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**Autor:** Chang, S.Y.A. / Wilson, J.M. / Wolff, T.H.  
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## Some weighted norm inequalities concerning the Schrödinger operators

S. Y. A. CHANG, J. M. WILSON and T. H. WOLFF

### Introduction

Let  $v$  be a nonnegative, locally integrable function on  $\mathbb{R}^d$ . Let  $L = -\Delta - v$  be the associated Schrödinger operator. If  $L$  is essentially selfadjoint on  $C_0^\infty$ , then positivity of  $L$  is equivalent via an integration by parts with

$$\int_{\mathbb{R}^d} |u|^2 v \, dx \leq \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \quad \forall u \in C_0^\infty. \quad (0.1)$$

In [8], C. Fefferman asks for conditions on  $v \geq 0$  which imply

$$\int_{\mathbb{R}^d} |u|^2 v \, dx \leq c \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \quad (0.2)$$

for some constant  $c$ . By considering translates and dilates of a fixed bump function it is clear that a necessary condition for (0.2) is

$$\frac{1}{|Q|} \int_Q v \, dx \leq c' l(Q)^{-2} \quad (0.3)$$

for some  $c'$ , for all cubes  $Q \subset \mathbb{R}^d$ . ( $|Q|$  and  $l(Q)$  denote the Lebesgue measure and side length of  $Q$  respectively.) Letting  $v \, dx$  be (approximately) Lebesgue measure on a codimension 2 hyperplane, we see that (0.3) is not sufficient. In [8], it is shown that a sufficient condition is:

$$\left( \frac{1}{|Q|} \int_Q v^p \, dx \right)^{1/p} \leq c_p l(Q)^{-2} \quad (0.4)$$

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for some  $p > 1$  and  $c_p < \infty$ . Comparing (0.3) and (0.4) suggests the following question. Let  $\varphi : [0, \infty) \rightarrow [1, \infty)$  be increasing. When it is the case that

$$(*)_\varphi: \sup_Q \frac{1}{|Q|} \int_Q v(x) l(Q)^{+2} \varphi(v(x) l(Q)^{+2}) dx < \infty \quad (0.5)$$

implies (0.2)? We will show that if

$$\int_1^\infty \frac{dx}{x \varphi(x)} < \infty \quad (0.6)$$

then (0.5) implies (0.2), and that this is essentially best possible. In [8], some joint work of C. Fefferman and D. H. Phong is presented in which (0.4)  $\Rightarrow$  (0.2) is used also to give bounds on the number of bound states of Schrödinger operators. Using (0.5), (0.6) instead of (0.4), we get a slight sharpening of their results.

We will now sketch the proof given in [8] that (0.4) implies (0.2). First it is shown that (0.3) implies

$$\int S^2(u) v dx \leq c \int |\nabla u|^2 dx \quad (0.7)$$

where  $S$  is a variant of the Lusin area function. It is well-known that

$$\int |u|^2 w dx \leq \int S^2(u) w dx \quad (0.8)$$

if  $w \geq 0$  satisfies the Muckenhoupt  $A_\infty$ -condition. Set

$$M_p v(x) = \sup_{x \in Q} \left( \frac{1}{|Q|} \int_Q v^p dx \right)^{1/p}.$$

A key point is that  $M_p v$  satisfies the  $A_\infty$ -condition for any  $v$ . If (0.4) holds, then  $M_p v$  also satisfies (0.3). Hence

$$\begin{aligned} \int |u|^2 v dx &\leq \int |u|^2 M_p v dx \\ &\leq c \int S^2(u) M_p v dx \\ &\leq c \int |\nabla u|^2 dx. \end{aligned}$$

A natural question arises from this: for what maximal functions  $\tilde{M}$  do we have

$$\int |u|^2 v \, dx \leq c \int S^2(u) \tilde{M}v \, dx \quad (0.9)$$

with  $c$  independent of  $u$  and  $v$ ? It was suggested in [8] that (0.9) might hold for  $\tilde{M} = M$ , the Hardy–Littlewood maximal function. If this were the case, then (0.5) would imply (0.2) when  $\varphi(x) = 1 + \log^+ x$ . Much of this paper is concerned with inequality (0.9). In Section 1, we show (0.9) does not hold for the Hardy–Littlewood maximal function. We also show that the “converse” inequality

$$\int S^2(u)v \, dx \leq c \int |u|^2 Mv \, dx$$

is true. In Section 2, we give the above mentioned results relating (0.2), (0.5) and (0.6). The examples showing that (0.6) is best possible are based on the counterexample to (0.9) in Section 1. In Section 3, we consider another question, raised by E. Stein (see [7] for related considerations). What is the sharp order of local integrability of a function which has a pointwise bounded  $S$  function? It is easy to see that if  $S(f) \in L^\infty$  then  $f \in BMO$ . Hence  $e^{\alpha |f|/\|S(f)\|_\infty} \in L^1_{loc}$  for a suitable constant  $\alpha > 0$ . But (0.9), if true with  $\tilde{M} = M$ , would have implied  $e^{\alpha |f|^2/\|S(f)\|_\infty^2} \in L^1_{loc}$  for suitable  $\alpha > 0$ . As it turns out this last statement is true despite the failure of (0.9).

Very recently, Kerman and Sawyer [12] gave a real variable necessary and sufficient condition on  $v$  for (0.2) to be true. Define

$$M_1 f(x) = \sup_{x \in Q} \frac{l(Q)}{|Q|} \int_Q |f| \, dt.$$

Then (0.2) holds if and only if

$$\int_Q (M_1(\chi_Q v))^2 \, dx \leq c \int_Q v \, dx \quad \text{for all cube } Q. \quad (0.10)$$

It is clear that our conditions (0.5), (0.6) must imply (0.10), and therefore that our results in Section 2 could be derived from theirs. However, it does not seem trivial to show directly that (0.5), (0.6) implies (0.10) (in fact we do not know how to do this!). So we will give our original arguments. In Section 4, we make some remarks about (0.10) and conditions of type (0.5). Specifically, if  $v$  satisfies (0.2) and  $A_\infty$ , then  $v$  satisfies the Kerman–Sawyer condition (0.10).

The argument we give for Theorem 3.1 is due to H. Rubin; we are grateful for his permission to include it. An alternate proof and generalization of our results in Section 2 has recently been given by S. Chanillo [5].

Some additional notation is as follows.  $Q$  will always be a dyadic cube in  $\mathbb{R}^d$ .  $Q_0$  is the unit cube  $\{(x_1, x_2, \dots, x_d) : 0 \leq x_i \leq 1\}$ .

We are grateful to C. Fefferman, who pointed out the questions in this paper to us.

## Section 1

Let  $\psi \in C^\infty(\mathbb{R}^d)$  be real, radial,  $\text{supp } \psi \subseteq \{|x| \leq 1\}$  and  $\int \psi = 0$ . For  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ , we set:

$$S_\psi^2(f)(x) = \int_{|x-t|<|y|} |f * \psi_y(t)|^2 \frac{dt dy}{y^{d+1}}.$$

Let  $M$  denote the Hardy–Littlewood maximal function. We prove the following:

**THEOREM 1.1.** *There exists a  $C = C(d)$  so that for all  $w \geq 0$  in  $L^1_{\text{loc}}(\mathbb{R}^d)$  and all  $f \in L^1_{\text{loc}}(\mathbb{R}^d)$  we have:*

$$\int S_\psi^2(f)w dx \leq C \int |f|^2 Mw dx.$$

**THEOREM 1.2.** *There exists no  $C$  such that*

$$\int |f|^2 w dx \leq C \int S_\psi^2(f)Mw dx$$

for all  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $w \geq 0$  in  $L^1_{\text{loc}}(\mathbb{R}^d)$ .

**Remark.** The proof of Theorem 1 still works if we replace  $S$  by the “polydisc”  $S$ -function, and  $M$  by the corresponding “strong” maximal function as described in [4].

**Proof of Theorem 1.1.** For  $k = 0, \pm 1, \pm 2, \dots$  let  $E^k = \{Mw > 2^k\}$ . Let  $B(x, r)$  denote the ball in  $\mathbb{R}^d$  centered at  $x$  of radius  $r$ . Define

$$A^k = \{(t, y) \in \mathbb{R}_+^{d+1}, B(t, y) \subset E^k\}.$$

Set

$$\Phi(x, y) = \frac{1}{|B(x, y)|} \int_{B(x, y)} w \, dt$$

Observe that  $\Phi \leq 2^{k+1}$  in  $A^k \setminus A^{k+1}$ . We have

$$\begin{aligned} \int S_\psi^2(f) w \, dx &= \int_{\mathbb{R}^d} w(x) \left( \int_{|x-t|< y} |f * \psi_y(t)|^2 \frac{dt \, dy}{y^{d+1}} \right) dx \\ &= C \int_{\mathbb{R}^{d+1}} |f * \psi_y(t)|^2 \Phi(t, y) \frac{dt \, dy}{y} \\ &= C \sum_{-\infty}^{\infty} \int_{A^k \setminus A^{k+1}} |f * \psi_y(t)|^2 \Phi(t, y) \frac{dt \, dy}{y} \\ &\leq C \sum_{-\infty}^{\infty} 2^k \int_{A^k} |f * \psi_y(t)|^2 \frac{dt \, dy}{y} \end{aligned}$$

Now note that when  $(t, y) \in A^k$ ,  $\text{supp } \psi_y(t - \cdot) \subset E^k$ . Thus:

$$\begin{aligned} \int_{A^k} |f * \psi_y(t)|^2 \frac{dt \, dy}{y} &= \int_{A^k} |f \chi_{E^k} * \psi_y(t)|^2 \frac{dt \, dy}{y} \\ &\leq \int_{\mathbb{R}_+^{d+1}} |f \chi_{E^k} * \psi_y(t)|^2 \frac{dt \, dy}{y} \\ &= C \|f \chi_{E^k}\|_2^2 = C \int_{E^k} |f|^2 \, dx \end{aligned}$$

where the next to last inequality is by Plancherel's theorem.

