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## The Euler and Pontrjagin numbers of an $n$ -manifold in $\mathbb{C}^n$

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### Introduction

According to a theorem of H. Whitney, every smooth  $n$ -dimensional manifold  $M^n$  can be smoothly embedded in the Euclidean space  $\mathbb{R}^{2n}$ . Viewing  $\mathbb{R}^{2n}$  as  $\mathbb{C}^n$ , one may ask for embeddings which have nice properties relative to the complex structure. The simplest properties relate to complex tangents. If  $M$  has no complex tangents, the embedding is said to be totally real. In general there are global obstructions to finding totally real embeddings. For example, if  $M$  is compact, orientable and totally real, then its Euler number and Pontrjagin classes must vanish, a result due to R. Wells [11].

In this paper we shall give an explicit formula for the Euler number of a compact real  $n$ -manifold  $M$  suitably immersed in a complex  $n$ -manifold. (The requirements on  $M$  hold generically if  $n \leq 5$ .) We shall also give a formula for the Pontrjagin number of a compact, orientable  $M^4$  generically immersed in  $\mathbb{C}^4$ . We must assume that  $M$  has only one-dimensional complex tangents which are non-degenerate in a certain sense, and occur along a smooth, compact, codimension two submanifold  $N \subset M$ . There is a smooth invariant function  $\gamma$  on  $N$ ,  $0 \leq \gamma \leq +\infty$ . In section 5 we derive a relation among the Euler numbers  $\chi(M)$ ,  $e = \chi[\gamma < \frac{1}{2}]$ ,  $h = (-1)^n \chi[\gamma > \frac{1}{2}]$ ,  $\chi(M^\perp)$  (normal bundle), and the parabolic index  $p$ , which is described in section 1. As a special case we have the following.

**THEOREM (0.1).** *Let the compact, orientable  $n$ -manifold  $M$  be embedded in  $\mathbb{C}^n$  as just described. Then its Euler number satisfies*

$$\chi(M) = e - h + p. \quad (0.1)$$

When  $n = 2$  we have  $p = 0$ , since there are no parabolic points. In this case the

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theorem is due to E. Bishop [1], who reduced it to a theorem of Chern and Spanier [3]. Our method of proof is different, being based on the Poincaré–Hopf formula for the Euler number. If  $M$  is totally real, (0.1) implies  $\chi(M) = 0$ . For a direct proof see [11]. If the 2-sphere is embedded in  $\mathbb{C}^2$  as above, then it must have at least 2 elliptic points by (0.1). Elliptic points are of interest since they contribute to the local hull of holomorphy of  $M$ . They have been studied most recently in [6], [7], and [10]. It is not known whether a 4-sphere generically embedded in  $\mathbb{C}^4$  must have an elliptic point. If not then (0.1) gives  $h = -2$ , which puts some restriction on the topology of  $N = N_h$ .

In some cases the Pontrjagin numbers give more information about the complex-tangent structure. Let  $M$  be immersed in  $\mathbb{C}^n$ , and  $J$  denote the real operator corresponding to multiplication by  $\sqrt{-1}$ . Then  $H_m = T_m(M) \cap JT_m(M)$  for  $m \in N$ , defines a complex line bundle  $H$  over  $N$ .  $N$  inherits a natural orientation from  $M$  (section 1).

**THEOREM (0.2).** *Suppose the compact, orientable 4-manifold  $M$  is generically immersed in  $\mathbb{C}^4$ . Then its Pontrjagin number satisfies*

$$p_1(M) = \chi(H). \quad (0.2)$$

This is proved in section 3, where we also show that  $\chi(H) = 0$  if  $M$  has no parabolic points. Thus if  $p_1(M) \neq 0$ , then  $M$  has a parabolic point, hence nearby elliptic and hyperbolic points. A generic embedding of  $\mathbb{CP}_2$  in  $\mathbb{C}^4$  therefore has a non-trivial hull of holomorphy.

In [8] H. F. Lai has given general formulas for the Euler and Pontrjagin classes of a wider class of submanifolds of  $\mathbb{C}^n$ . The relation of his work to the present paper is not clear. His formulas do not yield (0.1) or (0.2). In section 1 we describe the local properties of  $M^n \subset \mathbb{C}^n$  near a complex tangent. In section 2 we study the real Grassmannian and give a transversality argument. Section 3 contains a general result about the intersection properties of Schubert varieties needed for Theorem (0.2). Section 4 is devoted to deriving suitable local equations for  $M$  near a complex tangent.

## 1. Complex tangents and the parabolic index

Let  $M^n$  be a smooth real  $n$ -manifold immersed in the complex  $n$ -manifold  $\tilde{M}$ . In a local holomorphic coordinate system  $z = x + iy = (z^1, \dots, z^n)$ ,  $M$  is given by

$$M: R = (r^1, \dots, r^n) = 0, \quad R = \bar{R}, \quad dr^1 \wedge \dots \wedge dr^n \neq 0, \\ \partial r^1 \wedge \dots \wedge \partial r^n = B dz^1 \wedge \dots \wedge dz^n. \quad (1.1)$$

Under a change of holomorphic coordinates  $z \rightarrow z'$  and defining function  $R \rightarrow R'$ , the factor  $B$  changes by

$$B \rightarrow B' = \frac{\partial R'}{\partial R} \left( \frac{\partial z'}{\partial z} \right)^{-1} B. \quad (1.2)$$

Let  $F$  denote the normal bundle of  $M$  in  $\tilde{M}$ , and  $F^*$  its dual. Also, let  $K$  denote the canonical line bundle of  $\tilde{M}$ , and  $L$  and  $L^*$  the real line bundles  $\Lambda^n F$  and  $\Lambda^n F^*$ , respectively. Then (1.2) says that the collection  $\{B, B', \dots\}$ , which we denote simply by  $B$ , defines a section of the complex line bundle  $K \otimes L^{*-1} = K \otimes L$  over  $M$ . In particular, the set

$$N = \{m \in M : B(m) = 0\} \quad (1.3)$$

is well defined and is precisely the set of points  $m$  at which  $M$  has a (non-trivial) complex tangent space  $H_m$ . If  $J_m$  denotes the real linear operator on the real tangent space  $T_m \tilde{M}$  corresponding to multiplication by  $\sqrt{-1}$ , then  $H_m = T_m \cap J T_m$ , where  $T = T(M)$  is the real tangent bundle of  $M$ .

We assume that  $\dim_{\mathbb{C}} H_m = 1$ , so that  $H_m \otimes \mathbb{C} = H'_m \oplus H''_m$ ,  $H''_m = \bar{H}'_m$ , and  $H'_m$  is spanned by

$$X = \sum \xi^i \partial / \partial z^i, \quad XR = \sum \xi^i (\partial R / \partial z^i)|_m = 0. \quad (1.4)$$

We further assume that the complex tangent  $H_m$  is non-degenerate, in that

$$\text{either } XB \neq 0 \quad \text{or} \quad \bar{X}B \neq 0. \quad (1.5)$$

Since  $X$  is determined up to  $X \rightarrow cX$ ,  $c$  a non-zero complex constant, it follows that

$$\gamma(m) = \frac{1}{2} |XB / \bar{X}B| \in [0, \infty] \quad (1.6)$$

is a well defined biholomorphic invariant of  $M$  at  $m$ . It was first found by Bishop [1] in the following form.

We suppose that  $m$  is the origin of a coordinate system  $(z_1, z^\alpha, 2 \leq \alpha \leq n-1, z_n)$  in which  $T_m$  is the  $(z_1, x^\alpha)$ -space.  $M$  is given locally as the graph  $R = 0$ ,

$$\begin{aligned} r &\equiv -z_n + F(z_1, x), & F &= q + x \cdot O(1) + O(2), \\ r^\alpha &\equiv -y^\alpha + f^\alpha(z_1, x), & f^\alpha &= \bar{f}^\alpha = O(2), & 2 \leq \alpha \leq n-1, \\ x &= (x^\alpha), & q &= az_1^2 + bz_1\bar{z}_1 + c\bar{z}_1^2. \end{aligned} \quad (1.7)$$

Here and elsewhere  $O(k)$  indicates a term which vanishes to order  $k$  at the origin. Then, up to a constant,  $X = \partial/\partial z_1$ , and

$$B = \frac{\partial(r, \bar{r}, r^\alpha)}{\partial(z_1, z_n, z^\beta)} = (1/2)^{n-2} \partial \bar{F} / \partial z_1 + O(2), \quad XB = 2c, \quad \bar{X}B = \bar{b},$$

$$\gamma(m) = |c/b|. \quad (1.8)$$

We also introduce

$$\Delta = |XB|^2 - |\bar{X}B|^2. \quad (1.9)$$

$N$  is partitioned into

$$N_e = [\gamma < \tfrac{1}{2}], \quad N_p = [\gamma = \tfrac{1}{2}], \quad N_h = [\gamma > \tfrac{1}{2}], \quad (1.10)$$

the sets of elliptic, parabolic, and hyperbolic points. These sets correspond to  $\Delta < 0$ ,  $\Delta = 0$ ,  $\Delta > 0$ , respectively.

