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# Non-rotational minimal spheres and minimizing cones

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Recently the study of minimal and constant curvature hypersurfaces in space forms has produced many new examples of geometric interest, see [4, 5, 6, 8]. They are constructed by rotation of planar curves with rotation groups of cohomogeneity 2. Hence they carry a foliation with homogeneous leaves of codimension one (except for some singular leaves), which projects radially onto a homogeneous isoparametric family in the unit sphere.

In this paper we shall instead "blow up" isoparametric families of the sphere to obtain in particular minimally embedded hyperspheres in the sphere. Our emphasis, however, is on *inhomogeneous* isoparametric families (see [3] for details), and we call the resulting hypersurfaces *non-rotational*, since they cannot be obtained by rotation group actions. For example, for $n=16$ there is at least one, for $n=8k\geqslant 24$ there are at least $1+[k/2]$, and for $n=16k\geqslant 32$ there are at least $5+k+[k/2]$ such non-rotational minimal hyperspheres in $S^{n-1}$. We also foliate $\mathbb{R}^n$ by complete minimal hypersurfaces which are regular except for one absolutely minimizing cone [2]. Constant curvature examples can also be obtained in this way, but the emerging differential equations are somewhat more complicated.

The content of this paper is as follows. In the first three sections we describe the process of deforming spheres such that a given isoparametric family is respected, and the situation is controlled by a "generating curve". We compute the curvatures of the resulting hypersurface, and show that prescribed mean curvature leads to a second order differential equation for the generating curve. This equation is, up to certain "multiplicity constants", the same as obtained in [4] and [8] by different arguments. Section 3 contains the precise statements of our main results. In section 4 we begin the analysis of the differential equation (in the minimal case) with a study of the boundary behaviour of the solutions. In section 5 we transform the equation to a vector field in 3-space, of which we then give a rather detailed geometric description. As an application we construct in section 6 the minimal foliation of $\mathbb{R}^n$ mentioned above. In the rest of the paper we

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finally use the shooting method to find solutions which give minimal hypersurfaces in the sphere. In the course of this argument we need information about the rotation of solutions around special ones, and for that purpose extend Sturm–Liouville results for the linearized equation by a very neat geometric argument to the non-linear case.

We thank H. B. Lawson for drawing our attention to the minimal cones.

## 1. The basic construction

An isoparametric family in the sphere is a family of compact, parallel, constant curvature hypersurfaces, which fill the sphere up to some focal manifolds. For details which we need about these families in the sequel, we refer to [3]. The reader should carry in mind the example

$$\cos\varphi S^p \times \sin\varphi S^q \subset S^{p+q+1} \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1},$$

which is given by the levels $F^{-1}(\{\cos 2\varphi\})$ of the quadratic polynomial

$$F(x, y) := \langle x, x\rangle - \langle y, y\rangle, \qquad x \in \mathbb{R}^{p+1}, \qquad y \in \mathbb{R}^{q+1}$$

restricted to the sphere. Given functions $r(s) > 0$. $\varphi(s)$ on an interval $J$, the union of blown up levels

$$r(s)(\cos\varphi(s)S^p \times \sin\varphi(s)S^q) \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$$

is an immersed hypersurface (possibly with singularities, where $2\varphi(s)$ is a multiple of $\pi$).

Our hypersurfaces are constructed in a similar way, but based on a quartic polynomial

$$F(x) := \langle x, x\rangle^2 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2, \qquad m \geqslant 1, \tag{1}$$

where the $P_i : \mathbb{R}^n \to \mathbb{R}^n$ are self-adjoint endomorphisms which satisfy $P_iP_j + P_jP_i = 2\delta_{ij}Id$. Such sets of endomorphisms are obtained from each orthogonal representation of the Clifford algebra of $(\mathbb{R}^m, -\langle\,.\,,.\rangle)$ on $\mathbb{R}^{n/2}$. The levels $F^{-1}(\{\cos 4\varphi\}) \cap S^{n-1}$, $0 < \varphi < \pi/4$, form an isoparametric family, and have constant principal curvatures

$$\lambda_1 = \cot \varphi, \qquad \lambda_3 = -\tan \varphi \text{ with multiplicities } m_1 := m,$$
$$\lambda_2 = \cot\left(\varphi + \frac{\pi}{4}\right), \qquad \lambda_4 = -\tan\left(\varphi + \frac{\pi}{4}\right) \text{ with multiplicities } m_2 := \frac{n}{2} - m - 1, \tag{2}$$

and mean curvature $h(\varphi)/(n-2)$, where

$$h(\varphi) := 2m_1 \cot 2\varphi - 2m_2 \tan 2\varphi. \tag{3}$$

"Most" of these families (beginning with $n = 16$) are inhomogeneous, i.e. not the orbits of a subgroup of $Iso(S^{n-1})$. The only exceptions occur for $(m_1, m_2) = (5, 2)$, $(6, 1)$, $(9, 6)$, $(1, k)$, $(2, 2k-1)$, $(4, 4k-1)$, where in the last case there exist inequivalent representations giving both, one homogeneous and several inhomogeneous families. The case $(1, k)$ is announced in [8]. The extremal levels $F^{-1}(\{\pm 1\}) \cap S^{n-1} =: M_\pm$ are the focal manifolds of the family. $M_+ = \{x \in S^{n-1} \mid \langle P_i x, x\rangle = 0 \text{ for all } i\}$ has a trivial normal bundle, and $M_-$ is a sphere bundle $S^{m_1+m_2} \to M_- \to S^{m_1}$ over $S^{m_1}$, which is in most cases differentiably, though not metrically, a product bundle.

In the space forms $S^n$, $\mathbb{R}^n$, $H^n$ we shall always use polar coordinates $[0, \pi] \times S^{n-1}$ resp. $[0, \infty[ \times S^{n-1}$ with the metric of curvature $K$ given by

$$g = dr^2 + G^2(r)\, d\omega^2, \tag{4}$$

where

$$G'' + GK = 0, \qquad G(0) = 0, \qquad G'(0) = 1, \tag{5}$$

and where $d\omega^2$ denotes the standard metric on $S^{n-1}$. Consider now $S^{n-1}$ and therefore all distance spheres $\{r\} \times S^{n-1}$ endowed with an isoparametric family of type (1). From any differentiable curve in $S^2$, $\mathbb{R}^2$, or $H^2$ with polar coordinate representation $(r, \varphi)$ and arc-length parametrization

$$(r')^2 + G^2(r)(\varphi')^2 = 1, \qquad 0 \leq \varphi(s) \leq \pi/4, \tag{6}$$

we obtain a hypersurface in $S^n$, $\mathbb{R}^n$ or $H^n$ by taking

$$M := \bigcup_{s \in J} \{r(s)\} \times (F^{-1}(\{\cos 4\varphi(s)\}) \cap S^{n-1}). \tag{7}$$

$M$ is immersed except that it may have conical singularities over the focal manifolds $\varphi=0$, $\varphi=\pi/4$. It *is* immersed, if either $\varphi(J)\in\,]0,\pi/4[$, or if $r'(s)=0$ whenever $\varphi(s)\in\{0,\pi/4\}$. It is *embedded*, if moreover the curve $(r,\varphi)$ is injective. Its topological type is as follows:

(i) $M\sim S^1\times$regular level, resp. $\mathbb{R}\times$regular level, if $\varphi(J)\subset\,]0,\pi/4[$.