Then, the sum is bounded by:

$$\begin{aligned} &\leq C \sum_{-\infty}^{\infty} 2^k \int_{E^k} |f|^2 \, dx \\ &= C \int |f|^2 \sum_{-\infty}^{\infty} 2^k \chi_{E^k}(x) \, dx \\ &= C \int |f|^2 Mw \, dx. \end{aligned}$$

*Proof of Theorem 1.2.* We give the counterexample for  $d = 1$ . Let  $N$  be large.

Set  $w(x) = 2^N \chi_{\{|x| \leq 2^{-N}\}}$ . For  $k = 0, 1, 2, \dots$  define:

$$a_k(x) = \begin{cases} 1 & |x| \leq 2^{-k-1} \\ -1 & 2^{-k-1} < |x| \leq 2^{-k} \\ 0 & \text{otherwise} \end{cases}$$

Let

$$f(x) = \sum_{k=1}^{N-1} \frac{a_k(x)}{N-k}$$

(this  $f \notin \mathcal{S}$ , but that is unimportant:  $f$  is smooth enough to be nicely approximable). Clearly:

$$\int |f|^2 w dx \sim (\log N)^2$$

We now estimate  $\int S_\psi^2(f) Mw dx$ . Let

$$R(t, y) = \frac{1}{2y} \int_{t-y}^{t+y} Mw dx$$

For  $k = 0, 1, \dots, N$  let  $T^k = \{(t, y) : (t - y, t + y) \subset [-2^{-k}, 2^{-k}]\}$ , i.e., the “tent” over  $\{|x| \leq 2^{-k}\}$ . Note that  $R(t, y) \leq C(N - k + 1)2^k$  in  $T^k \setminus T^{k+1}$ ,  $k < N$ ,  $R(t, n) \leq C2^N$  in  $T^{N+1}$ . Thus:

$$\begin{aligned} \int S^2(f) Mw dx &= C \int_{\mathbb{R}_+^2} |f * \psi_y(t)|^2 R(t, y) \frac{dt dy}{y} \\ &= C \int_{(T^0)^c} |f * \psi_y(t)|^2 R(t, y) \frac{dt dy}{y} \\ &\quad + C \sum_{k=0}^N \int_{T^k \setminus T^{k+1}} |f * \psi_y(t)|^2 R(t, y) \frac{dt dy}{y} \\ &\quad + \int_{T^{N+1}} (|f * \psi_y(t)|^2 R(t, y)) \frac{dt dy}{y}. \end{aligned}$$

The last term is 0, since  $f * \psi_y(t) = 0$  for  $(t, y) \in T^{N+1}$  ( $f$  is constant on  $\text{supp } \psi_y(t - \cdot)$  and  $\int \psi = 0$ ).

For the first term we note that

$$\|f\|_2^2 = \sum_{k=0}^{N-1} \frac{2^{-k+1}}{(N-k)^2} \leq \frac{C}{N^2}$$

$$R(t, n) \leq CN \quad \text{on } (T^0)^c$$

Thus:

$$\int_{(T_0^c)} |f * \psi_y(t)|^2 R(t, y) \frac{dt dy}{y} \leq \frac{C}{N}$$

So we only need consider the middle terms. For each  $k$ , when  $(t, y) \in T^k \setminus T^{k+1}$ , we have

$$f * \psi_y(t) = \left( \sum_{j=k}^{N-1} \frac{a_j(\cdot)}{N-j} \right) * \psi_y(t)$$

since the terms for  $j < k$  are constant on  $[t-y, t+y]$ . Therefore:

$$\int_{T^k \setminus T^{k+1}} |f * \psi_y(t)|^2 \frac{dt dy}{y} \leq \sum_{j=k}^{N-1} \frac{2^{-j+1}}{(N-j)^2} \leq \frac{C 2^{-k}}{(N-k)^2}$$

The estimate for  $R(t, y)$  implies that:

$$\begin{aligned} \sum_{k=0}^N \int_{T^k \setminus T^{k+1}} |f * \psi_y(t)|^2 R(t, y) \frac{dt dy}{y} &\leq C \sum_{k=0}^{N-1} \frac{1}{N-k} \\ &\leq C \log N \end{aligned}$$

which proves Theorem 2.

The proof works because the coefficients of the  $a_k(x)$  run “backwards”: they get larger as the “frequencies” of the  $a_k$  get higher. This is also the idea (sort of) behind the counterexample in Section 2.

## Section 2

In this section, we will establish a theorem which indicates that we may put an average condition on  $v$  ( $v \geq 0$ ) which is slightly stronger than the  $L \log L$  condi-

tion as stated in the introduction (yet weaker than the  $L^p$ -condition for all  $p > 1$ ), with the Schrödinger operator  $L = -\Delta - v$  a positive operator.

**THEOREM 2.1.** *Suppose  $\varphi$  is an increasing function  $[0, \infty) \rightarrow [1, \infty)$  with  $\int_1^\infty \frac{dx}{x\varphi(x)} \leq c < \infty$ , then for every positive function  $v$  on  $\mathbb{R}^d$  ( $d \geq 3$ ) which satisfies:*

$$(*)_\varphi: \frac{1}{|Q|} \int_Q v(x) l(Q)^2 \varphi(v(x) l(Q)^2) dx \leq 1$$

for all cube  $Q$  in  $\mathbb{R}^d$  then

$$\int_{\mathbb{R}^d} u^2(x) v(x) dx \leq C_1 \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \quad \text{for all } u \in C_0^\infty(\mathbb{R}^d) \quad (2.1)$$

where  $C_1$  is a constant depends only on  $C_0$ .

**Remark.** The function  $\varphi(x) = 1 + (\log^+ x)^{1+\varepsilon}$ , or  $1 + (\log^+ x)(\log \log^+ x)^{1+\varepsilon}$ ,  $\varepsilon > 0$ ; all satisfy the condition  $\int_1^\infty \frac{dx}{x\varphi(x)} < \infty$ , but not the function  $\varphi(x) = 1 + (\log^+ x)$ .

The dyadic analogue of the theorem is a little easier to verify, to pass from the dyadic case to the “continuous” case, we will apply the following lemma of C. Fefferman ([8]; Lecture II, Lemma B).

First we will explain some notations used in [8]. For  $Q$  a dyadic cube, define

$H_+^Q$  = space of functions supported in  $Q$ , and linear + constant on each of the dyadic subcubes obtained by bisecting  $Q$ .

$H_0^Q$  = space of functions supported in  $Q$ , and linear + constant on all of  $Q$ .

$H^Q$  = the orthogonal complement of  $H_0^Q$  in  $H_+^Q$ .

For every  $u \in L^2(\mathbb{R}^d)$ , we write  $u = \sum_Q \hat{u}(Q)$  where  $\hat{u}(Q)$  denotes the orthogonal projection of  $L^2$  onto  $H^Q$ . Define  $\|u\| = \sum_Q (\text{diam } Q)^{-2} \|\hat{u}(Q)\|_2^2$  then

**LEMMA [8].**  $\|u\|_2^2 \leq C \|\nabla u\|_2^2$  for all  $u \in C_0^\infty(\mathbb{R}^d)$ ,  $C$  a universal constant.

**Proof of Theorem 2.1.** For each integer  $n$ , we let  $E_n$  denote the set  $\{x \in \mathbb{R}^d, 2^{2(n-1)} < v(x) \leq 2^{2n}\}$ . Let  $\mathfrak{I}_n$  denote the collection of dyadic cubes of length

$l(Q) = 2^{-n}$ . Then for each  $u \in C_0^\infty(\mathbb{R}^d)$ , we have

$$\begin{aligned} \int_{\mathbb{R}^d} u^2(x)v(x) dx &\leq \sum_{n=-\infty}^{\infty} 2^{2n} \int_{E_n} u^2(x) dx \\ &= \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \int_{E_n \cap Q_n} u^2(x) dx \\ &\leq 2 \sum_{-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \int_{E_n \cap Q_n} \left( \sum_{Q \subseteq Q_n} \hat{u}(Q) \right)^2 + \left( \sum_{Q \supseteq Q_n} \hat{u}(Q) \right)^2 \\ &= I + II \end{aligned}$$

where

$$\begin{aligned} I &= 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \int_{E_n \cap Q_n} \left( \sum_{Q \subseteq Q_n} \hat{u}(Q) \right)^2 dx \\ II &= 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \int_{E_n \cap Q_n} \left( \sum_{Q \supsetneq Q_n} \hat{u}(Q) \right)^2 dx. \end{aligned}$$

For the term  $I$ , we have the direct estimate:

$$\begin{aligned} I &\leq 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \sum_{Q \subseteq Q_n} \|\hat{u}(Q)\|_2^2 \\ &= 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{l(Q) \leq 2^{-n}} \|\hat{u}(Q)\|_2^2 \\ &= 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{k=0}^{\infty} \sum_{l(Q)=2^{-(n+k)}} \|\hat{u}(Q)\|_2^2 l(Q)^{-2} 2^{-2(n+k)} \\ &= 2 \sum_{k=0}^{\infty} 2^{-2k} \sum_{n} \sum_{Q_0 \in \mathfrak{D}_{n+k}} \|\hat{u}(Q)\|_2^2 l(Q)^{-2} \\ &\leq 2 \left( \sum_{k=0}^{\infty} 2^{-2k} \right) \sum_Q \|\hat{u}(Q)\|_2^2 l(Q)^{-2} \leq C \|\nabla u\|_2^2 \quad \text{by the lemma.} \end{aligned}$$

To estimate the second term, we notice that for each fixed dyadic cube  $Q_n \in \mathfrak{D}_n$ , if  $Q$  is dyadic and  $Q \supsetneq Q_n$  then  $\hat{u}(Q)$  is a linear function on  $Q_n$  thus  $(\sup_{x \in Q_n} |\hat{u}(Q)(x)|)^2 = \|\hat{u}(Q)|_{Q_n}\|_\infty^2 \leq C_d \|\hat{u}(Q)\|_2^2 l(Q)^{-d}$  for each such  $Q$  and for some constant  $C$  depending only on the dimension  $d$ . Let  $a_Q = \|\hat{u}(Q)|_{Q_n}\|_\infty$  then

$$\begin{aligned} II &\leq 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} \int_{E_n \cap Q_n} \left( \sum_{Q \supsetneq Q_n} a_Q \right)^2 dx \\ &= 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{D}_n} |E_n \cap Q_n| \left( \sum_{Q \supsetneq Q_n} a_Q \right)^2 \end{aligned}$$

