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Comment. Math. Helvetici 60 (1985) 312–318 0010–2571/85/020312–07$01.50 + 0.20/0

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§1. Introduction

Let α be an algebraic number and let k be a real number field with $\alpha \notin k$. The effective measure of irrationality of α over k, or effective type $\mu_{\text{eff}}(\alpha, k)$, is defined to be the infimum of all μ's for which

$$|\alpha - \beta| > c(\alpha) H_k(\beta)^{-\mu}$$

holds for every $\beta \in k$ and some effectively computable $c(\alpha) > 0$. Here $H_k(\beta)$ is the field height of β. We say $\mu_{\text{eff}}(\alpha, k)$ is non-trivial if $\mu_{\text{eff}}(\alpha, k) < \deg \alpha$.

Thue's principle was originally formulated by Thue in 1909. It deals with rational approximations to an algebraic number α. Roughly speaking, it says that given an algebraic number α, $\mu_{\text{eff}}(\alpha, Q)$ can be shown to be non-trivial if α has at least one "exceedingly good" rational approximation in Q. In 1982, Bombieri [1] succeeded in applying this principle to obtain non-trivial measures of irrationality for a new class of algebraic numbers. For example, he showed ([1], p. 294) that if α is the positive root of

$$x^r - mx^{r-1} + 1 = 0 \tag{1}$$

with $r \geqq 40$ and $m \geqq m_0(r)$, where $m_0(r)$ is effectively computable, then $\beta = m$ is a sufficiently good approximation to α to obtain the result $\mu_{\text{eff}}(\alpha, Q) \leqq 39.2$. We remark that this result has recently been improved by Bombieri and Masser (unpublished) to $\mu_{\text{eff}}(\alpha, Q) \leqq 9.5$. On p. 295 of the same paper he also remarked that similar results can be obtained by taking α to be the root of

$$f(x, m) = R(x) \tag{2}$$

where $f(x, m)$ is a polynomial of fixed degree and of height m, and where $R(x)$ is a rational function with a zero of sufficiently high order at ∞ and of bounded

[1] Research partially supported by NSF grant MCS82-05253.

height. Equation (2) may be regarded as a deformation of the equation $f(x, m) = 0$. Several other applications have followed since, for example in [2], Bombieri and Mueller obtained new non-trivial measures of irrationality for numbers of the type $\sqrt[r]{\xi}$, where $\xi \in k$. More recently, the same authors [4] showed that for a given real algebraic number α and $\varepsilon > 0$ there exist infinitely many real number fields k such that $\mu_{\text{eff}}(\alpha, k)$ is non-trivial and more precisely such that $\mu_{\text{eff}}(\alpha, k) < 2 + \varepsilon$.

A natural question arises: given an irreducible polynomial $F(x)$ of degree r and with rational integer coefficients, and given a number field k, how can we assert that F has a real root α with $\mu_{\text{eff}}(\alpha, k) < r$? In view of Thue's principle, an assertion of this sort will result from having exceptionally good approximations β to α in k. It is our intention to provide, in this note, some general criterions for constructing such pairs (α, β). In particular, our results extend both examples given by (1) and (2).

Since our focus here is to present such general principles as simply as possible, we have not tried to formulate our results in sharpest form. We hope to extend and refine such results in the future.

I would like to thank Professor Enrico Bombieri for his very helpful advice.

§2. Main results

Let $F(x)$ be a primitive irreducible polynomial of degree, r, $r \geqq 3$, with rational integer coefficients. Denote by $H(F)$ the height of F, that is the largest of the coefficients of F in absolute value. For a given positive integer $g_1 < r$, let $G_1(x)$ be the polynomial of degree g_1 defined by

$$F(x) = x^{r-g_1} G_1(x) + G_2(x), \tag{3}$$

where $\deg G_2 = g_2 < r - g_1$.

Let k be a real number field of degree s and let $\beta \in k$ be such that $k = Q(\beta)$, $F(\beta) \neq 0$. Let θ be defined by $|\beta| = H(\beta)^{\theta}$ where $H(\beta)$ is the maximum of the coefficients in absolute value of the primitive defining polynomial of β. We also define

$$b = \max\left(1, \frac{\log H(F)}{\log H(\beta)}\right). \tag{4}$$

THEOREM 1. *Let F, G_1, G_2 and β be given as before and suppose the*

following conditions hold:

(i) $|\beta| \geqq r$,

(ii) $G_1(\beta) = 0$

(iii) *for* $|u| < 1$, *we have*

$$|G_1'(\beta + u)| \geqq 8\,|\beta|^{g_1 - 1 - \delta},$$

for some δ *such that* $0 \leqq \delta < 1$ *and* $(r - g_2 - 1 - \delta)\theta > b$. *Then* F *has a real root* α *such that*

$$\mu_{\text{eff}}(\alpha, k) \leqq \frac{(4sb+1)(8r+2)}{\rho} + \frac{sb(8r+4)^2}{\rho \log H(F)} \tag{5}$$

where ρ *is given by*

$$\rho = (r - g_2 - 1 - \delta)\theta - b. \tag{6}$$

We remark that the right hand side of (5) also holds for $\mu_{\text{eff}}(\alpha', k)$ for every $\alpha' \in k$, $\alpha' \notin Q$.