Next we examine more closely a parabolic point  $m$ . Since the determinant (1.9) vanishes, (1.5) implies that the system

$$aXB + b\bar{X}B = 0, \quad aX\bar{B} + b\bar{X}\bar{B} = 0,$$

has a non-trivial solution  $(a, b)$  unique up to  $(a, b) \rightarrow (a', b') = (\mu a, \mu b)$ . Complex conjugation of these equations shows that  $\bar{b} = \lambda a$ ,  $\bar{a} = \lambda b$ , for some  $\lambda$ ,  $|\lambda| = 1$ . The factor  $\mu$  can be adjusted so that  $b' = \bar{a}'$ . Hence, there is an  $a$ , unique up to  $a \rightarrow \rho a$ ,  $\rho \in \mathbb{R}$ , for which

$$YB = 0, \quad Y = aX + \bar{a}\bar{X}. \quad (1.11)$$

It's easily to be seen that the changes  $X \rightarrow cX$  and (1.2) can change  $Y$  by at most a real factor. Thus  $Y$  spans an intrinsic real line  $l_m \subset H_m$ , which we call the *parabolic line* at  $m$ .

Now we assume that

$$dB \wedge d\bar{B} \neq 0 \quad \text{on} \quad B = 0, \quad (1.12)$$

so that  $N$  is a real submanifold of  $M$  of codimension 2. The conormal bundle  $S^*$  of  $N$  in  $M$  is spanned by the coframes  $dB$  along  $N$ , and hence is the restriction of  $(K \otimes L)^*$  to  $N$ . So the *normal bundle*  $S$  of  $N$  in  $M$  is the complex line bundle  $K \otimes L$  restricted to  $N$ . The non-degeneracy condition (1.5) precludes  $H_m$  being contained in  $T_m(N)$ . It follows from (1.11) and (1.9) that  $m$  is a parabolic point

precisely when  $T_m N \cap H_m = l_m$  is one dimensional. We denote by  $\varphi$  the composite vector bundle mapping

$$\varphi : H \hookrightarrow T(M)|_N \rightarrow (T(M)|_N)/T(N) \equiv S. \quad (1.13)$$

Since  $dB$  is local coframe for  $S$  and  $X$  is a  $(1, 0)$ -frame for  $H_m$ , consideration of the sign of (1.9) shows that  $\varphi_m$  is orientation reversing if  $m$  is elliptic and orientation preserving if  $m$  is hyperbolic. It is singular of real rank one if  $m$  is parabolic.

We assume still further that  $d\gamma \neq 0$  when  $\gamma = \frac{1}{2}$ , so that  $N_p$  is a smooth  $(n-3)$ -dimensional manifold. We may now proceed to define the parabolic index. For each  $m \in N_p$  we have two lines in  $T_m(N)$ , the parabolic line  $l_m$  and the line  $k_m$  determined by the normal vector  $\nabla\gamma$  (gradient relative to any convenient metric on  $N$ ). This gives us two sections  $l, k$  of the projective bundle  $P \rightarrow N_p$ , which has as fiber over  $m \in N_p$  all lines through the origin in  $T_m(N)$ .

We next describe an orientation on the open subset of  $P$  consisting of lines not tangent to  $N_p$ . Let  $x^\alpha$ ,  $3 \leq \alpha \leq n-1$ , be local coordinates on  $N_p$ . Let  $y$  be a local defining function for  $N_p$ :  $y=0$ , with  $\partial\gamma/\partial y > 0$ . Then  $(x, y)$  are local coordinates on  $N$ , and any line  $L \in P$ , not tangent to  $N_p$ , is spanned by a unique vector

$$\frac{\partial}{\partial y} + \sum_{\alpha=3}^{n-1} w^\alpha \frac{\partial}{\partial x^\alpha} \in L. \quad (1.14)$$

Thus  $(x, w)$  are coordinates for  $L$ , and

$$\Omega_P = dx^3 \wedge \cdots \wedge dx^{n-1} \wedge dw^3 \wedge \cdots \wedge dw^{n-1} \quad (1.15)$$

defines a local volume form on the  $(2n-6)$ -dimensional manifold  $P$ . If  $(\tilde{x}, \tilde{y})$  is another such coordinate system, then one easily sees that

$$\tilde{\Omega}_P = (\det \partial \tilde{x} / \partial x)^2 (\partial \tilde{y} / \partial y)^{-n+3} \Omega_P.$$

Since  $\partial \tilde{y} / \partial y > 0$ , we have a well defined orientation.

The *parabolic index* is defined as the intersection number of  $l(N_p)$  and  $k(N_p)$  relative to this orientation. This is possible since  $k_m$  is never tangent to  $N_p$ . More precisely, we take a slight perturbation of  $k$ , if necessary, so that  $k(N_p)$  and  $l(N_p)$  intersect transversely in  $P$  at a finite number of points. At a point  $m \in N_p$  where  $l_m = k_m$ , we choose local coordinates as in the previous paragraph. Then  $\partial l / \partial x^\alpha$ ,  $\partial k / \partial x^\alpha$ ,  $3 \leq \alpha \leq n-1$ , give frames for  $l(N_p)$  and  $k(N_p)$  at  $m$ . The intersection index

at  $m$  is given by the sign

$$\text{ind}_{P,m}(l, k) = \text{sgn } \Omega_P(\partial l / \partial x, \partial k / \partial x). \quad (1.16)$$

This is well defined since a change of orientation of the local coordinates  $x$  changes the orientations on *both*  $l(N_p)$  and  $k(N_p)$ , so that (1.16) remains unchanged. The parabolic index is

$$p = \sum \{\text{ind}_{P,m}(l, k) : m \in N_p, l_m = k_m\}. \quad (1.17)$$

## 2. The Grassmann manifold and transversality

In this section we consider an  $n$ -manifold  $M$  immersed in  $\mathbb{C}^n$ . Its Gauss map  $g$  associates to each  $m \in M$  the real tangent plane  $T_m(M)$ . It is a smooth mapping of  $M$  into  $Gr(n; n) \equiv Gr$ , the  $n^2$ -dimensional Grassmann manifold of real  $n$ -planes through the origin of  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ . We define

$$\begin{aligned} C_k &= \{V \in Gr : \dim_{\mathbb{C}} V \cap JV = k\}, \quad 0 \leq k \leq n/2; \\ C &= C_1 \cup \cdots \cup C_{[n/2]}. \end{aligned} \quad (2.1)$$

$C_0$  is the dense open subset of totally real planes, and  $Gr = C_0 \cup \cdots \cup C_{[n/2]}$  is a disjoint union. For each  $k$ ,  $0 < k \leq n/2$ ,  $C_k$  fibers over the complex Grassmannian  $Gc(k, n-k)$  of complex  $k$ -planes in  $\mathbb{C}^n$ . In fact, for  $V$  in  $C_k$ ,  $V \cap JV \cong \mathbb{C}^k$  and  $V \cap (V \cap JV)^\perp$  is a totally real  $(n-2k)$ -plane in  $\mathbb{C}^{n-k}$ . So the fiber is an open subset of  $Gr(n-2k; n-k)$ , which has dimension  $n(n-2k)$ , while the base has real dimension  $2k(n-k)$ . Thus  $C_k$  is a real submanifold of codimension  $2k^2$ . If  $n = 2m$  is even,  $C_m = Gc(m; n)$ ; while if  $n = 2m+1$ ,  $C_m$  is a bundle over  $Gc(m; n)$  with fiber the real projective space  $\mathbb{RP}(2m+1)$ .

We repeat some of the constructions of section one in this “universal” setting. For  $V \in Gr$  there are two natural vector spaces;  $V$  itself and  $F_V = \mathbb{C}^n / V$ . We also have the two real line bundles  $L = \bigwedge^n F$ ,  $L^* = \bigwedge^n F^*$ , and the complex line bundle  $K = \bigwedge^n (Gr \times \mathbb{C}^{n*}) \cong Gr \times \mathbb{C}$ . We refer back to (1.1) where now the  $dr^i$  are  $n$  independent real linear forms on  $\mathbb{C}^n$  annihilating a fixed  $V$  in  $Gr$ , and the  $dz^i$  are a basis of complex linear functions on  $\mathbb{C}^n$ . As before it follows that  $B$  is a section of  $K \otimes L$ , having as zero set precisely  $C$ . We restrict this bundle to  $C_1$ , where it is identified with the normal bundle  $S$  of  $C_1$  in  $Gr$ . Its dual bundle has local frames  $dB$  restricted to  $C_1$ . Also, we have the complex line bundle  $H \rightarrow C_1$ ,  $H_V = V \cap JV$ ,  $H \otimes \mathbb{C} = H' \oplus H''$ .

In addition we have a real 4-plane bundle  $E \rightarrow C_1$  defined by  $E_V = \text{Hom}_{\mathbb{R}}(H_V, S_V)$ . Each element of  $E_V$  may be described by an equation

$$dB = XB\theta + \bar{X}B\bar{\theta},$$

where  $X$  is a frame vector for  $H'$  and  $\theta$  is its dual. If we denote by  $E_0$  the zero section of  $E$ , which corresponds to  $XB = \bar{X}B = 0$ , then we have a well defined function  $\gamma : (E - E_0) \rightarrow [0, \infty]$  given by (1.6). Parallel to (1.10) we have the disjoint union  $E = E_0 \cup E_e \cup E_p \cup E_h$ , where  $E_e, E_p, E_h$  are the sets where  $\gamma < \frac{1}{2}$ ,  $\gamma = \frac{1}{2}$ ,  $\gamma > \frac{1}{2}$ , respectively.