We now notice that since  $v$  satisfies the  $(*)_\varphi$  condition, we have for every dyadic  $Q$  in  $\mathbb{R}^d$

$$\frac{1}{(l(Q))^{d-2}} \sum_{n=-\infty}^{\infty} 2^{2n} |Q \cap E_n| \varphi(2^{2n}l(Q)^2) \leq 1. \quad (2.2)$$

Thus if we let  $C_{Q,n} = \varphi(2^{2n}l(Q)^2)$ , then

$$\begin{aligned} \sum_{R \supseteq Q_n} \frac{1}{C_{R,n}} &= \sum_{R \supseteq Q_n} \frac{1}{\varphi(2^{2n}l^2(R))} \\ &= \sum_{j=1}^{\infty} \frac{1}{\varphi(2^{2j})} \leq \int_1^{\infty} \frac{dx}{x\varphi(x)} \leq C_0. \end{aligned}$$

Hence

$$\begin{aligned} II &\leq 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{Q}_n} |E_n \cap Q_n| \left( \sum_{Q \supseteq Q_n} a_Q^2 C_{Q,n} \right) \sum_{R \supseteq Q_n} \frac{1}{C_{R,n}} \\ &\leq 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{Q}_n} |E_n \cap Q_n| \sum_{Q \supseteq Q_n} a_Q^2 C_{Q,n} \\ &= 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q \text{ dyadic}} a_Q^2 C_{Q,n} \sum_{\substack{Q_n \in \mathfrak{Q}_n \\ Q_n \subset Q}} |E_n \cap Q_n| \\ &= 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_Q a_Q^2 C_{Q,n} |Q \cap E_n| \\ &= 2C_0 \sum_Q a_Q^2 \sum_{n=-\infty}^{\infty} 2^{2n} \varphi(2^{2n}l(Q)^2) |Q \cap E_n| \\ &\leq 2C_0 \sum_Q a_Q^2 l(Q)^{d-2} \quad \text{by (2.2)} \\ &\leq 2C_0 C_d \sum_Q \|\hat{u}(Q)\|_2^2 l(Q)^{-d} l(Q)^{d-2} \\ &\leq 2C_0 C_d \sum_Q \|\hat{u}(Q)\|_2^2 l(Q)^{-2} \\ &= 2C_0 C_d \|u\|_2^2 \leq \tilde{C}_1 \|\nabla u\|_2^2 \quad (\text{by the lemma}) \quad \tilde{C}_1 = \text{constant} \cdot C_0 C_d. \end{aligned}$$

Combining the estimate in  $I$  and  $II$ , we get

$$\int_{\mathbb{R}^d} u^2(x)v(x) dx \leq I + II \leq C_1 \|\nabla u\|_2^2 \quad (C_1 = \tilde{C}_1 + \text{constant}).$$

This finishes the proof of the theorem.

As in ([8], p. 145) we have

**COROLLARY 2.1.** *Suppose  $\varphi$  is an increasing function with  $\varphi(x) \geq 1$ ,  $\int_1^\infty \frac{dx}{x\varphi(x)} = C_0 < \infty$ , and also  $\varphi(4x) \leq C_2\varphi(x)$  for all  $x \in \mathbb{R}^d$  for some  $C_2$ . Then there exists some constant  $C_3$ , such that if  $Av_Q(vl(Q)^2\varphi(vl(Q)^2)) \leq C_3$  for all cube  $Q$  in  $\mathbb{R}^d$ , then  $L = -\Delta - v \geq 0$ .*

*Proof.*  $L$  is positive is equivalent to the inequality  $\int u^2 v \, dx \leq \int |\nabla u|^2$  for all  $u \in C_0^\infty(\mathbb{R}^d)$ . Thus it follows from the result of the theorem (and the fact  $\tilde{C}_1 = C_1 + \text{constant}$ ) that there exists some constant  $C_4, C_5$  (depending only on  $C_0$  and the dimension  $d$ ) such that if  $\tilde{v}$  satisfies

$$(*)_\varphi C_4: \quad Av_Q(\tilde{v}l(Q)^2\varphi(\tilde{v}l(Q)^2)) \leq C_4$$

then  $L_1 = -\Delta - C_5\tilde{v}$  is positive. Thus if  $\varphi$  satisfies the additional hypothesis  $\varphi(4x) \leq C_2\varphi(x)$ , then we can choose some constant  $C_3$  ( $C_3 \leq C_4 C_5 C_2^{\log_4 C_5}$ ) such that if  $v$  satisfies the condition  $(*)_\varphi(C_3)$ , and if we set  $\tilde{v} = (1/C_5)v$  then for each cube  $Q$  in  $\mathbb{R}^d$ ,

$$\begin{aligned} Av_Q(\tilde{v}l(Q)^2\varphi(\tilde{v}l(Q)^2)) &= \frac{1}{C_5} Av_Q\left(vl(Q)^2\varphi\left(\frac{1}{C_5}vl(Q)^2\right)\right) \\ &\leq \frac{1}{C_5} C_2^{-\log_4 C_5} Av_Q(vl(Q)^2\varphi(vl(Q)^2)) \\ &\leq \frac{1}{C_5} C_2^{-\log_4 C_5} C_3 \leq C_4. \end{aligned}$$

Thus  $L = -\Delta - v = -\Delta - C_5\tilde{v} = L_1$  is a positive operator. If we translate the result in Corollary 1, we obtain the following estimate of  $\lambda_1(L)$  (the first non-positive eigenvalue of  $L$ ) as in the case of Theorem 5 in ([8], p. 145).

**COROLLARY 2.2.** *With the same assumption as in Corollary 2.1 about  $\varphi$ , there are constants  $C, c$  depending only on  $d, \varphi$  such that  $cE_{\text{sm}} \leq -\lambda_1(L) \leq CE_{\text{big}}$ , where*

$$E_{\text{sm}} = \sup_Q [Av_Q v - cl(Q)^{-2}]$$

$$E_{\text{big}} = \sup_Q [Av_Q v\varphi(vl(Q)^2) - cl(Q)^{-2}].$$

With some minor changes, one can check that the proof of Theorem 6 in ([8], page 154 on) works for the  $(*)_\varphi$  version exactly as in the  $L^p$ -version, and obtain the following parallel result:

**PROPOSITION 2.3.** *With the same assumption about  $\varphi$  as in Corollary 2.1, if we assume in addition that  $\varphi(4x) \leq C_2\varphi(x)$  with  $C_2 < 2^{d-2}$ . Then there exists constant  $c$ ,  $C$  (depending on  $\varphi$ , and  $d$ ) so that if  $L = -\Delta - v$  has at least  $CN$  negative eigenvalues, then there is a collection of pairwise disjoint cubes  $Q_1, \dots, Q_N$  for which*

$$Av_{Q_i}(v\varphi(vl(Q_i)^2)) \geq Cl(Q_i)^{-2}.$$

*Remark.*  $\varphi(x) = (\log(c_d + x))^{1+\varepsilon}$  for  $\varepsilon > 0$  would satisfy the assumption of the proposition for some constant  $c_d$  depending on dimension  $d$ .

In the second part of this section, we will describe a counterexample for the question posed in the introduction.

We first remark that in  $\mathbb{R}^d$ , when the dimension  $d$  is one or two, the condition  $(*)$   $((*)_\varphi$  is the  $(*)_\varphi$  condition for  $\varphi(x) = \log^+ x$ )

$$(*) \quad \frac{1}{|Q|} \int_Q v(x)l(Q)^2 \log^+ (v(x)l(Q)^2) dx \leq C$$

for all cubes  $Q$  in  $\mathbb{R}^d$ , with  $l(Q)$  = side length of  $Q$ , implies that  $v \equiv 0$ . To see this fact, we notice that the condition  $(*)$  is equivalent to

$$\int_Q v(x) \log^+ (v(x)l^2(Q)) dx \leq \frac{C}{l(Q)^{2-d}}.$$

Thus for  $d = 1$  or  $2$ , and for cubes  $Q$  with  $l(Q) \geq 1$  we have  $\int_Q v(x) \log^+ (v(x)l^2(Q)) dx \leq C$ . If for each positive integer  $k$  we let  $E_k = \left\{x \in \mathbb{R}^d : v(x) \geq \frac{1}{k}\right\}$ . Then we get  $\int_{Q \cap E_k} v(x) dx \leq \frac{C}{\log l(Q)}$  for all cubes  $Q$  with  $l(Q) \geq k$ . Choosing an increasing sequence of cubes  $Q$  to cover  $E_k$  we may conclude  $E_k = \emptyset$  for each  $k$ . Thus  $v \equiv 0$ . In fact, when  $d = 1$  or  $2$ ;  $\lambda \geq 0$  and  $v \geq 0$  satisfies some mild decay conditions the operator  $-\Delta - \lambda v$  always has a negative eigenvalue. (Unique negative eigenvalue if  $\lambda$  is small enough.) In particular it is never positive. For more precise statements of this fact and other results of properties of eigenvalues of  $-\Delta - \lambda v$  when  $d = 1, 2$ , the reader is referred to [13], [16]. (The authors would like to thank the referee for pointing out these for us.)

