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Some weighted norm inequalities concerning the Schrödinger operators

S. Y. A. CHANG, J. M. WILSON and T. H. WOLFF

Introduction

Let v be a nonnegative, locally integrable function on $\mathbb{R}^d$. Let $L = -\Delta - v$ be the associated Schrödinger operator. If L is essentially selfadjoint on C_0^∞, then positivity of L is equivalent via an integration by parts with

$$\int_{\mathbb{R}^d} |u|^2 v\,dx \le \int_{\mathbb{R}^d} |\nabla u|^2\,dx \qquad \forall\, u \in C_0^\infty. \tag{0.1}$$

In [8], C. Fefferman asks for conditions on $v \ge 0$ which imply

$$\int_{\mathbb{R}^d} |u|^2 v\,dx \le c \int_{\mathbb{R}^d} |\nabla u|^2\,dx \tag{0.2}$$

for some constant c. By considering translates and dilates of a fixed bump function it is clear that a necessary condition for (0.2) is

$$\frac{1}{|Q|}\int_Q v\,dx \le c' l(Q)^{-2} \tag{0.3}$$

for some c', for all cubes $Q \subset \mathbb{R}^d$. ($|Q|$ and $l(Q)$ denote the Lebesgue measure and side length of Q respectively.) Letting $v\,dx$ be (approximately) Lebesgue measure on a codimension 2 hyperplane, we see that (0.3) is not sufficient. In [8], it is shown that a sufficient condition is:

$$\left(\frac{1}{|Q|}\int_Q v^p\,dx\right)^{1/p} \le c_p l(Q)^{-2} \tag{0.4}$$

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for some $p>1$ and $c_p<\infty$. Comparing (0.3) and (0.4) suggests the following question. Let $\varphi:[0,\infty)\to[1,\infty)$ be increasing. When it is the case that

$$(*)_\varphi\colon\ \sup_Q \frac{1}{|Q|}\int_Q v(x)l(Q)^{+2}\varphi(v(x)l(Q)^{+2})\,dx<\infty \tag{0.5}$$

implies (0.2)? We will show that if

$$\int_1^\infty \frac{dx}{x\varphi(x)}<\infty \tag{0.6}$$

then (0.5) implies (0.2), and that this is essentially best possible. In [8], some joint work of C. Fefferman and D. H. Phong is presented in which (0.4) $\Rightarrow$ (0.2) is used also to give bounds on the number of bound states of Schrödinger operators. Using (0.5), (0.6) instead of (0.4), we get a slight sharpening of their results.

We will now sketch the proof given in [8] that (0.4) implies (0.2). First it is shown that (0.3) implies

$$\int S^2(u)v\,dx\le c\int|\nabla u|^2\,dx \tag{0.7}$$

where S is a variant of the Lusin area function. It is well-known that

$$\int |u|^2\,w\,dx\le\int S^2(u)w\,dx \tag{0.8}$$

if $w\ge 0$ satisfies the Muckenhoupt A_∞-condition. Set

$$M_pv(x)=\sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q v^p\,dx\right)^{1/p}.$$

A key point is that M_pv satisfies the A_∞-condition for any v. If (0.4) holds, then M_pv also satisfies (0.3). Hence

$$\begin{aligned}\int|u|^2\,v\,dx&\le\int|u|^2M_pv\,dx\\&\le c\int S^2(u)M_pv\,dx\\&\le c\int|\nabla u|^2\,dx.\end{aligned}$$

A natural question arises from this: for what maximal functions $\tilde{M}$ do we have

$$\int |u|^2 v\,dx \leq c \int S^2(u)\tilde{M}v\,dx \tag{0.9}$$

with c independent of u and v? It was suggested in [8] that (0.9) might hold for $\tilde{M}=M$, the Hardy–Littlewood maximal function. If this were the case, then (0.5) would imply (0.2) when $\varphi(x)=1+\log^+ x$. Much of this paper is concerned with inequality (0.9). In Section 1, we show (0.9) does not hold for the Hardy–Littlewood maximal function. We also show that the "converse" inequality

$$\int S^2(u)v\,dx \leq c \int |u|^2 Mv\,dx$$

is true. In Section 2, we give the above mentioned results relating (0.2), (0.5) and (0.6). The examples showing that (0.6) is best possible are based on the counter-example to (0.9) in Section 1. In Section 3, we consider another question, raised by E. Stein (see [7] for related considerations). What is the sharp order of local integrability of a function which has a pointwise bounded S function? It is easy to see that if $S(f)\in L^\infty$ then $f\in BMO$. Hence $e^{\alpha |f|/\|S(f)\|_\infty}\in L^1_{\mathrm{loc}}$ for a suitable constant $\alpha>0$. But (0.9), if true with $\tilde{M}=M$, would have implied $e^{\alpha |f|^2/\|S(f)\|_\infty^2}\in L^1_{\mathrm{loc}}$ for suitable $\alpha>0$. As it turns out this last statement is true despite the failure of (0.9).

Very recently, Kerman and Sawyer [12] gave a real variable necessary and sufficient condition on v for (0.2) to be true. Define

$$M_1 f(x)=\sup_{x\in Q}\frac{l(Q)}{|Q|}\int_Q |f|\,dt.$$

Then (0.2) holds if and only if

$$\int_Q (M_1(\chi_Q v))^2\,dx \leq c\int_Q v\,dx \quad \text{for all cube} \quad Q. \tag{0.10}$$

It is clear that our conditions (0.5), (0.6) must imply (0.10), and therefore that our results in Section 2 could be derived from theirs. However, it does not seem trivial to show directly that (0.5), (0.6) implies (0.10) (in fact we do not know how to do this!). So we will give our original arguments. In Section 4, we make some remarks about (0.10) and conditions of type (0.5). Specifically, if v satisfies (0.2) and A_∞, then v satisfies the Kerman–Sawyer condition (0.10).

The argument we give for Theorem 3.1 is due to H. Rubin; we are grateful for his permission to include it. An alternate proof and generalization of our results in Section 2 has recently been given by S. Chanillo [5].

Some additional notation is as follows. Q will always be a dyadic cube in $\mathbb{R}^d$. Q_0 is the unit cube $\{(x_1, x_2, \ldots, x_d): 0 \le x_i \le 1\}$.

We are grateful to C. Fefferman, who pointed out the questions in this paper to us.

Section 1

Let $\psi \in C^\infty(\mathbb{R}^d)$ be real, radial, supp $\psi \subseteq \{|x| \le 1\}$ and $\int \psi = 0$. For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we set:

$$S^2_\psi(f)(x) = \int_{|x-t|<y} |f * \psi_y(t)|^2 \frac{dt\, dy}{y^{d+1}}.$$

Let M denote the Hardy–Littlewood maximal function. We prove the following:

THEOREM 1.1. *There exists a $C = C(d)$ so that for all $w \ge 0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ and all $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ we have:*

$$\int S^2_\psi(f) w \, dx \le C \int |f|^2 \, Mw \, dx.$$

THEOREM 1.2. *There exists no C such that*

$$\int |f|^2 \, w \, dx \le C \int S^2_\psi(f) Mw \, dx$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $w \ge 0$ in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Remark. The proof of Theorem 1 still works if we replace S by the "polydisc" S-function, and M by the corresponding "strong" maximal function as described in [4].

Proof of Theorem 1.1. For $k = 0, \pm 1, \pm 2, \ldots$ let $E^k = \{Mw > 2^k\}$. Let $B(x, r)$ denote the ball in $\mathbb{R}^d$ centered at x of radius r. Define

$$A^k = \{(t, y) \in \mathbb{R}^{d+1}_+, B(t, y) \subset E^k\}.$$

Set

$$\Phi(x, y) = \frac{1}{|B(x, y)|} \int_{B(x,y)} w \, dt$$

Observe that $\Phi \le 2^{k+1}$ in $A^k \backslash A^{k+1}$. We have

$$\begin{aligned} \int S_\psi^2(f) w \, dx &= \int_{\mathbb{R}^d} w(x) \left(\int_{|x-t|<y} |f * \psi_y(t)|^2 \frac{dt\, dy}{y^{d+1}} \right) dx \\ &= C \int_{\mathbb{R}^{d+1}} |f * \psi_y(t)|^2 \, \Phi(t, y) \frac{dt\, dy}{y} \\ &= C \sum_{-\infty}^{\infty} \int_{A^k \backslash A^{k+1}} |f * \psi_y(t)|^2 \, \Phi(t, y) \frac{dt\, dy}{y} \\ &\le C \sum_{-\infty}^{\infty} 2^k \int_{A^k} |f * \psi_y(t)|^2 \frac{dt\, dy}{y} \end{aligned}$$

Now note that when $(t, y) \in A^k$, supp $\psi_y(t - \cdot) \subset E^k$. Thus:

$$\begin{aligned} \int_{A^k} |f * \psi_y(t)|^2 \frac{dt\, dy}{y} &= \int_{A^k} |f \chi_{E^k} * \psi_y(t)|^2 \frac{dt\, dy}{y} \\ &\le \int_{\mathbb{R}_+^{d+1}} |f \chi_{E^k} * \psi_y(t)|^2 \frac{dt\, dy}{y} \\ &= C \, \|f \chi_{E_k}\|_2^2 = C \int_{E_k} |f|^2 \, dx \end{aligned}$$

where the next to last inequality is by Plancherel's theorem.