(ii) $M\sim$normal bundle of $M_+$ or $M_-$, if just one end of the curve reaches 0 or $\pi/4$ with $r'=0$.

(iii) $M\sim S^{n-1}$, if the curve goes from $\varphi=0$ to $\varphi=\pi/4$ with $r'=0$ at the ends.

(iv) $M\sim$normal disc bundle of $M_+$ or $M_-$ glued to itself, if the curve goes from $\varphi=0$ to $\varphi=0$, or from $\varphi=\pi/4$ to $\varphi=\pi/4$ with $r'=0$ at both ends.

## 2. Curvature computations

We shall compute the principal curvatures of $M$. Since the isoparametric hypersurfaces in $S^{n-1}$ are parallel to each other, they carry a family of normal great circles. The pre-images of these great circles under the radial projection

$$[0,\pi]\times S^{n-1}\supset M\to S^{n-1}$$

or

$$[0,\infty[\times S^{n-1}\supset M\to S^{n-1}$$

are planar curves congruent to $(r(s),\varphi(s))$, and perpendicular to the isoparametric levels of the distance spheres. The principal normals

$$\nu=\left(-G(r(s))\varphi'(s),\frac{r'(s)}{G(r(s))}\right)$$

of these planar curves are hypersurface normals, and the curves are lines of curvature: like meridians of ordinary surfaces of revolution. Their geodesic curvature $\kappa$ with respect to the ambient space form is a principal curvature of $M$, and computed as

$$\kappa=\langle\nu,\nabla_{d/ds}(r',\varphi')\rangle.$$

The covariant derivative in the space form is

$$\nabla_{d/ds}(r',\varphi')=(r'',\varphi'')+\left(-(\varphi')^2GG',2\frac{G'r'\varphi'}{G}\right),$$

where $G=G(r)$, and $G'=(dG/dr)(r)$. Hence

$$\kappa=-\frac{r''}{G\varphi'}+G'\varphi'.$$

In terms of

$$\sin\alpha:=r',\qquad \cos\alpha:=G\varphi' \tag{8}$$

we obtain

$$\kappa(s)=-\alpha'(s)+\frac{G'(r(s))}{G(r(s))}\cos\alpha(s). \tag{9}$$

The other principal curvatures equal those of the isoparametric level in $\{r(s)\}\times S^{n-1}$ with respect to the space form and to the normal $\nu$. We decompose

$$\nu=g\left(\nu,-\frac{\partial}{\partial r}\right)\left(-\frac{\partial}{\partial r}\right)+g(\nu,\xi)\xi, \tag{10}$$

where $\xi$ is the unit normal of the isoparametric family used in (2). With respect to $-\partial/\partial r$ the distance sphere $\{r(s)\}\times S^{n-1}$ has principal curvature $(G'/G)(r(s))$ in the ambient space, and it has sectional curvature $G^{-2}(r(s))$. The $\lambda_i$'s from (2) therefore change by a factor $G^{-1}(r(s))$, and from (10) and

$$g\left(\nu,-\frac{\partial}{\partial r}\right)=G\varphi'=\cos\alpha,\qquad g(\nu,\xi)=-r'=-\sin\alpha,$$

we obtain the principal curvatures

$$\cos\alpha(s)\frac{G'(r(s))}{G(r(s))}-\sin\alpha(s)\frac{\lambda_i}{G(r(s))}$$

with multiplicities $m_i$. Therefore $\tilde{h}:=(n-1)\times$mean curvature of $M$ is given by

$$\tilde{h}=-\alpha'(s)+(n-1)\frac{G'(r(s))}{G(r(s))}\cos\alpha(s)-\frac{h(\varphi)}{G(r(s))}\sin\alpha(s), \tag{11}$$

where $h$ is as in (3).

## 3. The differential equation. Main results

Given a constant $\tilde{h}$, any solution of the following 3-dimensional first-order differential equation produces by the previous section a hyper-surface of constant mean curvature $\tilde{h}/(n-1)$ in the space $S^n$, $\mathbb{R}^n$, or $H^n$ (related to the equation by its curvature $K$ via (4)).

$$
\begin{aligned}
r' &= \sin\alpha \\
\varphi' &= \frac{\cos\alpha}{G(r)} \\
\alpha' &= -\tilde{h} + (n-1)\frac{G'(r)}{G(r)}\cos\alpha - \frac{h(\varphi)}{G(r)}\sin\alpha
\end{aligned}
\tag{12}
$$

Special solutions are the distance spheres ($r = \text{const.}$, $\alpha = 0$), where $\tilde{h} = (n-1)G'(r)/G(r)$, and the so-called *minimal cone* with $\tilde{h} = 0$ and $(\varphi = \varphi_0, \alpha = \pi/2)$, where $\varphi_0$ is the unique zero of $h$ in $]0, \pi/4[$, characterized by

$$
\tan^2 2\varphi_0 = \frac{m_1}{m_2}. \tag{13}
$$

Note that (12) is the same differential equation as derived from orbital geometry in [4, 8], except that $h$ contains different $m_i$'s for inhomogeneous families. By proving the existence of suitable solutions of (12) we shall obtain:

THEOREM 1. *Each isoparametric family of Clifford type* (1) *in* $S^{n-1}$ *yields by the above construction a minimally embedded* $S^{n-1}$ *in* $S^n$, *which is not an equator.* (Proof in sections 7–9.)

THEOREM 2. *Each isoparametric family of Clifford type* (1) *in* $S^{n-1}$ *yields by the above construction a foliation of* $\mathbb{R}^n$ *by complete minimal hypersurfaces, which are regular except for one absolutely minimizing cone. The foliation is invariant under homotheties of* $\mathbb{R}^n$. (Proof in section 6.)

