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Total positivity and algebraic Witt classes

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This paper is in three parts. In the first part we give a criterion for an element of an algebraic number field F to be totally positive. Part two contains simple reformulations of this criterion in terms of Brauer groups, the Milnor K -Group $K_2(F)$, and sums of squares. Part three contains an application, due to P. E. Conner. It characterizes the totally positive elements of F as those elements α for which the rank one quadratic form αX^2 is Witt equivalent to the trace form of some finite extension E of F . As a corollary, it is proved that every Witt class in the Witt ring $W(F)$ is represented by a trace form when the base field F is purely imaginary.

We take this opportunity to acknowledge the generous contribution of P. E. Conner, and we thank him for many discussions.

I. The Norm Theorem

Let F be an algebraic number field. An element α in F^* is said to be *totally positive* (relative to F) if α is positive in every possible ordering of F . In particular, if F has no real embeddings, then every element of F^* is totally positive.

NORM THEOREM. *Let $\alpha \neq 0$ be an element of an algebraic number field F . Then there is a positive rational number q such that $-q$ is a norm from $F(\sqrt{\alpha})/F$ if and only if α is totally positive. Moreover, the existence of one positive rational number q with $-q$ a norm from $F(\sqrt{\alpha})/F$ is equivalent with the existence of infinitely many rational primes q with $-q$ a norm from $F(\sqrt{\alpha})/F$.*

Proof. We may replace α by αt^2 with $t \neq 0$ in \mathbb{Z} without affecting the statement of the theorem, and therefore we can assume that α is an algebraic integer. Suppose that α is totally positive, and set $m = 8 \cdot N_{F/\mathbb{Q}}(\alpha)$. Then m is a positive integer. Let ζ_m be a primitive m -th root of unity, and let N be the normal closure over \mathbb{Q} of $F(\sqrt{\alpha}, \zeta_m)$. Fix an embedding of N into the field of complex numbers, so we can talk about complex conjugation acting on N . By Čebotarev's Density

Theorem, there are infinitely many prime numbers q , unramified in N , having a prime factor Q in N whose Frobenius automorphism is complex conjugation. Of these infinitely many q take any one which is relatively prime to m . We claim that $-q$ is a norm from $F(\sqrt{\alpha})/F$.

This can be checked locally. Let P denote a prime of F . If P is infinite, then $F_P(\sqrt{\alpha}) = F_P$; this is obvious if P is complex, while if P is real this follows from the fact that α , being totally positive, is positive in the real embedding of F associated with P . In either case, we see that $-q$ is a norm from the trivial local extension. Now consider finite primes P of F . There are several cases. If P does not divide mq , then $-q$ is a unit in the local unramified extension $F_P(\sqrt{\alpha})/F_P$ and therefore $-q$ is a local norm (see [Lang], Lemma 4, p. 188). It remains to consider finite primes P dividing mq .

First, we claim that $-q \equiv 1 \pmod{m}$. For this let Q , from above, be the prime of N lying over q whose Frobenius automorphism Φ_Q is complex conjugation. Then we have

$$(\zeta_m)^{-1} = \Phi_Q(\zeta_m) \equiv (\zeta_m)^q \pmod{Q}.$$

Since $(q, m) = 1$, the m -th roots of unity are distinct mod Q , and it follows that $\zeta_m^{-1} = \zeta_m^q$ in F ; that is, $-q \equiv 1 \pmod{m}$.