Then, the sum is bounded by:

$$\begin{aligned} &\le C \sum_{-\infty}^{\infty} 2^k \int_{E^k} |f|^2 \, dx \\ &= C \int |f|^2 \sum_{-\infty}^{\infty} 2^k \chi_{E^k}(x) \, dx \\ &= C \int |f|^2 \, Mw \, dx. \end{aligned}$$

Proof of Theorem 1.2. We give the counterexample for $d = 1$. Let N be large.

Set $w(x)=2^N\chi_{\{|x|\le 2^{-N}\}}$. For $k=0,1,2,\ldots$ define:

$$a_k(x)=\begin{cases}1 & |x|\le 2^{-k-1}\\ -1 & 2^{-k-1}<|x|\le 2^{-k}\\ 0 & \text{otherwise}\end{cases}$$

Let

$$f(x)=\sum_{k=1}^{N-1}\frac{a_k(x)}{N-k}$$

(this $f\notin\mathcal{S}$, but that is unimportant: f is smooth enough to be nicely approximable). Clearly:

$$\int |f|^2\, w\, dx\sim(\log N)^2$$

We now estimate $\int S_\psi^2(f)Mw\,dx$. Let

$$R(t,y)=\frac{1}{2y}\int_{t-y}^{t+y}Mw\,dx$$

For $k=0,1,\ldots,N$ let $T^k=\{(t,y):(t-y,t+y)\subset[-2^{-k},2^{-k}]\}$, i.e., the "tent" over $\{|x|\le 2^{-k}\}$. Note that $R(t,y)\le C(N-k+1)2^k$ in $T^k\setminus T^{k+1}$, $k<N$, $R(t,n)\le C2^N$ in T^{N+1}. Thus:

$$\begin{aligned}\int S^2(f)Mw\,dx&=C\int_{\mathbb{R}_+^2}|f*\psi_y(t)|^2\,R(t,y)\frac{dt\,dy}{y}\\&=C\int_{(T^0)^c}|f*\psi_y(t)|^2\,R(t,y)\frac{dt\,dy}{y}\\&\quad+C\sum_{k=0}^{N}\int_{T^k\setminus T^{k+1}}|f*\psi_y(t)|^2\,R(t,y)\frac{dt\,dy}{y}\\&\quad+\int_{T^{N+1}}(|f*\psi_y(t)|^2\,R(t,y))\frac{dt\,dy}{y}.\end{aligned}$$

The last term is 0, since $f*\psi_y(t)=0$ for $(t,y)\in T^{N+1}$ (f is constant on $\operatorname{supp}\psi_y(t-\cdot)$ and $\int\psi=0$).

For the first term we note that

$$\|f\|_2^2=\sum_{k=0}^{N-1}\frac{2^{-k+1}}{(N-k)^2}\le\frac{C}{N^2}$$

$$R(t,n)\le CN \quad \text{on} \quad (T^0)^c$$

Thus:

$$\int_{(T_0^c)}|f*\psi_y(t)|^2\,R(t,y)\frac{dt\,dy}{y}\le\frac{C}{N}$$

So we only need consider the middle terms. For each k, when $(t,y)\in T^k\setminus T^{k+1}$, we have

$$f*\psi_y(t)=\left(\sum_{j=k}^{N-1}\frac{a_j(\cdot)}{N-j}\right)*\psi_y(t)$$

since the terms for $j<k$ are constant on $[t-y,t+y]$. Therefore:

$$\int_{T^k\setminus T^{k+1}}|f*\psi_y(t)|^2\frac{dt\,dy}{y}\le\sum_{j=k}^{N-1}\frac{2^{-j+1}}{(N-j)^2}\le\frac{C2^{-k}}{(N-k)^2}$$

The estimate for $R(t,y)$ implies that:

$$\begin{aligned}\sum_{k=0}^{N}\int_{T^k\setminus T^{k+1}}|f*\psi_y(t)|^2\,R(t,y)\frac{dt\,dy}{y}&\le C\sum_{k=0}^{N-1}\frac{1}{N-k}\\&\le C\log N\end{aligned}$$

which proves Theorem 2.

The proof works because the coefficients of the $a_k(x)$ run "backwards": they get larger as the "frequencies" of the a_k get higher. This is also the idea (sort of) behind the counterexample in Section 2.

Section 2

In this section, we will establish a theorem which indicates that we may put an average condition on v $(v\ge 0)$ which is slightly stronger than the $L\log L$ condition as stated in the introduction (yet weaker than the L^p-condition for all $p>1$), with the Schrödinger operator $L=-\Delta-v$ a positive operator.

THEOREM 2.1. *Suppose* φ *is an increasing function* $[0,\infty)\to[1,\infty)$ *with* $\int_1^\infty \frac{dx}{x\varphi(x)}\le c<\infty$, *then for every positive function* v *on* $\mathbb{R}^d$ $(d\ge 3)$ *which satisfies:*

$$(*)_\varphi:\ \frac{1}{|Q|}\int_Q v(x)l(Q)^2\varphi(v(x)l(Q)^2)\,dx\le 1$$

for all cube Q *in* $\mathbb{R}^d$ *then*

$$\int_{\mathbb{R}^d} u^2(x)v(x)\,dx\le C_1\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx \quad \text{for all} \quad u\in C_0^\infty(\mathbb{R}^d) \tag{2.1}$$

where C_1 *is a constant depends only on* C_0.

Remark. The function $\varphi(x)=1+(\log^+ x)^{1+\varepsilon}$, or $1+(\log^+ x)(\log\log^+ x)^{1+\varepsilon}$, $\varepsilon>0$; all satisfy the condition $\int_1^\infty \frac{dx}{x\varphi(x)}<\infty$, but not the function $\varphi(x)=1+(\log^+ x)$.

The dyadic analogue of the theorem is a little easier to verify, to pass from the dyadic case to the "continuous" case, we will apply the following lemma of C. Fefferman ([8]; Lecture II, Lemma B).

First we will explain some notations used in [8]. For Q a dyadic cube, define

H_+^Q = space of functions supported in Q, and linear + constant on each of the dyadic subcubes obtained by bisecting Q.
H_0^Q = space of functions supported in Q, and linear + constant on all of Q.
H^Q = the orthogonal complement of H_0^Q in H_+^Q.

For every $u\in L^2(\mathbb{R}^d)$, we write $u=\sum_Q \hat{u}(Q)$ where $\hat{u}(Q)$ denotes the orthogonal projection of L^2 onto H^Q. Define $|||u|||=\sum_Q (\operatorname{diam} Q)^{-2}\,\|\hat{u}(Q)\|_2^2$ then

LEMMA [8]. $|||u|||_2^2\le C\,\|\nabla u\|_2^2$ for all $u\in C_0^\infty(\mathbb{R}^d)$, C a universal constant.

Proof of Theorem 2.1. For each integer n, we let E_n denote the set $\{x\in\mathbb{R}^d, 2^{2(n-1)}<v(x)\le 2^{2n}\}$. Let $\mathfrak{I}_n$ denote the collection of dyadic cubes of length

$l(Q)=2^{-n}$. Then for each $u\in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} u^2(x)v(x)\,dx &\le \sum_{n=-\infty}^{\infty} 2^{2n}\int_{E_n} u^2(x)\,dx\\ &= \sum_{n=-\infty}^{\infty} 2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\int_{E_n\cap Q_n} u^2(x)\,dx\\ &\le 2\sum_{-\infty}^{\infty} 2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\int_{E_n\cap Q_n}\Big(\sum_{Q\subseteq Q_n}\hat u(Q)\Big)^2+\Big(\sum_{Q\supsetneqq Q_n}\hat u(Q)\Big)^2\\ &= I+II\end{aligned}$$

where

$$I=2\sum_{-\infty}^{\infty}2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\int_{E_n\cap Q_n}\Big(\sum_{Q\subseteq Q_n}\hat u(Q)\Big)^2dx$$

$$II=2\sum_{-\infty}^{\infty}2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\int_{E_n\cap Q_n}\Big(\sum_{Q\supsetneqq Q_n}\hat u(Q)\Big)^2dx.$$

For the term I, we have the direct estimate:

$$\begin{aligned}I&\le 2\sum_{n=-\infty}^{\infty}2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\sum_{Q\subseteq Q_n}\|\hat u(Q)\|_2^2\\ &=2\sum_{n=-\infty}^{\infty}2^{2n}\sum_{l(Q)\le 2^{-n}}\|\hat u(Q)\|_2^2\\ &=2\sum_{n=-\infty}^{\infty}2^{2n}\sum_{k=0}^{\infty}\sum_{l(Q)=2^{-(n+k)}}\|\hat u(Q)\|_2^2\,l(Q)^{-2}2^{-2(n+k)}\\ &=2\sum_{k=0}^{\infty}2^{-2k}\sum_{n}\sum_{Q_0\in\mathfrak{I}_{n+k}}\|\hat u(Q)\|_2^2\,l(Q)^{-2}\\ &\le 2\Big(\sum_{k=0}^{\infty}2^{-2k}\Big)\sum_{Q}\|\hat u(Q)\|_2^2\,l(Q)^{-2}\le C\,\|\nabla u\|_2^2\quad\text{by the lemma.}\end{aligned}$$