Thus for all  $v$  satisfies (\*),

$$\int_{\mathbb{R}^d} u^2(x)v(x) dx \leq C \int_{\mathbb{R}^d} |\nabla u|^2 dx$$

for all  $u \in C_0^\infty(\mathbb{R}^d)$  for the trivial reason. If in this case ( $d = 1$  or  $2$ ), we restrict the condition (\*) to a bounded subset  $D$  in  $\mathbb{R}$  (or  $\mathbb{R}^2$ ), that is if we assume:

$$(*)' \quad \frac{1}{|Q|} \int_Q v(x)l(Q)^2 \log^+ (v(x)l(Q)^2) dx \leq C$$

for all cubes  $Q$  in  $\mathbb{R}^d$  contained in a fixed bounded set  $D \subset \mathbb{R}^d$  then  $(*)'$  implies in particular (when  $d = 1$ , or  $2$ )

$$(*)'' \quad \int_D v(x) \log^+ v(x) dx \leq C.$$

Thus for all  $u$ ,  $C^\infty$  with compact support contained in  $D$ , we have

$$\begin{aligned} \int u^2(x)v(x) dx &\leq \left( \int_D e^{\alpha u^2(x)/\int_D |\nabla u(x)|^2 dx} + \frac{1}{\alpha} \int_D v(x) \log^+ v(x) dx \right) \int_D |\nabla u(x)|^2 dx \\ &\leq C' \int_D |\nabla u(x)|^2 dx \end{aligned}$$

for some universal constant  $\alpha$  (which depends only on the Lebesgue measure  $D$ ) and for some constant  $C$ . The existence of the constants  $\alpha$ ,  $C$  is a special case of the work of N. Trudinger [17] on the sharper form of the Sobolov inequality.

In  $\mathbb{R}^d$  with  $d \geq 3$ , we will construct some  $v$  which satisfies the condition (\*), yet  $\int_{\mathbb{R}^d} u^2(x)v(x) dx / \int_{\mathbb{R}^d} |\nabla u|^2 dx$  fails to be bounded for all  $u \in C_0^\infty(\mathbb{R}^d)$ . The construction is a variation of the counterexample for the inequality (1.2) given in the previous section. This example also shows that the condition  $\int_1^\infty \frac{dx}{x\varphi(x)} < \infty$  stated in Theorem 2.1 is essentially the best possible condition.

**EXAMPLE I.** Suppose  $\varphi : [1, \infty) \rightarrow [1, \infty)$  is an increasing function with  $\varphi(4x) \leq 2^{d-2}\varphi(x)$  and with  $\int_1^\infty \frac{dx}{x\varphi(x)}$  divergent. Then there exists a positive

function  $v$  on  $\mathbb{R}^d$  which satisfies

$$(*)_{\varphi} \quad \sup_{\substack{Q \\ \text{cubes in } \mathbb{R}^d}} \frac{1}{|Q|} \int_Q v(x) l^2(Q) \varphi(v(x) l^2(Q)) \, dx \leq 1.$$

Yet the weak type (2.2) bound

$$v(\{u \geq n\}) \leq \frac{A}{n^2} \int_{\mathbb{R}^d} |\nabla u|^2 \quad (2.3)$$

fails for all positive constant  $A < \infty$  and for some function  $u$  (which depends on  $A$ ) in  $C_0^\infty(\mathbb{R}^d)$ .

*Construction of the example.* The assumption  $\int_1^\infty \frac{dx}{x\varphi(x)}$  diverges is equivalent to the condition that  $\sum_{n=1}^\infty \frac{1}{\varphi(2^{2n})}$  diverges. We will also assume without loss of generality that  $\varphi(1) = 1$ , and  $d = 3$  (the example works for any  $d \geq 3$  with suitable changes of constants). Let  $\tau(n) = \varphi(2^{2n})$   $n = 0, 1, 2, \dots$ , and  $m$  be a fixed constant (so that  $\sum_{j < m} \frac{1}{\tau(j)}$  is big). For each  $0 \leq j \leq m$ , define a collection  $\mathfrak{G}_j$  of dyadic cubes in  $\mathbb{R}^3$  recursively as follows:

$$\mathfrak{G}_0 = \{\text{unit cube } Q_0 \text{ in } \mathbb{R}^3\}, \quad N_0 = 1$$

$\mathfrak{G}_j$  is a collection of  $N_j$  dyadic cubes in  $\mathbb{R}^3$ ,

each with side length  $2^{-j}$ , satisfying the following properties:

- (a) Each cube in  $\mathfrak{G}_j$  is a subcube of some cube in  $\mathfrak{G}_{j-1}$ .
- (b) Each  $\mathfrak{G}_{j-1}$  cube contains  $c_j$   $\mathfrak{G}_j$ -cubes, where  $c_j$  is either one or two and depends only on  $j$ .
- (c)  $c_j$  is chosen in such a way so that  $\frac{1}{2} \cdot 2^j \frac{\tau(m-j)}{\tau(m)} \leq N_j \leq 2 \cdot 2^j \frac{\tau(m-j)}{\tau(m)}$ .
- (c) can be done because by our assumption on  $\varphi$ ,  $\tau(m-j) \leq \tau(m-(j-1)) \leq 2\tau(m-j)$ , e.g. if  $\mathfrak{G}_{j-1}$  has been defined, let  $c_j = 1$  if  $N_{j-1} > 2^j \frac{\tau(m-j)}{\tau(m)}$  and  $c_j = 2$

otherwise. Then if  $c_j = 1$ , we get

$$\begin{aligned} 2^j \frac{\tau(m-j)}{\tau(m)} &\leq 2^j \frac{\tau(m-(j-1))}{\tau(m)} \leq N_{j-1} \leq 2 \cdot 2^{j-1} \frac{\tau(m-(j-1))}{\tau(m)} \\ &\leq 2 \cdot 2^j \frac{\tau(m-j)}{\tau(m)} \quad \text{and} \quad N_j = N_{j-1}. \end{aligned}$$

If  $c_j = 2$ , then  $N_j = 2N_{j-1}$  and

$$\frac{1}{2} \cdot 2^j \frac{\tau(m-j)}{\tau(m)} \leq 2^{j-1} \frac{\tau(m-(j-1))}{\tau(m)} \leq 2N_{j-1} \leq 2 \cdot 2^j \frac{\tau(m-j)}{\tau(m)}.$$

We terminate the process when  $j = m$  and define  $v = 2^{2m}$  on those  $\mathfrak{G}_m$  cubes and  $v = 0$  otherwise. We will now verify that  $\frac{v}{c}$  satisfies the condition  $(*)_\varphi$  for some constant  $c$ . Note that it suffices (up to a change of the constant  $c$ ) to verify  $(*)_\varphi$  for all dyadic cubes. To see  $(*)_\varphi$  for dyadic cubes, fix a dyadic cube  $Q$  in  $\mathbb{R}^3$ , and assume, say,  $Q$  has side length  $l(Q) = 2^{-j} \leq 1$ . Then  $Q$  contains either 0 or  $\frac{N_m}{N_j}$   $\mathfrak{G}_m$ -cubes. Thus it follows from condition (c) that

$$\begin{aligned} \frac{1}{|Q|} \int_Q v(x) l_Q^2 \varphi(v(x) l_Q^2) dx &= 2^{3j} \cdot 2^{2m} \cdot 2^{-2j} \frac{N_m}{N_j} 2^{-3m} \varphi(2^{2(m-j)}) \\ &= \frac{2^{-m} N_m}{2^{-j} N_j} \tau(m-j) \leq 4. \end{aligned}$$

For dyadic cubes  $Q$  with the side length  $2^j \geq 1$  ( $j \geq 0$ ), either  $Q$  contains 0 or (if  $Q$  contains the unit cube in  $\mathbb{R}^3$ )  $N_m$   $\mathfrak{G}_m$ -cubes. So

$$\begin{aligned} \frac{1}{|Q|} \int_Q v(x) l_Q^2 \varphi(v(x) l_Q^2) dx &= \frac{1}{2^j} \cdot 2^{-m} N_m \varphi(2^{2(m+j)}) \\ &\leq \frac{2}{\tau(m)} \cdot \frac{2^j \varphi(2^{2m})}{2^j} = \frac{2}{\tau(m)} \cdot \tau(m) = 2. \end{aligned}$$

(The last inequality follows by our assumption on  $\varphi$ .) Then  $\frac{v}{4}$  satisfies the condition  $(*)_\varphi$  for all dyadic cubes in  $\mathbb{R}^3$ .

Fix a constant  $n > 0$ , and constant  $A > 0$ , we will now define some function  $u$ . Let

$$d_j = \frac{n}{\sum_{1 \leq l \leq m} (\tau(l))^{-1}} (\tau(m-j))^{-1} \quad 0 \leq j \leq m-1.$$

We now define  $u$  as follows. For each cube  $Q \in \mathfrak{I}_j$  ( $0 \leq j \leq m-1$ ), let  $u_j^Q$  be a  $C^\infty$  function with compact support contained in  $\tilde{Q}$  (recall that  $\tilde{Q} = 3Q$ ) satisfying:  $u_j^Q(x) = 1$  on  $\bigcup_{Q' \in \mathfrak{I}_{j+1}} \tilde{Q}'$ , and  $|\nabla u_j^Q(x)| \leq C2^j$ ,  $0 \leq u_j^Q \leq 1$ . Define

$$u_j = \sum_{l=1}^{N_j} \left( \prod_{k < l} (1 - \varphi_j^{Q_k}) \right) \varphi_j^{Q_l}$$

where  $Q_1, Q_2, \dots, Q_{N_j}$  are the collection of cubes in  $\mathfrak{I}_j$ . Since only finitely many  $\varphi_j^{Q_k}$ 's can be simultaneously nonzero,  $|\nabla u_j|$  is bounded by  $C2^j$  ( $c$  depends only on the dimension  $d$ ), and  $u_j = 1$  on  $\bigcup_{Q' \in \mathfrak{I}_{j+1}} \tilde{Q}'$ ,  $u_j = 0$  on  $\bigcup_{Q \in \mathfrak{I}_j} \tilde{Q}$ . Define  $u = \sum_{j=0}^{m-1} d_j u_j$ , then on the set  $\bigcup_{Q \in \mathfrak{I}_j} \tilde{Q} \setminus \bigcup_{Q \in \mathfrak{I}_{j+1}} \tilde{Q}$ , only  $u_j$  is not a constant, hence  $|\nabla u(x)| \leq d_j$ ,  $|\nabla u_j(x)| \leq Cd_j 2^j$  on  $\bigcup_{Q \in \mathfrak{I}_j} \tilde{Q} \setminus \bigcup_{Q \in \mathfrak{I}_{j+1}} \tilde{Q}$ . And