For  $M$  immersed in  $\mathbb{C}^n$  with at most one dimensional complex tangents, it is clear that  $g^*H$  and  $g^*S$  are the corresponding bundles of section 1. For each  $m \in N$  we have

$$dg_m : H_m \cong H_{g(m)} \rightarrow T_{g(m)}Gr \rightarrow T_{g(m)}Gr/T_{g(m)}C_1 \cong S_{g(m)},$$

which defines a map  $dg : N \rightarrow E$ . The degenerate points form the set  $dg^{-1}(E_0)$ . Clearly,  $\gamma \circ dg$  is the invariant (1.6).

**PROPOSITION (2.1).** *Let  $f : M^n \rightarrow \mathbb{C}^n$  be a smooth immersion. a) A generic small perturbation of  $f$  results in an immersion with the following properties:  $M$  has no complex  $k$ -dimensional tangents if  $2k^2 > n$ , while if  $2k^2 \leq n$ , the points with such tangents form a submanifold of codimension  $2k^2$ . b) Suppose in addition that the immersion  $f$  has only one dimensional complex tangents which occur along the set  $N$ . After a generic small perturbation,  $N$  is a compact smooth  $(n-2)$ -manifold, and the set  $N_0$  of degenerate points is a smooth  $(n-6)$ -manifold. c) Assume further that  $N_0$  is empty. Then after a generic small perturbation the parabolic set  $N_p$  forms a compact smooth  $(n-3)$ -manifold along which  $d\gamma \neq 0$ .*

**Remark.** As a consequence a generic  $M^n$  in  $\mathbb{C}^n$  has the following characteristics:

- i)  $n = 2$  —isolated elliptic or hyperbolic points;
  - ii)  $n = 3, 4, 5$  —at most one-dimensional non-degenerate complex tangents with  $N_p$  smooth;
  - iii)  $n = 6$  —at most one-dimensional complex tangents and at most isolated degenerate points;
  - iv)  $n = 8$  —at most isolated 2-dimensional complex tangents.
- Case i) is due to Hunt and Wells [5].

The main ingredient in the proof is the parametric transversality theorem (see

e.g. Hirsch [4], p. 79). Let  $f$  be any immersion of  $M$  with Gauss map  $g = f_*$ . Let  $Q$  denote the set of all real affine transformations  $A(z) = A'(z) + a$ ,  $A' \in Gl(2n, \mathbb{R})$ ,  $a \in \mathbb{C}^n$ . Then  $G(A, z) \equiv g_{Af}(z) = A' \circ g_f(z)$  defines a smooth map

$$G : Q \times M \rightarrow Gr. \quad (2.2)$$

We claim that the mapping  $G$  is transverse to  $C_k$  for every  $k \leq [n/2]$ . This means that  $TC_k + T\mathcal{J}_m G = TGr$ , at every point  $G(A, z) \in C_k$ , where  $T\mathcal{J}_m G \equiv DG(T(Q \times M))$ . This will follow from  $T\mathcal{J}_m G = TGr$ , which has nothing to do with complex tangents. We let  $\mathbb{C}^n = V \oplus V^\perp$  with coordinates  $(x_1, x_2)$ , and restrict to the submanifold of  $Q$  of maps of the form  $(x_1, x_2) \rightarrow (x_1, x_2 - Bx_1)$ ,  $B \in \text{Hom}_{\mathbb{R}}(V, V^\perp)$ . It is clear that  $DG$  maps the tangent space of this submanifold at  $B = 0$  onto  $T_V Gr$ .

By the parametric transversality theorem, the set  $Q_k$  of  $A$ 's for which  $z \rightarrow G(A, z)$  is transverse to  $C_k$  is residual. It follows that  $Q_1 \cap \dots \cap Q_{[n/2]}$  is also residual and therefore dense. Thus there are affine (or even linear) mappings  $A$ , arbitrarily close to the identity, for which  $A \circ f$  is transverse to every  $C_k$ . Part a) follows by transversality. Under the additional assumption of b)  $N$  is a compact smooth  $(n-2)$ -manifold.

For parts b) and c) we must use mappings quadratic in  $(z, \bar{z})$ :  $A(z) = a + A'(z) + A''(z^2)$ . We first replace  $f$  by a perturbation as in a), so that the Gaussian image  $g(M)$  remains disjoint from  $C_k$ ,  $k > 1$ , and intersects  $C_1$  transversely. We then restrict to quadratic mappings  $A$  with  $A'$  unitary and  $A''$  so small that this situation is preserved. We let  $Q$  denote this set of maps. For  $A''$  small enough, perturbation by  $A \in Q$  will result in a new manifold  $N_A$  close to the original  $N$  in the following sense.  $N_A$  will lie in a small tubular neighborhood  $u = U\{D_m : m \in N\}$  and will intersect each normal 2-disc  $D_m$  in a unique point  $\eta_A(m)$ , giving a diffeomorphism  $\eta_A : N \rightarrow N_A$ . We consider the composite map.

$$G'(A, m) = dg_{Af} \circ \eta_A(m), \quad G' : Q \times N \rightarrow E.$$

We claim that the map  $G'$  is transverse to  $E_0$  under the assumptions of b) and transverse to  $E_p$  under those of c). Assume  $G'(A, m) \in E_0$ . After an affine unitary coordinate change we may assume that  $m = 0$  and that  $M$  is given as in (1.7). We then restrict  $G'$  to the submanifold of  $Q$  consisting of mappings of the special form

$$A : \begin{cases} z_1 \rightarrow z_1, z^\alpha \rightarrow z^\alpha, & 2 \leq \alpha \leq n-1 \\ z_n \rightarrow z_n + Bz_1\bar{z}_1 + C\bar{z}_1^2, \end{cases}$$

which result in  $(b, c) \rightarrow (b-B, c-C)$  in (1.7). Locally  $E \cong C_1 \times \mathbb{R}^4$  and  $E_0 \cong C_1 \times \{0\}$ . The normal space to  $E_0$  at  $m$  is  $\{m\} \times \mathbb{R}^4 \cong \{m\} \times \mathbb{C}^2$  with coordinates  $(b, c)$ . If we restrict  $G'$  further to  $(A, m)$  with  $A$  as described and  $m = 0$ , then it is clear that  $DG'$  at  $(A, m) = (I, 0)$  maps onto the normal space. Thus  $G'$  is transverse to  $E_0$ . A similar argument shows that  $G'$  is transverse to  $E_p \cong C_1 \times \text{Cone}$ . Now b) and c) follow by the parametric transversality theorem since  $E_0$  has codimension 4 and  $E_p$  codimension one in  $E$ .

### 3. The Pontrjagin number of a 4-manifold in $\mathbb{C}^4$

By a well-known theorem [9] the Pontrjagin class  $p_k(T(M))$  equals  $(-1)^k c_{2k}(T(M) \otimes \mathbb{C})$ , where  $c$  denotes the Chern class. For  $M^n \subset \mathbb{C}^n$  with Gauss map  $g$ ,  $p_k(T(M)) = g^* p_k(VGr)$ , where  $VGr \rightarrow Gr$  is the universal bundle.  $Gr \subset Gc(n, n)$  and  $(VGr) \otimes \mathbb{C}$  is the restriction of the complex universal bundle  $VGc \rightarrow Gc$ . Thus we must consider the Chern classes  $c_k(VGc)$ .

We begin by recalling some facts [2] about  $Gc(n, r)$ , the space of all complex  $n$ -planes  $Z^n \subset \mathbb{C}^{n+r}$ . For  $0 \leq k_1 \leq \dots \leq k_n \leq r$  and linear spaces  $L_1 \subset \dots \subset L_n$ ,  $\dim L_j = k_j + j$ , the Schubert variety is defined by

$$Z(k_1, \dots, k_n) = \{Z \in Gc(n, r) \mid \dim Z \cap L_j \geq j\},$$

and has complex dimension  $k_1 + \dots + k_n$ . The Chern class  $c_k(VGc(n, r))$  is dual to  $Z_k(n, r) \equiv Z(r-1, \dots, r-1, r, \dots, r)$ , where  $r-1$  appears  $k$ -times. It may also be defined as

$$Z_k(n, r) = \{Z \in Gc(n, r) \mid \dim Z \cap L \geq k\}, \quad \dim L = r-1+k, \quad (3.1)$$

and decomposes into the disjoint union of

$$Z_k^0(n, r) = \{Z : \dim Z \cap L = k\},$$

and

$$Z_k^s(n, r) = \{Z : \dim Z \cap L > k\}.$$

$Z_k^0(n, r)$  is a complex manifold of complex codimension  $k$ , fibering over  $Gc(k, r-1)$ , and  $Z_k^s(n, r) = Z(r-2, \dots, r-2, r, \dots, r)$  ( $k+1(r-2)$ 's) has codimension  $2(k+1)$ . Thus a generic compact orientable real  $2k$ -manifold in  $Gc(n, r)$ , which we also denote by  $M$ , will be disjoint from  $Z_k^s(n, r)$  and intersect  $Z_k^0(n, r)$

transversely in finitely many points.  $c_k(\text{VGc}(n, r))[M]$  is the sum of the intersection indices at these points.