## 4. Solutions extending smoothly over a focal manifold

We want to find solutions of the differential equation (12), for which $r' \to 0$ as $\varphi \to 0$ or $\pi/4$. Necessarily then $\lim \alpha \equiv 0 \bmod \pi$. To be specific, let us assume

$$
\varphi(0) := \lim \varphi(s) = 0, \qquad \alpha(0) := \lim \alpha(s) = 0, \qquad r(0) := \lim r(s) = r_0, \quad \text{for} \quad s \searrow 0. \tag{14}
$$

Then, using (12) and (13)

$$\varphi'(0) := \lim \varphi'(s) = \frac{1}{G(r_0)} \tag{15}$$

and

$$\alpha'(0) := \lim \alpha'(s) = -\tilde{h} + (n-1)\frac{G'(r_0)}{G(r_0)} - \frac{m_1}{G(r_0)} \lim \frac{\alpha(s)}{\varphi(s)},$$

whence

$$\alpha'(0) = \left(-\tilde{h} + (n-1)\frac{G'(r_0)}{G(r_0)}\right)(1+m_1)^{-1}. \tag{16}$$

It follows that the initial derivatives of arbitrary order are uniquely determined, and Hsiang states in his papers that one can show convergence of the resulting power series. We shall prove somewhat less, but by a hopefully simpler argument: For each $r_0$ there exists a unique smooth solution of (12), (14) which depends continuously on $r_0$. We restrict ourselves to $\tilde{h} = 0$, $K \in \{0, 1\}$, but the general case can be handled similarly.

First note that near $s = 0$ one can take $\varphi$ instead of $s$ as independent variable. Then (12), (14) become

$$\begin{aligned} &\frac{dr}{d\varphi} = G(r(\varphi)) \tan \alpha(\varphi) \\ &\frac{d\alpha}{d\varphi} = (n-1)G'(r(\varphi)) - \tan \alpha(\varphi) h(\varphi) \\ &r(0) = r_0, \qquad \alpha(0) = 0. \end{aligned} \tag{17}$$

To this we apply a modified Picard iteration. We fix a natural number $N$, and let $r_N(\varphi)$, $\alpha_N(\varphi)$ be $N$-th order polynomials, the coefficients of which are determined by the differential equation as above, starting with $r_N(0) = r_0$, $\alpha_N(0) = 0$. In other words, $(r_N, \alpha_N)$ is the $N$-th order Taylor polynomial of a prospective solution of (17). For a fixed positive $\Phi$ we consider the space

$$\mathcal{R} = \{(\rho, \omega) : [0, \Phi] \to \mathbb{R}^2 \text{ smooth} \mid \varphi^{(i)}(0) = \omega^{(i)}(0) = 0, \text{ for } 1 \leq i \leq N\}. \tag{18}$$

The operator

$$\mathcal{L}(\rho, \omega)(\varphi) := (\mathcal{L}_1(\rho, \omega)(\varphi), \mathcal{L}_2(\rho, \omega)(\varphi)) \tag{19}$$

with

$$\begin{aligned} \mathscr{L}_1(\rho, \omega)(\varphi) &:= r_0 - r_N(\varphi) + \int_0^\varphi G(r_N+\rho)\tan(\alpha_N+\omega) \\ \mathscr{L}_2(\rho, \omega)(\varphi) &:= -\alpha_N(\varphi) + \int_0^\varphi \{(n-1)G'(r_N+\rho) - \tan(\alpha_N+\omega)h\} \end{aligned} \tag{20}$$

maps $\mathscr{R}$ into itself, and a fixed point $(\rho, \omega)$ of $\mathscr{L}$ will obviously provide a solution $(r = r_N + \rho, \alpha = \alpha_N + \omega)$ of (17). On $\mathscr{R}$ we use the $L^\infty$-norm determined by

$$\|\rho\| := \sup\left|\frac{\rho(\varphi)}{\varphi^{N+1}}\right|, \qquad \|\omega\| := \sup\left|\frac{\omega(\varphi)}{\varphi^{N+1}}\right|, \tag{21}$$

where $\varphi \in [0, \phi]$. Note that this implies

$$|\rho(\varphi)| \leq \|\rho\|\,\varphi^{N+1}, |\omega(\varphi)| \leq \|\omega\|\varphi^{N+1} \quad \text{on} \quad [0, \Phi]. \tag{22}$$

We now choose $\Phi$ small enough to guarantee

$$|r_N(\varphi) - r_0| \leq 1, \qquad |\alpha_N(\varphi)| \leq \frac{\pi}{8}$$

on $[0, \Phi]$, and make the *a priori* assumptions

$$|\rho_i(\varphi)| \leq 1, \qquad |\omega_i(\varphi)| \leq \frac{\pi}{8} \tag{23}$$

on $[0, \Phi]$. If moreover $\Phi \leq \frac{1}{2}$, then a few lines of straight-forward computation give

$$\|\mathscr{L}_1(\rho_1, \omega_1) - \mathscr{L}_1(\rho_2, \omega_2)\| \leq a\frac{\Phi}{N+2}\|(\rho_1, \omega_1) - (\rho_2, \omega_2)\|,$$

where $a = 2$ for $K = 1$, and $a = 2(r_0 + 2)$ for $K = 0$, and

$$\|\mathscr{L}_2(\rho_1, \omega_1) - \mathscr{L}_2(\rho_2, \omega_2)\| \leq 2\frac{n-1}{N+1}\|(\rho_1, \omega_1) - (\rho_2, \omega_2)\|.$$

If we choose $N$ sufficiently large, we obtain

$$\|\mathscr{L}(\rho_1, \omega_1) - \mathscr{L}(\rho_2, \omega_2)\| \leq q\,\|(\rho_1, \omega_1) - (\rho_2, \omega_2)\| \tag{24}$$

with $q<1$. Starting the iteration with $(\rho_0, \omega_0)=(0,0)$, and assuming (23) for all $i\leq j$, we obtain from (22), (24)

$$|\rho_{j+1}(\varphi)|+|\omega_{j+1}(\varphi)|\leq\|(\rho_{j+1}, \omega_{j+1})\|\, \Phi^{N+1}\leq\|(\rho_1, \omega_1)\|\frac{\Phi^{N+1}}{1-q}.$$

Note that $\|.\|$ depends on $\Phi$ monotonically. Thus, choosing $\Phi$ small enough we have (23) for all $i$, and $\mathscr{L}$ is contracting on this subset of $\mathscr{R}$.

This gives the existence of a unique solution. As for the continuous dependence: The Taylor ploynomials $r_N$, $\alpha_N$ depend differentiably on $r_0$, and therefore the operator $\mathscr{L}$ depends (in the above norm!) differentiably on $r_0$, and is uniformly contracting with respect to $r_0$. Hence the solutions are Lipschitz dependent on $r_0$. Differentiable dependence can be proved, because the singularity of $h$ in the integrand does not get worse in the linearized equation, but we shall not need this.

*Remark.* The above considerations give unique existence and continuity of solutions of (17) on the interval $0\leq r_0<\pi$ (resp. $\infty$) including 0. This will be needed later.

