic closure and  $N$  the integer defined in §3 for which  $G^N = G^{N+1}$ . By Corollary 3.7 the group  $\Gamma^{N+1}$  is not distal, so there exists  $f \in \Gamma^{N+1}$  with an eigenvalue of modulus greater than one. Let  $\mu_1, \dots, \mu_s$  be the eigenvalues of  $f$  and set

$$\mu = \max\{|\mu_1|, |\mu_1^{-1}|, \dots, |\mu_s|, |\mu_s^{-1}|\}.$$

By reordering the  $\mu_i$  and replacing  $f$  with  $f^{-1}$  if necessary, one can assume

$$\mu = |\mu_1| = \dots = |\mu_r| > \lambda = |\mu_{r+1}| \geq \dots \geq |\mu_s|.$$

Let  $\{v(i, j) \mid 1 \leq i \leq s; 1 \leq j \leq r(i)\}$  be a basis of  $\mathbb{C}^n$  in which  $f$  has Jordan form:

$$\begin{aligned} f v(i, 1) &= \mu_i \cdot v(i, 1) \\ f v(i, j) &= \mu_i [v(i, j) + v(i, j-1)] \quad \text{for } 1 \leq j \leq r(i). \end{aligned} \tag{4.1}$$

We also require that  $v(i, 1) = \overline{v(j, 1)}$  if  $\mu_i = \overline{\mu_j}$ , where  $\overline{\phantom{x}}$  denotes complex conjugate. Set

$$V(i) = \text{Span} \{v(i, j) \mid 1 \leq j \leq r(i)\}.$$

Note that  $V(i)$  is stable under  $f$ , and there is a superdiagonal nilpotent matrix  $N(i)$  so that

$$f|_{V(i)} = \mu_i [Id + N(i)].$$

Let  $V_{\mathbb{C}} = \bigoplus_{i=1}^s V(i)$  and  $W_{\mathbb{C}} = \bigoplus_{i=r+1}^s V(i)$  so that  $\mathbb{C}^n = V_{\mathbb{C}} \oplus W_{\mathbb{C}}$ . Since  $f$  is real, both  $V_{\mathbb{C}}$  and  $W_{\mathbb{C}}$  are the complexifications of the real subspaces  $V = V_{\mathbb{C}} \cap \mathbb{R}^n$  and  $W = W_{\mathbb{C}} \cap \mathbb{R}^n$ .

Endow  $\mathbb{C}^n$  with the Hermitian metric for which the vectors  $\{v(i, j)\}$  are orthonormal. Let  $|v|$  denote the length of  $v \in \mathbb{C}^n$ , and for  $A \in GL(n, \mathbb{C})$  we set

$$|A| = \sup_{|v|=1} |Av|.$$

Define  $\pi : \mathbb{R}^n - \{0\} \rightarrow S^{n-1}$  by  $\pi(v) = v/|v|$ . For a subspace  $Z \subset \mathbb{R}^n$ , let  $Z^1$  denote the set of unit vectors in  $Z$ .

Note that (4.1) implies for all  $k > 0$  and  $1 \leq i \leq s$ ,

$$[f|V(i)]^k = \mu_i^k \left[ Id + \binom{k}{1} N(i) + \cdots + \binom{k}{n} N(i)^n \right] \quad (4.2)$$

where  $N(i)^j = 0$  for  $j \geq r(i)$ . Let  $q(k) = \sum_{j=0}^{n-1} \binom{k}{j}$ , a polynomial of degree  $(n-1)$  in  $k$ . Then (4.2) and our choice of metric yields:

#### LEMMA 4.3

a) For  $v \in V$ ,

$$|v| \mu^k \leq |f^k(v)| \leq \mu^k q(k) |v|.$$

b) For  $w \in W$ ,

$$|w| |\mu_s|^k \leq |f^k(w)| \leq \lambda^k q(k) |w|. \quad \square$$

Define the arctangent function  $a : \mathbb{R}^n - W \rightarrow \mathbb{R}^+$  between  $V$  and  $W$  by the rule:

For  $y \in \mathbb{R}^n$  with  $y = v + w$ ,  $v \in V$ ,  $w \in W$ ,  $0 \neq v$ ,

$$a(y) = \frac{|w|}{|v|}.$$

LEMMA 4.4. For all  $y \in \mathbb{R}^n - W$  and  $k > 0$ ,

$$\left( \frac{|\mu_s|}{\mu} \right)^k \frac{1}{q(k)} a(y) \leq a(f^k(y)) \leq \left( \frac{\lambda}{\mu} \right)^k q(k) a(y).$$