To estimate the second term, we notice that for each fixed dyadic cube $Q_n\in\mathfrak{I}_n$, if Q is dyadic and $Q\supsetneqq Q_n$ then $\hat u(Q)$ is a linear function on Q_n thus $(\sup_{x\in Q_n}|\hat u(Q)(x)|)^2=\|\hat u(Q)|_{Q_n}\|_\infty^2\le C_d\,\|\hat u(Q)\|_2^2\,l(Q)^{-d}$ for each such Q and for some constant C depending only on the dimension d. Let $a_Q=\|\hat u(Q)|_{Q_n}\|_\infty$ then

$$\begin{aligned}II&\le 2\sum_{n=-\infty}^{\infty}2^{2n}\sum_{Q_n\in\mathfrak{I}_n}\int_{E_n\cap Q_n}\Big(\sum_{Q\supsetneqq Q_n}a_Q\Big)^2dx\\ &=2\sum_{n=-\infty}^{\infty}2^{2n}\sum_{Q_n\in\mathfrak{I}_n}|E_n\cap Q_n|\Big(\sum_{Q\supsetneqq Q_n}a_Q\Big)^2\end{aligned}$$

We now notice that since v satisfies the $(*)_\varphi$ condition, we have for every dyadic Q in $\mathbb{R}^d$

$$\frac{1}{(l(Q))^{d-2}} \sum_{n=-\infty}^{\infty} 2^{2n} |Q \cap E_n| \, \varphi(2^{2n} l(Q)^2) \leq 1. \tag{2.2}$$

Thus if we let $C_{Q,n} = \varphi(2^{2n} l(Q)^2)$, then

$$\sum_{R \supsetneq Q_n} \frac{1}{C_{R,n}} = \sum_{R \supsetneq Q_n} \frac{1}{\varphi(2^{2n} l^2(R))}$$
$$= \sum_{j=1}^{\infty} \frac{1}{\varphi(2^{2j})} \leq \int_1^{\infty} \frac{dx}{x\varphi(x)} \leq C_0.$$

Hence

$$II \leq 2 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{I}_n} |E_n \cap Q_n| \Big(\sum_{Q \supsetneq Q_n} a_Q^2 C_{Q,n} \Big) \sum_{R \supsetneq Q_n} \frac{1}{C_{R,n}}$$
$$\leq 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q_n \in \mathfrak{I}_n} |E_n \cap Q_n| \sum_{Q \supsetneq Q_n} a_Q^2 C_{Q,n}$$
$$= 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q \text{ dyadic}} a_Q^2 C_{Q,n} \sum_{\substack{Q_n \in \mathfrak{I}_n \\ Q_n \subset Q}} |E_n \cap Q_n|$$
$$= 2C_0 \sum_{n=-\infty}^{\infty} 2^{2n} \sum_{Q} a_Q^2 C_{Q,n} \, |Q \cap E_n|$$
$$= 2C_0 \sum_{Q} a_Q^2 \sum_{n=-\infty}^{\infty} 2^{2n} \varphi(2^{2n} l(Q)^2) \, |Q \cap E_n|$$
$$\leq 2C_0 \sum_{Q} a_Q^2 l(Q)^{d-2} \quad \text{by (2.2)}$$
$$\leq 2C_0 C_d \sum_{Q} \|\hat{u}(Q)\|_2^2 \, l(Q)^{-d} l(Q)^{d-2}$$
$$\leq 2C_0 C_d \sum_{Q} \|\hat{u}(Q)\|_2^2 \, l(Q)^{-2}$$
$$= 2C_0 C_d \, |||u|||_2^2 \leq \tilde{C}_1 \, \|\nabla u\|_2^2 \text{ (by the lemma)} \qquad \tilde{C}_1 = \text{constant} \cdot C_0 C_d.$$

Combining the estimate in I and II, we get

$$\int_{\mathbb{R}^d} u^2(x) v(x) \, dx \leq I + II \leq C_1 \, \|\nabla u\|_2^2 \qquad (C_1 = \tilde{C}_1 + \text{constant}).$$

This finishes the proof of the theorem.

As in ([8], p. 145) we have

COROLLARY 2.1. *Suppose* φ *is an increasing function with* $\varphi(x)\geq 1$, $\int_1^\infty \frac{dx}{x\varphi(x)} = C_0 < \infty$, *and also* $\varphi(4x)\leq C_2\varphi(x)$ *for all* $x\in\mathbb{R}^d$ *for some* C_2. *Then there exists some constant* C_3, *such that if* $Av_Q(vl(Q)^2\varphi(vl(Q)^2))\leq C_3$ *for all cube* Q *in* $\mathbb{R}^d$, *then* $L=-\Delta-v\geq 0$.

Proof. L is positive is equivalent to the inequality $\int u^2 v\,dx \leq \int |\nabla u|^2$ for all $u\in C_0^\infty(\mathbb{R}^d)$. Thus it follows from the result of the theorem (and the fact $\tilde{C}_1 = C_1 +$ constant) that there exists some constant C_4, C_5 (depending only on C_0 and the dimension d) such that if $\tilde{v}$ satisfies

$$(*)_\varphi C_4\colon \qquad Av_Q(\tilde{v}l(Q)^2\varphi(\tilde{v}l(Q)^2))\leq C_4$$

then $L_1=-\Delta-C_5\tilde{v}$ is positive. Thus if φ satisfies the additional hypothesis $\varphi(4x)\leq C_2\varphi(x)$, then we can choose some constant C_3 ($C_3\leq C_4C_5C_2^{\log_4 C_5}$) such that if v satisfies the condition $(*)_\varphi(C_3)$, and if we set $\tilde{v}=(1/C_5)v$ then for each cube Q in $\mathbb{R}^d$,

$$\begin{aligned} Av_Q(\tilde{v}l(Q)^2\varphi(\tilde{v}l(Q)^2)) &= \frac{1}{C_5}Av_Q\left(vl(Q)^2\varphi\left(\frac{1}{C_5}vl(Q)^2\right)\right)\\ &\leq \frac{1}{C_5}C_2^{-\log_4 C_5}Av_Q(vl(Q)^2\varphi(vl(Q)^2))\\ &\leq \frac{1}{C_5}C_2^{-\log_4 C_5}C_3\leq C_4. \end{aligned}$$

Thus $L=-\Delta-v=-\Delta-C_5\tilde{v}=L_1$ is a positive operator. If we translate the result in Corollary 1, we obtain the following estimate of $\lambda_1(L)$ (the first non-positive eigenvalue of L) as in the case of Theorem 5 in ([8], p. 145).

COROLLARY 2.2. *With the same assumption as in Corollary* 2.1 *about* φ, *there are constants* C, c *depending only on* d, φ *such that* $cE_{\text{sm}}\leq -\lambda_1(L)\leq CE_{\text{big}}$, *where*

$$E_{\text{sm}} = \sup_Q\,[Av_Qv - cl(Q)^{-2}]$$

$$E_{\text{big}} = \sup_Q\,[Av_Qv\varphi(vl(Q)^2) - cl(Q)^{-2}].$$

With some minor changes, one can check that the proof of Theorem 6 in ([8], page 154 on) works for the $(*)_\varphi$ version exactly as in the L^p-version, and obtain the following parallel result:

PROPOSITION 2.3. *With the same assumption about φ as in Corollary 2.1, if we assume in addition that $\varphi(4x) \le C_2\varphi(x)$ with $C_2 < 2^{d-2}$. Then there exists constant c, C (depending on φ, and d) so that if $L = -\Delta - v$ has at least CN negative eigenvalues, then there is a collection of pairwise disjoint cubes $Q_1, \dots, Q_N$ for which*

$$Av_{Q_j}(v\varphi(vl(Q_j)^2)) \ge Cl(Q_j)^{-2}.$$

Remark. $\varphi(x) = (\log(c_d + x))^{1+\varepsilon}$ for $\varepsilon > 0$ would satisfy the assumption of the proposition for some constant c_d depending on dimension d.

In the second part of this section, we will describe a counterexample for the question posed in the introduction.

We first remark that in $\mathbb{R}^d$, when the dimension d is one or two, the condition $(*)$ ($(*)$ is the $(*)_\varphi$ condition for $\varphi(x) = \log^+ x$)

$$(*) \qquad \frac{1}{|Q|}\int_Q v(x)l(Q)^2 \log^+(v(x)l(Q)^2)\,dx \le C$$

for all cubes Q in $\mathbb{R}^d$, with $l(Q) =$ side length of Q, implies that $v \equiv 0$. To see this fact, we notice that the condition $(*)$ is equivalent to

$$\int_Q v(x)\log^+(v(x)l^2(Q))\,dx \le \frac{C}{l(Q)^{2-d}}.$$

Thus for $d = 1$ or 2, and for cubes Q with $l(Q) \ge 1$ we have $\int_Q v(x)\log^+(v(x)l^2(Q))\,dx \le C$. If for each positive integer k we let $E_k = \left\{x \in \mathbb{R}^d : v(x) \ge \frac{1}{k}\right\}$. Then we get $\int_{Q\cap E_k} v(x)\,dx \le \frac{C}{\log l(Q)}$ for all cubes Q with $l(Q) \ge k$. Choosing an increasing sequence of cubes Q to cover E_k we may conclude $E_k = \varnothing$ for each k. Thus $v \equiv 0$. In fact, when $d = 1$ or 2; $\lambda \ge 0$ and $v \ge 0$ satisfies some mild decay conditions the operator $-\Delta - \lambda v$ always has a negative eigenvalue. (*Unique* negative eigenvalue if λ is small enough.) In particular it is never positive. For more precise statements of this fact and other results of properties of eigenvalues of $-\Delta - \lambda v$ when $d = 1, 2$, the reader is referred to [13], [16]. (The authors would like to thank the referee for pointing out these for us.)