We fix  $k$  and  $L_{r-2+k} \subset L_{r-1+k}$ , subscripts denoting dimension, and consider the corresponding varieties  $Z_{k-1}(n, r) \supset Z_k(n, r)$ ,  $Z_{k-1}^s(n, r) \supset Z_k^s(n, r)$ . A generic  $M^{2k}$  will miss  $Z_{k-1}^s(n, r)$ , since it has real codimension  $4k$ . Thus

$$M \cap Z_{k-1}(n, r) = M \cap Z_{k-1}^0(n, r) \equiv N$$

is a smooth compact oriented 2-manifold containing the finite set  $Z_k(n, r) \cap M$ . For  $Z \in Z_{k-1}^0(n, r)$ , we set

$$A_Z = Z \cap L_{r-2+k} \quad \text{and} \quad B_Z = Z \cap A_Z^\perp,$$

so that  $Z = A_Z \oplus B_Z$  is an orthogonal direct sum relative to the standard hermitian inner product on  $\mathbb{C}^{n+r}$ . We define a smooth map  $\beta$  by

$$\beta: Z_{k-1}^0(n, r) \rightarrow \text{Gc}(n-k+1, r+k-1), \quad \beta(Z) = B_Z. \quad (3.2)$$

For  $Z \in Z_{k-1}^0(n, r)$ ,  $Z \in Z_k^0(n, r)$  if and only if  $B_Z \in Z_1(n-k+1, r+k-1)$ ; i.e.

$$Z_k(n, r) \cap Z_{k-1}^0(n, r) = \beta^{-1}(Z_1(n-k+1, r+k-1)).$$

LEMMA (3.1)

$$c_k(\text{VGc}(n, r))[M] = c_1(\text{VGc}(n-k+1, r+k-1))[\beta N]. \quad (3.3)$$

*Proof.* This comes down to comparing two intersection indices. First, we have the equality of oriented vector spaces at  $m \in M \cap Z_k^0(n, r)$

$$T_m Z_k^0(n, r) \oplus T_m M = c_k T_m \text{Gc}(n, r),$$

where  $c_k = \pm 1$ . If  $S$  is the normal bundle of  $Z_{k-1}^0(n, r)$  in  $\text{Gc}(n, r)$ , then its restriction to  $N$  is that of  $N$  in  $M$ , so

$$T_m \text{Gc} = T_m Z_{k-1}^0 \oplus S_m \quad \text{and} \quad T_m M = T_m N \oplus S_m.$$

It follows that

$$T_m Z_k^0(n, r) \oplus T_m N = c_k T_m Z_{k-1}^0(n, r). \quad (3.4)$$

The map  $\beta$  is not holomorphic since it involves the orthogonal complement  $A_Z^\perp$ . However, a slight local deformation of it is. For all  $Z \in Z_{k-1}^0(n, r)$  sufficiently near  $m$  we may replace  $A_Z^\perp$  by  $A_m^\perp$ , then  $\alpha(z) = Z \cap A_m^\perp$  is holomorphic and approximates  $\beta$  near  $m$ . Clearly,  $\alpha^{-1}(Z_1^0(n-k+1, r+k-1)) = Z_k^0(n, r)$ . By (3.4)

$$T_{\alpha m}(\alpha Z_k^0(n, r)) \oplus T_{\alpha m}(\alpha N) = c_k T_{\alpha m}(\alpha Z_{k-1}^0(n, r)), \quad (3.5)$$

where the orientations agree with those from  $Gc(n-k+1, r+k-1)$ , since  $\alpha$  is holomorphic. If  $\tilde{S}$  is the normal bundle of  $\alpha Z_{k-1}^0(n, r)$  in  $Gc(n-k+1, r+k-1)$  it is also the normal bundle of  $\alpha Z_k^0(n, r)$  in  $Z_1^0(n-k+1, r+k-1)$ . Adding  $\tilde{S}_{\alpha m}$  to both sides of (3.5) gives

$$T_{\alpha m} Z_1^0(n-k+1, r+k-1) \oplus T_{\alpha m}(\alpha N) = c_k T_{\alpha m} Gc(n-k+1, r+k-1).$$

Now  $\alpha m = \beta m$ , and if we continuously deform  $\alpha$  back to  $\beta$ , we see that  $c_k = c_1$ , where  $c_1$  is the intersection index at  $\beta m$  entering into the right hand side of (3.3). Summing over all such  $m$  in  $M \cap Z_k^0(n, r)$  gives (3.3).

*Note.* The same argument gives

$$c_k(VGc(n, r)[M^{2k}]) = c_{k-l}(VGc(n-l, r+l))[M \cap Z_l^0(n, r)],$$

when the intersections are nice, which is generically so when  $2l+2 > k$ .

We return to the study of  $Gr(n; n)$ . We set  $\mathbb{C}^n = (W, J)$ ,  $W \otimes \mathbb{C} \equiv W^c \equiv W' \oplus W''$ ,  $\bar{W}'' = W' = \{w \in W^c : Jw = iw\}$ . For a subspace  $V \subset W$ ,  $V^c \subset W^c$ ,  $V^c \supset V' \oplus V''$ , where  $V' = V^c \cap W'$ ,  $V'' = V^c \cap W''$ . The map  $V \rightarrow V^c$  embeds  $Gr(n; n)$  in  $Gc(n, n)$  as a totally real submanifold.

LEMMA (3.2). a)  $V^c = V' \oplus V''$  if and only if  $JV = V$ .

b)  $V' = H'$  if  $H = V \cap JV$ .

c)  $C = Gr(n; n) \cap Z_1(n, n)$ .

*Proof.* a) If  $JV = V$ , then  $JV^c = V^c$ , and any  $w \in V^c$  is the sum  $\frac{1}{2}(w - iJw) + \frac{1}{2}(w + iJw) \in V' \oplus V''$ . If  $V^c = V' \oplus V''$ , then  $JV^c = JV' + JV'' = V' \oplus V'' = V^c$ ; so  $JV = V$ . b)  $V' \supset H'$  is clear since  $V^c \supset H^c$ . If  $w \in V'$ , then  $Jw = iw$ . So if  $w = u + iv$ ,  $u, v \in V$ , then  $u = Jv$ ,  $v = -Ju$ . Hence,  $u, v \in H$  and b) holds. If we apply a) to  $H$ , then it follows that  $V \in C$  if and only if  $\dim_{\mathbb{C}} V^c \cap W' \geq 1$ . So c) follows by taking  $n = r$ ,  $k = 1$ , and  $L_n = W'$  in (3.1).

We now turn to the proof of Theorem (0.2) of the introduction. Since  $M^4$  is generically immersed in  $\mathbb{C}^4$  it has the properties of Remark (ii) following Proposi-

tion (2.1). We have

$$\begin{aligned} p_1(M) &= p_1(T(M))[M] = p_1(VGr(4; 4))[gM] \\ &= -c_2(VGc(4, 4))[gM] = -c_1(VGc(3, 5))[\beta gN], \end{aligned}$$

by (3.3) with  $n=r=4$ ,  $k=2$ . Now over the surface  $N$  (or rather  $\beta gN$ )  $VGc(4, 4) = A \oplus B = H' \oplus VGc(3, 5)$  by Lemma 3.2). Thus, the total Chern class  $c = 1 + c_1$  satisfies  $[9] \quad c(VGc(4, 4)) = c(H')c(VGc(3, 5))$  or  $c_1(VGc(4, 4)) = c_1(H') + c_1(VGc(3, 5))$ . Over  $Gr(4; 4)$   $VGc(4, 4) = VGr(4; 4) \otimes \mathbb{C}$ , hence its first Chern class is a 2-torsion element. When pulled back to the compact orientable surface  $N$  it vanishes; thus  $c_1(VGc(3, 5)) = -c_1(H') = -c(H)$ . Hence,  $p_1(M) = \chi(H)$ , since  $\chi(H) = c_1(H)[N]$ .

The bundle mapping  $\varphi$  (1.13) can be used to get a formula for  $\chi(H)$ . Since  $\tilde{M} = \mathbb{C}^n$ , the canonical bundle  $K$  is trivial. Since  $M$  is orientable, so is its normal bundle, hence the line bundle  $L$  is trivial. It follows that  $S = K \otimes L$  is trivial.  $\varphi$  is a bundle isomorphism over  $N_h$  and an anti-isomorphism over  $N_e$ . Therefore  $H$  is trivial over any connected component of  $N$  which does not meet  $N_p$ . If we let  $N^0$  be the union of the components of  $N$  which meet  $N_p$ , and  $H^0$  the restriction of  $H$  to  $N^0$ , then  $\chi(H) = \chi(H^0)$ . We choose a section  $v$  of  $H^0$  which does not vanish on  $N_e^0 \cup N_p$  and has only isolated non-degenerate zeros in  $N_h^0$ .  $v$  gives a trivialization of  $H$  over  $N_p$ ; hence, the parabolic line  $l$  gives a map  $\tilde{l}: N_p \rightarrow \mathbb{RP}_1$ , where  $N_p$  and  $\mathbb{RP}_1$  have naturally induced orientations. We define the  $H$ -parabolic index  $p_H$  to be the degree of this mapping  $\tilde{l}$ . If  $w$  is a piecewise smooth section of  $H$  over  $N_p$  which spans  $l$  at each point, then  $w = \mu v$ ,  $\mu \neq 0$  and piecewise smooth. We have

$$p_H = \operatorname{Re} \frac{1}{\pi i} \int_{N_p} \frac{d\mu}{\mu}. \quad (3.6)$$

Note that  $w$  is determined up to  $w \rightarrow \rho w$ , with  $\rho \neq 0$ , real and piecewise smooth. It follows that (3.6) is not affected by this change. Also,  $v$  may be changed by  $v \rightarrow \xi v$ ,  $\xi \neq 0$  and smooth on  $N_p \cup N_e^0$ . Applying Stokes's theorem to  $d\xi/\xi$  on  $N_e^0$  shows that the integral in (3.6) remains unchanged. Thus  $p_H$  is well defined.

