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Autor: Eklof, Paul C. / Huber, Martin

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On ω-filtered vector spaces and their application to abelian p-groups: I

PAUL C. EKLOF(*) and MARTIN HUBER

0. Introduction

Let ω denote the first infinite ordinal. An *ω-filtered vector space* is an ordinary vector space, X, together with a descending chain of subspaces

$$X = X^0 \supseteq X^1 \supseteq \cdots \supseteq X^n \supseteq \cdots \qquad (n < \omega).$$

A *morphism* between ω-filtered vector spaces X and Y is a linear map $f: X \to Y$ such that for all n, $f(X^n) \subseteq Y^n$. The first treatment of ω-filtered vector spaces, in a somewhat more restricted sense, was given by Charles [C], who was studying abelian p-groups of length $\leq \omega$. If G is any abelian p-group, there is associated with it an ω-filtered vector space over $\mathbb{Z}(p)$, the field of p elements, called the *socle* of G:

$$X = G[p] \overset{\text{def}}{=} \{x \in G : px = 0\}$$

$$X^n = (p^n G)[p] = \{x \in p^n G : px = 0\}.$$

In general, the socle of G does not determine G up to isomorphism. But it is possible to identify Σ-cyclic groups and torsion-complete groups from their socles (cf. Corollaries 1.9 and 1.11). Moreover, Fuchs and Irwin [FI] showed that $p^{\omega+1}$-projective p-groups are determined by their socles (cf. Section 5). Richman [R] used ω-filtered vector spaces to study extensions of p-bounded groups; this led to a classification of $p^{\omega+1}$-injective p-groups by ω-filtered vector spaces (cf. Section 4). Filtered vector spaces have also arisen in the work of Gross on quadratic forms on infinite dimensional vector spaces (see e.g., [GK]).

In recent years, more general filtered vector spaces – with a subspace filtration of arbitrary ordinal length – have been studied; these have usually been considered as *valuated* vector spaces, where the correspondence between filtered

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vector spaces and valuated vector spaces is given by:

$$v(x)=\sigma \Leftrightarrow x \in X^{\sigma}-X^{\sigma+1} \quad \text{(cf. [F2], [F3], [Hi], [HW]).}$$

Much of this work has focused on the use of valuated vector spaces to study p-groups of length $>\omega$. However, recently, set-theoretic methods have been proved effective in the study of separable p-groups, most notably in Megibben's work on Crawley's problem [Meg1] and the work of Megibben [Meg2], Eklof–Mekler [EM], and Huber [Hu] on (weakly) ω_1-separable p-groups.

Thus, in this paper we begin a systematic investigation of ω-filtered vector spaces, over an arbitrary countable field, making use of set-theoretic methods to obtain new results about the structure and classification of such spaces, and new applications to p-groups.

In Section 1 we review the work of Charles, Gabriel and others on the categorical properties of ω-filtered vector spaces and the relevance of such spaces to the study of abelian p-groups. In Section 2 we begin our investigation of ω-filtered vector spaces of uncountable dimension by introducing a set-theoretic invariant, Γ, analogous to that used in the study of groups. (In fact, for a separable p-group G, $\Gamma(G[p])=\Gamma(G)$: see 2.10.)

Two of the main concerns of the paper are: the classification problem for (weakly) ω_1-separable spaces; and the number of dense subspaces in a given space. (A weakly ω_1-separable (resp. ω_1-separable) space is one s.t. $\bigcap_{n<\omega} X^n=0$ and every countable subset is contained in a countable closed subspace (resp. countable direct summand)). In Section 2 we begin the study of the first question by showing the existence of a large number of ω_1-separable spaces of dimension $\aleph_1$ with the same Γ-invariant and even the same basic subspace (Theorem 2.8). In Section 4 we obtain, as a consequence, the existence of large numbers of non-isomorphic $p^{\omega+1}$-injective groups.

In Section 3 we take up the second question; the main theorem (3.4) characterizes exactly which Γ-invariants can be realized by (dense) subspaces of a given space X, in terms of a new invariant $\Sigma(X)$. This leads to the identification of an interesting new property of ω-filtered vector spaces (and hence of p-groups): the SCC (for "smooth chain of closures") property (cf. 3.7). One consequence for p-groups is the following (3.11): if a separable p-group G of cardinality $\aleph_1$ contains at least one pure subgroup which is weakly ω_1-separable but not Σ-cyclic, then it contains $2^{\aleph_1}$ such subgroups, which are pure and dense. Another consequence is a strengthening of a theorem of Warfield about the number of ω-elongations of certain separable p-groups by elementary p-groups (Theorem 3.13).