Now suppose that the prime P of F divides mq . If P divides m and is nondyadic, then the fact $-q \equiv 1 \pmod{P}$ implies that $-q$ is a square in F_P , and is therefore a norm from $F_P(\sqrt{\alpha})/F_P$. If P is a dyadic prime dividing m , then $-q \equiv 1 \pmod{m}$ implies $-q \equiv 1 \pmod{8}$ by the definition of m , so $-q$ is already a square in the subfield \mathbb{Q}_2 of F_P , and therefore $-q$ is a norm from $F_P(\sqrt{\alpha})/F_P$. Finally, suppose that P divides q . Again, let Q be the chosen factor of q in N whose Frobenius automorphism equals complex conjugation, and let $\Phi_{Q'}$ be the Frobenius automorphism of a prime factor Q' of P in N . Then $\Phi_{Q'} = \sigma^{-1}\Phi_Q\sigma$ for some σ in $\text{Gal}(N/\mathbb{Q})$. We claim that the extension $F_P(\sqrt{\alpha})/F_P$ is trivial. Now this extension is sandwiched in the quadratic extension $N_{Q'}/\mathbb{Q}_q$. Note that the Galois group of this latter extension is generated by $\Phi_{Q'}$.

Therefore $F_P(\sqrt{\alpha}) = F_P$ if and only if $\Phi_{Q'}$ has the same restriction to both of these fields. But this is detected in the dense subfields $F(\sqrt{\alpha})$ and F . If $\Phi_{Q'}$ acts non-trivially on F then there is nothing to show, so we may assume that $\Phi_{Q'}$ is trivial on F . Then for each x in F we see that $\sigma(x)$ is fixed by complex conjugation, so σ is a real embedding of F . Since α is totally positive, $\sigma(\sqrt{\alpha})$ is real, and it follows that $\Phi_{Q'}$ is also trivial on $F(\sqrt{\alpha})$. Hence $F_P(\sqrt{\alpha}) = F_P$, so $-q$ is a norm from $F_P(\sqrt{\alpha})/F_P$. This being true for every P , Hasse's Norm Theorem implies that $-q$ is a norm from $F(\sqrt{\alpha})/F$.

Conversely, if α is not totally positive then α is not a square in F . If q is a positive rational number and $-q$ is a norm from $F(\sqrt{\alpha})/F$ then $-q = x^2 - \alpha y^2$ for appropriate x and y in F . But then for some real embedding of F we would have $x^2 - \alpha y^2$ to be positive, while $-q$ is negative. Hence $-q$ is not a norm.

We finish this first part with a small remark. While we started it for algebraic number fields, the Norm Theorem can be interpreted for any field F of characteristic 0. It is easy to see that the Norm Theorem remains true when F is any p -adic field. However, for $F = \mathbb{R}(X_1, X_2, X_3, X_4)(\sqrt{-d})$ with $d = X_1^2 + X_2^2 + X_3^2 + X_4^2$ one can show that the Norm Theorem is false for the choice $\alpha = d$.

II. Reformulations

Since $-q$ is represented over F by the binary quadratic form $\langle 1, -\alpha \rangle$ if and only if α is represented over F by $\langle 1, q \rangle$, the Norm Theorem can be restated as

REFORMULATION 1. *Let F be a number field and α in F^* . Then there exists a positive q in \mathbb{Z} such that α is represented over F by the form $\langle 1, q \rangle$ if and only if α is totally positive.*

Recall that an element of F^* is totally positive if and only if it is a sum of squares of elements in F^* . Hence even for sums of squares in F which require more than two squares (i.e. three or four in the number field case) we obtain

REFORMULATION 2. *Let α be an element in a number field F . Then α is a sum of squares in F if and only if α is a single square plus a sum of equal squares of elements of F , i.e. $\alpha = x^2 + y^2 + \cdots + y^2$ for certain elements x, y in F .*

Since $-q$ is a norm from $F(\sqrt{\alpha})/F$ if and only if the quaternion algebra $\left(\frac{\alpha, -q}{F}\right)$ is isomorphic to a full matrix algebra $M_2(F)$, if and only if the class of $\left(\frac{\alpha, -q}{F}\right)$ is trivial in the Brauer group $Br(F)$, we have

REFORMULATION 3. *Let F be a number field and α in F^* . Then there exists a rational prime q such that $\left(\frac{\alpha, -q}{F}\right) = 1$ in $Br(F)$ if and only if α is totally positive.*