Thus for all v satisfies (*),

$$\int_{\mathbb{R}^d} u^2(x)v(x)\,dx \le C\int_{\mathbb{R}^d} |\nabla u|^2\,dx$$

for all $u \in C_0^\infty(\mathbb{R}^d)$ for the trivial reason. If in this case ($d = 1$ or 2), we restrict the condition (*) to a bounded subset D in $\mathbb{R}$ (or $\mathbb{R}^2$), that is if we assume:

$$(*)' \qquad \frac{1}{|Q|}\int_Q v(x)l(Q)^2 \log^+ (v(x)l(Q)^2)\,dx \le C$$

for all cubes Q in $\mathbb{R}^d$ contained in a fixed bounded set $D \subset \mathbb{R}^d$ then (*)′ implies in particular (when $d = 1$, or 2)

$$(*)'' \qquad \int_D v(x)\log^+ v(x)\,dx \le C.$$

Thus for all u, C^∞ with compact support contained in D, we have

$$\begin{aligned}\int u^2(x)v(x)\,dx &\le \left(\left(\int_D e^{\alpha u^2(x)/\int_D |\nabla u(x)|^2\,dx} + \frac{1}{\alpha}\int_D v(x)\log^+ v(x)\,dx\right)\right)\int_D |\nabla u(x)|^2\,dx \\ &\le C'\int_D |\nabla u(x)|^2\,dx\end{aligned}$$

for some universal constant α (which depends only on the Lebesgue measure D) and for some constant C. The existence of the constants α, c is a special case of the work of N. Trudinger [17] on the sharper form of the Sobolov inequality.

In $\mathbb{R}^d$ with $d \ge 3$, we will construct some v which satisfies the condition (*), yet $\int_{\mathbb{R}^d} u^2(x)v(x)\,dx/\int_{\mathbb{R}^d} |\nabla u|^2\,dx$ fails to be bounded for all $u \in C_0^\infty(\mathbb{R}^d)$. The construction is a variation of the counterexample for the inequality (1.2) given in the previous section. This example also shows that the condition $\int_1^\infty \frac{dx}{x\varphi(x)} < \infty$ stated in Theorem 2.1 is essentially the best possible condition.

EXAMPLE I. Suppose $\varphi : [1, \infty) \to [1, \infty)$ is an increasing function with $\varphi(4x) \le 2^{d-2}\varphi(x)$ and with $\int_1^\infty \frac{dx}{x\varphi(x)}$ divergent. Then there exists a positive function v on $\mathbb{R}^d$ which satisfies

$$(*)_\varphi \qquad \sup_{\substack{Q \\ \text{cubes in } \mathbb{R}^d}} \frac{1}{|Q|}\int_Q v(x)l^2(Q)\varphi(v(x)l^2(Q))\,dx \le 1.$$

Yet the weak type (2.2) bound

$$v(\{u \ge n\}) \le \frac{A}{n^2}\int_{\mathbb{R}^d} |\nabla u|^2 \tag{2.3}$$

fails for all positive constant $A<\infty$ and for some function u (which depends on A) in $C_0^\infty(\mathbb{R}^d)$.

Construction of the example. The assumption $\int_1^\infty \frac{dx}{x\varphi(x)}$ diverges is equivalent to the condition that $\sum_{n=1}^\infty \frac{1}{\varphi(2^{2n})}$ diverges. We will also assume without loss of generality that $\varphi(1)=1$, and $d=3$ (the example works for any $d\ge 3$ with suitable changes of constants). Let $\tau(n)=\varphi(2^{2n})$ $n=0,1,2,\ldots$, and m be a fixed constant $\left(\text{so that } \sum_{j<m} \frac{1}{\tau(j)} \text{ is big}\right)$. For each $0\le j\le m$, define a collection $\mathfrak{G}_j$ of dyadic cubes in $\mathbb{R}^3$ recursively as follows:

$\mathfrak{G}_0=\{\text{unit cube } Q_0 \text{ in } \mathbb{R}^3\}, \qquad N_0=1$

$\mathfrak{G}_j$ is a collection of N_j dyadic cubes in $\mathbb{R}^3$,

each with side length 2^{-j}, satisfying the following properties:

(a) Each cube in $\mathfrak{G}_j$ is a subcube of some cube in $\mathfrak{G}_{j-1}$.
(b) Each $\mathfrak{G}_{j-1}$ cube contains c_j $\mathfrak{G}_j$-cubes, where c_j is either one or two and depends only on j.
(c) c_j is chosen in such a way so that $\frac{1}{2}\cdot 2^j\frac{\tau(m-j)}{\tau(m)} \le N_j \le 2\cdot 2^j\frac{\tau(m-j)}{\tau(m)}$.

(c) can be done because by our assumption on φ, $\tau(m-j)\le\tau(m-(j-1))\le 2\tau(m-j)$, e.g. if $\mathfrak{G}_{j-1}$ has been defined, let $c_j=1$ if $N_{j-1}>2^j\frac{\tau(m-j)}{\tau(m)}$ and $c_j=2$

otherwise. Then if $c_j = 1$, we get

$$2^j \frac{\tau(m-j)}{\tau(m)} \le 2^j \frac{\tau(m-(j-1))}{\tau(m)} \le N_{j-1} \le 2 \cdot 2^{j-1} \frac{\tau(m-(j-1))}{\tau(m)}$$

$$\le 2 \cdot 2^j \frac{\tau(m-j)}{\tau(m)} \quad \text{and} \quad N_j = N_{j-1}.$$

If $c_j = 2$, then $N_j = 2N_{j-1}$ and

$$\tfrac{1}{2} \cdot 2^j \frac{\tau(m-j)}{\tau(m)} \le 2^{j-1} \frac{\tau(m-(j-1))}{\tau(m)} \le 2N_{j-1} \le 2 \cdot 2^j \frac{\tau(m-j)}{\tau(m)}.$$

We terminate the process when $j = m$ and define $v = 2^{2m}$ on those $\mathfrak{G}_m$ cubes and $v = 0$ otherwise. We will now verify that $\frac{v}{c}$ satisfies the condition $(*)_\varphi$ for some constant c. Note that it suffices (up to a change of the constant c) to verify $(*)_\varphi$ for all dyadic cubes. To see $(*)_\varphi$ for dyadic cubes, fix a dyadic cube Q in $\mathbb{R}^3$, and assume, say, Q has side length $l(Q) = 2^{-j} \le 1$. Then Q contains either 0 or $\frac{N_m}{N_j}$ $\mathfrak{G}_m$-cubes. Thus it follows from condition (c) that

$$\frac{1}{|Q|} \int_Q v(x) l_Q^2 \varphi(v(x) l_Q^2)\, dx = 2^{3j} \cdot 2^{2m} \cdot 2^{-2j} \frac{N_m}{N_j} 2^{-3m} \varphi(2^{2(m-j)})$$

$$= \frac{2^{-m} N_m}{2^{-j} N_j} \tau(m-j) \le 4.$$

For dyadic cubes Q with the side length $2^j \ge 1$ $(j \ge 0)$, either Q contains 0 or (if Q contains the unit cube in $\mathbb{R}^3$) N_m $\mathfrak{G}_m$-cubes. So

$$\frac{1}{|Q|} \int_Q v(x) l_Q^2 \varphi(v(x) l_Q^2)\, dx = \frac{1}{2^j} \cdot 2^{-m} N_m \varphi(2^{2(m+j)})$$

$$\le \frac{2}{\tau(m)} \cdot \frac{2^j \varphi(2^{2m})}{2^j} = \frac{2}{\tau(m)} \cdot \tau(m) = 2.$$

(The last inequality follows by our assumption on φ.) Then $\frac{v}{4}$ satisfies the condition $(*)_\varphi$ for all dyadic cubes in $\mathbb{R}^3$.

Fix a constant $n>0$, and constant $A>0$, we will now define some function u. Let

$$d_j=\frac{n}{\sum_{1\le l\le m}(\tau(l))^{-1}}(\tau(m-j))^{-1}\qquad 0\le j\le m-1.$$

We now define u as follows. For each cube $Q\in\mathfrak{I}_j$ $(0\le j\le m-1)$, let u_j^Q be a C^∞ function with compact support contained in $\tilde{Q}$ (recall that $\tilde{Q}=3Q$) satisfying: $u_j^Q(x)=1$ on $\bigcup_{Q'\subseteq Q}^{Q'\in\mathfrak{I}_{j+1}}\tilde{Q}'$, and $|\nabla u_j^Q(x)|\le C2^j$, $0\le u_j^Q\le 1$. Define

$$u_j=\sum_{l=1}^{N_j}\Big(\prod_{k<l}(1-\varphi_j^{Q_k})\Big)\varphi_j^{Q_l}$$

where $Q_1, Q_2, \dots, Q_{N_j}$ are the collection of cubes in $\mathfrak{I}_j$. Since only finitely many $\varphi_j^{Q_k}$'s can be simultaneously nonzero, $|\nabla u_j|$ is bounded by $C2^j$ (c depends only on the dimension d), and $u_j=1$ on $\bigcup_{Q'\in\mathfrak{I}_{j+1}}\tilde{Q}'$, $u_j=0$ on $\bigcup_{Q\in\mathfrak{I}_j}\tilde{Q}$. Define $u=\sum_{j=0}^{m-1}d_ju_j$, then on the set $\bigcup_{Q\in\mathfrak{I}_j}\tilde{Q}\backslash\bigcup_{Q\in\mathfrak{I}_{j+1}}\tilde{Q}$, only u_j is not a constant, hence $|\nabla u(x)|\le d_j\,|\nabla u_j(x)|\le Cd_j2^j$ on $\bigcup_{Q\in\mathfrak{I}_j}\tilde{Q}\backslash\bigcup_{Q\in\mathfrak{I}_{j+1}}\tilde{Q}$. And