## 5. Qualitative description of the vector field

Our aim is to get a qualitative picture of the solution curves of (12). We restrict ourselves to $\tilde{h}=0$. It is obviously irrelevant that the solutions are parametrized by arc-length. The description becomes simpler, if no vanishing denominators occur, and we therefore discuss the vector field $Y$ determined by the equivalent system

$$\begin{aligned}\dot{r}&=G(r)\sin 4\varphi\sin\alpha\\ \dot{\varphi}&=\sin 4\varphi\cos\alpha\\ \dot{\alpha}&=(n-1)G'(r)\sin 4\varphi\cos\alpha-\sin\alpha(4m_1\cos^2 2\varphi-4m_2\sin^2 2\varphi)\\ &=(n-1)G'(r)\sin 4\varphi\cos\alpha-(n-2)\sin\alpha(\cos 4\varphi-\cos 4\varphi_0),\end{aligned}\tag{25}$$

where $\varphi_0$ is given by (13).

To simplify notation we normalize the curvature in the case $K>0$ to $K=1$. We think of the $r$-axis as pointing vertically upward, and consider the following "fundamental domain":

$$[0, \infty[\times\left[0, \frac{\pi}{4}\right]\times[0, \pi], \quad \text{if} \quad K\leq 0$$

$$[0, \pi]\times\left[0, \frac{\pi}{4}\right]\times[0, \pi], \quad \text{if} \quad K=1.$$

In its interior we have $\dot{r}>0$. In the neighbouring domains $-\pi\leq\alpha\leq 0$ and $\pi\leq\alpha\leq 2\pi$ all essential features are the same, but $\dot{r}<0$.

a) The vector field vanishes on the four *vertical edges* $\varphi\in\{0, \pi/4\}$, $\alpha\in\{0, \pi\}$, which correspond to points over the focal submanifolds with $\sin\alpha=0$, i.e. with $dr/d\varphi=0$. The results of section 4 about the singular initial value problem show: Solution curves of (25) in the interior of the fundamental domain will – in the limit, of course, – start from (or end at) these vertical edges if and only if the initial vector (or the negative of the "final" vector) of the corresponding solution of (17) points into the interior. Therefore, if $K\leq 0$, solutions start into the interior from $(\varphi=0, \alpha=0)$ and $(\varphi=\pi/4, \alpha=\pi)$ for arbitrary positive $r$, but no solutions from the interior end on the vertical edges. If $K=1$, we have interior solutions starting at $(0<r<\pi/2, \varphi=0, \alpha=0)$ and $(0<r<\pi/2, \varphi=\pi/4, \alpha=\pi)$, and ending at $(\pi/2<r<\pi, \varphi=\pi/4, \alpha=0)$ and $(\pi/2<r<\pi, \varphi=0, \alpha=\pi)$.

b) The *faces* $\varphi=0$ and $\varphi=\pi/4$ are filled with straight solutions without geometric interest: $r=\text{const.}$, $\varphi\in\{0, \pi/4\}$, $\dot{\alpha}=\text{const.}\sin\alpha$.

c) On the *horizontal face*(*s*) $r=0$ (and $r=\pi$, if $K=1$) the vector field $Y$ is horizontal ($\dot{r}=0$), with a horizontally attractive fixed point $(r, \varphi, \alpha)=(0, \varphi_0, \pi/2)$. For $K=1$ $(\pi, \varphi_0, \pi/2)$ is a horizontally repulsive fixed point. The *minimal cone solution* starts vertically upward from the bottom fixed point:

$$\dot{r}(t)=\sin 4\varphi_0 G(r(t)), \qquad \varphi(t)=\varphi_0, \qquad \alpha(t)=\frac{\pi}{2}.$$

If $K=1$, it ends at the top fixed point.

d) On the two *remaining faces* $\alpha=0$ and $\alpha=\pi$ the vector field is in the case $K\leq 0$ transversally inward. For $K=1$ one has equator sphere solutions $r(t)=\pi/2$, $\dot{\varphi}(t)=\cos\alpha\sin 4\varphi(t)$, $\alpha\in\{0, \pi\}$, dividing the faces into the lower halves, where $Y$ is transversally inward, and the upper halves, where it is transversally outward.

To describe $Y$ inside the fundamental domain, we study its behaviour along several surfaces.

e) The planes $\varphi=\text{const.}$: For $\alpha\in\,]0, \pi/2[$ the field is transversal toward growing $\varphi$, for $\alpha\in\,]\pi/2, \pi[$ toward decreasing $\varphi$.

f) The planar pieces

$$0<\varphi<\varphi_0, \qquad \alpha=\frac{\pi}{2} \tag{26}$$

$$\varphi_0<\varphi<\frac{\pi}{4}, \qquad \alpha=\frac{\pi}{2}. \tag{27}$$

The flow is transversal toward increasing $\alpha$ on (26), and toward decreasing $\alpha$ on (27).

Therefore *the flow moves around the minimal cone solution.*

g) The cylindrical levels

$$L(\varphi,\alpha):=\sin^{m_1}2\varphi\cos^{m_2}2\varphi\sin\alpha=\text{const.} \tag{28}$$

The function $L$ is zero on the vertical boundary of the fundamental domain, and maximal on the minimal cone solution. We have

$$\frac{d}{dt}L(\varphi(t),\alpha(t))=(n-1)G'(r(t))\sin 4\varphi(t)\frac{\cos^2\alpha(t)}{\sin\alpha(t)}\cdot L$$

If $K\leqslant 0$, then $G'\geqslant 1$, and we have $dL/dt\geqslant 0$ inside the fundamental domain, with equality only for $\alpha=\pi/2$. But to the latter set the flow is transversal. This shows: For $K\leqslant 0$ the function $L$ is strictly increasing along the solution curves inside the fundamental domain, i.e. the minimal cone solution attracts the others. If $K=1$, then $G'(r)=\cos r$. Therefore the minimal cone solution attracts the others below $r=\pi/2$, and repells them above.

h) The cylindrical level

$$f(\alpha,\varphi)=(\cos 4\varphi_0-\cos 4\varphi)^n-k^n\cos\alpha=0, \quad \varphi_0<\varphi<\frac{\pi}{4}, \quad \text{and} \quad 0<\alpha<\frac{\pi}{2}, \tag{29}$$

where $k:=(1+\cos 4\varphi_0)$, connects the minimal cone solution to the edge $\varphi=\pi/4$, $\alpha=0$. For $K=1$ we shall eventually count the intersections of a solution $(r(t),\varphi(t),\alpha(t))$ with this surface. Now, on it

$$\begin{aligned}\frac{d}{dt}f(\alpha(t),\varphi(t))&=n(\cos 4\varphi_0-\cos 4\varphi)^{n-1}k^{-n}4\sin^2 4\varphi\, k^n\cos\alpha\\&\quad+(n-1)k^n\sin 4\varphi\sin\alpha\cos\alpha\cos r\\&\quad+(n-2)k^n\sin^2\alpha(\cos 4\varphi_0-\cos 4\varphi)\end{aligned}$$

$$= (\cos 4\varphi_0 - \cos 4\varphi)\{(2(nk^{-n})^{1/2}(\cos 4\varphi_0 - \cos 4\varphi)^{n-1} \sin 4\varphi - ((n-2)k^n)^{1/2} \sin \alpha)^2 + (4(n(n-2))^{1/2} + (n-1)\cos r) \times (\cos 4\varphi_0 - \cos 4\varphi)^{n-1} \sin 4\varphi \sin \alpha\} > 0.$$

Hence the solutions intersect (29) transversally and always in the same direction.