In Section 5 we discuss the existence of certain projective resolutions; as a consequence we are able to characterize the socles of $p^{\omega+1}$-projective p-groups (5.5).

In a second paper on the same subject we shall continue to discuss the two themes proposed above. In there we shall deal with results not provable in ordinary (Zermelo–Fraenkel) set theory. Using different additional hypotheses, we shall establish a classification theorem for ω_1-separable ω-filtered vector spaces of dimension $\aleph_1$ on the one hand, and the existence of a large number of non-isomorphic dense subspaces of codimension one on the other.

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1. The category of ω-filtered vector spaces

In this section we focus on categorial properties of ω-filtered vector spaces and review some known facts about socles of abelian p-groups. To some extent the present discussion follows an unpublished note by Gabriel [G].

The category of ω-filtered K-vector spaces and their morphisms, as defined in the introduction, will be denoted by $\mathscr{FV}$ or, more precisely, $\mathscr{FV}_K$; it is additive and has kernels and cokernels. We say that Y is a *(filtered) subspace* of $X \in \mathscr{FV}$ if the inclusion map $f: Y \hookrightarrow X$ is an *embedding*, i.e., $f^{-1}(X^n) = Y^n$ for all n. The quotient space X/Y will always be equipped with the filtration given by $(X/Y)^n = (X^n + Y)/Y$, $n < \omega$, so that the natural map $\pi: X \to X/Y$ is a cokernel. Not every monomorphism is a kernel and not every epimorphism is a cokernel; thus $\mathscr{FV}$ is pre-abelian but not abelian.

Furthermore, $\mathscr{FV}$ has arbitrary products and coproducts. The symbols "Π" and "$\oplus$" will be used for products and coproducts in $\mathscr{FV}$, and the latter will be called direct sums. If X and Y are subspaces of an ω-filtered vector space, $X+Y$ will denote the obvious subspace. Note that $X+Y = X \oplus Y$ if and only if $X \cap Y = 0$ and for all $n < \omega$, $(X+Y)^n = X^n + Y^n$.

Given any $X \in \mathscr{FV}$, we note that for all n, X^n is a direct summand of X: if C is any vector space complement of X^n in X, we have $X = C \oplus X^n$. Also $X^\infty \overset{\text{def}}{=} \bigcap_{n \in \omega} X^n$ is a direct summand of X.

We remark that an ω-filtered vector space X may as well be regarded as a valuated vector space, where the valuation $v: X \to \omega \cup \{\infty\}$ is given by

$$v(x) = \begin{cases} n & \text{if} \quad x \in X^n - X^{n+1} \\ \infty & \text{if} \quad x \in X^\infty \end{cases}.$$

In contrast to [F2, F3] we allow $v(x) = \infty$ also for $x \neq 0$.

An ω-filtered vector space X is called *homogeneous* (*of value* n where $n \in \omega \cup \{\infty\}$) if $v(x) = n$ for all $x \in X - \{0\}$ or, equivalently, $X = X^n$ for some n and (if $n \neq \infty$) $X^{n+1} = 0$. A space X is *Σ-homogeneous* if it is a direct sum of homogeneous subspaces; X is *Π-homogeneous* if it is a product of homogeneous spaces. The subspaces X^n, $n < \omega$, of $X \in \mathscr{FV}$ form a neighborhood basis at zero of a linear topology. All topological notions will refer to this topology. Thus X is *separated* (Hausdorff) if and only if $X^\infty = 0$. Note that a discrete space is a finite direct sum (product) of homogeneous spaces.

In the category $\mathscr{FV}$ a sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is *exact* if f is the kernel of g and g is the cokernel of f. Every kernel and cokernel is semi-stable (in the sense of [RW]), so every exact sequence in $\mathscr{FV}$ is stable exact (cf. also [Mi]). We thus obtain in $\mathscr{FV}$ a homology theory w.r.t. all exact sequences. We aim to determine the projectives and injectives. Variants of the following results are well known (cf. [F2]); we therefore omit many of the proofs.