*Proof.* For  $y = v + w$ ,  $f^k(y) = v_k + w_k$  where  $w_k = f^k w \in W$  and  $v_k = f^k v \in V$ . Thus,  $a(f^k(y)) = |w_k|/|v_k|$  and Lemma 4.3 yields the estimate.  $\square$

The last result needed for the constructions in §5 asserts that  $\Gamma^N$  contains enough elements to map all of the strong expanding manifold  $V$  of  $f$  into the domain  $\mathbb{R}^n - (V \cup W)$ . We remark that if it were possible to find a single  $g \in \Gamma$  for which  $gV \cap V = \{0\}$  and  $gV \cap W = \{0\}$ , then a much simplified proof of Theorem 2 would be possible along the lines of [5]. As it is, we make do with the following:

**PROPOSITION 4.5.** *Let  $v \in V$  be a non-zero vector. Then there exists  $g \in \Gamma^N$  such that*

$$gv \notin V \text{ and } v \cdot gv \neq 0 \tag{4.6}$$

and hence  $gv \notin W$ .

*Proof.* Suppose to the contrary that for all  $g \in \Gamma^N$ , either  $gv \in V$  or  $v \cdot gv = 0$ . These are algebraic conditions on  $\Gamma$ , so by Lemma 3.2 they also hold for all  $g \in G^N$ . Now  $G^N$  is irreducible as it is a connected algebraic group, so either  $gv \in V$  for all  $g \in G^N$ , or  $v \cdot gv = 0$  for all  $g \in G^N$ . Clearly we must have the first case, so  $G^N \cdot v \subset V$ . Let  $\tilde{V}$  denote the span of  $G^N v$ . Then  $\tilde{V}$  is a subspace of  $V$  stable under  $\Gamma^N$ , hence  $f|_{\tilde{V}}$  is in the commutator group of  $\Gamma^N|_{\tilde{V}}$ . But the determinant of  $f|_{\tilde{V}}$  is  $\mu^{\dim \tilde{V}} > 1$ , which contradicts  $f|_{\tilde{V}}$  being a product of commutators.  $\square$

The condition (4.6) is open for  $v \in V$ , so given any  $v \in V^1$  and  $g_v \in \Gamma^N$  satisfying (4.6), there is a  $\delta(v) > 0$  so that for the closed  $2\delta(v)$ -ball  $B(v, 2\delta(v))$  in  $\mathbb{R}^n$  centered at  $v$ , we have (4.6) is satisfied for  $g_v$  and all  $y \in B(v, 2\delta(v))$ . Since  $V^1$  is compact, we can choose a finite set  $\{g_1, \dots, g_d\} \subset \Gamma^N$  and radii  $\{\delta_1, \dots, \delta_d\}$  so that the balls  $B_i = B(v_i, \delta_i) \cap V^1$  cover  $V^1$ , and (4.6) is satisfied for each  $g_i$  with  $y \in B(v_i, 2\delta_i)$ . Note this implies that for  $1 \leq i \leq d$ , the arctangent  $a$  is defined and bounded away from zero on the set  $g_i B(v_i, 2\delta_i)$ .

Finally, replacing  $f$  with a positive multiple if necessary, we can assume that  $\mu > 3$ , and for all  $1 \leq i \leq d$  both  $\mu > |g_i|$  and  $\mu > |g_i^{-1}|$ . By our choice of metric on  $\mathbb{C}^n$  and (4.1), we also have both  $|f| < 2\mu$  and  $|f^{-1}| < 2\mu$ .

**§5. Exponential growth on the expanding manifold.**

Let  $\mathcal{G}$  be the groupoid given in Theorem 2 and  $\Gamma$  the linear group of Jacobians at 0. Assume that  $\Gamma$  is not amenable. Let  $f \in \Gamma^{N+1}$  and  $\{g_1, \dots, g_d\} \subset \Gamma^N$  be chosen as in §4. Choose  $\gamma \in \mathcal{G}$  with  $J_0 \gamma = f$ , and for each  $1 \leq i \leq d$  choose  $\gamma_i \in \mathcal{G}$  with  $J_0 \gamma_i = g_i$ . For notational convenience, set  $\gamma_0 = \gamma$ . Let  $D \subset \mathbb{R}^n$  be an open neighborhood of 0 on which all of the  $\gamma_i$  are defined. Let  $\mathcal{G}_0$  denote the

subgroupoid of  $\mathcal{G}$  generated by the set  $\{\gamma_0, \dots, \gamma_d\}$ . We will show  $\mathcal{G}_0$  has a continuum of orbits with exponential growth.