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx &= \sum_{j=0}^{m-1} \int_{\bigcup_{Q \in \mathfrak{I}_j} \tilde{Q} \setminus \bigcup_{Q \in \mathfrak{I}_{j+1}} \tilde{Q}} |\nabla u(x)|^2 \\ &\leq C \sum_{j=0}^{m-1} (d_j 2^j)^2 \cdot N_j \cdot 2^{-3j} \\ &= C \sum_{j=0}^{m-1} \frac{n^2}{\left( \sum_{l=1}^m (\tau(l))^{-1} \right)^2} (\tau(m-j))^{-2} N_j 2^{-j} \\ &\leq 2C \sum_{j=0}^{m-1} \frac{n^2}{\left( \sum_{l=1}^m (\tau(l))^{-1} \right)^2} (\tau(m-j))^{-2} \frac{\tau(m-j)}{\tau(m)} \\ &= 2cn^2 \frac{1}{\tau(m)} \frac{1}{\left( \sum_{l=1}^m (\tau(l))^{-1} \right)^2}. \end{aligned}$$

While

$$u = \sum_{j=0}^{m-1} d_j = n \quad \text{on} \quad \bigcup_{Q \in \mathfrak{I}_m} \tilde{Q}.$$

Thus  $v(\{u \geq n\}) \geq 2^{2m} N_m 2^{-3m} \geq \frac{1}{2} \frac{1}{\tau(m)}$ . So if we choose  $m$  large enough, with  $\sum_{l=1}^m (\tau(l))^{-1} \geq 5cA$ , then the inequality  $v(\{u \geq n\}) \leq A \frac{1}{n^2} \int |\nabla u|^2$  would fail for such functions  $u$ , which finishes the construction of the example.

It turns out the constant  $2^{d-2}$  is critical in Example I in the following sense.

**EXAMPLE II.** For each  $C > 2^{d-2}$ , there exists an increasing function  $\varphi$  with  $\varphi(4x) \leq C\varphi(x)$  for all  $x$  and  $\int_1^\infty \frac{dx}{x\varphi(x)} = \infty$ . Yet for all positive function  $v$  which satisfies the  $(*)_\varphi$  condition, we have

$$\int_{\mathbb{R}^d} u^2(x)v(x) dx \leq A \int_{\mathbb{R}^d} |\nabla u(x)|^2$$

for some finite constant  $A$  (which depends on  $C$ ), and for all  $u \in C_0^\infty(\mathbb{R}^d)$ .

*Construction.* We will assume  $d = 3$ , and  $C > 2$ . The construction is based on the following observation: Suppose  $v$  satisfies the  $(*)_\varphi$  condition, then for each integer  $k \geq 1$ , we have

$$\frac{1}{|Q|} \int_Q v(x)l(Q)^2 \frac{\varphi(4^k v(x)l(Q)^2)}{2^k} dx \leq 1$$

for each cube  $Q$  (which was obtained from the  $(*)_\varphi$  condition by looking at cube  $Q^*$  containing  $Q$  with  $l(Q^*) = 2^k l(Q)$ ). Thus if we can construct some  $\varphi$  with

$$\sum_{k=0}^{\infty} \left(\frac{2}{C}\right)^k \frac{\varphi(4^k x)}{2^k}$$

converges uniformly, and call the function  $\psi(x)$ , then  $v \in (*)&_\varphi$  implies  $v \in (*)&_{(C-2/C)\psi}$ . For the purpose of our example, we will construct some  $\varphi$  with

- (a)  $\sum_{k=0}^{\infty} \frac{1}{\varphi(4^k)} = \infty$
- (b)  $\varphi(4^{k+1}) \leq c\varphi(4^k)$  for each  $k$
- (c)  $\psi(x) = \sum_{k=0}^{\infty} \left(\frac{2}{C}\right)^k \frac{\varphi(4^k x)}{2^k}$  converges uniformly and satisfies  $\sum_{k=0}^{\infty} \frac{1}{\psi(4^k)} < \infty$ .

Applying Theorem 2.1, we can then conclude that for such  $\varphi$

$$\int u^2 v \leq A \int |\nabla u|^2 \quad \forall u \in C_0^\infty(\mathbb{R}^3).$$

Construct  $\varphi$  as follows: Choose sequence of integers  $0 = k_0 < k_1 < \dots < k_n < \dots$ ; with  $E_m = [k_{2m}, k_{2m+1}]$ ,  $0_m = [k_{2m+1}, k_{2m+2}]$  even and odd intervals respectively for  $m = 0, 1, 2, \dots$ . And define  $\varphi(4^k) = C^{|E_0|+|E_1|+\dots+|E_m|}$  for  $k \in 0_m$  and  $\varphi(4^k) = C^{|E_0|+|E_1|+\dots+|E_{m-1}|} C^{k-k_{2m}}$  and define  $\varphi$  to be linear in between the interval  $4^k$  to  $4^{k+1}$ . Then (b) is satisfied by the way  $\varphi$  is defined. To check (a), we let  $|E_0| = 1$ ,  $|0_0| = [C^{|E_0|}] + 1$ , and  $|0_m| = [C^{|E_0|+\dots+|E_m|}] + 1$ ,  $|E_{m+1}| = [C^{|0_m|}] + 1$  respectively for  $m = 0, 1, 2, \dots$  ( $[x]$  denote the greatest integer  $\leq x$ .) Then

$$(a) \quad \sum_{\substack{k \in \bigcup E_m \\ 0}} \frac{1}{\varphi(4^k)} = \sum_{m=0}^{\infty} \sum_{k \in E_m} \frac{1}{C^{k-k_{2m}}} \frac{1}{C^{|E_0|+|E_1|+\dots+|E_{m-1}|}} \\ \leq 2 \sum_{m=0}^{\infty} \frac{1}{C^{m-1}} < \infty$$

$$\sum_{\substack{k \in \bigcup 0_m \\ 0}} \frac{1}{\varphi(4^k)} = \sum_{m=0}^{\infty} \frac{|0_m|}{C^{|E_0|+\dots+|E_m|}} = \infty \quad \text{by our choice of } 0_m.$$

And

$$(c) \quad \sum_{k=0}^{\infty} \frac{\varphi(4^k)}{C^k} = \sum_{m=0}^{\infty} \sum_{k \in E_m} \frac{C^{k-k_{2m}} C^{|E_0|+\dots+|E_{m-1}|}}{C^k} + \sum_{m=0}^{\infty} \sum_{k \in 0_m} \frac{C^{|E_0|+\dots+|E_m|}}{C^k} \\ \leq \sum_{m=0}^{\infty} \frac{|E_m|}{C^{|0_0|+|0_1|+\dots+|0_{m-1}|}} \\ + \sum_{m=0}^{\infty} \frac{1}{C^{|0_0|+\dots+|0_{m-1}|}} \left( 1 + \frac{1}{C} + \dots + \frac{1}{C^{|0_m|}} \right) \\ \leq \frac{2C}{C-1} \left( \sum_{m=0}^{\infty} \frac{1}{C^m} \right) \leq 2 \left( \frac{C}{C-1} \right)^2.$$

Thus

$$\psi(4^n) = \sum_{k=0}^{\infty} \frac{\varphi(4^k 4^n)}{C^k} \leq C^n \sum_{k=0}^{\infty} \frac{\varphi(4^k)}{C^k} < \infty$$

for each  $n$  and  $\sum_{k=0}^{\infty} \frac{\varphi(4^k x)}{C^k}$  converges uniformly to  $\psi(x)$  for all  $x \geq 1$  while  $\psi \geq \varphi$

and

$$\sum_{\substack{n \in \bigcup E_m \\ 0}} \frac{1}{\psi(4^n)} \leq \sum_{\substack{n \in \bigcup E_m \\ 0}} \frac{1}{\varphi(4^n)} < \infty.$$

For  $n \in 0_m$ , we have

$$\begin{aligned}
 \psi(4^n) &= \sum_{k=0}^{\infty} \frac{\varphi(4^{k+n})}{C^k} \geq \sum_{k+n \in E_{m+1}} \frac{\varphi(4^{k+n})}{C^k} \\
 &\geq \sum_{k+n \in E_{m+1}} C^{|E_0|+|E_1|+\dots+|E_m|} \frac{C^{k+n-k_{2m+2}}}{C^k} \\
 &\geq (|0_m|-1) \sum_{\substack{k \\ k+n \in E_{m+1}}} C^{n-k_{2m+2}} \\
 &\geq |0_m| C^{|0_m|} C^{n-k_{2m+2}} = |0_m| C^{n-k_{2m+1}}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 (c) \quad \sum_{\substack{n \in \bigcup_{m=0}^{\infty} 0_m}} \frac{1}{\psi(4^n)} &= \sum_{m=0}^{\infty} \sum_{n \in 0_m} \frac{1}{\psi(4^n)} \\
 &\leq \sum_{m=0}^{\infty} \sum_{n \in 0_m} \frac{1}{|0_m|} \frac{1}{C^{n-k_{2m+1}}} \\
 &= \sum_{m=0}^{\infty} \frac{1}{|0_m|} \left( 1 + \frac{1}{C} + \dots + \frac{1}{C^{|0_m|}} \right) \\
 &\leq \frac{C}{C-1} \sum_{m=0}^{\infty} \frac{1}{|0_m|} < \infty.
 \end{aligned}$$

Thus  $\varphi$  and  $\psi$  satisfies the estimates (a), (b), (c) and we have established the example as desired.