LEMMA 1.1. *Every Π-homogeneous ω-filtered vector space is injective. In particular, every discrete and every finite-dimensional space is injective.* $\square$

Let $X \in \mathscr{FV}$ be of countable dimension. Then $X = \bigcup_{n<\omega} X_n$ where $\{X_n \mid n < \omega\}$ is an increasing chain of finite-dimensional subspaces and $X_0 = 0$. It follows that for all n, $X_{n+1} = X_n \oplus C_n$ for some finite-dimensional C_n, and hence $X = \bigoplus_{n<\omega} C_n$. Thus we have proved:

PROPOSITION 1.2. *Every countable dimensional ω-filtered vector space is Σ-homogeneous.* $\square$

PROPOSITION 1.3. *There are enough projectives in $\mathscr{FV}$; they are precisely the separated Σ-homogeneous ω-filtered vector spaces.*

Proof. It is not hard to construct a cokernel $f: Y \to X$ where Y is separated Σ-homogeneous and thus projective. This shows that there are enough projectives and that every projective is a direct summand of a separated Σ-homogeneous space. That every projective is itself Σ-homogeneous follows from 1.2 and Theorem 4 of [WW]. □

PROPOSITION 1.4. *If $f: X \to Y$ is a monomorphism in $\mathscr{F}\mathscr{V}$ and Y is projective, then X is projective as well.*

Proof: Write $Y = \bigoplus_{n<\omega} Y_n$ where Y_n is homogeneous of value n. Let $X_n = f^{-1}(\bigoplus_{k<n} Y_k)$ so that $X = \bigcup_{n<\omega} X_n$. Now since $Y^n \cap (\bigoplus_{k<n} Y_k) = 0$, it follows that $X^n \cap X_n = 0$. Therefore each X_n is discrete and hence injective, and by the same reasoning as in 1.2 we conclude that X is projective. □

Thus the dimension of the homology theory in $\mathscr{F}\mathscr{V}$ is one.

Remark. One could expect that our projectives would agree with the projectives of length $\leq \omega + 1$ in [F2]. This is not the case, however, because Fuchs considers projectives relative to nice exact sequences (which agree with the proper projectives in [HW]).

Given $X \in \mathscr{F}\mathscr{V}$ we define its nth Ulm invariant $f_n(X)$ by

$$f_n(X) = \dim_K (X^n / X^{n+1}), \qquad n < \omega;$$

furthermore we let $f_\infty(X) = \dim_K (X^\infty)$. Note that for Σ-homogeneous spaces X the cardinals $f_n(X)$, $n < \omega$, and $f_\infty(x)$ form a complete set of invariants. We call a subspace B of X *basic* if B is projective and dense in X. (Note again the difference between this definition and that in [F2].)

PROPOSITION 1.5. *Every ω-filtered vector space X contains a basic subspace. Any two basic subspaces of X are isomorphic.* □

For any $X \in \mathscr{F}\mathscr{V}$ let $\hat{X} = \varprojlim_{n<\omega} X/X^n$, the *completion* of X, be equipped with the obvious filtration. The completion map $\gamma_X : X \to \hat{X}$ is a morphism in $\mathscr{F}\mathscr{V}$; its kernel is X^∞. If X is projective, say $X = \bigoplus_{k<\omega} X_k$ where X_k is homogeneous of value k, we clearly have $\hat{X} = \prod_{k<\omega} X_k$. If X is any space and B a basic subspace of X, then the inclusion $h: B \to X$ induces isomorphisms $B/B^n \xrightarrow{\sim} X/X^n$ $(n < \omega)$, hence the induced map $\hat{h}: \hat{B} \to \hat{X}$ is likewise an isomorphism.

PROPOSITION 1.6. *There are enough injectives in $\mathscr{F}\mathscr{V}$; they are precisely the Π-homogeneous ω-filtered vector spaces.*

Proof. Each $X \in \mathscr{F}\mathscr{V}$ can be embedded in $\hat{X} \oplus X^\infty$ which by the preceding considerations is Π-homogeneous and thus injective. This proves the first assertion. It follows that any injective X is a summand of a Π-homogeneous space. Consequently, X/X^∞ is complete and hence X itself is Π-homogeneous. □

Our next aim is to indicate some basic facts concerning the relevance of ω-filtered vector spaces to abelian p-groups. In what follows we adopt Gabriel's point of view [G], which differs somewhat from the usual one (as given in [C] or [F3]). With any abelian p-group G we associate its socle $S = G[p]$ which is an ω-filtered $\mathbb{Z}(p)$-vector space where $S^n = p^nG \cap S$. Furthermore, to any homomorphism $f: G \to H$ between p-groups we assign its restriction $f_*: G[p] \to H[p]$, which is a morphism in $\mathscr{FV}_{\mathbb{Z}(p)}$.