LEMMA (3.3).  $\chi(H) = -p_H$ .

*Proof.* This follows by comparing the index sums for  $v$  and  $\varphi(v)$ . We assume that  $v$  has been chosen so that  $l_m$  contains  $v_m$  at only a finite number of points  $m$  in  $N_p$  and that  $l$  crosses  $v$  transversely at such points. In otherwords  $1 \in \mathbb{RP}_1$  is a regular value of  $\tilde{l}$ . At such a point  $m$  we choose local coordinates  $(x, y)$  on  $N$  so that  $m = (0, 0)$ ,  $N_p$  is given by  $y = 0$ ,  $N_h$  by  $y > 0$ , and  $N_e$  by  $y < 0$ . We let  $\zeta = \xi + i\eta$

be a local fiber coordinate on  $H^0$  relative to  $v$  and choose a local frame  $v'$  and related coordinate  $\zeta' = \xi' + i\eta'$  for  $S$  near  $m$ . We may assume that  $\varphi(v)$  is a positive multiple of  $iv'$  at  $m$ . Then

$$\begin{aligned} \varphi: \quad \xi' &= a\xi + b\eta, & a(0) &= b(0) = c(0) = 0, \\ \eta' &= c\xi + d\eta, & d(0) &> 0. \end{aligned}$$

$\Delta = ad - bc$ ,  $\Delta(x, 0) = 0$ ,  $\Delta_y(x, 0) > 0$ . We let  $(\xi, \eta) = (1, \lambda(x))$ ,  $\lambda(0) = 0$ , span  $l$  along  $N_p$ ; then the sign of  $\lambda_x(0)$  gives the intersection index of  $l$  with respect to  $v$ . Since  $\varphi(l) = 0$ , we have  $c + d\lambda = 0$ , so  $\lambda_x(0) = -c_x(0)/d(0)$ . Also,  $a(x, 0) = (bc/d)(x, 0)$ , so  $a_x(0) = 0$ , and  $\Delta_y(0, 0) = a_y(0)d(0)$ , so  $a_y(0) > 0$ . Finally,  $\varphi(v) = \varphi(1, 0) = (a, c)$  has index at  $m = (0, 0)$  given by the sign of

$$\frac{\partial(a, c)}{\partial(x, y)}(0, 0) = -(a_y c_x)(0, 0).$$

Thus the index of  $\varphi(v)$  at  $m$  is the same as the  $H$ -parabolic index at  $m$ . Since  $\varphi(v)$  has the same index as  $v$  at any zero of  $v$  (in  $N_h$ ), we have  $\chi(S) = \chi(H) + p_H$ . But  $\chi(S) = 0$ , since  $S$  is trivial, and the lemma follows.

Theorem (0.2) and Lemma (3.3) give

**COROLLARY (3.4).** *If  $M^4$  is compact, orientable and generically immersed in  $\mathbb{C}^4$ , then  $p_1(M) = -p_H$ . If  $p_1(M) \neq 0$ , then  $M$  must have elliptic, parabolic, and hyperbolic points.*

#### 4. Local equations for $M$

To facilitate the study of  $M$  near a complex tangent, we shall simplify the presentation (1.7) by means of a local holomorphic coordinate change. In this section we prove the following.

**PROPOSITION (4.1).** *Suppose  $M$  has a non-degenerate one-dimensional complex tangent at a point  $m$ . Then holomorphic coordinates  $z = (z_1, z^\alpha, 2 \leq \alpha \leq n-1, z_n)$  can be chosen so that  $m = 0$  and  $M$  is given locally by*

$$\begin{aligned} M: \quad z_n &= F(z_1, x), & x &= (x^2, \dots, x^{n-1}), \\ y^\alpha &= f^\alpha(z_1, x), & f^\alpha &= \bar{f}^\alpha, 2 \leq \alpha \leq n-1. \end{aligned} \tag{4.1}$$

If  $m$  is an elliptic or hyperbolic point, then

$$\begin{aligned} F &= q + H, \quad q = az_1^2 + bz_1\bar{z}_1 + a\bar{z}_1^2, \quad H = O(3), \\ f^\alpha &= b^\alpha z_1\bar{z}_1 + h^\alpha, \quad h^\alpha = O(3), \end{aligned} \quad (4.2)$$

where  $a \geq 0$ , and  $b, b^\alpha$  are either 0 or 1. If  $m$  is a parabolic point, then

$$\begin{aligned} F &= Q + H, \quad f^\alpha = O(4) \\ Q &= \frac{1}{2}(z_1 + \bar{z}_1)^2 + i(z_1 - \bar{z}_1)c(x), \quad c(x) = c_\beta x^\beta, \\ H &= (-i\eta(z_1 + \bar{z}_1) + \eta_\beta x^\beta)z_1\bar{z}_1 + O(4), \end{aligned} \quad (4.3)$$

where  $\beta$  is summed from 2 to  $n-1$ ,  $c_\beta, \eta_\beta$  are real, and  $\eta$  is either 0 or 1. If the transversality condition  $dB \wedge d\bar{B} \neq 0$  holds, then  $c(x) \neq 0$ . In this case the parabolic line at  $m$ , which is the  $y_1$ -axis, is tangent to  $N_p$  if and only if  $\eta = 0$ .

We remark that (4.2) is already known [1], [10].

We begin with  $M$  in the form (1.7). If  $b \neq 0$ , we replace  $z_n$  by  $bz_n$  to make  $b = 1$ . By a rotation  $z_1 \rightarrow \mu z_1$ ,  $\mu\bar{\mu} = 1$ , we can make  $c \geq 0$ . Then by a change of the form

$$\begin{aligned} z_n &\rightarrow z_n + (c - a)z_1^2 + e_\alpha z_1 z^\alpha + f_{\beta\alpha} z^\alpha z^\beta, \\ z^\alpha &\rightarrow z^\alpha + 2i(a^\alpha z_1^2 + d_\beta^\alpha z_1 z^\beta + f_{\beta\gamma}^\alpha z^\beta z^\gamma), \end{aligned} \quad (4.4)$$

we can achieve (4.2) but with

$$H = c(x)z_1 + \bar{c}(x)\bar{z}_1 + O(3), \quad c(x) = c_\beta x^\beta. \quad (4.5)$$

The  $b^\alpha$  are either 0 or can be made 1 by  $z^\alpha \rightarrow b^\alpha z^\alpha$ .

We make the further change

$$z_1 \rightarrow z_1 + A(z), \quad A(z) = A_\alpha z^\alpha, \quad (4.6)$$

under which  $c(x)$  in (4.5) changes by

$$\begin{aligned} c(x) &\rightarrow c(x) + 2aA(x) + b\bar{A}(x), \\ \bar{c}(y) &\rightarrow \bar{c}(x) + bA(x) + 2a\bar{A}(x). \end{aligned}$$

If  $\gamma = |a/b| \neq \frac{1}{2}$ , then the determinant  $b^2 - 4a^2 \neq 0$ , and  $A(z)$  can be chosen

uniquely to make  $c(x) \rightarrow 0$ . If  $\gamma = \frac{1}{2}$ , we take  $b = 1$  and  $a = \frac{1}{2}$ . Then (4.6) results in

$$c(x) \rightarrow c(x) + 2 \operatorname{Re} A(x),$$

which may be used to make the  $c(x)$  in (4.5) purely imaginary. The  $x^\alpha x^\beta$  terms introduced by (4.6) can then be removed by a transformation of the form (4.4). This gives (4.2) and the form (4.3) for the quadratic term  $Q$ .

We must investigate the third order terms in the parabolic case. Since  $b = 1$ , the change  $z^\alpha \rightarrow z^\alpha - b^\alpha z_n$ , followed by one of the type (4.4), makes  $f^\alpha \equiv h^\alpha = O(3)$  in (4.1). We put

$$h^\alpha = h_0^\alpha + h_{\beta}^\alpha x^\beta + h_{\beta\gamma}^\alpha x^\beta x^\gamma + c_{\beta\gamma\rho}^\alpha x^\beta x^\gamma x^\rho + O(4),$$

$$h_0^\alpha = c^\alpha z_1^3 + e^\alpha z_1^2 \bar{z}_1 + \bar{e}^\alpha z_1 \bar{z}_1^2 + \bar{c}^\alpha \bar{z}_1^3,$$

$$h_{\beta}^\alpha = c_{\beta}^\alpha z_1^2 + e_{\beta}^\alpha z_1 \bar{z}_1 + \bar{c}_{\beta}^\alpha \bar{z}_1^2, e_{\beta}^\alpha \text{ real},$$

$$h_{\beta\gamma}^\alpha = c_{\beta\gamma}^\alpha z_1 + \bar{c}_{\beta\gamma}^\alpha \bar{z}_1, c_{\beta\gamma\rho}^\alpha \text{ real}.$$