A simple computation shows that $\mu_{\text{eff}}(\alpha, k) < r$ if for example

$$\log H(F) \geqq (2r+1)^2 \tag{7}$$

and $g_2 < D$, with

$$D = (r - 1 - \delta) - \frac{8(r+1)(4sb+1) + rb}{r\theta}.$$

THEOREM 2. *Let* F, G_1, G_2 *and* β *be given as before and suppose the following conditions hold*:

(i) $|\beta| \geqq r$ *and* $H(\beta) \geqq r$;

(ii) $G_2(\beta^{-1}) = 0$

(iii) *for* $|\mu| < 1/|\beta|^2$ *we have*

$$|G_2'(\beta^{-1} + u)| \geqq 8\,|\beta|^{-\delta}$$

for some δ *such that* $\delta > 0$ *and* $(r - g_1 - \delta)\theta > b + 4$. *Then* F *has a real root* α *such that* (5) *holds with*

$$\rho = (r - g_1 - \delta)\theta - b. \tag{8}$$

We remark that, as in Theorem 1, $\mu_{\text{eff}}(\alpha, k) < r$ if (7) holds and $g_1 < D + 1$.

We note that (1) is a special case of (3) where $G_1(x)=x-m$ and $G_2(x)=1$. Also (3) is an extension of (2). That is, by replacing $f(x, m)$ by $G_1(x)/G_2(x)$ and by taking $R(x)=-x^{-r+g_1}$ in (2), we get (3). Thus Theorems 1 and 2 may be viewed as explicit formulations of Bombieri's remark in this context.

We illustrate our results with two examples. In our first example, k is the rational field Q,

$$F(x)=x^r-mx^{r-1}+G(x).$$

We assume that $F(x)$ is irreducible, $r\geqq 44$, deg $G\leqq r-43$ and $H(F)=|m|$, $|m|>\exp((2r+1)^2)$.

We apply Theorem 1 choosing $G_1(x)=x-m$, $G_2(x)=G(x)$. Clearly $b=1$, $s=1$, $\theta=1$, $g_1=1$, $g_2\leqq r-43$ and thus $\rho\geqq 41-\delta$. Now (5) yields

$$\mu_{\text{eff}}(\alpha, Q)\leqq\frac{40r+10}{41-\delta}+\frac{(8r+4)^2}{(41-\delta)\log|m|}.$$

If $|m|$ is large enough, for example $|m|>\exp((2r+1)^2)$, and δ is sufficiently small, for example $\delta=2/5$, we obtain $\mu_{\text{eff}}(\alpha, Q)<r$. It remains to verify condition (iii), which in our case becomes $1\geqq 8\,|m|^{-2/5}$, that is $|m|\geqq 1024$, which is verified. This proves that our theorem yields a non-trivial effective measure of irrationality for the real root α near to m of the polynomial F.

Our second example is

$$F(x)=x^r-mx^{r-1}+mx^2-2$$

where m is an even natural number, $m>\exp(32r)$. By Eisenstein's criterion, F is irreducible.

We take $G_1(x)=x-m$, $G_2(x)=mx^2-2$ and consider the three cases $\beta=m$, $\beta=\sqrt{2}/m$, $\beta=-\sqrt{2}/m$. Theorem 1 applies to the first case, while Theorem 2 applies to the second and third cases. We omit the easy calculation and state only the result.

If α is the real root of F near m then we have

$$\mu_{\text{eff}}(\alpha, Q)<47 \quad \text{if} \quad r\geqq 50;$$

if instead α is the real root of F close to either $\sqrt{2}/m$ or $-\sqrt{2}/m$ then we have

$$\mu_{\text{eff}}(\alpha, Q(\sqrt{2}m))<170 \quad \text{if} \quad r\geqq 170.$$

§3. Proof of Theorems 1 and 2

The proof of Theorem 1 (and also of Theorem 2) is a direct application of a simplified version of Thue's principle. A derivation of this from a more general formulation can be found in [3].