Now consider the quadratic norm residue homomorphism from the Milnor

K -group $K_2(F)$ to $Br(F)$, which maps every Steinberg symbol $\{a, b\}$ in $K_2(F)$ to the class of $\left(\frac{a, b}{F}\right)$ in $Br(F)$. The kernel of this map is the subgroup of squares in $K_2(F)$ (see [Tate], Theorem 2, p. 207 for number fields F , or [Mer] for arbitrary fields F). Thus we see

REFORMULATION 4. *Let F be a number field and α in F^* . Then there exists a rational prime q such that $\{\alpha, -q\}$ is a square in $K_2(F)$ if and only if α is totally positive.*

III. Algebraic Witt classes

We will use the results of [C-P] to obtain another characterization of total positivity. Let E be a finite extension of the algebraic number field F . The *trace form* of the extension E/F is the quadratic form $tr_{E/F}(X^2)$, and the Witt class of this form in the Witt ring $W(F)$ is denoted $\langle E \rangle$. The Witt classes in $W(F)$ arising in this way from algebraic extensions E/F are said to be *algebraic* classes. For an element α of F^* , the Witt class of the rank one form αX^2 is denoted $\langle \alpha \rangle$.

COROLLARY 1. *The element α in F^* is totally positive if and only if the Witt class $\langle \alpha \rangle$ in $W(F)$ is algebraic.*

We use three lemmas from [C-P].

LEMMA 1. *Let $f(t) = t^m + at + b$ be an irreducible polynomial in $F[t]$, with odd degree $m \geq 3$. Let $E = F[t]/(f(t))$ be the associated extension of F , and let $d = \text{dis}\langle E \rangle$ be the discriminant of the Witt class $\langle E \rangle$. Then in $W(F)$*

$$\langle E \rangle = \langle d \rangle + (\langle d \rangle - \langle 1 \rangle)(\langle 1 - m \rangle - \langle 1 \rangle).$$

This is proved in [C-P], Theorem VI.2.1 for the field $F = \mathbb{Q}$, but the proof is valid for any field F of characteristic 0.

LEMMA 2. *For any odd $m \geq 3$ and for any α in F^* there is an irreducible trinomial $f(t) = t^m + at + b$ in $F[t]$ for which the resulting extension E has $\text{dis}\langle E \rangle = \alpha$, modulo squares in F^* .*

This was shown in [C-P], Theorem VI.2.8, again for the field $F = \mathbb{Q}$. However, the argument is entirely local in character, and extends at once to any algebraic number field F .

LEMMA 3. *Let E be a finite extension of the algebraic number field F . Then in any ordering of F the corresponding signature of the Witt class $\langle E \rangle$ equals the number of extensions of that ordering to an ordering of E . Hence if X is an algebraic Witt class in $W(F)$, then every signature of X is non-negative.*

This is proved in [C-P], Theorem I.5.2 when $F = \mathbb{Q}$, and again the proof remains true without change when F is an algebraic number field.

Proof of Corollary 1. Take α in F^* and assume that $\langle \alpha \rangle$ is algebraic. By Lemma 3, every signature of $\langle \alpha \rangle$ is non-negative, so α is non-negative and hence positive in every ordering of F . So α is totally positive.

Conversely, suppose α is totally positive. By the Norm Theorem we can find a positive rational integer whose negative is a relative norm from $F(\sqrt{\alpha})/F$. Multiplying by the square 4, which is clearly a relative norm, we may assume our rational integer to be even, say $2n$. Take $m = 2n + 1$. Then using Lemmas 1 and 2 we find an extension E/F of degree m for which