By the stable manifold theorem (cf. [11]) applied to  $\gamma^{-1}$ , there is a connected submanifold  $S \subset D$  with  $0 \in S$ , the tangent space  $T_0S$  at 0 is equal  $V$ , and  $\gamma^{-1}$  is uniformly contracting on  $S$ . In particular,  $\gamma^{-1}S \subset S$ . By a change of coordinates on  $\mathbb{R}^n$ , we can assume  $S$  is an open neighborhood of 0 in  $V$ .

Before entering into the details of the proof of Theorem 2, a brief overview of the argument may help the reader. We first define an open cone  $C \subset S$  whose points satisfy  $\lim_{k \rightarrow \infty} \pi(\gamma^{-k}y) \in V^1$  and  $|\gamma^{-k}y| < \mu^{-k/2}$ . For an appropriate constant  $e_0$ , we set  $y_p = \gamma^{-pe_0}y$  for a given  $y \in C$ . For each  $p > 0$  we construct a subset  $\mathcal{R}_p \subset \mathcal{G}_0$  consisting of  $2^p$  words of length  $\leq m_0 \cdot p$ , such that the linear parts of the words in  $\mathcal{R}_p$  move  $y_p$  to  $2^p$  distinct points. We furthermore obtain an exponentially decreasing lower bound on the distance between these  $2^p$  points. Using Taylor’s theorem for  $C^2$ -maps, and for  $e_0$  sufficiently large so that  $y_p$  is sufficiently small, we conclude that  $\mathcal{R}_p \cdot y_p$  consists of  $2^p$  distinct points. The last remark is that in constructing  $\mathcal{R}_p$ , we use a version of the “ping-pong” lemma of [5]. In our version, the orbits are repeatedly returned to the attractor  $V$  by applying high powers of  $f$ , and are then scattered back into  $\mathbb{R}^n - (V \cup W)$  by the elements of  $\{g_1, \dots, g_d\}$ . Thus, all of the orbits we build concentrate on the subspace  $V$ , and one does not have the bilateral symmetry inherent in the method of Tits. Instead of 2 players, one can think of this as an instructor with many students.

Recall that for a  $C^2$ -diffeomorphism  $\phi$  with  $\phi(0) = 0$ , Taylor’s Theorem gives an estimate on the spherical error between  $\phi$  and  $J_0\phi$ , and the estimate is linear in  $y$ :

For all  $\epsilon > 0$  sufficiently small, there exists  $k(\phi, \epsilon) > 0$  so that

$$\frac{|\phi y - J_0\phi y|}{|y|} < k(\phi, \epsilon) \cdot |y| \quad \text{for all } |y| < \epsilon. \tag{5.1}$$

As an immediate consequence we have:

**LEMMA 5.2.** *Let  $\mathcal{R} = \{\phi_1, \dots, \phi_p\}$  be a set of local  $C^2$ -diffeomorphisms of an open neighborhood  $U$  of  $0 \in \mathbb{R}^n$  into  $\mathbb{R}^n$  with  $\phi_i(0) = 0$  for all  $i$ . Let  $\epsilon > 0$  be sufficiently small so that there exists constants  $k(\phi_i, \epsilon)$  for which (5.1) holds. Then for  $K = \max_{1 \leq i \leq p} k(\phi_i, \epsilon)$  and  $y \in U$  with  $|y| < \epsilon$ , suppose that*

$$|J_0\phi_i y - J_0\phi_j y| > 2 \cdot K \cdot |y|^2 \quad \text{for all } i \neq j.$$

*Then the set  $\mathcal{R} \cdot y = \{\phi_i y \mid 1 \leq i \leq p\}$  consist of  $p$  distinct points.  $\square$*

**LEMMA 5.3.** *There exists  $\delta > 0$  and an integer  $b > 0$  such that  $|\gamma^{-b}y| < \mu^{-b/2} |y|$  for all  $y \in S$  with  $|y| < \delta$ .*