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx&=\sum_{j=0}^{m-1}\int_{\bigcup_{Q\in\mathfrak{I}_j}\tilde{Q}\backslash\bigcup_{Q\in\mathfrak{I}_{j+1}}\tilde{Q}}|\nabla u(x)|^2\\
&\le C\sum_{j=0}^{m-1}(d_j2^j)^2\cdot N_j\cdot 2^{-3j}\\
&=C\sum_{j=0}^{m-1}\frac{n^2}{\Big(\sum_{l=1}^{m}\tau(l)^{-1}\Big)^2}(\tau(m-j))^{-2}N_j2^{-j}\\
&\le 2C\sum_{j=0}^{m-1}\frac{n^2}{\Big(\sum_{l=1}^{m}(\tau(l))^{-1}\Big)^2}(\tau(m-j))^{-2}\frac{\tau(m-j)}{\tau(m)}\\
&=2cn^2\frac{1}{\tau(m)}\frac{1}{\Big(\sum_{l=1}^{m}(\tau(l))^{-1}\Big)}.\end{aligned}$$

While

$$u=\sum_{j=0}^{m-1}d_j=n\quad\text{on}\quad\bigcup_{Q\in\mathfrak{I}_m}\tilde{Q}.$$

Thus $v(\{u\geq n\})\geq 2^{2m}N_m 2^{-3m}\geq\frac{1}{2}\frac{1}{\tau(m)}$. So if we choose m large enough, with $\sum_{l=1}^{m}(\tau(l))^{-1}\geq 5cA$, then the inequality $v(\{u\geq n\})\leq A\frac{1}{n^2}\int|\nabla u|^2$ would fail for such functions u, which finishes the construction of the example.

It turns out the constant 2^{d-2} is critical in Example I in the following sense.

EXAMPLE II. For each $C>2^{d-2}$, there exists an increasing function φ with $\varphi(4x)\leq C\varphi(x)$ for all x and $\int_1^\infty \frac{dx}{x\varphi(x)}=\infty$. Yet for all positive function v which satisfies the $(*)_\varphi$ condition, we have

$$\int_{\mathbb{R}^d} u^2(x)v(x)\,dx\leq A\int_{\mathbb{R}^d}|\nabla u(x)|^2$$

for some finite constant A (which depends on C), and for all $u\in C_0^\infty(\mathbb{R}^d)$.

Construction. We will assume $d=3$, and $C>2$. The construction is based on the following observation: Suppose v satisfies the $(*)_\varphi$ condition, then for each integer $k\geq 1$, we have

$$\frac{1}{|Q|}\int_Q v(x)l(Q)^2\frac{\varphi(4^k v(x)l(Q)^2)}{2^k}\,dx\leq 1$$

for each cube Q (which was obtained from the $(*)_\varphi$ condition by looking at cube Q^* containing Q with $l(Q^*)=2^k l(Q)$). Thus if we can construct some φ with

$$\sum_{k=0}^{\infty}\left(\frac{2}{C}\right)^k\frac{\varphi(4^k x)}{2^k}$$

converges uniformly, and call the function $\psi(x)$, then $v\in(*)_\varphi$ implies $v\in(*)_{(C-2/C)\psi}$. For the purpose of our example, we will construct some φ with

(a) $\sum_{k=0}^{\infty}\frac{1}{\varphi(4^k)}=\infty$

(b) $\varphi(4^{k+1})\leq c\varphi(4^k)$ for each k

(c) $\psi(x)=\sum_{k=0}^{\infty}\left(\frac{2}{C}\right)^k\frac{\varphi(4^k x)}{2^k}$ converges uniformly and satisfies $\sum_{k=0}^{\infty}\frac{1}{\psi(4^k)}<\infty$.

Applying Theorem 2.1, we can then conclude that for such φ

$$\int u^2 v\leq A\int|\nabla u|^2\qquad\forall\, u\in C_0^\infty(\mathbb{R}^3).$$

Construct φ as follows: Choose sequence of integers $0=k_0<k_1<\cdots<k_n<\cdots$; with $E_m=[k_{2m}, k_{2m+1}]$, $0_m=[k_{2m+1}, k_{2m+2}]$ even and odd intervals respectively for $m=0,1,2,\ldots$. And define $\varphi(4^k)=C^{|E_0|+|E_1|+\cdots+|E_m|}$ for $k\in 0_m$ and $\varphi(4^k)=C^{|E_0|+|E_1|+\cdots+|E_{m-1}|}C^{k-k_{2m}}$ and define φ to be linear in between the interval 4^k to 4^{k+1}. Then (b) is satisfied by the way φ is defined. To check (a), we let $|E_0|=1$, $|0_0|=[C^{|E_0|}]+1$, and $|0_m|=[C^{|E_0|+\cdots+|E_m|}]+1$, $|E_{m+1}|=[C^{|0_m|}]+1$ respectively for $m=0,1,2,\ldots$. ([x] denote the greatest integer $\leq x$.) Then

$$\text{(a)}\quad \sum_{k\in\bigcup_0^\infty E_m}\frac{1}{\varphi(4^k)}=\sum_{m=0}^\infty\sum_{k\in E_m}\frac{1}{C^{k-k_{2m}}}\frac{1}{C^{|E_0|+|E_1|+\cdots+|E_{m-1}|}}$$

$$\leq 2\sum_{m=0}^\infty\frac{1}{C^{m-1}}<\infty$$

$$\sum_{k\in\bigcup_0^\infty 0_m}\frac{1}{\varphi(4^k)}=\sum_{m=0}^\infty\frac{|0_m|}{C^{|E_0|+\cdots+|E_m|}}=\infty \quad \text{by our choice of } 0_m.$$

And

$$\text{(c)}\quad \sum_{k=0}^\infty\frac{\varphi(4^k)}{C^k}=\sum_{m=0}^\infty\sum_{k\in E_m}\frac{C^{k-k_{2m}}C^{|E_0|+\cdots+|E_{m-1}|}}{C^k}+\sum_{m=0}^\infty\sum_{k\in 0_m}\frac{C^{|E_0|+\cdots+|E_m|}}{C^k}$$

$$\leq\sum_{m=0}^\infty\frac{|E_m|}{C^{|0_0|+|0_1|+\cdots+|0_{m-1}|}}$$

$$+\sum_{m=0}^\infty\frac{1}{C^{|0_1|+\cdots+|0_{m-1}|}}\left(1+\frac{1}{C}+\cdots+\frac{1}{C^{|0_m|}}\right)$$

$$\leq\frac{2C}{C-1}\left(\sum_{m=0}^\infty\frac{1}{C^m}\right)\leq 2\left(\frac{C}{C-1}\right)^2.$$

Thus

$$\psi(4^n)=\sum_{k=0}^\infty\frac{\varphi(4^k4^n)}{C^k}\leq C^n\sum_{k=0}^\infty\frac{\varphi(4^k)}{C^k}<\infty$$

for each n and $\sum_{k=0}^\infty\frac{\varphi(4^kx)}{C^k}$ converges uniformly to $\psi(x)$ for all $x\geq 1$ while $\psi\geq\varphi$ and

$$\sum_{n\in\bigcup_0^\infty E_m}\frac{1}{\psi(4^n)}\leq\sum_{n\in\bigcup_0^\infty E_m}\frac{1}{\varphi(4^n)}<\infty.$$

For $n \in 0_m$, we have

$$\psi(4^n) = \sum_{k=0}^{\infty} \frac{\varphi(4^{k+n})}{C^k} \geq \sum_{k+n \in E_{m+1}} \frac{\varphi(4^{k+n})}{C^k}$$

$$\geq \sum_{k+n \in E_{m+1}} C^{|E_0|+|E_1|+\cdots+|E_m|} \frac{C^{k+n-k_{2m+2}}}{C^k}$$

$$\geq (|0_m|-1) \sum_{\substack{k \\ k+n \in E_{m+1}}} C^{n-k_{2m+2}}$$

$$\geq |0_m| \, C^{|0_m|} C^{n-k_{2m+2}} = |0_m| \, C^{n-k_{2m+1}}.$$

Thus

$$\text{(c)} \quad \sum_{n \in \bigcup_{m=0}^{\infty} 0_m} \frac{1}{\psi(4^n)} = \sum_{m=0}^{\infty} \sum_{n \in 0_m} \frac{1}{\psi(4^n)}$$

$$\leq \sum_{m=0}^{\infty} \sum_{n \in 0_m} \frac{1}{|0_m|} \frac{1}{C^{n-k_{2m+1}}}$$

$$= \sum_{m=0}^{\infty} \frac{1}{|0_m|} \left(1 + \frac{1}{C} + \cdots + \frac{1}{C^{|0_m|}}\right)$$

$$\leq \frac{C}{C-1} \sum_{m=0}^{\infty} \frac{1}{|0_m|} < \infty.$$

Thus φ and ψ satisfies the estimates (a), (b), (c) and we have established the example as desired.