The transformation

$$z^\alpha \rightarrow z^\alpha + 2i\{c^\alpha z_1^3 + c_{\beta}^\alpha z_1^2 z^\beta + c_{\beta\gamma}^\alpha z_1 z^\beta z^\gamma + \frac{1}{2}c_{\beta\gamma\rho}^\alpha z^\beta z^\gamma z^\rho\} \quad (4.7)$$

reduces  $h^\alpha$  to the form

$$h^\alpha = (c^\alpha z_1 + \bar{c}^\alpha \bar{z}_1 + c_{\beta}^\alpha x^\beta) z_1 \bar{z}_1 + O(4). \quad (4.8)$$

The substitution

$$z^\alpha \rightarrow z^\alpha + 2i\{c^\alpha z_1 + \frac{1}{2}c_{\beta}^\alpha z^\beta\} z_n, \quad (4.9)$$

followed by another one of type (4.7) (to remove any newly introduced third order terms already removed by (4.7)) results in

$$\begin{aligned} z_n &= Q + H, & H &= O(3) \\ y^\alpha &= h^\alpha, & h^\alpha &= O(4). \end{aligned} \quad (4.10)$$

Next we consider the third order terms in  $h$ ,

$$\begin{aligned} H &= H_0 + H_\alpha x^\alpha + H_{\alpha\beta} x^\alpha x^\beta + K_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma + O(4), \\ H_0 &= K_0 z_1^3 + K_1 z_1^2 \bar{z}_1 + K_2 z_1 \bar{z}_1^2 + K_3 \bar{z}_1^3, \\ H_\alpha &= K_{\alpha 0} z_1^2 + K_{\alpha 1} z_1 \bar{z}_1 + K_{\alpha 2} \bar{z}_1^2, \\ H_{\alpha\beta} &= K_{\alpha\beta 0} z_1 + K_{\alpha\beta 1} \bar{z}_1. \end{aligned} \quad (4.11)$$

We shall simplify this by means of a transformation of the form

$$\begin{aligned}
 z_1 &\rightarrow z_1 + A(z_1, z^\alpha, z_n), & A &= A_2 + A_0 z_n, \\
 z_n &\rightarrow z_n + B(z_1, z^\alpha, z_n), & B &= B_3 + B_1 z_n, \\
 A_2 &= A_{20} z_1^2 + A_{2\alpha} z_1 z^\alpha + A_{2\alpha\beta} z^\alpha z^\beta, & A_0 &= \text{const.}, \\
 B_3 &= B_{30} z_1^3 + B_{3\alpha} z_1^2 z^\alpha + B_{3\alpha\beta} z_1 z^\alpha z^\beta + B_{3\alpha\beta\gamma} z^\alpha z^\beta z^\gamma, \\
 B_1 &= B_{10} z_1 + B_{1\alpha} z^\alpha.
 \end{aligned} \tag{4.12}$$

This will not alter any of the previous normalizations. Note that

$$Q(z_1 + A, x) = Q(z_1, x) + (z_1' + \bar{z}_1)(A + \bar{A}) + Q(A, x).$$

Therefore, when we substitute (4.12) into (4.10), we get

$$H \rightarrow H + (z_1 + \bar{z}_1)(A + \bar{A}) - B + i(A - \bar{A})c(x) + \frac{1}{2}(A + \bar{A})^2, \tag{4.13}$$

in which we must make the substitution (4.10). We shall simplify the terms of  $H$  in order of increasing degree in  $x^\alpha$ . This allows us to ignore the term  $i(A - \bar{A})c(x)$ , and hence  $Q(A, x)$ , since  $(A + \bar{A})^2$  is of fourth order.

In simplifying  $H_0$  we ignore terms in  $x^\alpha$  and  $z^\alpha = x^\alpha + O(4)$ , so that

$$H_0 \rightarrow H_0 + (z_1 + \bar{z}_1)(A_2 + \bar{A}_2 + (A_0 + \bar{A}_0)Q) - B_3 - B_1 Q,$$

with  $A_2 \equiv A_{20} z_1^2$ ,  $B_3 = B_{30} z_1^3$ ,  $B_1 \equiv B_{10} z_1$ ,  $Q \equiv \frac{1}{2}(z_1 + \bar{z}_1)^2$ . Comparison of coefficients shows that

$$\begin{aligned}
 K_0 &\rightarrow K_0 + A_{20} + \frac{1}{2}(A_0 + \bar{A}_0) - B_{30} - \frac{1}{2}B_{10}, \\
 K_1 &\rightarrow K_1 + A_{20} + \frac{3}{2}(A_0 + \bar{A}_0) - B_{10}, \\
 K_2 &\rightarrow K_2 + \bar{A}_{20} + \frac{3}{2}(A_0 + \bar{A}_0) - \frac{1}{2}B_{10}, \\
 K_3 &\rightarrow K_3 + \bar{A}_{20} + \frac{1}{2}(A_0 + \bar{A}_0).
 \end{aligned}$$

By proper choice of  $A_{20}$  and  $B_{30}$  we can realize  $K_0 = K_3 = 0$ , after which  $A_{20} = -\text{Re } A_0$ ,  $B_{30} = -\frac{1}{2}B_{10}$ . Then  $K_1 - K_2 \rightarrow K_1 - K_2 - \frac{1}{2}B_{10}$ , so that we can make  $K_1 = K_2$ , and restrict to  $B_{10} = 0$ . This leaves the change  $K_1 \rightarrow K_1 + 2 \text{Re } A_0$ , by which we make  $K_1 = -i\eta$ , purely imaginary.

To simplify  $H_\alpha x^\alpha$  in (4.11), we set  $A_{20} = A_0 = B_{30} = B_{10} = 0$  in (4.12) and work mod  $x^\alpha x^\beta$ ,  $z^\alpha z^\beta$ . With  $A_0 = A_{20} = 0$ ,  $i(A - \bar{A})c(x) \equiv 0$ , mod  $x^\alpha x^\beta$ , so

$$H_\alpha x^\alpha \rightarrow H_\alpha x^\alpha + (z_1 + \bar{z}_1)(A + \bar{A}) - B_3 - B_1 Q,$$

with  $A \equiv A_{2\alpha} z_1 z^\alpha$ ,  $B_3 \equiv B_{3\alpha} z_1^2 z^\alpha$ ,  $B_1 \equiv B_{1\alpha} z_1^\alpha$ , and  $Q \equiv \frac{1}{2}(z_1 + \bar{z}_1)^2$ . Comparison of coefficients gives

$$K_{\alpha 0} \rightarrow K_{\alpha 0} + A_{2\alpha} - B_{3\alpha} - \frac{1}{2}B_{1\alpha},$$

$$K_{\alpha 1} \rightarrow K_{\alpha 1} + A_{2\alpha} + \bar{A}_{2\alpha} - B_{1\alpha},$$

$$K_{\alpha 2} \rightarrow K_{\alpha 2} + \bar{A}_{2\alpha} - \frac{1}{2}B_{1\alpha}.$$

So we normalize to  $K_{\alpha 0} = K_{\alpha 2} = 0$  and restrict to  $A_{2\alpha} = B_{3\alpha} + \frac{1}{2}B_{1\alpha} = \frac{1}{2}\bar{B}_{1\alpha}$ . It follows that  $K_{\alpha 1} \rightarrow K_{\alpha 1} + \frac{1}{2}(\bar{B}_{1\alpha} - B_{1\alpha})$ , so that we can make  $K_{\alpha 1} = \eta_\alpha = \bar{\eta}_\alpha$ , real.

Now we further restrict to  $A_{2\alpha} = B_{3\alpha} = B_{1\alpha} = 0$  in (4.12) and work mod  $x^\alpha x^\beta x^\gamma$ ,  $z^\alpha z^\beta z^\gamma$ . Again  $i(A - \bar{A})c(x)$  can be ignored in (4.13). We have

$$H_{\alpha\beta} x^\alpha x^\beta \rightarrow H_{\alpha\beta} x^\alpha x^\beta + (z_1 + \bar{z}_1)(A + \bar{A}) - B_3,$$

where  $A \equiv A_{2\alpha\beta} z^\alpha x^\beta$ ,  $B_3 \equiv B_{3\alpha\beta} z_1 z^\alpha z^\beta$ . This results in the change

$$K_{\alpha\beta 0} \rightarrow K_{\alpha\beta 0} + A_{2\alpha\beta} - B_{3\alpha\beta},$$

$$K_{\alpha\beta 1} \rightarrow K_{\alpha\beta 1} + \bar{A}_{2\alpha\beta}.$$

It's clear that we can make  $K_{\alpha\beta 0} = K_{\alpha\beta 1} = 0$ . Finally, we remove the term  $K_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma$  by a transformation

$$z_n \rightarrow z_n + B_{3\alpha\beta\gamma} z^\alpha z^\beta z^\gamma.$$

This achieves the form (4.3). If  $\eta \neq 0$ , a dilation  $(z_1, z^\alpha, z_n) \rightarrow (\lambda z_1, \lambda z^\alpha, \lambda^2 z_n)$  with  $\lambda$  real results in  $\eta \rightarrow \lambda\eta$ , so we can make  $\eta = 1$ .

