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**Autor:** Sinai, Ya. G.  
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## An answer to a question by J. Milnor

YA. G. SINAI

We consider two commuting automorphisms  $T_1, T_2$  of the Lebesgue space  $(M, \mathcal{M}, \mu)$  such that  $h_{m,n} = h(T_1^m T_2^n) < \infty$  where  $h$  is the measure-theoretic entropy. Under additional assumptions we show the existence of the limits  $\lim (1/m)h_{m,n}$  where  $m \rightarrow \infty, n \rightarrow \infty, m/n \rightarrow \omega$  and  $\omega$  is an irrational number.

### §1. Formulation of the problem and the result

Let  $X = \{x^{(1)}, \dots, x^{(\kappa)}\}$  be a finite alphabet and  $M$  be the space of double-infinite sequences  $x = \{x_n\}_{-\infty}^{\infty}, x_n \in X$ ,  $S$  is the shift in  $M$ , i.e.  $Sx = x' = \{x'_n\}, x'_n = x_{n+1}$ . Then  $M$  is a compact topological space in topology of direct product and  $S$  is a homeomorphism of  $M$ . Assume that a function  $f(x_{-r}, \dots, x_r)$  with values in  $X$  is given. It generates a homomorphism  $T$  of  $M$  by the formula:  $Tx = y = \{y_n\}_{-\infty}^{\infty}, y_n = f(x_{n-r}, \dots, x_{n+r})$ .  $S$  and  $T$  commute and we assume that they generate an action of the group  $\mathbb{Z}^2$  on  $M$ : for  $(m, n) \in \mathbb{Z}^2$  the corresponding transformation is  $T_{m,n} = S^m T^n$ . The described situation was considered by Professor J. Milnor in his talk “Cellular automata as discrete dynamical systems” during the celebration of the 20-th anniversary of the Forschungsinstitut für Mathematik, ETH in Zurich. He formulated the following question. Assume that  $\mu$  is a normed ergodic measure invariant under the action of  $\mathbb{Z}^2$ . Denote  $h_{m,n} = h(S^m T^n)$  measure-theoretic entropy of  $T_{m,n}$  with respect to  $\mu$ . It is easy to show that  $h_{m,0} < \infty$  for all  $-\infty < m < \infty$ . We shall consider the case when  $h_{m,n} < \infty$  for all  $-\infty < m, n < \infty$ . From the properties of entropy (see [1]) it follows that the function  $h_{m,n}$  is an homogeneous function of the first degree, i.e.  $h_{\kappa m, \kappa n} = |\kappa| h_{m,n}$ . Fix an irrational number  $\omega_0 > 0$  and choose a sequence  $(m_i, n_i) \in \mathbb{Z}^2, m_i \rightarrow \infty, n_i \rightarrow \infty, m_i/n_i \rightarrow \omega_0$  as  $i \rightarrow \infty$ . The question is whether there exists a limit  $\lim_{i \rightarrow \infty} (1/\sqrt{m_i^2 + n_i^2})h_{m_i, n_i}$  which can be called as entropy per unit of length in the direction  $\omega_0$ . The aim of this paper is to give an affirmative answer to this question. It will be more convenient to show the existence of the limit  $\lim_{i \rightarrow \infty} (1/n_i)h_{m_i, n_i}$  which is equivalent to the first one.

We introduce the partition  $\xi$  into  $\kappa$  sets  $C_\kappa, 1 \leq i \leq \kappa, C_i = \{x \mid x_0 = x^{(i)}\}, \xi_{m,n} = T_{m,n}\xi$ . We shall use later standard notations and facts of the theory of measurable

partitions and measure-theoretic entropy (there are many good references, we shall mention only few of them, [1], [2], [3]). By  $I = I(a, \omega)$  we denote the segment on the plane joining the points  $(a, 0)$  and  $(a + \omega^{-1}, 1)$  and  $\Gamma(a, \omega)$  is the half-line  $y = \omega(x - a)$ ,  $y \leq 1$ . It is clear that  $I(a, \omega) \subset \Gamma(a, \omega)$ . We shall always consider the case  $\omega > 0$ . The main role in our analysis play the conditional entropies

$$\mathcal{H}_r(I) = H\left(\bigvee_{m \geq a + \omega^{-1}} \xi_{m,1} \mid \bigvee_{n=0}^{\infty} \bigvee_{m \geq a + \omega^{-1}n} \xi_{m,-n}\right)$$

$$\mathcal{H}_l(I) = H\left(\bigvee_{m \leq a + \omega^{-1}} \xi_{m,1} \mid \bigvee_{n=0}^{\infty} \bigvee_{m \leq a + \omega^{-1}n} \xi_{m,-n}\right)$$

$$\mathcal{H}(I) = \mathcal{H}_r(I) + \mathcal{H}_l(I).$$

It is easy to see that both  $\mathcal{H}_r(I)$ ,  $\mathcal{H}_l(I)$  are finite. We shall list three properties of them which will be used later:

1.  $\mathcal{H}_r(I)$ ,  $\mathcal{H}_l(I)$  are periodic functions of  $a$  with the period 1 for each fixed  $\omega$ ;
2. if  $\omega$  is a rational number,  $\omega = p/q$ , then  $\mathcal{H}_r(I)$ ,  $\mathcal{H}_l(I)$  are constants on each interval of  $a$  of the length  $1/p$  where the half-lines  $\Gamma(a, \omega)$  do not pass through points of the lattice  $\mathbb{Z}^2$ .
3. if  $\omega$  is irrational and  $\Gamma(a, \omega)$  does not pass through points of the lattice  $\mathbb{Z}^2$  then  $\mathcal{H}_r(I)$ ,  $\mathcal{H}_l(I)$  are continuous at the point  $(a, \omega)$ .

The last property follows easily from the properties of continuity of conditional entropy. We shall use also a transformation  $Q$  in the space of segments  $I(a, \omega)$ , where  $Q(I(a, \omega)) = I(a', \omega)$ ,  $a' = a + \omega^{-1}$ .

Our first result is the following theorem.

**THEOREM 1.** *Let  $p > 0$ ,  $q > 0$  have no common factor. Then  $h_{p,q} = \sum_{i=0}^{p-1} \mathcal{H}(Q^i(I)) = p \int_0^1 \mathcal{H}(I) da$  for any interval  $I = I(a, -q/p)$ .*

The proof of Theorem 1 is given in §2.

**THEOREM 2.** *Let  $\omega_0$  be an irrational number,  $(m_i, n_i)$  be a sequence of points of the lattice  $\mathbb{Z}^2$ ,  $m_i \cdot n_i \rightarrow \infty$  and  $m_i/n_i \rightarrow \omega_0$  as  $i \rightarrow \infty$ . Then*

$$\lim_{i \rightarrow \infty} \frac{1}{n_i} h_{m_i, n_i} = \int_0^1 \mathcal{H}(I(a, \omega_0)) da.$$

*Proof of Theorem 2.* We have from Theorem 1

$$\frac{1}{n_i} h_{m_i, n_i} = \int_0^1 \mathcal{H}(I(a, m_i/n_i)) da.$$

All functions  $\mathcal{H}(I(a, m_i/n_i))$  are uniformly bounded and non-negative. It follows from the property 3 that for almost every  $a$

$$\lim_{i \rightarrow \infty} \mathcal{H}(I(a, m_i/n_i)) = \mathcal{H}(I(a, \omega_0)).$$

Thus in view of Lebesgue dominance theorem we have the desired result. Q.E.D.

In §3 we make some additional remarks.

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## §2. Proof of Theorem 1

It follows from the properties of measure-theoretic entropy that

$$h_{q,p} = \lim_{s \rightarrow \infty} H \left( \bigvee_{n=1}^p \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n} \right), \quad \omega = q/p.$$

The last conditional entropy is equal to

$$\begin{aligned} & \sum_{l=1}^p H \left( \bigvee_{a+\omega^{-1}l-s \leq m \leq a+\omega^{-1}l+s} \xi_{m,l} \mid \bigvee_{n < l} \bigvee_{|m-a-\omega^{-1}n| \leq s} \xi_{m,n} \right) \\ &= \sum_{l=1}^p H \left( \bigvee_{a+\omega^{-1}l-s \leq m \leq a+\omega^{-1}l+s} \xi_{m,l} \mid \bigvee_{n < 0} \bigvee_{|m-a-\omega^{-1}n| \leq s} \xi_{m,n} \right). \end{aligned}$$

We shall show that the  $l$ -th term converges as  $s \rightarrow \infty$  to  $\mathcal{H}(Q^l(I))$ . It is sufficient to consider  $l = 1$ , other terms are treated in the same way. From the description of

our system it follows easily that

$$\begin{aligned}
 & H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \right. \\
 & \qquad \qquad \qquad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 & \quad + H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \right. \\
 & \qquad \qquad \qquad \left. \times \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \quad (1)
 \end{aligned}$$

The first term in (1) is equal to

$$H\left(\bigvee_{a+\omega^{-1}-[a+\omega^{-1}]-r \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]} \xi_{m,1} \mid \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-2s-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}n-[a+\omega^{-1}]} \xi_{m,n}\right)$$