$$\langle E \rangle = \langle \alpha \rangle + X$$

with $X = (\langle \alpha \rangle - \langle 1 \rangle)(\langle -2n \rangle - \langle 1 \rangle)$ in $W(F)$. We contend that $X = 0$. For this it suffices to show that the invariants of X equal the corresponding invariants of the 0 class in $W(F)$, namely: $\text{rank}(0) \equiv 0 \pmod{2}$; $\text{sgn}(0) = 0$ in any ordering, $\text{dis}(0) \equiv 1$ modulo squares in F^* , and every Hasse-Witt symbol $c_p(0) = 1$. Clearly $\text{rank}(X) \equiv 0 \pmod{2}$. Since α is totally positive, the presence of the factor $\langle \alpha \rangle - \langle 1 \rangle$ guarantees that every signature of X is 0. Being the product of two classes of even rank, $\text{dis}(X)$ is a square in F^* (see [C-P], p. 12), so $\text{dis}(X) \equiv 1$ modulo squares. Finally we compute the Hasse-Witt symbols $c_p(X)$. By multiplying the factors in X and adding two copies of the trivial class $\langle 1, -1 \rangle$ we obtain the rank 8 representative $\langle -2\alpha, -\alpha, 2n, 1, 1, -1, 1, -1 \rangle$ of X . Then the Hasse-Witt symbol $c_p(X)$ is just the Hasse symbol of this rank 8 representative, and using the definition (see [C-P], p. 15) we see at once that $c_p(X) = (-2n, \alpha)_p$. But since $-2n$ is a relative norm from $F(\sqrt{\alpha})/F$ this latter symbol is 1, as desired. So $X = 0$, and $\langle \alpha \rangle = \langle E \rangle$ is an algebraic class, proving Corollary 1.

In the extreme case when the number field F is totally complex there are no orderings at all, so every element α in F^* is totally positive, and the Witt class $\langle \alpha \rangle$ of every rank one form αX^2 is algebraic. In fact we can show more.

COROLLARY 2. *If the algebraic number field F is totally complex then every Witt class in $W(F)$ is algebraic.*

Proof. Take X in $W(F)$ and suppose first that X has even rank. Since F has no orderings, it follows that X is algebraic by [C–P], Theorem II.9.5. So we must consider classes of odd rank.

Since F is an algebraic number field with no orderings, any quadratic form over F of rank exceeding four is isotropic. Hence the odd-rank Witt class X is represented by a rank three form, which may still be isotropic. The matrix of this form, after diagonalizing, is a non-singular 3×3 diagonal matrix over F . Then Lemmas III.5.4 and III.5.2 of [C–P] show the existence of a cubic extension L of F and an element α in L^* such that X can be written

$$X = T_{L/F}\langle\alpha\rangle_L$$

as the Scharlau Transfer of the Witt class $\langle\alpha\rangle_L$ in $W(L)$. (Since we will deal with several fields, we have appended a subscript on the Witt classes). Now the field L is also totally complex, so we can apply Corollary 1 to $\langle\alpha\rangle_L$ in $W(L)$ to find an extension E/L for which $\langle E\rangle_L = \langle\alpha\rangle_L$ in $W(L)$. Note that the class $\langle E\rangle_L$ in $W(L)$ is just the image under the Scharlau Transfer $T_{E/L}$ of the class $\langle 1\rangle_E$ in $W(E)$. If we then transfer this class all the way down to $W(F)$ we obtain

$$\langle E\rangle_F = T_{E/F}\langle 1\rangle_E = T_{L/F}\langle\alpha\rangle_L = X,$$

so X is algebraic. This proves Corollary 2.

In general it is a difficult problem to determine the algebraic classes in the Witt ring of an algebraic number field F . By Lemma 3, any algebraic class necessarily has non-negative signature in every possible ordering of F . When $F = \mathbb{Q}$ is the field of rational numbers, it is proved in [C–P] that non-negative signature is not only a necessary but also a sufficient condition for a Witt class in $W(\mathbb{Q})$ to be algebraic. This result together with Corollary 2 makes it reasonable to ask:

Question. Let F be an algebraic number field. Is it true that a Witt class X in $W(F)$ is algebraic if and only if the signature of X with respect to every ordering of F is non-negative?

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