At a parabolic point (1.8) and (4.3) give

$$B = (i/2)^{n-2} (z_1 + \bar{z}_1 + ic(x) + i\eta(\bar{z}_1^2 + 2z_1\bar{z}_1) + \eta_\beta x^\beta \bar{z}_1) + O(3). \quad (4.14)$$

It follows that  $dB \wedge d\bar{B} = 4^{1-n} ic(dx) \wedge dx_1 + O(1)$ , so that  $c(x) \neq 0$  if the transversality condition holds. We make a linear change in the coordinates  $(x^2, \dots, x^{n-1})$  so that  $c(x) = x^2$ , then  $N$  has the local equations

$$\begin{aligned} x_1 &= O(3), \\ x^2 &= -\eta y_1^2 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta y_1 + O(3). \end{aligned} \quad (4.15)$$

The conditions  $Xr = X\bar{r} = Xr^\alpha = 0$ , which determine  $X$  give

$$X = \frac{\partial}{\partial z_1} + (Q + H)_{z_1} \frac{\partial}{\partial z_n} + O(3). \quad (4.16)$$

Also,

$$\begin{aligned} XB &= B_{z_1} + O(3) = 1 + \eta y_1 + O(2), \\ \bar{X}B &= B_{\bar{z}_1} + O(3) = 1 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2). \end{aligned} \quad (4.17)$$

The condition (1.11) gives  $a(0) + \bar{a}(0) = 0$ , so we may take  $a = a' + i$ ,  $a'$  real,  $a'(0) = 0$ . Then

$$YB = a'(2 + \eta y_1 + \eta_\beta x^\beta) + i(\eta y_1 - \eta_\beta x^\beta) + O(2),$$

so that  $a' = O(2)$ . Thus, in coordinates  $(y_1, x^2, \dots, x^{n-1})$

$$Y = \partial/\partial y_1 + O(2). \quad (4.18)$$

From (4.17) and (1.9)

$$\Delta = 2\eta y_1 - 2 \sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2); \quad (4.19)$$

so that  $Y[\Delta] = 2\eta + O(1)$ . It follows that  $Y$  is tangent to  $N_p : \Delta = 0$  if and only if  $\eta = 0$ . If  $\eta = 1$ , then  $Y[\Delta] > 0$  implies that  $Y$  points toward  $N_h$ .

## 5. A formula for the Euler number

To derive our formula we shall make use of the Poincaré–Hopf theorem characterizing the Euler number  $\chi(M)$  as the sum of the indices of the zeros of a vector field tangent to  $M$ . This does not require  $M$  to be orientable and is applicable to compact manifolds with boundary, provided the vector field points outward along the boundary. For  $M^n$  immersed in the complex  $n$ -manifold  $\tilde{M}$  with normal bundle  $F$ ,  $\chi(F)$  denotes the sum of the indices of the zeros of a suitable section of  $F$ . The index at an isolated zero  $m \in M$  is well defined since  $T_m \tilde{M} = T_m M \oplus F_m$  as oriented vector spaces locally. A reversal of the local orientation of  $M$  near  $m$  results in a reversal of that of  $F$  as well as of  $TM$ .

In this section we prove the following, which does not require  $M$  to be orientable.

**THEOREM (5.1).** *Suppose that the compact  $n$ -dimensional manifold  $M$  is immersed in the complex  $n$ -dimensional manifold  $\tilde{M}$  with at most nondegenerate, one-dimensional complex tangents as in section 1 and Proposition (2.1c). Then*

$$\chi(M) = \varepsilon_n \chi(F) + e - h + p, \quad \varepsilon_n = (-1)^{(n-1)n/2}, \quad (5.1)$$

where  $e = \chi(N_e)$ ,  $h = (-1)^n \chi(N_h)$ , and  $p$  is the parabolic index.

If  $M$  is also orientable and embedded in  $\mathbb{C}^n$ , then a theorem of Whitney (see [4] or [9]) asserts that  $\chi(F) = 0$ . Theorem (0.1) follows immediately from this. As mentioned after Proposition (2.1) the assumptions of Theorem (5.1) are generic if  $n \leq 5$ . The remainder of this section is devoted to the proof of Theorem (5.1).

We choose some convenient hermitian metric on  $\tilde{M}$  and denote by  $\pi_m : T_m \tilde{M} \rightarrow F_m$ , the orthogonal projection onto  $F_m$  along  $T_m \equiv T_m M$ . Then  $\pi_m \circ J_m$  gives a linear mapping from  $T_m$  to  $F_m$ , which will be a linear isomorphism if  $m$  is a totally real point of  $M$ . If  $v$  is a vector field tangent to  $M$ , then  $\pi Jv$  is a section of  $F$ . The idea of the proof is to relate the index sum of  $\pi Jv$  to that of  $v$  for a suitable choice of  $v$ .

About any particular  $m$  in  $M$  we choose holomorphic coordinates  $z = x + iy$  for  $\tilde{M}$  centered at  $m$ . The orientation of  $\tilde{M}$  is given by the local form

$$\tilde{\Omega} = \prod_{\alpha=1}^n \left( \frac{i}{2} dz^\alpha \wedge d\bar{z}^\alpha \right) = \varepsilon_n dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n, \quad (5.2)$$

and the operator  $J$  is identified with  $(x, y) \rightarrow (-y, x)$ . Suppose  $m$  is a totally real point of  $M$ . Then the coordinates may be chosen so that  $T_m$  is the  $x$ -space and  $F_m$  is the  $y$ -space, which by (5.2) have the orientations

$$\Omega_T = dx^1 \wedge \cdots \wedge dx^n, \quad \Omega_F = \varepsilon_n dy^1 \wedge \cdots \wedge dy^n.$$

Since  $\pi$  is smoothly deformable to  $(x, y) \rightarrow (0, y)$  and  $\pi \circ J$  to  $(x, 0) \rightarrow (0, x)$ , we have

$$(\pi \circ J)^* \Omega_F = c \Omega_T, \quad \text{sgn } c = \varepsilon_n.$$

It follows that the effect of  $\pi J$  on the index of a vector field  $v$  with isolated zero at

$m$  is

$$\text{ind}_{F,m}(\pi Jv) = \varepsilon_n \text{ind}_{M,m}(v),$$

so that

$$\sum_{m \notin N} \text{ind}_{F,m}(\pi Jv) = \varepsilon_n \chi(M). \quad (5.3)$$

This proves (5.1) if  $M$  is totally real.

In the general case we start with a smooth vector field  $v_0$  tangent to  $N$  with the following properties. It is to have only finitely many zeros  $m_j$ ,  $1 \leq j \leq l$ , which are non-degenerate and lie in  $N_e \cup N_h$ , and is to be transverse to  $N_p$  and point toward  $N_h$  along  $N_p$ . Furthermore, the line field  $k$  along  $N_p$  spanned by  $v_0$  is to satisfy  $k_m = l_m$  for only finitely many  $m \in N_p$ , and at such  $m$  this intersection is transverse in the space  $P$  (see (1.15)). We find disjoint neighborhoods  $U_j$  of  $m_j$  in  $N - N_p$  and smooth sections  $v_j$  of  $H$ , compactly supported in  $U_j$ , with  $v_j(m_j) \neq 0$ . Then we smoothly extend  $v_0 + \sum v_j$  to a vector field  $v$  on  $M$  having a finite number of non-degenerate zeros. By construction  $v$  does not vanish on  $N$ ; however  $\pi Jv$  will have a zero at each  $m_j$  and at each  $m$  in  $N_p$  where  $v(m) \in l_m \subset H_m$ , as well as at each zero of  $v$ . There is much freedom in the choice of such a  $v$ , which we shall specify more precisely later.

Let  $m_j$  be one of the zeros of  $v_0$ , and choose coordinates as in (4.1), (4.2), so that  $(z_1, x^\alpha)$  are coordinates on  $M$ . We may assume that the hermitian metric on  $\tilde{M}$  has been chosen so that  $F_m$  coincides with the  $(y^\alpha, z_n)$ -space for all  $m$  near  $m_j$ . The local orientations are given by

$$\Omega_T = \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1}, \quad (5.4)$$

$$\Omega_F = \varepsilon_{n-2} dy^2 \wedge \cdots \wedge dy^{n-1} \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n. \quad (5.5)$$

We set  $G(z_1, x^\alpha) = (z_1, x^\alpha + if^\alpha, F)$ , so that  $G_{x_1}, G_{y_1}, G_{x^\alpha}$  span  $T(M)$ . In the local coordinates  $(z_1, x^\alpha)$  on  $M$  we have

$$v = v_1 \partial / \partial z_1 + \bar{v}_1 \partial / \partial \bar{z}_1 + v^\alpha \partial / \partial x^\alpha, \quad (5.6)$$

so that as a vector in  $\mathbb{C}^n$

$$v \equiv v[G] = (v_1, v^\alpha + iv[f^\alpha], v[F]),$$

where  $v[\cdot]$  denotes directional derivative. It follows that

$$Jv \equiv iv[G] = (iv_1, -v[f^\alpha] + iv^\alpha, iv[F]),$$

so that

$$\begin{aligned} \pi Jv &\equiv iv[G] - c'G_{x_1} - c''G_{y_1} - c^\alpha G_{x^\alpha} \\ &= iv[G] - cG_{z_1} - \bar{c}G_{\bar{z}_1} - c^\alpha G_{x^\alpha} = (0, 0 + i*, *). \end{aligned}$$