Section 3

We recall that Q_0 is the unit dyadic cube in $\mathbb{R}^d$. First we consider dyadic martingales on Q_0. Let $\mathscr{G}_n$ be the σ-algebra generated by the 2nd dyadic subcubes on Q_0 of sidelength 2^{-n} and let $E(f, \mathscr{G}_n)$ be the conditional expectation of f on $\mathscr{G}_n$ $\left(\text{i.e. } E(f, \mathscr{G}_n)(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} f \text{ where } x \in Q_n(x) \text{ and } l(Q_n(x)) = 2^{-n}\right)$. By a dyadic martingale we mean a sequence of functions $\{f_n\}$ from Q_0 to $\mathbb{R}$ such that f_n is $\mathscr{G}_n$-measurable and $E(f_{n+1}, \mathscr{G}_n) = f_n$. Our martingales will (almost) all be L^2 bounded and we denote the limit function by f. We define the martingale difference functions s_n $(n \geq 0)$ by $s_n = f_{n+1} - f_n$. Thus s_n is $\mathcal{G}_{n+1}$-measurable. The square functions of the martingale $\{f_n\}$ is defined by

$$Sf(x) = \sum_{n>0} \|\chi_{Q_n(x)} S_n\|_\infty^2$$

where $Q_n(x)$ is the unique dyadic cube of length 2^{-n} containing x. This is not standard. One generally sees $Sf(x) = (\sum_n E(S_n^2, \mathcal{G}_n)(x))^{1/2}$; this definition agrees with ours only when $d = 1$, although when d is fixed the two are always comparable. Our results work out nicer with our definition of Sf; with the other definition the "$\frac{1}{2}$" in (3.1) below would have to be replaced by a constant depending on d. Our result for dyadic martingales is as follows.

THEOREM 3.1. *Let $\{f_n\}$ be a dyadic martingale on $Q_0 \subset \mathbb{R}^d$ with limit function f. Suppose $\|Sf\|_\infty < \infty$. Then for $\lambda \geq 0$,*

$$|\{x \in Q_0 : f(x) - f_0(x) \geq \lambda\}| \leq \exp\left(-\tfrac{1}{2}\lambda^2/\|Sf\|_\infty^2\right). \tag{3.1}$$

The proof we give for this is due to Herman Rubin; it replaces a much longer argument of the authors. It is based on the fundamental identity of sequential analysis in statistics.

Proof of (3.1) (Rubin). We can assume $f_0 = 0$. Fix $t > 0$. Define $q_n : Q_0 \to R$ by

$$q_n = e^{tf_n}\left(\prod_{j=1}^{n-1} E(e^{tS_j}, \mathcal{G}_j)\right)^{-1}.$$

These q_n form a martingale! Clearly q_n is $\mathcal{G}_n$-measurable, and we have

$$\begin{aligned}
E(q_{n+1}, \mathcal{G}_n) &= E\left(e^{tf_{n+1}}\left(\prod_{j=1}^{n} E(e^{tS_j}, \mathcal{G}_j)\right)^{-1}, \mathcal{G}_n\right) \\
&= E\left(e^{tS_n}e^{tf_n}\left(\prod_{j=1}^{n} E(e^{tS_j}, \mathcal{G}_j)\right)^{-1}, \mathcal{G}_n\right) \\
&= E(e^{tS_n}(E(e^{tS_n}, \mathcal{G}_n))^{-1}q_n, \mathcal{G}_n) \\
&= E(e^{tS_n}, \mathcal{G}_n)(E(e^{tS_n}, \mathcal{G}_n))^{-1}q_n \\
&= q_n.
\end{aligned}$$

It follows that $\int q_n = 1$ for all n. (This is the fundamental identity of sequential analysis.) Using the elementary inequalities

$$\int e^{\phi}\, d\mu \le \cosh(\|\phi\|_\infty) \le \exp(\tfrac{1}{2}\|\phi\|_\infty^2)$$

valid when μ is a probability measure and $\int \phi\, d\mu = 0$, we find that

$$E(e^{tS_j}, \mathcal{G}_j)(x) \le \exp(\tfrac{1}{2}t^2 \|\chi_{Q_j(x)} S_j\|_\infty^2).$$

So $\prod_{j=1}^{n-1} E(e^{tS_j}, \mathcal{G}_j)(x) \le e^{t^2/2\|Sf\|_\infty^2}$ for all n and x, and the fundamental identity now implies that $\int_{Q_0} e^{tf_n} \le \exp(\tfrac{1}{2}t^2\|Sf\|_\infty^2)$ for all n. Letting n go to ∞ gives $\int_{Q_0} e^{tf} \le \exp(\tfrac{1}{2}t^2\|Sf\|_\infty^2)$. Now take $t = \dfrac{\lambda}{\|Sf\|_\infty}$ and apply Tsebyshev's inequality to get (3.1).

We record some corollaries, all of which follow from Theorem 3.1 by standard arguments.

COROLLARY 3.1. *We have the good λ inequality*

$$|\{x \in Q_0 : f^*(x) > 2\lambda,\ Sf(x) < \varepsilon\lambda\}| \le C \exp -\frac{1}{2}\frac{(1-\varepsilon)^2}{\varepsilon^2}\, |\{x \in Q_0 : f^*(x) > \lambda\}|,$$

with C a universal constant.

Proof [1]. Define a stopping time $\tau_\lambda = \min(\{n : \sum_{j\le n}(f_{j+1}-f_j)^2 \ge \varepsilon^2\lambda^2\})$. Let $\mathfrak{G} = \{Q\}$ be the maximal dyadic cubes with $f_Q > \lambda$. For each $Q \in \mathfrak{G}$ consider $\{x \in Q : f^*(x) > 2\lambda,\ Sf(x) < \varepsilon\lambda\}$. If this is non-empty then $f_Q < (1+\varepsilon)\lambda$. Also

$$\begin{aligned} |\{x \in Q : f^*(x) > 2\lambda,\ Sf(x) < \varepsilon\lambda\}| &= |\{x \in Q : f^*(x) > 2\lambda \text{ and } \tau_\lambda = \infty\}| \\ &\le |\{x \in Q : (f(\tau_\lambda) - f_Q)^* > (1-\varepsilon)\lambda\}| \end{aligned}$$

and $\|S(f(\tau_\lambda))\|_\infty < \varepsilon\lambda$ pointwise. By (3.1) and Lemma 3.1,

$$|\{x \in Q : f^*(x) > 2\lambda,\ Sf(x) < \varepsilon\lambda\}| \le C \exp\left(-\frac{1}{2}\frac{(1-\varepsilon)^2\lambda^2}{\varepsilon^2\lambda^2}\right)|Q|$$

and if we sum over Q we are done.

Integrating out the good $-\lambda$ inequality gives the bizarre estimate

$$\int_{Q_0} \exp\left(c_1(f^*)^2/1+(Sf)^2\right) \le c_2 \int_{Q_0} (1+\log^+ f^*)$$

with $c_1>0$, $c_2<\infty$. We can also obtain the following "law of the iterated logarithm" for dyadic martingales.

COROLLARY 3.2. *If $\{f_n\}$ is a dyadic martingale on Q_0, then*

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S^n f(x)\sqrt{2\log\log S^n f(x)}} \le 1$$

a.e. *on the set where $\{f_n\}$ is unbounded. (Here $(S^n f(x))^2 = \sum_{k=0}^{n-1} \|\chi_{Q_k(x)} S_k\|_\infty^2$.)*

This may be proved like [6], Section 7.5 using Theorem 3.1 instead of the central limit theorem. We omit the proof since Corollary 3.2 is a minor modification of known results (due to W. F. Stout; see [15] and the references there). We're grateful to R. Banuelos and C. Mueller for pointing out the known results, especially to Banuelos for reference [15]. We note that the corresponding lower bound

$$\limsup_{n\to\infty} \frac{|f_n|}{S^n\sqrt{2\log\log S^n}} \ge 1$$

is false in our context.

Now we prove a version of Theorem 3.1 in which the Lusin area integral replaces S. Let

$$p(x) = \Gamma\left(\frac{d+1}{2}\right) \Big/ \left(\pi^{(d+1/2)}(1+|x|^2)^{(d+1)/2}\right)$$

and $p_t(x) = t^{-d}p(x/t)$, $t>0$, so that $F(x,t) = p_t * f(x)$ is the harmonic extension of f to $\mathbb{R}_+^{d+1}$. Let ψ be the vector valued function $\nabla_x p$ and $\psi_t(x) = t^{-d}\psi(x/t) = t\,\nabla_x p_t(x)$. For $0<\gamma<\infty$ define

$$\begin{aligned} A_\gamma f(x) &= \left(\int_{\Gamma_\gamma(x)} |\nabla_y f(y,t)|^2\, t^{1-d}\, dy\, dt\right)^{1/2} \\ &= \left(\int_{\Gamma_\gamma(x)} |\psi_t * f(y)|^2\, t^{-1-d}\, dy\, dt\right)^{1/2} \end{aligned}$$

where $\Gamma_\gamma(x)=\{(y,t)\in\mathbb{R}_+^{d+1}:|x-y|<\gamma t\}$. We then have

THEOREM 3.2. *Suppose $A_\gamma f\in L^\infty$. Then*

$$\sup_{Q:\,\text{cube}}\frac{1}{|Q|}\int_Q \exp\left(c_1\frac{|f-f_Q|^2}{\|A_\gamma f\|_\infty^2}\right)<c_2,$$

where $c_1>0$ and $c_2<\infty$ depend on d and γ.