Thue's principle. Let k be a real number field of degree s and let α be a real algebraic number of degree r, $\alpha \notin k$. Suppose there exists β in k with $k = Q(\beta)$ and

$$|\alpha - \beta| < H(\beta)^{-\rho}, \qquad \rho > 0. \tag{9}$$

Define

$$b = \max\left(1, \frac{\log H(\alpha)}{\log H(\beta)}\right),$$

then

$$\mu_{\text{eff}}(\alpha, k) \leqq \frac{(8r+2)(4sb+1)}{\rho} + \frac{(8r+4)^2 sb}{\rho \log H(\alpha)}. \tag{10}$$

Moreover, the right hand side of (10) is a bound for $\mu_{\text{eff}}(\alpha', k)$ for every element α' of $k(\alpha)$.

To prove Theorem 1, it suffices to show that if β satisfies conditions (i), (ii) and (iii) of Theorem 1, then F has a real root α such that (9) and (10) hold with ρ given by (6).

Let $y = H(\beta)^{-\rho}$ where ρ is given by (6); then the interval $(\beta - y, \beta + y)$ will have a real root α if $F(\beta - y)$ and $F(\beta + y)$ have opposite signs. Since $G_1(\beta) = 0$, we apply the mean value theorem to G_1 to obtain

$$G_1(\beta + y) = G_1(\beta + y) - G_1(\beta) = yG'(\beta + y_1)$$

for some y_1, $0 < y_1 < y$.

It follows that

$$F(\beta + y) = y(\beta + y)^{r-g_1} \cdot G_1'(\beta + y_1) + G_2(\beta + y).$$

In the same way, we have

$$F(\beta - y) = -y(\beta - y)^{r-g_1} \cdot G_1'(\beta - y_2) + G_2(\beta - y),$$

for some y_2, $-y < y_2 < 0$.

Suppose now that $|G_1'(\beta+\mu)|>C>0$ in the interval $-1\leqq u\leqq 1$. Then $G_1'(\beta+u)$ does not change sign there and thus $y(\beta+y)^{r-g_1}G_1'(\beta+y_1)$ and $-y(\beta-y)^{r-g_1}G_1'(\beta-y_2)$ have opposite signs. We conclude that $F(\beta+y)$ and $F(\beta-y)$ also have opposite sign if

$$y\,|\beta\pm y|^{r-g_1}>|G_2(\beta\pm y)| \tag{11}$$

for both choices of the sign $\pm$. We have

$$|G_2(\beta\pm y)|\leqq H(F)(|\beta|+1)^{g_2}$$

hence (11) will follow if

$$H(F)(|\beta|+1)^{g_2}\leqq y(|\beta|-1)^{r-g_1}C.$$

Since $|\beta|\geqq r>2$ we have

$$\frac{(|\beta|+1)^{g_2}}{(|\beta|-1)^{r-g_1}}\leqq|\beta|^{g_1+g_2-r}\left(\frac{r+1}{r-1}\right)^r\leqq 8\,|\beta|^{g_1+g_2-r},$$

and also

$$y=H(\beta)^{-\rho},\quad H(F)\leqq H(\beta)^b,\quad H(\beta)=|\beta|^{1/\theta}$$

by our definitions. It is clear now that (11) follows if we choose $C=8\,|\beta|^{g_1-1-\delta}$, which yields condition (iii) in Theorem 1.

The proof of Theorem 2 follows very much the same lines, operating with G_2 rather than G_1, with the following change. We suppose now that $|G_2'(\beta^{-1}+u)|>C>0$ in the interval $-1/|\beta|^2<u<1/|\beta|^2$ and we want to determine C so that $F(\beta-y)$ and $F(\beta+y)$ have opposite signs, where $y=H(\beta)^{-\rho}$. The value of ρ is to be determined later, once we have determined C. This reasoning requires $H(\beta)^{-\rho}\leqq|\beta|^{-2}$ and, since $|\beta|\leqq(g_1+1)H(\beta)$, it suffices that $\rho\geqq 4$ and $H(\beta)\geqq r$; these conditions have been incorporated in the statement of Theorem 2. The rest of the proof is entirely analogous to our preceding argument and therefore it will be omitted.

REFERENCES

[1] Bombieri, E., *On the Thue–Siegel–Dyson theorem*, Acta Mathematica, Vol. **148**, 1982, p. 255–296.

[2] BOMBIERI, E., MUELLER, J., *On effective measures of irrationality for $\sqrt[d]{a/b}$ and related numbers*, J. reine angew., Math. *342* (1983), p. 173–196.
[3] MUELLER, J., On Thue's principle and its applications, Proceedings of Number Theory, Noordwijkerhout 1983, Springer-Verlag Lecture Notes in Mathematics, No. 1068, p. 158–166.
[4] BOMBIERI, E., MUELLER, J., *Remarks on the approximation to an algebraic number by algebraic numbers*, to appear in Michigan J. of Math.

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Received April 6, 1984