It follows from the properties of continuity of conditional entropy that this expression converges to  $\mathcal{H}_l(I)$ . We shall show that the second term in (1) converges to  $\mathcal{H}_r(I)$ . We have

$$\begin{aligned}
 & H\left(\bigvee_{a+\omega^{-1}-s \leq m \leq a+\omega^{-1}+s+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+s-r \leq m \leq a+\omega^{-1}+s} \xi_{m,1} \right. \\
 & \quad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}n-s \leq m \leq a+\omega^{-1}n+s} \xi_{m,n}\right) \\
 &= H\left(\bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r} \xi_{m,1} \mid \bigvee_{a+\omega^{-1}+2s-r-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+2s} \xi_{m,1} \right. \\
 & \quad \left. \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n}\right).
 \end{aligned}$$

We denote

$$\eta = \bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r} \xi_{m,1}$$

and  $C_\eta(x)$  is an element containing  $x \in M$ . Also let us introduce the partitions

$$\begin{aligned} \zeta_s &= \bigvee_{a+\omega^{-1}-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+r+2s} \xi_{m,0} \\ &\vee \bigvee_{n < 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n}, \\ \zeta^+ &= \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m} \xi_{m,n} \end{aligned}$$

In view of Doob's theorem on convergence of conditional probabilities

$$\mu(C_\eta(x) | C_{\zeta_s}(x)) \rightarrow \mu(C_\eta(x) | C_{\zeta^+}(x)) \quad \text{a.e.},$$

where  $C_{\zeta_s}(x)$ ,  $C_{\zeta^+}(x)$  are elements of corresponding partitions containing  $x$ . But

$$\begin{aligned} \mu\left(C_\eta(x) \middle| \bigvee_{a+\omega^{-1}+2s-r-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}-[a+\omega^{-1}]+2s} \xi_{m,1} \right. \\ \left. \vee \bigvee_{n \leq 0} \bigvee_{a+\omega^{-1}(n+1)-[a+\omega^{-1}] \leq m \leq a+\omega^{-1}(n+1)-[a+\omega^{-1}]+2s} \xi_{m,n} \right) \quad (2) \end{aligned}$$

can be represented as finite linear combinations of  $\mu(C_\eta(x) | C_{\zeta_s}(x))$ . This shows easily that the conditional probabilities (2) also converge a.e. as  $s \rightarrow \infty$  to  $\mu(C_\eta(x) | C_{\zeta^+}(x))$ . This gives the desired result. Q.E.D.

### §3. Several general remarks

Let us consider two commuting automorphisms  $T_1, T_2$  of Lebesgue space  $(M, \mathcal{M}, \mu)$ . Then we have a measure-preserving action of the group  $\mathbb{Z}^2$  on  $M$  and we shall assume that it is ergodic and at least one of automorphisms  $T_1, T_2$  is also ergodic. Without any loss of generality we can assume  $T_1$  is ergodic. If the measure-theoretic entropy  $h(T_1)$  is finite one can find a finite generating partition  $\xi = \{C_1, \dots, C_\kappa\}$  in view of Krieger's theorem [4]. It means that  $T_1$  is isomorphic to the shift in the space of doubly-infinite sequences written in the alphabet of  $\kappa$  symbols. If  $T_2 x = y = \{y_n\}$  then  $y_n = f(x_n, x_{n\pm 1}, x_{n\pm 2}, \dots)$  where  $f$  is a measurable function with the values in the space  $\{1, 2, \dots, \kappa\}$ . Thus the pair  $(T_1, T_2)$  is represented as a system of cellular automata but maybe with an infinite memory. Our arguments presented above can be extended to the case when  $f$  can be approximated sufficiently well by functions of finite number of variables. However, the general case remains completely open. One can mention also an

interesting paper by G. A. Galperin [5] where some results concerning topological entropy of systems of cellular automata were established.

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*L. D. Landau Institute of Theoretical Physics,  
Academy of Sciences of USSR,  
Moscow, USSR.*

#### Added in proof:

It is clear that theorems 1 and 2 are easily generalized to the case of non-ergodic measures. Indeed, if  $\nu_{\text{inv}}$  is the measurable partition of  $M$  into ergodic components of the action of  $\mathbb{Z}^2$  corresponding to the measure  $\mu$ , then

$$\frac{1}{n} h_{m,n} = \int_{M | \nu_{\text{inv}}} \frac{1}{n} h(S^m T^n | C_{\nu_{\text{inv}}}) d\mu_{\text{inv}}$$

where  $\mu_{\text{inv}}$  is the induced measure on the factor-space  $M | \nu_{\text{inv}}$ . We showed already for a.e. element  $C_{\nu_{\text{inv}}}$  of  $\nu_{\text{inv}}$  the convergence of  $(1/n)h(S^m T^n | C_{\nu_{\text{inv}}})$ ,  $(m/n) \rightarrow \omega$ , which implies the convergence of  $(1/n)h_{m,n}$ .

Also in the same way one can consider the action of the semi-group  $\mathbb{Z}_+^2 = \{(m, n): -\infty < m < \infty, n \geq 0\}$ . In order to get the assertions of theorems 1 and 2 one should replace possible pasts by possible futures in all arguments.