### Section 3

We recall that  $Q_0$  is the unit dyadic cube in  $\mathbb{R}^d$ . First we consider dyadic martingales on  $Q_0$ . Let  $\mathcal{G}_n$  be the  $\sigma$ -algebra generated by the 2nd dyadic subcubes on  $Q_0$  of sidelength  $2^{-n}$  and let  $E(f, \mathcal{G}_n)$  be the conditional expectation of  $f$  on  $\mathcal{G}_n$  (i.e.  $E(f, \mathcal{G}_n)(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} f$  where  $x \in Q_n(x)$  and  $l(Q_n(x)) = 2^{-n}$ ). By a dyadic martingale we mean a sequence of functions  $\{f_n\}$  from  $Q_0$  to  $\mathbb{R}$  such that  $f_n$  is  $\mathcal{G}_n$ -measurable and  $E(f_{n+1}, \mathcal{G}_n) = f_n$ . Our martingales will (almost) all be  $L^2$  bounded and we denote the limit function by  $f$ . We define the martingale

difference functions  $s_n$  ( $n \geq 0$ ) by  $s_n = f_{n+1} - f_n$ . Thus  $s_n$  is  $\mathcal{G}_{n+1}$ -measurable. The square functions of the martingale  $\{f_n\}$  is defined by

$$Sf(x) = \sum_{n>0} \|\chi_{Q_n(x)} S_n\|_\infty^2$$

where  $Q_n(x)$  is the unique dyadic cube of length  $2^{-n}$  containing  $x$ . This is not standard. One generally sees  $Sf(x) = (\sum_n E(S_n^2, \mathcal{G}_n)(x))^{1/2}$ ; this definition agrees with ours only when  $d = 1$ , although when  $d$  is fixed the two are always comparable. Our results work out nicer with our definition of  $Sf$ ; with the other definition the “ $\frac{1}{2}$ ” in (3.1) below would have to be replaced by a constant depending on  $d$ . Our result for dyadic martingales is as follows.

**THEOREM 3.1.** *Let  $\{f_n\}$  be a dyadic martingale on  $Q_0 \subset \mathbb{R}^d$  with limit function  $f$ . Suppose  $\|Sf\|_\infty < \infty$ . Then for  $\lambda \geq 0$ ,*

$$|\{x \in Q_0 : f(x) - f_0(x) \geq \lambda\}| \leq \exp(-\frac{1}{2}\lambda^2/\|Sf\|_\infty^2). \quad (3.1)$$

The proof we give for this is due to Herman Rubin; it replaces a much longer argument of the authors. It is based on the fundamental identity of sequential analysis in statistics.

*Proof of (3.1) (Rubin).* We can assume  $f_0 = 0$ . Fix  $t > 0$ . Define  $q_n : Q_0 \rightarrow \mathbb{R}$  by

$$q_n = e^{tf_n} \left( \prod_{j=1}^{n-1} E(e^{ts_j}, \mathcal{G}_j) \right)^{-1}.$$

These  $q_n$  form a martingale! Clearly  $q_n$  is  $\mathcal{G}_n$ -measurable, and we have

$$\begin{aligned} E(q_{n+1}, \mathcal{G}_n) &= E\left(e^{tf_{n+1}} \left( \prod_{j=1}^n E(e^{ts_j}, \mathcal{G}_j) \right)^{-1}, \mathcal{G}_n\right) \\ &= E\left(e^{ts_n} e^{tf_n} \left( \prod_{j=1}^n E(e^{ts_j}, \mathcal{G}_j) \right)^{-1}, \mathcal{G}_n\right) \\ &= E(e^{ts_n} (E(e^{ts_n}, \mathcal{G}_n))^{-1} q_n, \mathcal{G}_n) \\ &= \dot{E}(e^{ts_n}, \mathcal{G}_n) (E(e^{ts_n}, \mathcal{G}_n))^{-1} q_n \\ &= q_n. \end{aligned}$$

It follows that  $\int q_n = 1$  for all  $n$ . (This is the fundamental identity of sequential

analysis.) Using the elementary inequalities

$$\int e^\phi d\mu \leq \cosh(\|\phi\|_\infty) \leq \exp(\frac{1}{2}\|\phi\|_\infty^2)$$

valid when  $\mu$  is a probability measure and  $\int \phi d\mu = 0$ , we find that

$$E(e^{tS_i}, \mathcal{G}_i)(x) \leq \exp(\frac{1}{2}t^2 \|\chi_{Q_i(x)} S_i\|_\infty^2).$$

So  $\prod_{i=1}^{n-1} E(e^{tS_i}, \mathcal{G}_i)(x) \leq e^{t^2/2 \|Sf\|_\infty^2}$  for all  $n$  and  $x$ , and the fundamental identity now implies that  $\int_{Q_0} e^{t f_n} \leq \exp(\frac{1}{2}t^2 \|Sf\|_\infty^2)$  for all  $n$ . Letting  $n$  go to  $\infty$  gives  $\int_{Q_0} e^{t f} \leq \exp(\frac{1}{2}t^2 \|Sf\|_\infty^2)$ . Now take  $t = \frac{\lambda}{\|Sf\|_\infty}$  and apply Tsebyshev's inequality to get (3.1).

We record some corollaries, all of which follow from Theorem 3.1 by standard arguments.

**COROLLARY 3.1.** *We have the good  $\lambda$  inequality*

$$|\{x \in Q_0 : f^*(x) > 2\lambda, Sf(x) < \varepsilon\lambda\}| \leq C \exp\left(-\frac{1}{2} \frac{(1-\varepsilon)^2}{\varepsilon^2} |\{x \in Q_0 : f^*(x) > \lambda\}|\right),$$

with  $C$  a universal constant.

*Proof* [1]. Define a stopping time  $\tau_\lambda = \min(\{n : \sum_{j \leq n} (f_{j+1} - f_j)^2 \geq \varepsilon^2 \lambda^2\})$ . Let  $\mathbb{G} = \{Q\}$  be the maximal dyadic cubes with  $f_Q > \lambda$ . For each  $Q \in \mathbb{G}$  consider  $\{x \in Q : f^*(x) > 2\lambda, Sf(x) < \varepsilon\lambda\}$ . If this is non-empty then  $f_Q < (1+\varepsilon)\lambda$ . Also

$$\begin{aligned} |\{x \in Q : f^*(x) > 2\lambda, Sf(x) < \varepsilon\lambda\}| &= |\{x \in Q : f^*(x) > 2\lambda \text{ and } \tau_\lambda = \infty\}| \\ &\leq |\{x \in Q : (f(\tau_\lambda) - f_Q)^* > (1-\varepsilon)\lambda\}| \end{aligned}$$

and  $\|S(f(\tau_\lambda))\|_\infty < \varepsilon\lambda$  pointwise. By (3.1) and Lemma 3.1,

$$|\{x \in Q : f^*(x) > 2\lambda, Sf(x) < \varepsilon\lambda\}| \leq C \exp\left(-\frac{1}{2} \frac{(1-\varepsilon)^2 \lambda^2}{\varepsilon^2 \lambda^2}\right) |Q|$$

and if we sum over  $Q$  we are done.

Integrating out the good  $-\lambda$  inequality gives the bizarre estimate

$$\int_{Q_0} \exp(c_1(f^*)^2/1 + (Sf)^2) \leq c_2 \int_{Q_0} (1 + \log^+ f^*)$$

with  $c_1 > 0$ ,  $c_2 < \infty$ . We can also obtain the following “law of the iterated logarithm” for dyadic martingales.

**COROLLARY 3.2.** *If  $\{f_n\}$  is a dyadic martingale on  $Q_0$ , then*

$$\limsup_{n \rightarrow \infty} \frac{|f_n(x)|}{S^n f(x) \sqrt{2 \log \log S^n f(x)}} \leq 1$$

a.e. on the set where  $\{f_n\}$  is unbounded. (Here  $(S^n f(x))^2 = \sum_{k=0}^{n-1} \|\chi_{Q_k(x)} S_k\|_\infty^2$ .)

This may be proved like [6], Section 7.5 using Theorem 3.1 instead of the central limit theorem. We omit the proof since Corollary 3.2 is a minor modification of known results (due to W. F. Stout; see [15] and the references there). We’re grateful to R. Banuelos and C. Mueller for pointing out the known results, especially to Banuelos for reference [15]. We note that the corresponding lower bound

$$\limsup_{n \rightarrow \infty} \frac{|f_n|}{S^n \sqrt{2 \log \log S^n}} \geq 1$$

is false in our context.

Now we prove a version of Theorem 3.1 in which the Lusin area integral replaces  $S$ . Let

$$p(x) = \Gamma\left(\frac{d+1}{2}\right) / (\pi^{(d+1)/2} (1 + |x|^2)^{(d+1)/2})$$

and  $p_t(x) = t^{-d} p(x/t)$ ,  $t > 0$ , so that  $F(x, t) = p_t * f(x)$  is the harmonic extension of  $f$  to  $\mathbb{R}_+^{d+1}$ . Let  $\psi$  be the vector valued function  $\nabla_x p$  and  $\psi_t(x) = t^{-d} \psi(x/t) = t \nabla_x p_t(x)$ . For  $0 < \gamma < \infty$  define

$$\begin{aligned} A_\gamma f(x) &= \left( \int_{T_\gamma(x)} |\nabla_y f(y, t)|^2 t^{1-d} dy dt \right)^{1/2} \\ &= \left( \int_{T_\gamma(x)} |\psi_t * f(y)|^2 t^{-1-d} dy dt \right)^{1/2} \end{aligned}$$

where  $\Gamma_\gamma(x) = \{(y, t) \in \mathbb{R}_+^{d+1} : |x - y| < \gamma t\}$ . We then have

**THEOREM 3.2.** *Suppose  $A_\gamma f \in L^\infty$ . Then*

$$\sup_{Q: \text{cube}} \frac{1}{|Q|} \int_Q \exp \left( c_1 \frac{|f - f_Q|^2}{\|A_\gamma f\|_\infty^2} \right) < c_2,$$

where  $c_1 > 0$  and  $c_2 < \infty$  depend on  $d$  and  $\gamma$ .

Some remarks about this. First, P. Jones [11] (private communication) has shown that it is possible for a function  $f$  to satisfy  $A_{\gamma_1} f \in L^\infty$ ,  $A_{\gamma_2} f \notin L^\infty$ , for two numbers  $0 < \gamma_1 < \gamma_2 < \infty$ . Second, regarding the proof of Theorem 3.2, it is a reduction to the dyadic case using one of the tools available for such purposes, a decomposition from [2] (see also [4]). This grind it out approach has the drawback that we do not obtain a sharp (or even dimension independent) constant like the  $\frac{1}{2}$  in Theorem 3.1. Also it does not seem to let us replace  $A_\gamma$  by the  $g$  function. On the other hand it does apply to square functions formed using fairly general kernels; for example, we could take  $p$  to be a nonnegative radial Schwarz function instead of the Poisson kernel and the same proof would work.