Some remarks about this. First, P. Jones [11] (private communication) has shown that it is possible for a function f to satisfy $A_{\gamma_1}f\in L^\infty$, $A_{\gamma_2}f\notin L^\infty$, for two numbers $0<\gamma_1<\gamma_2<\infty$. Second, regarding the proof of Theorem 3.2, it is a reduction to the dyadic case using one of the tools available for such purposes, a decomposition from [2] (see also [4]). This grind it out approach has the drawback that we do not obtain a sharp (or even dimension independent) constant like the $\frac{1}{2}$ in Theorem 3.1. Also it does not seem to let us replace A_γ by the g function. On the other hand it does apply to square functions formed using fairly general kernels; for example, we could take p to be a nonnegative radial Schwarz function instead of the Poisson kernel and the same proof would work.

We note that R. Banuelos (personal communication) has proved results similar to Theorem 3.2 by probabilistic arguments, which give sharp constants.

We give two lemmas. The first is a version of the Calderon–Torchinsky machinery [3] and the second is a variant of a trick due to S. Janson.

LEMMA 3.1. *If $\rho>0$ there is a smooth, radial function K supported in $\{x:|x|\le\rho\}$ such that if we define $q(x)=\nabla K(x)$ and $q_t(x)=t^{-d}q(x/t)$, then*

$$\lim_{T\to 0}\int_{\substack{t>T\\ y\in\mathbb{R}^d}}\langle\psi_t*f(x-y),q_t(y)\rangle\,dy\frac{dt}{t}=f(x)$$

for all $f\in L^2$ with compact support. The limit is (say) in the L^2 sense.

Proof [3]. Choose K_0 supported in $|x|\le\rho$, smooth and radial and such that $\int K_0(x)(-\Delta)^{1-d/2}p(x)\neq 0$. Let $K=cK_0$ where c is a suitable constant. Let $v(\xi)=(2\pi|\xi|)^2\int_{t=0}^\infty \hat{K}(t\xi)\hat{p}(t\xi)t\,dt$. Then v is radial and homogeneous of degree zero, hence constant. It follows from the Plancherel theorem that $\int_{|\xi|=1}v(\xi)\,d\xi\neq 0$; hence we can (and do) choose c so that $v\equiv 1$. Then

$$\left(\int_{t>T}\langle\psi_t*f(x-y),q_t(y)\rangle\,dy\frac{dt}{t}\right)^\wedge(\xi)=\hat{f}(\xi)\int_{t>T}(2\pi|\xi|)^2\hat{p}(t\xi)\hat{K}(t\xi)t\,dt\to\hat{f}$$

in L^2 as $T\to 0$.

Let $x \in \mathbb{R}^d$. In the next lemma and subsequently, we let $\mathfrak{I}^x$ denote the set of dyadic cubes translated by x, i.e. $\mathfrak{I}^x = \{Q \subseteq \mathbb{R}^d : \{y : y + x \in Q\} \in \mathfrak{I}\}$.

LEMMA 3.2. *There is a number $N = N(d)$ with the following property: if $A < \infty$, then the set $\mathfrak{G} = \{Q : Q \in \mathfrak{I}$ and $l(Q) \leq 2^A\}$ may be decomposed, $\mathfrak{G} = \bigcup_{j=1}^N \mathfrak{G}^{(j)}$, and there exist $x_1 \cdots x_N \in \mathbb{R}^d$ such that*

$$\text{If } Q \in \mathfrak{G}^{(j)} \text{ then } \tilde{Q} \text{ is contained in a cube } Q' \in \mathfrak{I}^{x_j} \text{ with } l(Q') = 8l(Q). \tag{3.1}$$

$$\text{If } Q_1, Q_2 \in \mathfrak{G}^{(j)} \text{ and } Q_1 \neq Q_2, \text{ then } Q_1' \neq Q_2'. \tag{3.2}$$

Proof. By scaling we can assume $A = -3$. We need only obtain (3.1) since if (3.1) holds and if $J \in \mathfrak{I}^{x_j}$ is given there can be at most 8^d cubes Q with $Q \in \mathfrak{G}^{(j)}$ and $Q' = J$. So we can get (3.2) by further decomposing each $\mathfrak{G}^{(j)}$.

We first show (3.1) when $d = 1$. Let $N = 2$, $x_1 = \frac{1}{3}$, $x_2 = 0$. If $Q \in \mathfrak{G}$ then $Q = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ for some $n \geq 3$, $K \in \mathbb{Z}$. Let $Q \in \mathfrak{G}^{(1)}$ if $K = 0$ or $1 \pmod 8$, $Q \in \mathfrak{G}^{(2)}$ otherwise. Then (3.1) is clear for $j = 2$. Suppose $Q \in \mathfrak{G}^{(1)}$. We must show $\tilde{Q}$ contains no point of the form $q2^{-(n-3)} + \frac{1}{3}$, $q \in \mathbb{Z}$. Suppose $\tilde{Q}$ did contain such a point. We know $\tilde{Q} \subset [p2^{-(n-3)} - 2^{-(n-1)}, p2^{-(n-3)} + 2^{-(n-1)}]$ for some $p \in \mathbb{Z}$. So $p - \frac{1}{4} \leq \frac{2^{n-3}}{2} + q \leq p + \frac{1}{4}$, contradiction.

When $d > 1$ any $Q \in \mathfrak{G}$ is uniquely $Q = I_1^{(Q)} \times \cdots \times I_d^{(Q)}$ where each $I_j^{(Q)} \subset \mathbb{R}$. If $\sigma \subset \{1, \ldots, d\}$ let $\mathfrak{G}^{(\sigma)} = \{Q \in \mathfrak{G} : I_i^{(Q)} \in \mathfrak{G}^{(1)}$ if $i \in \sigma$, $I_1^{(Q)} \in \mathfrak{G}^{(2)}$ if $i \notin \sigma\}$. Let x_σ be the point whose ith coordinate $x_\sigma(i)$ satisfies $x_\sigma(i) = \frac{1}{3}$ if $i \in \sigma$, $x_\sigma(i) = 0$ if $i \notin \sigma$. Then the decomposition $\mathfrak{G} = \bigcup \mathfrak{G}^{(\sigma)}$ and the points x_σ satisfy (3.1).

Proof of Theorem 3.2. We can assume $Q = Q_0$, $f_{Q_0} = 0$. Let $\varphi \in C_0^\infty$ with $\varphi(x) = 1$ if $x \in 5Q_0$, $\varphi(x) = 0$ if $x \notin 7Q_0$. Let $h = \varphi f$. We must show $\int_{Q_0} e^{c_1 h^2} \leq c_2$. Choose K by Lemma 3.1 with $\rho = \gamma/4$. Form the corresponding q. Let $S = \{(x, t) \in \mathbb{R}_+^{d+1} : x \in \tilde{Q}_0, t \leq \rho^{-1}\}$. Write

$$\begin{aligned} h(x) &= \int_S \langle \psi_t * h(y), q_t(x, y) \rangle \, dy \frac{dt}{t} + \int_{\mathbb{R}_+^{d+1}/S} \langle \psi_t * h(y), q_t(x, y) \rangle \, dy \frac{dt}{t} \\ &= h_1(x) + h_2(x). \end{aligned}$$

The integrand giving $h_2(x)$ vanishes when $x \in Q_0$ and t is sufficiently small.

Hence for $x \in Q_0$,

$$|h_2(x)| \le \int_{t \ge t_0} \|\psi_t * h\|_1 \, \|q_t\|_\infty \, dy \frac{dt}{t}$$
$$\le C \|h\|_1 \int_{t \ge t_0} t^{-d} \frac{dt}{t}$$
$$\le C \|f\|_{BMO} \le C \|A_\gamma f\|_\infty .$$

Also note $y \in \tilde{Q}_0$ implies by standard arguments that $|\psi_t * ((\varphi - 1)f)(y)| \le Ct^2 \|f\|_{BMO}$. So for any x,

$$\left| h_1(x) - \int_S \langle \psi_t * f(y), q_t(x-y) \rangle \, dy \frac{dt}{t} \right|$$
$$= \left| \int_S \langle \psi_t * ((\varphi - 1)f)(y), q_t(x-y) \rangle \, dy \frac{dt}{t} \right|$$
$$\le \|f\|_{BMO} \le C \|A_\gamma f\|_\infty .$$

So it suffices to estimate

$$\Lambda(x) = \int_S \langle \psi_t * f(y), q_t(x-y) \rangle \, dy \frac{dt}{t} .$$

For Q dyadic with $l(Q) \le 1$, let $T_Q = S \cap \left\{ (y,t) : y \in Q, \ \frac{l(Q)}{2\rho} \le t \le \frac{l(Q)}{\rho} \right\}$. Define $\lambda_Q(x) = \int_{T_Q} \langle \psi_t * f(y), q_t(x-y) \rangle \, dy \frac{dt}{t}$. Then

$$\Lambda(x) = \sum_{l(Q) \le 1} \lambda_Q(x). \tag{3.3}$$

It is easy to see that

$$\lambda_Q \text{ is supported on } \tilde{Q} \tag{3.4}$$

$$\int \lambda_Q = 0 \tag{3.5}$$

$$\|\lambda_Q\|^2_{\mathrm{Lip}\,\alpha} \, l(Q)^{2\alpha} \le C l(Q)^{-(d+1)} \int_{T_Q} |\psi_t * f|^2 \, dy \, dt, \quad \text{if} \quad 0 < \alpha < 1 \text{ (say).}$$

The last inequality together with the choice of ρ implies that for $x \in \mathbb{R}^d$ fixed,

$$\sum_{x \in Q} \|\lambda_Q\|^2_{\mathrm{Lip}\,\alpha}\, l(Q)^{2\alpha} \le CB^2, \qquad 0<\alpha<1$$

where $B = \|A_\gamma f\|_\infty$. So we are reduced to proving the following.