*Proof.* By Lemma 4.3 there exists an integer  $b > 0$  for which  $|f^{-b} | V| < \mu^{-3b/4}$ . Choose  $\delta > 0$  sufficiently small so that

$$\delta \cdot k(\epsilon, \gamma^{-b}) < \{\mu^{-b/2} - \mu^{-3b/4}\}$$

where  $\epsilon$  is such that (5.1) holds for  $\gamma^{-b}$ , and  $\delta < \epsilon$ . Then

$$\begin{aligned} |\gamma^{-b}y| &\leq |\gamma^{-b}y - f^{-b}y| + |f^{-b}y| \\ &\leq |y|^2 \cdot k(\epsilon, \gamma^{-b}) + \mu^{-3b/4} |y| \\ &\leq \mu^{-b/2} |y|. \quad \square \end{aligned}$$

For  $b, \delta$  as in (5.3) we replace  $f, \gamma$  and  $\mu$  with  $f^b, \gamma^b$  and  $\mu^b$ , so we can assume:

$$|\gamma^{-p}y| < \mu^{-p/2} |y| \quad \text{for all } p > 0, y \in S, |y| < \delta \tag{5.4}$$

Choose  $\epsilon > 0$  to satisfy  $\epsilon < \delta, \epsilon < \mu^{-1}$  and there exists a constant  $K_0$  so that for all  $\phi \in \{\gamma, \gamma^{-1}, \gamma_1, \dots, \gamma_d\}$ , condition (5.1) holds for all  $|y| < \epsilon$  and  $k(\phi, \epsilon) = K_0$ . Then set

$$C = \{y \in S \mid 0 < |y| < \epsilon\}$$

These remarks are then summarized by

**COROLLARY 5.5.**  $\gamma^{-1}C \subset C$ , and for all  $p > 0$  and  $y \in C$ ,

$$|\gamma^{-p}y| < \mu^{-p/2} \cdot \epsilon. \quad \square$$

Set  $K = \max\{K_0, 2\mu\}$  and  $\epsilon_p = \min\{\epsilon, K^{-p}\}$ . For a word  $\phi = \phi_1 \circ \dots \circ \phi_p$  of length  $\leq p$  with each  $\phi_i \in \{\gamma_0, \dots, \gamma_d\}$ , we estimate the constant  $k(\phi, \epsilon_p)$  required for (5.1):

**LEMMA 5.6.** For  $\phi, K$  and  $\epsilon_p$  as above

$$|\phi y - J_0 \phi y| < K^{2p} |y|^2 \quad \text{for } |y| < \epsilon_p \tag{5.7}$$

Thus,  $K(\phi, \epsilon_p) \leq K^{2p}$ .

*Proof.* For  $p = 1$ , (5.7) follows from the definition of  $K$ . Assume (5.7) holds for

$\phi$  of length  $(p-1)$ , and set  $\tilde{\phi}_2 = \phi_2 \circ \dots \circ \phi_p$ . Then

$$\begin{aligned} |\phi y - J_0 \phi y| &= |\phi_1 \circ \tilde{\phi}_2 y - J_0 \phi_1 \circ J_0 \tilde{\phi}_2 y| \\ &\leq |y|^2 \cdot \{|J_0 \phi_1| \cdot K^{2p-2} + |J_0 \tilde{\phi}_2|^2 \cdot K \\ &\quad + 2 |J_0 \tilde{\phi}_2| |y| K^{2p-1} + |y|^2 K^{4p-3}\}. \end{aligned}$$

From  $|J_0 \phi_1| \leq K$ ,  $|J_0 \tilde{\phi}_2| < K^{p-1}$  and  $|y| < K^{-p}$  we conclude

$$\begin{aligned} |\phi y - J_0 \phi y| &< |y|^2 \{K^{2p-1} + K^{2p-1} + 2K^{2p-2} + K^{2p-3}\} \\ &= |y|^2 \cdot K^{2p-1} \left\{ 2 + \frac{2}{K^2} + \frac{1}{K^3} \right\} \\ &\leq |y|^2 \cdot K^{2p} \end{aligned}$$

since  $K > \mu > 3$ .  $\square$

**LEMMA 5.8.** For  $g \in \{f, f^{-1}, g_1, \dots, g_d\}$  and all  $u_1, u_2 \in \mathbb{R}^n$ ,  $|gu_1| > \frac{1}{2\mu} |u_1|$  and  $|gu_1 - gu_2| > \frac{1}{2\mu} |u_1 - u_2|$ .