At this point the qualitative picture of the solutions of $Y$ is reasonably complete. To construct the minimal spheres we need quantitative estimates for the rotation of the solutions around the minimal cone. We postpone this, and prove Theorem 2 first.

## 6. Proof of Theorem 2: Foliations of $\mathbb{R}^n$ by complete minimal hypersurfaces

In this section we assume $K \leqslant 0$. For any $r_0 > 0$ there exists a solution of (12) with $\tilde{h} = 0$, which starts at ($\varphi = 0$, $\alpha = 0$) or at ($\varphi = \pi/4$, $\alpha = \pi$) into the interior of the fundamental domain. The solutions cross the cylinders (28) toward the minimal cone solution; therefore $r'(s) = \sin \alpha(s)$ is bounded away from 0, and $r$ increases monotonically to $+\infty$: We obtain complete minimal hypersurfaces, which are embeddings of the normal bundle of the focal manifold $M_+$ (resp. $M$) into $\mathbb{R}^n$ or $H^n$. For $\mathbb{R}^n$ we can show that they form a foliation: they do not intersect each other nor the minimal cone. To prove this, we need in addition to the qualitative picture of section 5 some further information obtained from an integration of (12). We first show: If $\varphi(0) = 0$, then $\varphi(t) < \varphi_0$ for all $t$. As long as $\varphi < \varphi_0$ we have $\alpha < \pi/2$ by f) of section 5. Therefore $\varphi' > 0$, and we can choose $\varphi$ as independent parameter. From (17) we obtain, using $K = 0$,

$$\frac{dr}{d\varphi} = r \tan \alpha$$

$$\frac{d\alpha}{d\varphi} = (n-1) - h \tan \alpha. \tag{30}$$

We substitute $\sigma := r^{-1}\, dr/d\varphi = \tan \alpha$, and obtain

$$\frac{d\sigma}{d\varphi} = (1+\sigma^2)(n-1-\sigma h(\varphi)) =: S(\sigma, \varphi)$$

$$\sigma(0) = 0.$$

As in (15), (16) we get

$$\frac{dr}{d\varphi}(0)=0, \qquad \frac{d\alpha}{d\varphi}(0)=\frac{n-1}{m_1+1}$$
$$\frac{d^2r}{d\varphi^2}(0)=r_0\frac{n-1}{m_1+1}.$$

whence

$$\sigma'(0)=\frac{n-1}{m_1+1}.$$

The function

$$f:=\frac{n-1}{2h}, \qquad f(0):=0$$

is a lower bound for $\sigma$, as we shall show below. Therefore

$$\ln(r(\varphi))-\ln(r(0))=\int_0^\varphi \sigma \geq \int_0^\varphi f, \tag{31}$$

and the right-hand side goes to $\infty$ as $\varphi$ goes to $\varphi_0$, see the definitions of $f$ and $h$. Hence $\varphi(t)<\varphi_0$ along the complete solution curve. A similar argument works for $\varphi(0)=\pi/4$.

Now $\sigma \geq f$ is a consequence of $f'(\varphi)<S(f(\varphi), \varphi)$, $\varphi \geq 0$. For $\varphi=0$ we have $f'(0)=(n-1)/2m_1<\sigma'(0)$. For $\varphi>0$ the inequality is equivalent with

$$4m_1(1+\cot^2 2\varphi)+4m_2(1+\tan^2 2\varphi)<h(\varphi)^2+\tfrac{1}{4}(1+2m_1+2m_2)$$

– remember $n-1=1+2m_1+2m_2$–, or

$$3(m_1+m_2)+6m_1m_2<4m_1(m_1-1)\cot^2 2\varphi+4m_2(m_2-1)\tan^2 2\varphi+\tfrac{1}{4}+m_1^2+m_2^2.$$

The inequality

$$m_1(m_1-1)\cot^2 2\varphi+m_2(m_2-1)\tan^2 2\varphi \geq 2(m_1m_2(m_1-1)(m_2-1))^{1/2}$$

gives the sufficient condition

$$3(m_1+m_2)+6m_1m_2<8(m_1m_2(m_1-1)(m_2-1))^{1/2}+\tfrac{1}{4}+m_1^2+m_2^2.$$

This is obviously satisfied for the lowest-dimensional inhomogeneous Clifford family $(m_1, m_2) = (3, 4)$. All others have $3 \leq m_1 < m_2$ and $m_1 + m_2 \geq 11$. Then

$$0 < (m_1 + m_2)^2 - 11(m_1 + m_2) + \tfrac{33}{4}$$

or, equivalently,

$$3(m_1 + m_2) + 6m_1 m_2 < 8(m_1 - 1)(m_2 - 1) + \tfrac{1}{4} + m_1^2 + m_2^2.$$

This *a fortiori* implies (31). Hence the minimal hypersurfaces given by initial conditions on the edges $(\varphi = 0, \alpha = 0)$ and $(\varphi = \pi/4, \alpha = \pi)$ do not meet the minimal cone. Equation (30) is invariant under $r \to \lambda r$, $\lambda \in \mathbb{R}_+$. Therefore the minimal hypersurfaces do not intersect each other, and together with the minimal cone form a foliation of $\mathbb{R}^n$. To show that the minimal cone is absolutely minimizing, we define the $(n-1)$-form

$$\omega(v_1, \ldots, v_{n-1}) := \det(\nu, v_1, \ldots, v_{n-1}),$$

where $\nu$ is the unit normal field of the foliation. Then, by minimality,

$$d\omega = -\tilde{h} \det = 0.$$

The $1/r$-singularity of $\omega$ at 0 is integrable. If $N^{n-1} \subset \mathbb{R}^n$ has unit normal $\eta$ and the same boundary as some portion $C$ of the minimal cone, then

$$\operatorname{vol}(C) = \int_C \omega = \int_N \omega = \int_N \langle \nu, \eta \rangle (\pm dN) \leq \operatorname{vol}(N),$$

where equality holds only for $\nu = \pm\eta$, i.e. for $N$ a leaf of the foliation.

