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An answer to a question by J. Milnor

YA. G. SINAI

We consider two commuting automorphisms T_1, T_2 of the Lebesgue space (M, \mathcal{M}, μ) such that $h_{m,n} = h(T_1^m T_2^n) < \infty$ where h is the measure-theoretic entropy. Under additional assumptions we show the existence of the limits $\lim (1/m)h_{m,n}$ where $m \rightarrow \infty, n \rightarrow \infty, m/n \rightarrow \omega$ and ω is an irrational number.

§1. Formulation of the problem and the result

Let $X = \{x^{(1)}, \dots, x^{(\kappa)}\}$ be a finite alphabet and M be the space of double-infinite sequences $x = \{x_n\}_{-\infty}^{\infty}, x_n \in X$, S is the shift in M , i.e. $Sx = x' = \{x'_n\}, x'_n = x_{n+1}$. Then M is a compact topological space in topology of direct product and S is a homeomorphism of M . Assume that a function $f(x_{-r}, \dots, x_r)$ with values in X is given. It generates a homomorphism T of M by the formula: $Tx = y = \{y_n\}_{-\infty}^{\infty}, y_n = f(x_{n-r}, \dots, x_{n+r})$. S and T commute and we assume that they generate an action of the group \mathbb{Z}^2 on M : for $(m, n) \in \mathbb{Z}^2$ the corresponding transformation is $T_{m,n} = S^m T^n$. The described situation was considered by Professor J. Milnor in his talk “Cellular automata as discrete dynamical systems” during the celebration of the 20-th anniversary of the Forschungsinstitut für Mathematik, ETH in Zurich. He formulated the following question. Assume that μ is a normed ergodic measure invariant under the action of \mathbb{Z}^2 . Denote $h_{m,n} = h(S^m T^n)$ measure-theoretic entropy of $T_{m,n}$ with respect to μ . It is easy to show that $h_{m,0} < \infty$ for all $-\infty < m < \infty$. We shall consider the case when $h_{m,n} < \infty$ for all $-\infty < m, n < \infty$. From the properties of entropy (see [1]) it follows that the function $h_{m,n}$ is an homogeneous function of the first degree, i.e. $h_{\kappa m, \kappa n} = |\kappa| h_{m,n}$. Fix an irrational number $\omega_0 > 0$ and choose a sequence $(m_i, n_i) \in \mathbb{Z}^2, m_i \rightarrow \infty, n_i \rightarrow \infty, m_i/n_i \rightarrow \omega_0$ as $i \rightarrow \infty$. The question is whether there exists a limit $\lim_{i \rightarrow \infty} (1/\sqrt{m_i^2 + n_i^2}) h_{m_i, n_i}$ which can be called as entropy per unit of length in the direction ω_0 . The aim of this paper is to give an affirmative answer to this question. It will be more convenient to show the existence of the limit $\lim_{i \rightarrow \infty} (1/n_i) h_{m_i, n_i}$ which is equivalent to the first one.

We introduce the partition ξ into κ sets $C_\kappa, 1 \leq i \leq \kappa, C_i = \{x \mid x_0 = x^{(i)}\}, \xi_{m,n} = T_{m,n}\xi$. We shall use later standard notations and facts of the theory of measurable

partitions and measure-theoretic entropy (there are many good references, we shall mention only few of them, [1], [2], [3]). By $I = I(a, \omega)$ we denote the segment on the plane joining the points $(a, 0)$ and $(a + \omega^{-1}, 1)$ and $\Gamma(a, \omega)$ is the half-line $y = \omega(x - a)$, $y \leq 1$. It is clear that $I(a, \omega) \subset \Gamma(a, \omega)$. We shall always consider the case $\omega > 0$. The main role in our analysis play the conditional entropies

$$\mathcal{H}_r(I) = H\left(\bigvee_{m \geq a + \omega^{-1}} \xi_{m,1} \left| \bigvee_{n=0}^{\infty} \bigvee_{m \geq a + \omega^{-1}n} \xi_{m,-n}\right.\right)$$

$$\mathcal{H}_l(I) = H\left(\bigvee_{m \leq a + \omega^{-1}} \xi_{m,1} \left| \bigvee_{n=0}^{\infty} \bigvee_{m \leq a + \omega^{-1}n} \xi_{m,-n}\right.\right)$$

$$\mathcal{H}(I) = \mathcal{H}_r(I) + \mathcal{H}_l(I).$$

It is easy to see that both $\mathcal{H}_r(I)$, $\mathcal{H}_l(I)$ are finite. We shall list three properties of them which will be used later:

1. $\mathcal{H}_r(I)$, $\mathcal{H}_l(I)$ are periodic functions of a with the period 1 for each fixed ω ;
2. if ω is a rational number, $\omega = p/q$, then $\mathcal{H}_r(I)$, $\mathcal{H}_l(I)$ are constants on each interval of a of the length $1/p$ where the half-lines $\Gamma(a, \omega)$ do not pass through points of the lattice \mathbb{Z}^2 .
3. if ω is irrational and $\Gamma(a, \omega)$ does not pass through points of the lattice \mathbb{Z}^2 then $\mathcal{H}_r(I)$, $\mathcal{H}_l(I)$ are continuous at the point (a, ω) .

The last property follows easily from the properties of continuity of conditional entropy. We shall use also a transformation Q in the space of segments $I(a, \omega)$, where $Q(I(a, \omega)) = I(a', \omega)$, $a' = a + \omega^{-1}$.

Our first result is the following theorem.

THEOREM 1. *Let $p > 0$, $q > 0$ have no common factor. Then $h_{p,q} = \sum_{i=0}^{p-1} \mathcal{H}(Q^i(I)) = p \int_0^1 \mathcal{H}(I) da$ for any interval $I = I(a, -q/p)$.*

The proof of Theorem 1 is given in §2.

THEOREM 2. *Let ω_0 be an irrational number, (m_i, n_i) be a sequence of points of the lattice \mathbb{Z}^2 , $m_i, n_i \rightarrow \infty$ and $m_i/n_i \rightarrow \omega_0$ as $i \rightarrow \infty$. Then*

$$\lim_{i \rightarrow \infty} \frac{1}{n_i} h_{m_i, n_i} = \int_0^1 \mathcal{H}(I(a, \omega_0)) da.$$

Proof of Theorem 2. We have from Theorem 1

$$\frac{1}{n_i} h_{m_i, n_i} = \int_0^1 \mathcal{H}(I(a, m_i/n_i)) da.$$

All functions $\mathcal{H}(I(a, m_i/n_i))$ are uniformly bounded and non-negative. It follows from the property 3 that for almost every a

$$\lim_{i \rightarrow \infty} \mathcal{H}(I(a, m_i/n_i)) = \mathcal{H}(I(a, \omega_0)).$$

Thus in view of Lebesgue dominance theorem we have the desired result. Q.E.D.

In §3 we make some additional remarks.

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§2. Proof of Theorem 1

It follows from the properties of measure-theoretic entropy that

$$h_{q,p} = \lim_{s \rightarrow \infty} H \left(\bigvee_{n=1}^p \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n} \right), \quad \omega = q/p.$$

The last conditional entropy is equal to

$$\begin{aligned} \sum_{l=1}^p H \left(\bigvee_{a+\omega^{-1}l-s \leq m \leq a+\omega^{-1}l+s} \xi_{m,l} \mid \bigvee_{n < l} \bigvee_{|m-a-\omega^{-1}n| \leq s} \xi_{m,n} \right) \\ = \sum_{l=1}^p H \left(\bigvee_{a+\omega^{-1}l-s \leq m \leq a+\omega^{-1}l+s} \xi_{m,l} \mid \bigvee_{n < 0} \bigvee_{|m-a-\omega^{-1}n| \leq s} \xi_{m,n} \right). \end{aligned}$$

We shall show that the l -th term converges as $s \rightarrow \infty$ to $\mathcal{H}(Q^l(I))$. It is sufficient to consider $l = 1$, other terms are treated in the same way. From the description of

our system it follows easily that

$$\begin{aligned}
 & H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \right. \\
 &\quad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &\quad + H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \right. \\
 &\quad \left. \times \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \quad (1)
 \end{aligned}$$