LEMMA 3.3. *If Λ has a decomposition* (3.3) *satisfying* (3.4), (3.5), *and* (3.6) *for some α, $0<\alpha<1$, then*

$$\int_{Q_0} e^{c_1 f^2/B^2} < c_2 < \infty.$$

Proof. Choose N, $\mathfrak{G}^{(j)}$, $\{x_j\}_1^N$ as in Lemma 3.2 with $A=1$. Write $\Lambda = \sum_1^N \Lambda_j$ where $\Lambda_j = \sum_{Q \in \mathfrak{G}^{(j)}} \lambda_Q$. It suffices to estimate Λ_j, so fix j and let $g(x) = \Lambda_j(x+x_j)$. Then $g = \sum_{l(Q') \le 8} g_{Q'}$, where $g_{Q'} = \lambda_Q$ with Q and Q' related as in (3.1), (3.2). These $g_{Q'}$ satisfy (3.5), (3.6) and a stronger form of (3.4): $g_{Q'}$ is supported on Q'. Write $g_{Q'}$ as a dyadic martingale, $g_{Q'} = \sum_{Q \subseteq Q'} g^Q_{Q'}$ where $g^Q_{Q'}$ is supported on Q and constant on cubes of length $\frac{1}{2}l(Q)$, and $\int g^Q_{Q'} = 0$. This gives the representation of g as dyadic martingale,

$$g = \sum_{l(Q) \le 8} \sum_{\substack{Q' \supseteq Q \\ l(Q') \le 8}} g^Q_{Q'} = \sum_{l(Q) \le 8} g^Q$$

By well-known properties of Lipschitz functions ([14]), $\|g^Q_{Q'}\|_\infty \le C\|g_{Q'}\|_{\mathrm{Lip}\,\alpha}\, l(Q)^{-\alpha}$. Let $\beta \in (0, \alpha)$. If x is fixed, then

$$\begin{aligned}
\sum_{x \in Q} \|g^Q\|^2_\infty &\le \sum_{x \in Q} \Big(\sum_{Q' \supseteq Q} \|g^Q_{Q'}\|_\infty\Big)^2 \\
&\le C \sum_{x \in Q} l(Q)^{2\alpha} \Big(\sum_{Q' \supseteq Q} \|g_{Q'}\|_{\mathrm{Lip}\,\alpha}\Big)^2 \\
&\le C \sum_{x \in Q} l(Q)^{2(\alpha-\beta)} \sum_{Q' \supseteq Q} \|g_{Q'}\|^2_{\mathrm{Lip}\,\alpha}\, l(Q')^{2\beta} \\
&\le C \sum_{x \in Q'} l(Q')^{2\beta} \|g_{Q'}\|^2_{\mathrm{Lip}\,\alpha} \sum_{x \in Q \subseteq Q'} l(Q)^{2(\alpha-\beta)} \\
&\le C \sum_{x \in Q'} l(Q')^{2\alpha} \|g_{Q'}\|^2_{\mathrm{Lip}\,\alpha} \\
&\le CB^2.
\end{aligned}$$

We can now apply Theorem 3.1 to conclude that $\int_{Q_0} e^{c_1 g^2/B^2} \leq c_2$, so the lemma is proved.

This finishes the proof of Theorem 3.2. Quite likely good λ inequalities analogous to Corollary 3.1 may also be obtained; however, we have not done this. But we want to point out two immediate corollaries.

COROLLARY 3.3. *If $A_\gamma f \in L^\infty$, then f belongs to the closure of L^∞ in BMO.*

This follows from [9] and Theorem 3.2. In fact if $A_\gamma f \in L^\infty$, then f exists in $[L^\infty, BMO]_{1/2,\infty}$: This follows from Theorem 3.2 and unpublished results of S. Janson [10]. Using that the full gradient is invariant by the Hilbert transform we also have the following: if F is bounded and harmonic on $\mathbb{R}^2_+$ and $\sup_{x\in\mathbb{R}} \int_{\Gamma_\gamma(x)} |\nabla f(y, t)|^2\, dy\, dt < \infty$, then f may be approximated in L^∞ norm by real parts of H^∞ functions. Back on $\mathbb{R}^d$, if $\tilde{S}f$ denotes either $A_\gamma f$ or Sf, we also have

COROLLARY 3.4. *If $w \geq 0$ and f are given, then*

$$\int_Q (f-f_Q)^2 w \leq C\, \|\tilde{S}f\|_\infty^2 \int_Q w\left(1+\log^+ \frac{w}{w_Q}\right)$$

for any cube Q for which the right hand side is finite.

This follows from Theorem 3.1 or 3.2 by Young's inequality. Note the formal similarity between Corollary 3.4 and the (false) analogue of Theorem 2.1 where $\varphi(x) = 1 + \log^+ x$.

Section 4. Appendix

In this section we show that if $v \geq 0$ has the Muckenhoupt A_∞ condition and satisfies

$$\sup_Q \frac{l(Q)^2}{|Q|} \int_Q v\, dx < \infty \tag{4.1}$$

then v has the Kerman–Sawyer property (0.10).

Recall the v is said to have A_∞ if $\forall\, \varepsilon > 0$ there is a $\delta > 0$ so that for all cubes Q

and all measurable sets $E\subset Q$,

$$\frac{|E|}{|Q|}<\delta \Rightarrow \frac{\int_E v}{\int_Q v}<\varepsilon.$$

An easy consequence of this is that there is a $C<\infty$ so that

$$\int_Q M(\chi_Q v)\,dx \le C\int_Q v\,dx \qquad \forall\, Q \tag{4.2}$$

where M is the Hardy–Littlewood maximal function. We shall assume that v satisfies (4.1) and (4.2).

A collection of dyadic cubes $\{Q_\alpha\}$ is said to satisfy the Carleson nesting condition if there is an $A<\infty$ so that for all cubes Q_α from the collection, we have:

$$\sum_{Q_{\alpha'}\subset Q_\alpha} |Q_{\alpha'}|\le A\,|Q_\alpha| \tag{4.3}$$

It is a fact that if $\{Q_\alpha\}$ has (4.3) then for all $f\ge 0$ is $L^1_{\text{loc}}(\mathbb{R}^d)$ and all cubes Q:

$$\sum_{Q_\alpha\subset Q}\left(\frac{1}{|Q_\alpha|}\int_{Q_\alpha} f\,dx\right)|Q_\alpha|\le CA\int_Q M(\chi_Q f)\,dx. \tag{4.4}$$

We wish to show that (4.1) and (4.2) imply

$$\int_Q (M_1\,|\chi_Q v|)^2\,dx \le c\int_Q v\,dx \tag{4.5}$$

where $M_1 f=\sup\limits_{x\in Q}\dfrac{l(Q)}{|Q|}\displaystyle\int_Q |f|\,dt.$ It is clearly enough to prove (4.5) when the supremum is taken over dyadic cubes. (We may also take Q to be dyadic.) Let $R>0$ be large. Let $\{Q_j^k\}$ be the maximal dyadic subcubes of Q such that

$$\frac{l(Q_j^k)}{|Q_j^k|}\int_{Q_j^k} v\,dx>R^k \tag{4.6}$$

Then:

$$\begin{aligned}\int_Q M_1(\chi_Q v)^2\,dx &= \sum_{Q_j^k}\left(\frac{l(Q_j^k)}{|Q_j^k|}\int_{Q_j^k} v\right)^2 |Q_j^k| \\ &= \sum_{Q_j^k}\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} v\right)\left(\frac{l(Q_j^k)^2}{|Q_j^k|}\int_{Q_j^k} v\right)|Q_j^k| \\ &\le C\sum_{Q_j^k}\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} v\right)|Q_j^k|\end{aligned}$$

by (4.1). Thus if the $\{Q_j^k\}$ had (4.3), then (4.2) and (4.4) would imply (4.5). They do. By their maximality, the $\{Q_j^k\}$ satisfy:

$$\frac{l(Q_j^k)}{|Q_j^k|}\int_{Q_j^k} v\,dx \le cR^k \tag{4.7}$$

where c depends on the dimension d. Let $Q_j^{k+1}\subset Q_l^k$. Combining (4.6) and (4.7) yields:

$$\frac{\dfrac{1}{|Q_j^{k+1}|}\displaystyle\int_{Q_j^{k+1}} v\,dx}{\dfrac{1}{|Q_l^k|}\displaystyle\int_{Q_l^k} v\,dx} > cR\frac{l(Q_l^k)}{l(Q_j^{k+1})} > 10 \tag{4.8}$$

for R large enough. This implies that:

$$\sum_{Q_l^{k+1}\subset Q_j^k} |Q_l^{k+1}| \le \tfrac{1}{10}|Q_j^k|$$

And by iterating we get:

$$\sum_{n=0}^{\infty}\sum_{Q_p^{k+n}\subset Q_j^k} |Q_p^{k+n}| \le (1+\tfrac{1}{9})|Q_j^k|$$

which is (4.3).

Remark (1). We note that at one point, namely (4.8), our argument is extremely wasteful. This would seem to indicate the delicacy of the Kerman–Sawyer condition.

(2). While this paper was in preparation we learned that Sawyer has independently found another proof that (4.1) and A_∞ imply (0.10). Sawyer has also found a direct proof that if $0 \leq V \leq W$ and W has (0.10), then so does V.

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA 90024

Department of Mathematics
University of Wisconsin
Madison, WI 53233

Department of Mathematics 253-37
California Institute of Technology
Pasadena, CA 91125

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