## 7. Proof of Theorem 1

We consider the case $K = 1$, $\tilde{h} = 0$. The fundamental domain is then a "fundamental cube". We concentrate on solutions of (12) with (singular) initial conditions $(r = r_0, \varphi = 0, \alpha = 0)$, $0 < r_0 < \pi/2$. (See section 4 for the handling of this singular initial value problem.) We show in section 8, that for $r_0$ sufficiently close to 0 these solutions intersect the planar piece (27) at least once below $r = \pi/2$, and at least once above. In particular they intersect (29) at least twice. By contrast, we show in section 9 that either there exist solutions which do not intersect (29) at all, or that for $r_0$ close enough to $\pi/2$ the first intersection of solutions with (27) is above $r=\pi/2$; since these intersections are transversal we find in particular a solution which reaches (27) at $r=\pi/2$ and then continues symmetrically to end at $\pi-r_0$ on the edge $(\varphi=0, \alpha=\pi)$ – clearly intersecting (29) only once.

How can a transversal intersection with (29) disappear? Limits of solutions – which we consider until they hit the boundary of the fundamental cube (necessarily above $\pi/2$, see (28)) – are again solution curves of the vector field, and the end points on the boundary of the cube are either singular points or points of transversal intersection.

By the above there exists $r^* \in ]0, \pi/2[$ such that solutions with $r_0 < r^*$ have at least two transversal intersections with (29) while for $r_0 = r^*$ there is at most one. The end point of this limit solution has to be on the compact surface (29) and the boundary of the cube – but bounded away from the minimal cone solution $(r^*>0)$. This forces the end point to be on the vertical edge $(\varphi=\pi/4, \alpha=0)$. The basic construction of section 1 performed with this solution of (12) gives a minimally immersed $(n-1)$-sphere in $S^n$ which is embedded since $r$ is strictly increasing along each solution in the fundamental cube.

## 8. The rotational behaviour near the minimal cone

First we study the linearization of (12) with $K=1$, $\tilde{h}=0$, along the minimal cone solution $\varphi=\varphi_0$, $\alpha=\pi/2$, and $r'=\sin 4\varphi_0 G(r)$. Here $r$ is a good independent parameter, and from

$$\begin{aligned} \frac{d\varphi}{dr} &= \frac{\cos\alpha}{G} \\ \frac{d\alpha}{dr} &= (n-1)\frac{G'}{G}\cot\alpha - \frac{h(\varphi)}{G} \end{aligned} \tag{32}$$

we obtain the linearized equation $(\Phi=\delta\varphi, A=\delta\alpha)$

$$\begin{aligned} \frac{d\Phi}{dr} &= -\frac{A}{G} \\ \frac{dA}{dr} &= -(n-1)\frac{G'}{G}A + 4(n-2)\frac{\Phi}{G}. \end{aligned} \tag{33}$$

For later comparison with (32) we need to change one coefficient slightly:

$$\frac{d\Phi}{dr} = -\frac{A}{G}$$

$$\frac{dA}{dr} = -(n-1)\frac{G'}{G}A + c(n-2)\frac{\Phi}{G}.$$

where $c$ is close to 4. This yields

$$\frac{d^2\Phi}{dr^2} = -n\frac{G'}{G}\frac{d\Phi}{dr} - c(n-2)\frac{\Phi}{G^2},$$

and, substituting $\Psi := G^k\Phi$ with $2k = n$ and $G = \sin$, we get

$$\Psi'' + (k + 2c(k-1) - (k-1)(k-2c)\cot^2 r)\Psi = 0. \tag{34}$$

LEMMA 1. *For* $k \geqslant 8$ (= *smallest value of interest for inhomogeneous families*) *and* $c$ *sufficiently close to* 4, *each solution of* (34) *has at least two zeros in each of the intervals* $](\pi/2) - 1, \pi/2[$ *and* $]\pi/2, (\pi/2) + 1[$.

*Proof.* By continuity we can restrict ourselves to $c = 4$. If $k = 8$, then (34) becomes $\Psi'' + 64\Psi = 0$, and every solution has at least two zeros in each open interval of length $> \pi/4$. For $k > 8$ ($k = 12$ is the next value of interest) we can reparametrize with $s := (9k-8)^{1/2}(r - (\pi/2))$, and want to show that each solution of

$$\Psi''(s) + \left(1 - \frac{k^2 - 9k + 8}{9k - 8}\tan^2\frac{s}{(9k-8)^{1/2}}\right)\Psi(s) = 0 \tag{35}$$

has at least two zeros in $]0, 8[$. This will imply the lemma. By Sturm–Liouville the number of zeros does not increase, if we decrease the coefficient. We use $\tan^2 x \leqslant x^2/1 - x^2$ to find

$$\frac{k^2 - 9k + 8}{9k - 8}\tan^2\frac{s}{(9k-8)^{1/2}} \leqslant \frac{(k-1)(k-8)s^2}{(9k-8)9(k-8)} \leqslant \frac{s^2}{81}$$

and we change (35) to

$$\Psi''(s) + \left(1 - \frac{s^2}{81}\right)\Psi(s) = 0. \tag{36}$$

Computer integration of (36) with $\Psi(0)=0$, $\Psi'(0)=1$ gives 3 zeros in $[0,7.1[$, and, using separation of zeros, therefore at least 2 zeros in $]0,8[$ for arbitrary initial conditions. For a theoretical estimate note that the difference between consecutive characteristic parameters, i.e. $s$-values with $\Psi(s)=0$, $\Psi'(s)=\sqrt{q_0}\Psi(s)$, $\Psi'(s)=0$, $\Psi'(s)=-\sqrt{q_0}\Psi(s)$ of the equation $\Psi''+q\Psi=0$ with $q_0:=\min q>0$ is less than $\pi/(4\sqrt{q_0})$. Apply this to (36) on the subintervals between 0, 0.79, 1.59, 2.41, 3.26, 4.15, 5.11, 6.20, 7.74. This proves Lemma 1.

From it we conclude that the solutions $(r, \Phi(r), A(r))$ of (33) rotate around the minimal cone solution $(\Phi=0, A=0)$ intersecting the surfaces (27) and (29) at least twice, once below $r=\pi/2$, and once above. We want to show that the same is true for those solutions of the non-linear equation (32), which between $(\pi/2)-1$ and $(\pi/2)+1$ stay close enough to the minimal cone solution. We control the distance from the minimal cone using the function $L$ of (28). To compare the linear with the non-linear case, we take a solution $(r, \Phi(r), A(r))$ of (33), and think of it as generating a helicoidal surface $\{(r, \lambda\Phi(r), \lambda A(r)) \mid \lambda>0\}$ around the minimal cone solution. Its forward normal is given by

$$\begin{aligned} N(r, \lambda\Phi(r), \lambda A(r)) &= (\Phi' A-\Phi A', -A, \Phi) \\ &= \left(-\frac{A^2}{G}+(n-1)\frac{G'}{G}\Phi A-c(n-2)\frac{\Phi^2}{G}, -A, \Phi\right). \end{aligned} \tag{37}$$

In coordinates $(r, \phi=\varphi-\varphi_0, A=\alpha-(\pi/2))$ the vector field of (32) reads

$$X(r, \Phi, A)=\left(1, -\frac{\tan A}{G(r)}, -(n-1)\frac{G'(r)}{G(r)}\tan A+\frac{H(\Phi)}{G(r)}\right). \tag{38}$$

where

$$H(\Phi):=2m_1\cot 2(\Phi+\varphi_0)-2m_2\tan 2(\Phi+\varphi_0).$$

If we can show that $\langle X, N\rangle>0$, then the solutions of (32) intersect the helicoidal surface *always in the forward direction*, and hence they spin around the axis at least as fast as the surface. They therefore intersect (27) and (29) once below, and once above $r=\pi/2$.