The first term in (1) is equal to

$$H\left(\bigvee_{a+\omega^{-1}-[a+\omega^{-1}]-r \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-2s-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}n-[a+\omega^{-1}]} \xi_{m,n}\right)$$

It follows from the properties of continuity of conditional entropy that this expression converges to $\mathcal{H}_l(I)$. We shall show that the second term in (1) converges to $\mathcal{H}_r(I)$. We have

$$\begin{aligned}
 & H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \right. \\
 &\quad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+2s-r-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+2s} \xi_{m,1} \right. \\
 &\quad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n}\right).
 \end{aligned}$$

We denote

$$\eta = \bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r} \xi_{m,1}$$

and $C_\eta(x)$ is an element containing $x \in M$. Also let us introduce the partitions

$$\begin{aligned}\zeta_s &= \bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r+2s} \xi_{m,0} \\ &\vee \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n}, \\ \zeta^+ &= \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m} \xi_{m,n}\end{aligned}$$

In view of Doob's theorem on convergence of conditional probabilities

$$\mu(C_\eta(x) | C_{\zeta_s}(x)) \rightarrow \mu(C_\eta(x) | C_{\zeta^+}(x)) \quad \text{a.e.},$$

where $C_{\zeta_s}(x)$, $C_{\zeta^+}(x)$ are elements of corresponding partitions containing x . But

$$\begin{aligned}\mu\left(C_\eta(x) \middle| \bigvee_{a+\omega^{-1}+2s-r-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+2s} \xi_{m,1} \right. \\ \left. \vee \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n} \right) \quad (2)\end{aligned}$$

can be represented as finite linear combinations of $\mu(C_\eta(x) | C_{\zeta_s}(x))$. This shows easily that the conditional probabilities (2) also converge a.e. as $s \rightarrow \infty$ to $\mu(C_\eta(x) | C_{\zeta^+}(x))$. This gives the desired result. Q.E.D.

§3. Several general remarks

Let us consider two commuting automorphisms T_1, T_2 of Lebesgue space (M, \mathcal{M}, μ) . Then we have a measure-preserving action of the group \mathbb{Z}^2 on M and we shall assume that it is ergodic and at least one of automorphisms T_1, T_2 is also ergodic. Without any loss of generality we can assume T_1 is ergodic. If the measure-theoretic entropy $h(T_1)$ is finite one can find a finite generating partition $\xi = \{C_1, \dots, C_\kappa\}$ in view of Krieger's theorem [4]. It means that T_1 is isomorphic to the shift in the space of doubly-infinite sequences written in the alphabet of κ symbols. If $T_2 x = y = \{y_n\}$ then $y_n = f(x_n, x_{n \pm 1}, x_{n \pm 2}, \dots)$ where f is a measurable function with the values in the space $\{1, 2, \dots, \kappa\}$. Thus the pair (T_1, T_2) is represented as a system of cellular automata but maybe with an infinite memory. Our arguments presented above can be extended to the case when f can be approximated sufficiently well by functions of finite number of variables. However, the general case remains completely open. One can mention also an

interesting paper by G. A. Galperin [5] where some results concerning topological entropy of systems of cellular automata were established.

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Added in proof:

It is clear that theorems 1 and 2 are easily generalized to the case of non-ergodic measures. Indeed, if ν_{inv} is the measurable partition of M into ergodic components of the action of \mathbb{Z}^2 corresponding to the measure μ , then

$$\frac{1}{n} h_{m,n} = \int_{M | \nu_{\text{inv}}} \frac{1}{n} h(S^m T^n | C_{\nu_{\text{inv}}}) d\mu_{\text{inv}}$$

where μ_{inv} is the induced measure on the factor-space $M | \nu_{\text{inv}}$. We showed already for a.e. element $C_{\nu_{\text{inv}}}$ of ν_{inv} the convergence of $(1/n)h(S^m T^n | C_{\nu_{\text{inv}}})$, $(m/n) \rightarrow \omega$, which implies the convergence of $(1/n)h_{m,n}$.

Also in the same way one can consider the action of the semi-group $\mathbb{Z}_+^2 = \{(m, n): -\infty < m < \infty, n \geq 0\}$. In order to get the assertions of theorems 1 and 2 one should replace possible pasts by possible futures in all arguments.