*Proof.*  $|g| < 2\mu$ , so  $|gw| < 2\mu \cdot |w|$  and hence for  $w = g^{-1}u_1$  or  $w = g^{-1}(u_1 - u_2)$  we get the estimate.  $\square$

Recall that  $\{B_i = B(v_i, \delta_i) \mid 1 \leq i \leq d\}$  is the covering of  $V^1$  by closed balls in  $V$  defined at the end of §4. By compactness of the sets  $g_i B_i(v_i, 2\delta_i)$  and the continuity of the arctangent function  $a$  on them, there exists constants  $0 < c_1 < c_2$  for which  $c_1 < a(g_i y) < c_2$  for all  $1 \leq i \leq d$  and  $y \in B(v_i, 2\delta_i)$ .

Set  $X = \{x \in \mathbb{R}^n \mid |x| = 1 \text{ and } c_1 \leq a(x) \leq c_2\}$ .

For  $\delta > 0$ , set

$$A(\delta) = \{x \in \mathbb{R}^n \mid |x| = 1 \text{ and } a(x) < \delta\}$$

$$A_i(\delta) = \{x \in A(\delta) \mid x = v + w, v \in B_i, w \in W\}$$

Note the sets  $\{A_1(\delta), \dots, A_d(\delta)\}$  cover  $A(\delta)$ . Choose  $\delta_0 > 0$  sufficiently small so that for all  $1 \leq i \leq d$ ,  $g_i A_i(2\delta_0) \subset X$ . Lemma 4.4 implies there exists an integer  $e$  for which  $f^p(X) \subset A(\delta_0)$  for all  $p \geq e$ . Set  $m_0 = 2d \cdot e + 1$ , and define

$$c_1 = \infimum_{\substack{y, z \in X \\ 1 \leq i < j \leq 2d}} |\pi f^{i \cdot e} y - \pi f^{j \cdot e} z|.$$

Choose  $e_0 > 1$  so that for all  $p \geq 1$ ,

$$\mu^{(2m_0 - e_0/2)p} < \frac{c_1}{2K^{2p} \cdot \epsilon \cdot 2^{2 \cdot p \cdot m_0}} \quad (5.9)$$

and

$$\mu^{e_0} > K^4. \quad (5.10)$$

For all non-zero  $y \in C$  we now show the groupoid  $\mathcal{R}_0$  has exponential orbit growth on  $y$ . Fix a choice of  $0 \neq y \in C$ . For  $p > 0$  set  $y_p = \gamma^{-p \cdot e_0} y$ . By Lemma 5.5 and (5.10) we have  $|y_p| < K^{-2p} \epsilon < \epsilon_p$ , and then (5.9) yields

$$\begin{aligned} 2|y_p|^2 K^{2p} &\leq 2K^{2p} \cdot |y_p| \cdot \mu^{-p \cdot e_0/2} \cdot \epsilon \\ &\leq \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \end{aligned}$$

We can now define the set  $\mathcal{R}_p$ , which consists of  $2^p$  words of length  $\leq p \cdot m_0$  in  $\mathcal{R}_0$ . The set  $\mathcal{R}_p$  will be chosen so that for all  $\phi \neq \psi \in \mathcal{R}_p$ ,

$$|J_0 \phi y_p - J_0 \psi y_p| > \frac{c_1}{(2\mu)^{2p \cdot m_0}} |y_p|. \quad (5.12)$$

By Lemma 5.2 and (5.11), the set  $\mathcal{R}_p y_p = \{\phi y_p \mid \phi \in \mathcal{R}_p\}$  consists of  $2^p$  distinct points. Thus,  $\mathcal{R}_p \cdot \gamma^{-p e_0}$  consists of words of length  $\leq (m_0 + e_0)p$ , and applied to  $y$  yields  $2^p$  distinct orbits. Since  $y_p \rightarrow 0$ , this will finish the proof of Theorem 2.

Fix  $p$ , choose  $i_0$  with  $\pi y_p \in B(i_0)$ , and consider the  $2d$  points

$$F_1 = \{\pi f^{e \cdot k} g_{i_0} y_p \mid 1 \leq k \leq 2d\} \subset A(\delta_0).$$

There exists an integer  $i_1$  with  $1 \leq i_1 \leq d$  for which  $Q_1 = F_1 \cap A_{i_1}$  contains at least 2 points.