We note that R. Banuelos (personal communication) has proved results similar to Theorem 3.2 by probabilistic arguments, which give sharp constants.

We give two lemmas. The first is a version of the Calderon–Torchinsky machinery [3] and the second is a variant of a trick due to S. Janson.

**LEMMA 3.1.** *If  $\rho > 0$  there is a smooth, radial function  $K$  supported in  $\{x : |x| \leq \rho\}$  such that if we define  $q(x) = \nabla K(x)$  and  $q_t(x) = t^{-d}q(x/t)$ , then*

$$\lim_{T \rightarrow 0} \int_{\substack{t > T \\ y \in \mathbb{R}^d}} \langle \psi_t * f(x - y), q_t(y) \rangle dy \frac{dt}{t} = f(x)$$

for all  $f \in L^2$  with compact support. The limit is (say) in the  $L^2$  sense.

*Proof* [3]. Choose  $K_0$  supported in  $|x| \leq \rho$ , smooth and radial and such that  $\int K_0(x)(-\Delta)^{1-d/2}p(x) \neq 0$ . Let  $K = cK_0$  where  $c$  is a suitable constant. Let  $v(\xi) = (2\pi |\xi|)^2 \int_{t=0}^\infty \hat{K}(t\xi) \hat{p}(t\xi) t dt$ . Then  $v$  is radial and homogeneous of degree zero, hence constant. It follows from the Plancherel theorem that  $\int_{|\xi|=1} v(\xi) d\xi \neq 0$ ; hence we can (and do) choose  $c$  so that  $v \equiv 1$ . Then

$$\left( \int_{t>T} \langle \psi_t * f(x - y), q_t(y) \rangle dy \frac{dt}{t} \right)^\wedge(\xi) = \hat{f}(\xi) \int_{t>T} (2\pi |\xi|)^2 \hat{p}(t\xi) \hat{K}(t\xi) t dt \rightarrow \hat{f}$$

in  $L^2$  as  $T \rightarrow 0$ .

Let  $x \in \mathbb{R}^d$ . In the next lemma and subsequently, we let  $\mathfrak{J}^x$  denote the set of dyadic cubes translated by  $x$ , i.e.  $\mathfrak{J}^x = \{Q \subseteq \mathbb{R}^d : \{y : y + x \in Q\} \in \mathfrak{J}\}$ .

**LEMMA 3.2.** *There is a number  $N = N(d)$  with the following property: if  $A < \infty$ , then the set  $\tilde{\mathfrak{G}} = \{Q : Q \in \mathfrak{J} \text{ and } l(Q) \leq 2^A\}$  may be decomposed,  $\tilde{\mathfrak{G}} = \bigcup_{j=1}^N \mathfrak{G}^{(j)}$ , and there exist  $x_1 \dots x_N \in \mathbb{R}^d$  such that*

*If  $Q \in \mathfrak{G}^{(j)}$  then  $\tilde{Q}$  is contained in a cube  $Q' \in \mathfrak{J}^{x_j}$  with  $l(Q') = 8l(Q)$ . (3.1)*

*If  $Q_1, Q_2 \in \mathfrak{G}^{(j)}$  and  $Q_1 \neq Q_2$ , then  $Q'_1 \neq Q'_2$ . (3.2)*

*Proof.* By scaling we can assume  $A = -3$ . We need only obtain (3.1) since if (3.1) holds and if  $J \in \mathfrak{J}^{x_j}$  is given there can be at most  $8^d$  cubes  $Q$  with  $Q \in \mathfrak{G}^{(j)}$  and  $Q' = J$ . So we can get (3.2) by further decomposing each  $\mathfrak{G}^{(j)}$ .

We first show (3.1) when  $d = 1$ . Let  $N = 2$ ,  $x_1 = \frac{1}{3}$ ,  $x_2 = 0$ . If  $Q \in \tilde{\mathfrak{G}}$  then  $Q = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$  for some  $n \geq 3$ ,  $K \in \mathbb{Z}$ . Let  $Q \in \mathfrak{G}^{(1)}$  if  $K = 0$  or  $1 \pmod{8}$ ,  $Q \in \mathfrak{G}^{(2)}$  otherwise. Then (3.1) is clear for  $j = 2$ . Suppose  $Q \in \mathfrak{G}^{(1)}$ . We must show  $\tilde{Q}$  contains no point of the form  $q2^{-(n-3)} + \frac{1}{3}$ ,  $q \in \mathbb{Z}$ . Suppose  $\tilde{Q}$  did contain such a point. We know  $\tilde{Q} \subset [p2^{-(n-3)} - 2^{-(n-1)}, p2^{-(n-3)} + 2^{-(n-1)}]$  for some  $p \in \mathbb{Z}$ . So  $p - \frac{1}{4} \leq \frac{2^{n-3}}{2} + q \leq p + \frac{1}{4}$ , contradiction.

When  $d > 1$  any  $Q \in \tilde{\mathfrak{G}}$  is uniquely  $Q = I_1^{(Q)} \times \dots \times I_d^{(Q)}$  where each  $I_i^{(Q)} \subset \mathbb{R}$ . If  $\sigma \subset \{1, \dots, d\}$  let  $\mathfrak{G}^{(\sigma)} = \{Q \in \tilde{\mathfrak{G}} : I_i^{(Q)} \in \mathfrak{G}^{(1)} \text{ if } i \in \sigma, I_i^{(Q)} \in \mathfrak{G}^{(2)} \text{ if } i \notin \sigma\}$ . Let  $x_\sigma$  be the point whose  $i$ th coordinate  $x_\sigma(i)$  satisfies  $x_\sigma(i) = \frac{1}{3}$  if  $i \in \sigma$ ,  $x_\sigma(i) = 0$  if  $i \notin \sigma$ . Then the decomposition  $\tilde{\mathfrak{G}} = \bigcup \mathfrak{G}^{(\sigma)}$  and the points  $x_\sigma$  satisfy (3.1).

*Proof of Theorem 3.2.* We can assume  $Q = Q_0$ ,  $f_{Q_0} = 0$ . Let  $\varphi \in C_0^\infty$  with  $\varphi(x) = 1$  if  $x \in 5Q_0$ ,  $\varphi(x) = 0$  if  $x \notin 7Q_0$ . Let  $h = \varphi f$ . We must show  $\int_{Q_0} e^{c_1 h^2} \leq c_2$ . Choose  $K$  by Lemma 3.1 with  $\rho = \gamma/4$ . Form the corresponding  $q$ . Let  $S = \{(x, t) \in \mathbb{R}_+^{d+1} : x \in \tilde{Q}_0, t \leq \rho^{-1}\}$ . Write

$$\begin{aligned} h(x) &= \int_S \langle \psi_t * h(y), q_t(x, y) \rangle dy \frac{dt}{t} + \int_{\mathbb{R}_+^{d+1} \setminus S} \langle \psi_t * h(y), q_t(x, y) \rangle dy \frac{dt}{t} \\ &= h_1(x) + h_2(x). \end{aligned}$$

The integrand giving  $h_2(x)$  vanishes when  $x \in Q_0$  and  $t$  is sufficiently small.

Hence for  $x \in Q_0$ ,

$$\begin{aligned} |h_2(x)| &\leq \int_{t \geq t_0} \|\psi_t * h\|_1 \|q_t\|_\infty dy \frac{dt}{t} \\ &\leq C \|h\|_1 \int_{t \geq t_0} t^{-d} \frac{dt}{t} \\ &\leq C \|f\|_{BMO} \leq C \|A_\gamma f\|_\infty. \end{aligned}$$

Also note  $y \in \tilde{Q}_0$  implies by standard arguments that  $|\psi_t * ((\varphi - 1)f)(y)| \leq Ct^2 \|f\|_{BMO}$ . So for any  $x$ ,

$$\begin{aligned} &\left| h_1(x) - \int_S \langle \psi_t * f(y), q_t(x-y) \rangle dy \frac{dt}{t} \right| \\ &= \left| \int_S \langle \psi_t * ((\varphi - 1)f)(y), q_t(x-y) \rangle dy \frac{dt}{t} \right| \\ &\leq \|f\|_{BMO} \leq C \|A_\gamma f\|_\infty. \end{aligned}$$

So it suffices to estimate

$$\Lambda(x) = \int_S \langle \psi_t * f(y), q_t(x-y) \rangle dy \frac{dt}{t}.$$

For  $Q$  dyadic with  $l(Q) \leq 1$ , let  $T_Q = S \cap \left\{ (y, t) : y \in Q, \frac{l(Q)}{2\rho} \leq t \leq \frac{l(Q)}{\rho} \right\}$ .

Define  $\lambda_Q(x) = \int_{T_Q} \langle \psi_t * f(y), q_t(x-y) \rangle dy \frac{dt}{t}$ . Then

$$\Lambda(x) = \sum_{l(Q) \leq 1} \lambda_Q(x). \tag{3.3}$$

It is easy to see that

$$\lambda_Q \text{ is supported on } \tilde{Q} \tag{3.4}$$

$$\int \lambda_Q = 0 \tag{3.5}$$

$$\|\lambda_Q\|_{\text{Lip } \alpha}^2 l(Q)^{2\alpha} \leq Cl(Q)^{-(d+1)} \int_{T_Q} |\psi_t * f|^2 dy dt, \quad \text{if } 0 < \alpha < 1 \text{ (say).}$$

The last inequality together with the choice of  $\rho$  implies that for  $x \in \mathbb{R}^d$  fixed,

$$\sum_{x \in Q} \|\lambda_Q\|_{\text{Lip } \alpha}^2 l(Q)^{2\alpha} \leq CB^2, \quad 0 < \alpha < 1$$

where  $B = \|A_\gamma f\|_\infty$ . So we are reduced to proving the following.