Now

$$\langle X, N\rangle = -\frac{A^2}{G} + (n-1)\frac{G'}{G}\Phi A - c(n-2)\frac{\Phi^2}{G} + \frac{A}{G}\tan A$$

$$-(n-1)\frac{G'}{G}\tan A + \frac{H(\Phi)}{G}\Phi$$

$$= \frac{1}{G}\{A(\tan A - A) + \Phi(H(\Phi) - c(n-2)\Phi) - (n-1)G'\Phi(\tan A - A)\}.$$

Since $H'(0) = 4(n-2)$, we can choose $c = 4 - 2\eta < 4$ such that Lemma 1 holds, and then choose $\varepsilon > 0$ such that

$$\left|L\left(\varphi_0 + \Phi, \frac{\pi}{2} + A\right) - L\left(\varphi_0, \frac{\pi}{2}\right)\right| < \varepsilon$$

implies

$$\frac{H(\Phi)}{\Phi} - c(n-2) \geqslant \eta(n-2).$$

and

$$A + \frac{A^3}{3} \leqslant \tan A \leqslant A + 2A^3.$$

Then

$$\langle X, N\rangle \geqslant \frac{1}{G}\left\{\frac{A^4}{3} + \eta(n-2)\phi^2 - 2(n-1)\,|G'\Phi A^3|\right\}.$$

Using

$$2(n-1)\,|\Phi A^3| \leqslant \eta(n-3)\Phi^2 + \frac{(n-1)^2}{\eta(n-3)}A^6$$

we see

$$\langle X, N\rangle > \frac{1}{G}\left\{\left(\frac{1}{3} - \frac{(n-1)^2}{\eta(n-3)}A^2\right)A^4 + \eta\phi^2\right\}$$

which is positive for $(A, \Phi) \neq (0, 0)$, if we choose $\varepsilon > 0$ sufficiently small to force $A^2 \leqslant \eta/4(n-1)$.

LEMMA 2. *Assume* $K=1, \tilde{h}=0$. *If* $r_0>0$ *is sufficiently small, then the solutions of* (12) *with initial conditions* $(r_0, \varphi=0, \alpha=0)$ *intersect the surfaces* (27) *and* (29) *at least twice, once below* $r=\pi/2$, *and once above.*

*Proof.* It follows from the remark at the end of section 4 that for $r_0 \to 0$ the solutions of (12) on $s>0$ converge to a solution in the bottom face, which by c) and g) of section 5 is attracted by the fixed point $(r=0, \varphi=\varphi_0, \alpha=\pi/2)$. Hence we can first select $\varepsilon>0$ sufficient for the above considerations, and then choose $r_0>0$ sufficiently small so that the solutions of (12) will enter the cylinder $L(\varphi, \alpha) \geq L(\varphi_0, \pi/2)-\varepsilon$ below $r=(\pi/2)-1$, and stay in it beyond $(\pi/2)+1$. The rest of the proof was given above.

## 9. Solutions with little rotation

In the previous section we proved the existence of solutions which intersect the surface (29) (at least) twice. Now we want to find solutions which intersect it at most once to guarantee the existence of (at least) one solution from the edge $(\varphi=0, \alpha=0)$ to the edge $(\varphi=\pi/4, \alpha=0)$, see section 7. As in section 8 the proof combines an *a priori* estimate for solutions of the non-linear equation with properties of the system obtained by linearizing along the equator solution $(r=\pi/2, \alpha=0)$. We still suppose $K=1$, $\tilde{h}=0$.

We know already that solutions which start with sufficiently small $r_0$ from the edge $(\varphi=0, \alpha=0)$ intersect the planar piece (27) at least once below $r=\pi/2$, see Lemma 2. If there should exist $r_0 \in \left]0, \pi/2\right[$ such that the corresponding solution does not meet (27) before leaving the fundamental cube through the face $\alpha=0$, then it would not intersect (29) at all, and we would find even two minimally embedded spheres. Since we have no indication for this better fact to be true, we may assume that all solutions (with $r_0 \in \left]0, \pi/2\right[$) meet (27). The first intersection $r^*(r_0)$ depends continuously on $r_0$, and we shall prove that $r^*(r_0)>\pi/2$ for some $r_0$ sufficiently close to $\pi/2$. This implies the existence of a solution with $r^*(r_0)=\pi/2$; the symmetry properties of (12) with respect to the axis $(r=\pi/2, \alpha=\pi/2)$ imply that this solution continues symmetrically and ends at $r=\pi-r_0$ on the edge $(\varphi=0, \alpha=\pi)$. Apart from giving another compact embedded minimal hypersurface in $S^n$ it also intersects (29) only once and therefore completes the proof of Theorem 1.

Either $r^*(r_0)>\pi/2$ for all $r_0$ sufficiently close to $\pi/2$, or else there are values $r_0<\pi/2$ arbitrarily close to $\pi/2$ such that the solutions of (12) starting at $r=r_0$ on the edge ($\varphi = 0, \alpha = 0$) satisfy

$$\alpha([0, s]) \subset \left[0, \frac{\pi}{2}\right] \Rightarrow r([0, s]) \subset \left[0, \frac{\pi}{2}\right[. \tag{39}$$

We can derive a contradiction from this last possibility.

LEMMA 3. (*A priori estimate for* $\alpha$ *assuming* (39)) *Given* $\varphi^* \in [0, \pi/4[$ *and* $\varepsilon \in ]0, \pi/2[$ *there exists* $\delta \in ]0, \pi/2[$ *such that for* $r_0 \in [(\pi/2 - \delta, \pi/2]$ *the corresponding solution satisfies* $\alpha(\varphi) \leqslant \varepsilon$ *on* $[0, \varphi^*]$. (This will be used for one fixed $\varepsilon$, e.g. $\varepsilon = \pi/4$.)