Now proceed inductively, and suppose  $i_{q-1}$ ,  $F_{q-1}$  and  $Q_{q-1}$  have been chosen with  $Q_{q-1} = F_{q-1} \cap A_{i_{q-1}}$  and  $\#Q_{q-1} \geq 2^{q-1}$ . The set

$$F_q = \{\pi f^{e \cdot k} g_{i_{q-1}} Q_{q-1} \mid 1 \leq k \leq 2d\} \subset A(\delta_0)$$

consists of at least  $2d \cdot 2^{q-1}$  points, since

$$g_{i_{q-1}} A_{i_{q-1}} \subset X \quad \text{and} \quad f^{e \cdot k}(X) \cap f^{e \cdot j}(X) = \emptyset \quad \text{for} \quad j \neq k.$$

Therefore, there exists  $i_q$  with  $Q_q = F_q \cap A_{i_q}$  containing at least  $2^q$  points. This completes the inductive step.

Let  $F_p$  be the set obtained in this inductive fashion; let  $\mathcal{R}_p$  be the set of words in  $\{\gamma_0, \dots, \gamma_d\}$  corresponding to the words in  $\{f, g_1, \dots, g_d\}$  which are applied to  $y_p$  to obtain the points in  $F_p$ . A typical element of  $\mathcal{R}_p$  has the form

$$\phi = y^{e \cdot k_p} \circ \gamma_{i_{p-1}} \circ y^{e \cdot k_{p-1}} \circ \gamma_{i_{p-2}} \circ \dots \circ \gamma_{i_0}$$

for some integers  $1 \leq k_1, \dots, k_p \leq 2d$ . The length of  $\phi$  is at most  $p \cdot m_0$  with respect to the set  $\{\gamma_0, \dots, \gamma_p\}$ , and  $\mathcal{R}_p y_p$  consists of at least  $2d \cdot 2^{p-1} \geq 2^p$  points, once we have established the estimate (5.12).

Let  $\phi \neq \psi \in \mathcal{R}_p$  and let  $g = J_0 \phi$ ,  $h = J_0 \psi$  be their linear parts. There are integers  $1 \leq k_1, \dots, k_p \leq 2d$  and  $1 \leq j_1, \dots, j_p \leq 2d$  for which

$$\begin{aligned} g &= f^{e \cdot j_p} g_{i_{p-1}} \cdots f^{e \cdot j_1} g_{i_0} \\ h &= f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_1} g_{i_0} \end{aligned}$$

Let  $q$  be the largest integer such that  $j_{q-1} \neq k_{q-1}$ . Set

$$\begin{aligned} \xi &= f^{e \cdot k_p} g_{i_{p-1}} \cdots f^{e \cdot k_q} g_{i_{q-1}} \\ g' &= \xi^{-1} g, \quad h' = \xi^{-1} h \end{aligned}$$

Apply Lemma 5.8 at most  $q \cdot m_0$  times to obtain

$$\begin{aligned} |gy_p - hy_p| &= |\xi(g'y_p - h'y_p)| \\ &\geq (2\mu)^{-qm_0} |g'y_p - h'y_p|. \end{aligned}$$

Next,  $g'$  and  $h'$  have length  $\leq pm_0$ , so Lemma 5.8 again yields

$$\min\{|g'y_p|, |h'y_p|\} \geq (2\mu)^{-pm_0} |y_p|.$$

Hence,

$$|g'y_p - h'y_p| \geq (2\mu)^{-pm_0} |y_p| \cdot |\pi g'y_p - \pi h'y_p| \geq (2\mu)^{-pm_0} \cdot |y_p| \cdot c_1$$

and so

$$|gy_p - hy_p| \geq (2\mu)^{-2pm_0} |y_p| \cdot c_1. \quad \square$$

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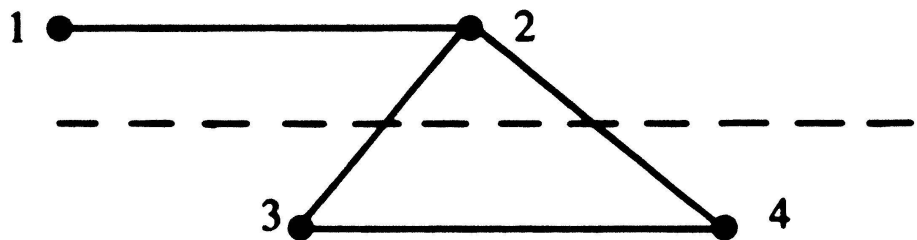
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