**LEMMA 3.3.** *If  $\Lambda$  has a decomposition (3.3) satisfying (3.4), (3.5), and (3.6) for some  $\alpha$ ,  $0 < \alpha < 1$ , then*

$$\int_{Q_0} e^{c_1 f^2/B^2} < c_2 < \infty.$$

*Proof.* Choose  $N$ ,  $\mathbb{G}^{(j)}$ ,  $\{x_j\}_1^N$  as in Lemma 3.2 with  $A = 1$ . Write  $\Lambda = \sum_1^N \Lambda_j$  where  $\Lambda_j = \sum_{Q \in \mathbb{G}^{(j)}} \lambda_Q$ . It suffices to estimate  $\Lambda_j$ , so fix  $j$  and let  $g(x) = \Lambda_j(x + x_j)$ . Then  $g = \sum_{l(Q') \leq 8} g_{Q'}$ , where  $g_{Q'} = \lambda_Q$  with  $Q$  and  $Q'$  related as in (3.1), (3.2). These  $g_{Q'}$  satisfy (3.5), (3.6) and a stronger form of (3.4):  $g_{Q'}$  is supported on  $Q'$ . Write  $g_{Q'}$  as a dyadic martingale,  $g_{Q'} = \sum_{Q \subseteq Q'} g_Q^Q$  where  $g_Q^Q$  is supported on  $Q$  and constant on cubes of length  $\frac{1}{2}l(Q)$ , and  $\int g_Q^Q = 0$ . This gives the representation of  $g$  as dyadic martingale,

$$g = \sum_{l(Q) \leq 8} \sum_{\substack{Q' \supseteq Q \\ l(Q') \leq 8}} g_{Q'}^Q = \sum_{l(Q) \leq 8} g^Q$$

By well-known properties of Lipschitz functions ([14]),  $\|g_{Q'}^Q\|_\infty \leq C \|g_{Q'}\|_{\text{Lip } \alpha} l(Q)^{-\alpha}$ . Let  $\beta \in (0, \alpha)$ . If  $x$  is fixed, then

$$\begin{aligned} \sum_{x \in Q} \|g^Q\|_\infty^2 &\leq \sum_{x \in Q} \left( \sum_{Q' \supseteq Q} \|g_{Q'}^Q\|_\infty \right)^2 \\ &\leq C \sum_{x \in Q} l(Q)^{2\alpha} \left( \sum_{Q' \supseteq Q} \|g_{Q'}\|_{\text{Lip } \alpha} \right)^2 \\ &\leq C \sum_{x \in Q} l(Q)^{2(\alpha-\beta)} \sum_{Q' \supseteq Q} \|g_{Q'}\|_{\text{Lip } \alpha}^2 l(Q')^{2\beta} \\ &\leq C \sum_{x \in Q'} l(Q')^{2\beta} \|g_{Q'}\|_{\text{Lip } \alpha}^2 \sum_{x \in Q \subseteq Q'} l(Q)^{2(\alpha-\beta)} \\ &\leq C \sum_{x \in Q'} l(Q')^{2\alpha} \|g_{Q'}\|_{\text{Lip } \alpha}^2 \\ &\leq CB^2. \end{aligned}$$

We can now apply Theorem 3.1 to conclude that  $\int_{Q_0} e^{c_1 g^2/B^2} \leq c_2$ , so the lemma is proved.

This finishes the proof of Theorem 3.2. Quite likely good  $\lambda$  inequalities analogous to Corollary 3.1 may also be obtained; however, we have not done this. But we want to point out two immediate corollaries.

**COROLLARY 3.3.** *If  $A_\gamma f \in L^\infty$ , then  $f$  belongs to the closure of  $L^\infty$  in BMO.*

This follows from [9] and Theorem 3.2. In fact if  $A_\gamma f \in L^\infty$ , then  $f$  exists in  $[L^\infty, BMO]_{1/2,\infty}$ : This follows from Theorem 3.2 and unpublished results of S. Janson [10]. Using that the full gradient is invariant by the Hilbert transform we also have the following: if  $F$  is bounded and harmonic on  $\mathbb{R}_+^2$  and  $\sup_{x \in \mathbb{R}} \int_{\Gamma_\gamma(x)} |\nabla f(y, t)|^2 dy dt < \infty$ , then  $f$  may be approximated in  $L^\infty$  norm by real parts of  $H^\infty$  functions. Back on  $\mathbb{R}^d$ , if  $\tilde{S}f$  denotes either  $A_\gamma f$  or  $Sf$ , we also have

**COROLLARY 3.4.** *If  $w \geq 0$  and  $f$  are given, then*

$$\int_Q (f - f_Q)^2 w \leq C \|\tilde{S}f\|_\infty^2 \int_Q w \left(1 + \log^+ \frac{w}{w_Q}\right)$$

for any cube  $Q$  for which the right hand side is finite.

This follows from Theorem 3.1 or 3.2 by Young's inequality. Note the formal similarity between Corollary 3.4 and the (false) analogue of Theorem 2.1 where  $\varphi(x) = 1 + \log^+ x$ .

## Section 4. Appendix

In this section we show that if  $v \geq 0$  has the Muckenhoupt  $A_\infty$  condition and satisfies

$$\sup_Q \frac{l(Q)^2}{|Q|} \int_Q v dx < \infty \tag{4.1}$$

then  $v$  has the Kerman–Sawyer property (0.10).

Recall the  $v$  is said to have  $A_\infty$  if  $\forall \varepsilon > 0$  there is a  $\delta > 0$  so that for all cubes  $Q$

and all measurable sets  $E \subset Q$ ,

$$\frac{|E|}{|Q|} < \delta \Rightarrow \frac{\int_E v}{\int_Q v} < \varepsilon.$$

An easy consequence of this is that there is a  $C < \infty$  so that

$$\int_Q M(\chi_Q v) dx \leq C \int_Q v dx \quad \forall Q \quad (4.2)$$

where  $M$  is the Hardy-Littlewood maximal function. We shall assume that  $v$  satisfies (4.1) and (4.2).

A collection of dyadic cubes  $\{Q_\alpha\}$  is said to satisfy the Carleson nesting condition if there is an  $A < \infty$  so that for all cubes  $Q_\alpha$  from the collection, we have:

$$\sum_{Q_{\alpha'} \subset Q_\alpha} |Q_{\alpha'}| \leq A |Q_\alpha| \quad (4.3)$$

It is a fact that if  $\{Q_\alpha\}$  has (4.3) then for all  $f \geq 0$  is  $L^1_{\text{loc}}(\mathbb{R}^d)$  and all cubes  $Q$ :

$$\sum_{Q_\alpha \subset Q} \left( \frac{1}{|Q_\alpha|} \int_{Q_\alpha} f dx \right) |Q_\alpha| \leq CA \int_Q M(\chi_Q f) dx. \quad (4.4)$$

We wish to show that (4.1) and (4.2) imply

$$\int_Q (M_1 |\chi_Q v|)^2 dx \leq c \int_Q v dx \quad (4.5)$$

where  $M_1 f = \sup_{x \in Q} \frac{l(Q)}{|Q|} \int_Q |f| dt$ . It is clearly enough to prove (4.5) when the supremum is taken over dyadic cubes. (We may also take  $Q$  to be dyadic.) Let  $R > 0$  be large. Let  $\{Q_j^k\}$  be the maximal dyadic subcubes of  $Q$  such that

$$\frac{l(Q_j^k)}{|Q_j^k|} \int_{Q_j^k} v dx > R^k \quad (4.6)$$

Then:

$$\begin{aligned}
 \int_Q M_1(\chi_Q v)^2 dx &= \sum_{Q_j^k} \left( \frac{l(Q_j^k)}{|Q_j^k|} \int_{Q_j^k} v \right)^2 |Q_j^k| \\
 &= \sum_{Q_j^k} \left( \frac{1}{|Q_j^k|} \int_{Q_j^k} v \right) \left( \frac{l(Q_j^k)^2}{|Q_j^k|} \int_{Q_j^k} v \right) |Q_j^k| \\
 &\leq C \sum_{Q_j^k} \left( \frac{1}{|Q_j^k|} \int_{Q_j^k} v \right) |Q_j^k|
 \end{aligned}$$

by (4.1). Thus if the  $\{Q_j^k\}$  had (4.3), then (4.2) and (4.4) would imply (4.5). They do. By their maximality, the  $\{Q_j^k\}$  satisfy:

$$\frac{l(Q_j^k)}{|Q_j^k|} \int_{Q_j^k} v dx \leq cR^k \tag{4.7}$$

where  $c$  depends on the dimension  $d$ . Let  $Q_j^{k+1} \subset Q_l^k$ . Combining (4.6) and (4.7) yields:

$$\frac{\frac{1}{|Q_j^{k+1}|} \int_{Q_j^{k+1}} v dx}{\frac{1}{|Q_l^k|} \int_{Q_l^k} v dx} > cR \frac{l(Q_l^k)}{l(Q_j^{k+1})} > 10 \tag{4.8}$$

for  $R$  large enough. This implies that:

$$\sum_{Q_l^{k+1} \subset Q_j^k} |Q_l^{k+1}| \leq \frac{1}{10} |Q_j^k|$$

And by iterating we get:

$$\sum_{n=0}^{\infty} \sum_{Q_p^{k+n} \subset Q_j^k} |Q_p^{k+n}| \leq (1 + \frac{1}{9}) |Q_j^k|$$

which is (4.3).

*Remark* (1). We note that at one point, namely (4.8), our argument is extremely wasteful. This would seem to indicate the delicacy of the Kerman-Sawyer condition.

(2). While this paper was in preparation we learned that Sawyer has independently found another proof that (4.1) and  $A_\infty$  imply (0.10). Sawyer has also found a direct proof that if  $0 \leq V \leq W$  and  $W$  has (0.10), then so does  $V$ .

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Department of Mathematics  
University of California, Los Angeles  
Los Angeles, CA 90024

Department of Mathematics  
University of Wisconsin  
Madison, WI 53233

Department of Mathematics 253-37  
California Institute of Technology  
Pasadena, CA 91125

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