Here  $G_{z_1} = (1, if_{z_1}^\alpha, F_{z_1})$ ,  $G_{\bar{z}_1} = (0, if_{\bar{z}_1}^\alpha, F_{\bar{z}_1})$ , and  $G_{x^\alpha} = (0, \delta_\alpha^\beta + if_{x^\alpha}^\beta, F_{x^\alpha})$ , so that  $c = iv_1$  and  $c^\alpha = -v[f^\alpha]$ . Hence, as a map from  $(z_1, x^\alpha)$ -space to  $(y^\alpha, z_n)$ -space,  $\pi Jv$  has the form

$$\begin{aligned} y^\alpha &= v^\alpha - iv_1 f_{z_1}^\alpha + i\bar{v}_1 f_{\bar{z}_1}^\alpha + v[f^\beta]f_{x^\beta}^\alpha, \\ z_n &= iv[F] - iv_1 F_{z_1} + i\bar{v}_1 F_{\bar{z}_1} + v[f^\beta]F_{x^\beta}. \end{aligned} \quad (5.7)$$

If we substitute (5.7) into (5.5), we get (5.4) multiplied by the Jacobian factor

$$\varepsilon_{n-2} \frac{\partial(y^\alpha, z_n, \bar{z}_n)}{\partial(z_1, \bar{z}_1, x^\beta)}, \quad (5.8)$$

the sign of which gives the index of  $\pi Jv$  at  $m_j$ . If we take into account (4.2), (5.7) becomes

$$\begin{aligned} y^\alpha &= v^\alpha - iv_1 b^\alpha \bar{z}_1 + i\bar{v}_1 b^\alpha z_1 + O(2), \\ z_n &= 2i\bar{v}_1 q_{\bar{z}_1} + O(2). \end{aligned}$$

We may assume that the  $H$ -component  $v_j$  added to  $v_0$  is such that  $v_1 \equiv 1$  near 0. Also, we assume that the extension of  $v$  from  $N$  to  $M$  is made so that the coefficients of  $v$  are locally independent of  $z_1$ . Then at the origin (5.8) has the value

$$4(b^2 - 4a^2) \det(\partial v_0^\alpha / \partial x^\beta)(0). \quad (5.9)$$

The sign of the determinant is the index of  $v_0$  at  $m_j$ , and  $b^2 - 4a^2$  is positive if  $m_j$  is elliptic and negative if  $m_j$  is hyperbolic. Hence,

$$\varepsilon_{n-2} \text{ind}_{F, m_j}(\pi Jv) = \delta \text{ind}_{N, m_j}(v_0), \quad (5.10)$$

where  $\delta = +1$  if  $m_i$  is elliptic or  $\delta = -1$  if  $m_i$  is hyperbolic. If we sum (5.10) over the  $m_i$  in  $N_e$ , the right hand side is  $\chi(N_e)$ . To get  $\chi(N_h)$  we must use  $-v_0$  which multiplies the determinant in (5.9) by  $(-1)^{n-2}$ . Thus we get

$$\varepsilon_{n-2} \sum_j \text{ind}_{F, m_j}(\pi J v) = \chi(N_e) - (-1)^n \chi(N_h), \quad (5.11)$$

which accounts for the term  $e - h$  in (5.1).

Finally, we consider a zero of  $\pi J v$  at a point  $m$  in  $N_p$  which arises when  $v(m)$ , which spans the line  $k_m$ , lies in  $l_m$ . We first elaborate further on the construction of  $v$  along  $N_p$ . It is initially defined so that  $k(N_p)$  intersects  $l(N_p)$  transversely at  $m$ . Then it will be extended to  $N$ . We take coordinates as in (4.1), (4.3) with  $c(x) = x^2$ , so that  $N$  is given by (4.15).  $(x^3, \dots, x^{n-1})$  gives coordinates on  $N_p$ , and  $(y_1, x^3, \dots, x^{n-1})$  coordinates on  $N$ . In (5.6) we take  $v_1 = v^1 + i$ ,  $v^1 = \bar{v}^1$ , so that

$$v = \partial/\partial y_1 + \sum_{j=1}^{n-1} v^j \partial/\partial x^j, \quad v^j(0) = 0. \quad (5.12)$$

The condition that  $v$  be tangent to  $N$  gives, via (4.15) and (5.12),

$$v^1 = O(2), \quad v^2 = -2\eta y_1 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2). \quad (5.13)$$

Thus we start with

$$v^\alpha = v^\alpha(x^3, \dots, x^{n-1}), \quad v^\alpha(0) = 0, \quad \det \frac{\partial v^\alpha}{\partial x^\beta}(0) \neq 0, \quad 3 \leq \alpha, \beta \leq n-1, \quad (5.14)$$

and determine  $v^1$  and  $v^2$  by (5.13). We then extend this vector  $v$  locally from  $N_p$  to  $N$  by keeping (5.14) independent of  $y_1$ , and from  $N$  to  $M$  by keeping (5.14) independent of  $x_1$  and  $x^2$ . Again we assume that  $F_m$  is the  $(y^\alpha, z_n)$ -space for  $m$  near 0. Note that we may take  $\eta = 1$ , since  $l_0 = k_0$  is transverse to  $N_p$ .

The parabolic index as defined in section 1 is computed relative to a coordinate system  $(x_\star^\alpha, y_\star)$  with  $y_\star = 0$  on  $N_p$ . Therefore we set (4.19)

$$y_\star = \frac{1}{2}\Delta = y_1 - \sum_{\beta=3}^{n-1} \eta_\beta x^\beta + O(2), \quad x_\star^\alpha = x^\alpha.$$

The chain rule in (1.14) gives

$$w_*^\alpha = w^\alpha (\partial y_*/\partial y_1 + w^\beta \partial y_*/\partial x^\beta)^{-1} = w^\alpha (1 - \eta_\beta w^\beta + O(1))^{-1}.$$

Since  $w^\alpha = O(1)$  for both  $Y$  (4.18) and  $v$  (5.12), and  $\partial/\partial x_*^\beta = \partial/\partial x^\beta$  for functions defined along  $N_p$ , we have  $\partial w_*^\alpha/\partial x_*^\beta(0) = \partial w^\alpha/\partial x^\beta(0)$ . Thus the parabolic intersection index at  $m = 0$  is given (see (1.15)) by the sign of

$$\Omega_P\left(\frac{\partial l}{\partial x}, \frac{\partial k}{\partial x}\right)(0) = \det \begin{bmatrix} \delta_{\alpha\beta} & 0 \\ \delta_{\alpha\beta} & \partial v^\alpha/\partial x^\beta(0) \end{bmatrix}.$$

Hence,

$$\text{ind}_{P,m}(l, k) = \text{sgn} \det (\partial v^\alpha/\partial x^\beta(0))_{3 \leq \alpha, \beta \leq n-1}. \quad (5.15)$$

For the index of  $\pi Jv$  at  $m$  we again compute the determinant (5.8). We substitute (4.3) into (5.7) and ignore second order terms. By (5.12) and (5.13) we get

$$\begin{aligned} y^2 &\equiv v^2 \equiv -2y_1 + \sum_{\beta=3}^{n-1} \eta_\beta x^\beta, & y^\alpha &\equiv v^\alpha(x^3, \dots, x^{n-1}), & 3 \leq \alpha \leq n-1, \\ z_n &\equiv 2i\bar{v}_1 Q_{z_1} \equiv 4x_1 + 2ix^2. \end{aligned}$$

Thus,

$$\frac{\partial(y^2, y^\alpha, z_n, \bar{z}_n)}{\partial(z_1, \bar{z}_1, x^2, x^\beta)}(0) = 16 \det [\partial v^\alpha/\partial x^\beta(0)]_{3 \leq \alpha, \beta \leq n-1}.$$

Comparison with (5.15) gives

$$\text{ind}_{F,m}(\pi Jv) = \varepsilon_{n-2} \text{ind}_{P,m}(l, k),$$

so that

$$\sum_{N_p} \text{ind}_{F,m}(\pi Jv) = \varepsilon_{n-2} p. \quad (5.16)$$

Combining (5.3), (5.11), and (5.16) gives (5.1), since  $\varepsilon_n \varepsilon_{n-2} = -1$ .

## REFERENCES

- [1] E. BISHOP, *Differentiable manifolds in complex Euclidean space*, Duke Math. Jour. 32 (1965) 1–22.
- [2] S. S. CHERN, *Characteristic classes of hermitian manifolds*, Ann. Math. 47 (1946) 85–121.
- [3] S. S. CHERN and E. H. SPANIER, *A theorem on orientable surfaces in four-dimensional space*, Comment. Math. Helv. 25 (1951) 1–5.
- [4] M. W. HIRSCH, *Differential Topology*, Springer-Verlag, New York, 1976.
- [5] L. R. HUNT and R. O. WELLS, *The envelope of holomorphy of a two-manifold in  $\mathbb{C}^2$* , Rice Univ. Studies 56 (1970) 51–62.
- [6] C. KENIG and S. WEBSTER, *The local hull of holomorphy of a surface in the space of two complex variables*, Invent. Math. 67 (1982) 1–21.
- [7] —, *On the hull of holomorphy of an  $n$ -manifold in  $\mathbb{C}^n$* , Ann. Sc. Norm. Sup. Pisa vol. 11, no. 2 (1984) 261–280.
- [8] H. F. LAI, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. AMS 172 (1972) 1–33.
- [9] J. W. MILNOR and J. D. STASHEFF, *Characteristic classes*, Annals of Math. Studies, Princeton Univ. Press, 1974.
- [10] J. K. MOSER and S. M. WEBSTER, *Normal forms for real surfaces in  $\mathbb{C}^2$  near complex tangents and hyperbolic surface transformations*, Acta Math 150 (1983) 255–296.
- [11] R. O. WELLS, *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. 179 (1969) 123–129.

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