*Proof.* Assuming $\alpha([0, s]) \subset [0, \pi/2[$ we can use $\varphi$ as independent parameter and obtain – first for $\varphi \leqslant \varphi_0$ – from (17)

$$\alpha(\varphi) = \int_0^\varphi \alpha' \leqslant \int_0^\varphi (n-1) \cos r(t)\, dt \leqslant (n-1) \cos r_0 \varphi =: \alpha^*(r_0, \varphi). \tag{40}$$

Secondly, on $[\varphi_0, \pi/4[$ we have

$$\begin{aligned}\frac{d}{d\varphi}(\sin \alpha \sin^{m_1} 2\varphi \cos^{m_2} 2\varphi) &= (n-1) \cos r \cos \alpha \sin^{m_1} 2\varphi \cos^{m_2} 2\varphi \\ &\leqslant (n-1) \cos r(\varphi_0) \sin^{m_1} 2\varphi_0 \cos^{m_2} 2\varphi_0,\end{aligned}$$

whence

$$\begin{aligned}\sin \alpha(\varphi) &\leqslant \left(\frac{\sin 2\varphi_0}{\sin 2\varphi}\right)^{m_1} \left(\frac{\cos 2\varphi_0}{\cos 2\varphi}\right)^{m_2} \{\sin \alpha(\varphi_0) + (n-1) \cos r(\varphi_0)\, (\varphi - \varphi_0)\} \\ &\leqslant \left(\frac{\sin 2\varphi_0}{\sin 2\varphi}\right)^{m_1} \left(\frac{\cos 2\varphi_0}{\cos 2\varphi}\right)^{m_2} \{\sin \alpha(\varphi_0) + (n-1) \cos r_0\, (\varphi - \varphi_0)\}\end{aligned} \tag{41}$$

Clearly (40), (41) prove the lemma with explicite estimates under the assumption $r(\varphi) < \pi/2$.

Again using (39) the solutions meet (27) below $r = \pi/2$ and therefore meet $\varphi = \varphi_0$ below $r = \pi/2$. We shall now for $\varphi \geqslant \varphi_0$ estimate the growth of $r$ (for $r_0$ close to $\pi/2$) by comparison with solutions of the linearized equation along the equator solution ($r = \pi/2, \alpha = 0$).

The linearization yields ($R = \delta r$, $A = \delta\alpha$)

$$\frac{dR}{d\varphi} = A$$

$$\frac{dA}{d\varphi} = -(n-1)R - hA.$$

As in section 8, to achieve a comparison we have to change the coefficients a little. Eventually the pole of $h$ is the essential feature; we multiply the other coefficients by $\frac{1}{2}$ (any other positive number $<1$ would do, too) and study the system

$$\begin{aligned} \frac{dR}{d\varphi} &= \tfrac{1}{2}A \\ \frac{dA}{d\varphi} &= -\frac{n-1}{2}R - hA \end{aligned} \tag{42}$$

with the initial conditions

$$R(\varphi_0) = -1, \qquad A(\varphi_0) = 0 \tag{43}$$

on the interval $[\varphi_0, \pi/4[$.

LEMMA 4. *There exists $\varphi^* \in ]\varphi_0, \pi/4[$ such that $R(\varphi^*) = 0$.*

*Proof.* (42), (43) imply

$$\begin{aligned} &\frac{d^2R}{d\varphi^2} = -h\frac{dR}{d\varphi} - \frac{n-1}{4}R \\ &R(\varphi_0) = -1, \qquad \frac{dR}{d\varphi}(\varphi_0) = 0, \qquad \frac{d^2R}{d\varphi^2}(\varphi_0) = \frac{n-1}{4}. \end{aligned} \tag{44}$$

Therefore $R(\varphi_0)$ is a local minimum and every other negative critical point of $R$ is also a local minimum: As long as $R$ stays negative, $R$ is strictly increasing. We choose $\varphi_1 > \varphi_0$ sufficiently close to $\varphi_0$ such that $R$ is negative on $[\varphi_0, \varphi_1]$; we have $dR/d\varphi > 0$ at $\varphi_1$. Again, as long as $R$ is negative

$$\frac{d^2R}{d\varphi^2} \geqslant -h\frac{dR}{d\varphi}$$

or

$$\frac{d}{d\varphi}\ln\left(\frac{dR}{d\varphi}\sin^{m_1}2\varphi\cos^{m_2}2\varphi\right)\geq 0,$$

hence for $\varphi \geq \varphi_1$

$$\frac{dR}{d\varphi}(\varphi_1)\left(\frac{\sin 2\varphi_1}{\sin 2\varphi}\right)^{m_1}\left(\frac{\cos 2\varphi_1}{\cos 2\varphi}\right)^{m_2}\leq\frac{dR}{d\varphi}.$$

The left-hand side has an *unbounded* integral as $\varphi$ goes to $\pi/4$. Therefore $R(\varphi)$ cannot stay negative on $[\varphi_0, \pi/4[$.

We call $\varphi^*$ the first zero. For this $\varphi^*$ we choose $\delta \in ]0, 1/2]$ according to Lemma 3 such that $\alpha \leq \pi/4$ on $[0, \varphi^*]$ for all $r_0 \in [(\pi/2)-\delta, \pi/2[$. Still assuming (39), we will finally show that solutions of (12) with $r_0 \in [(\pi/2)-\delta, \pi/2[$ reach $r = \pi/2$ on $[\varphi_0, \varphi^*]$; since they also have $\alpha \leq \pi/4$ this contradicts (39), as desired.

(42) corresponds to the vector field

$$X=\left(\tfrac{1}{2}A, 1, -\frac{n-1}{2}R-hA\right).$$

The helicoidal surface corresponding to proportional solutions (compare section 8) have the upward normal field

$$N=\left(A, -\tfrac{1}{2}A^2-\frac{n-1}{2}R^2-hAR, -R\right).$$

The non-linear system (12), written in appropriate coordinates ($R = r-(\pi/2)$, $A = \alpha$) is given by the vector field

$$Y=(\cos R\tan A, 1, -(n-1)\sin R-h\tan A). \tag{45}$$

Therefore

$$\begin{aligned}\langle N, Y\rangle &= (\cos R\tan A-\tfrac{1}{2}A)A+(n-1)(\tfrac{1}{2}R-\sin R)(-R)\\ &\quad +h(A-\tan A)(-R). \end{aligned}\tag{46}$$

As long as the solutions of (45) stay negative they satisfy $-\frac{1}{2}\leq-\delta\leq R<0$ and therefore

$$0 < \cos R - \tfrac{1}{2}, \qquad 0 > \tfrac{1}{2}R - \sin R.$$

Because of (39) they are at $\varphi = \varphi_0$ above the helicoidal surface determined by the initial conditions (43); they cannot intersect this surface from above because at intersection points $\langle N, Y\rangle > 0$ from (46). This shows that the solutions of (45) with $r_0 \in [(\pi/2) - \delta, \pi/2[$ reach $R(\varphi) = 0$ before the solution (43) of (42), i.e. before $\varphi = \varphi^*$, as we claimed.

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