LEMMA 1.7.
(a) *f is a monomorphism if and only if so is f_*;*
(b) *f is a pure monomorphism if and only if f_* is an embedding;*
(c) *f is a pure epimorphism if and only if f_* is a cokernel;*
(d) *f is an isomorphism if and only if so is f_*.*

Proof. Most of the statements are routine to check. We indicate only how to prove the "if" part of (b). Consider the induced map $\tilde{f}: p^nG \to \operatorname{Im}(f) \cap p^nH$. Since f_* is an embedding, $(\tilde{f})_*$ is an isomorphism. By (d) $\tilde{f}$ is an isomorphism, hence $\operatorname{Im}(f)$ is pure in H. □

COROLLARY 1.8. [G]. *Let $f: G \to H$, $g: H \to K$ be homomorphisms of p-groups such that $g \circ f = 0$. Then the sequence*

$$0 \longrightarrow G \xrightarrow{f} H \xrightarrow{g} K \longrightarrow 0$$

is pure exact if and only if the sequence

$$0 \longrightarrow G[p] \xrightarrow{f_*} H[p] \xrightarrow{g_*} K[p] \longrightarrow 0$$

is exact in $\mathscr{FV}$. □

COROLLARY 1.9 [C]. *A p-group G is Σ-cyclic (i.e., a direct sum of cyclic groups) if and only if $G[p]$ is projective.* □

COROLLARY 1.10 ([C], [G]). *Given any ω-filtered vector space X over $\mathbb{Z}(p)$, there exists an abelian p-group G and an isomorphism $G[p] \xrightarrow{\sim} X$ in $\mathscr{FV}$.*

Proof. Choose a projective resolution

$$0 \longrightarrow P_1 \xrightarrow{h} P_0 \longrightarrow X \longrightarrow 0$$

in $\mathscr{FV}$. There exists a homomorphism $f: F_1 \to F_0$ of Σ-cyclic groups such that $F_l[p] = P_l$ $(l = 0, 1)$, and f extends h. Now f is a pure monomorphism; hence by 1.8 $G = \text{Coker}\,(f)$ has the desired property. □

Given a p-group G, its p-adic completion is denoted by $\hat{G}_p$, and the *torsion completion* of G is $t(\hat{G}_p)$, the torsion subgroup of $\hat{G}_p$. The group G is *torsion-complete* if $G \cong t(\hat{G}_p)$.

COROLLARY 1.11 [F1, Theorem 70.6]. *A p-group G is torsion-complete if and only if $G[p]$ is complete.*

Proof. Obviously, if G is torsion-complete then $G[p]$ is complete. Conversely, suppose that $G[p]$ is complete. Let $\nu: G \to \bar{G}$ denote the natural map of G into its torsion completion. By [F1, Corollary 68.2] ν is a pure monomorphism whose image is dense in $\bar{G}$ (w.r.t. the p-adic topology). Thus $\nu_*: G[p] \to \bar{G}[p]$ is an embedding with dense image. But then by hypothesis ν_* is an isomorphism and hence so is ν. □

We conclude this section by quoting a result of Hill and Megibben which will be one of the main tools for the application of our results on ω-filtered vector spaces to p-groups.

PROPOSITION 1.12 [F1, Theorem 66.3]. *Let S be a dense subspace of the socle of a p-group G. Then S supports a subgroup H which is pure and dense in G.* □

2. The abundance of ω_1-separable spaces

In this section we begin our investigation of uncountable dimensional ω-filtered vector spaces by means of set-theoretic methods. For such spaces we introduce a set-theoretic invariant Γ and discuss its significance. We shall prove that for any prescribed value of Γ there exists a large family of non-isomorphic spaces. This discussion largely parallels that in [Hu].

Let us first recall some terminology and notation from set theory. As usual, an ordinal is identified with the set of its predecessors, and a cardinal is an ordinal which has greater cardinality than all its predecessors. Thus ω is the first infinite cardinal, also denoted $\aleph_0$. We shall use ω_1 and $\aleph_1$ interchangeably to denote the first uncountable cardinal. For any set X, $|X|$ denotes the cardinality of X. Given an infinite cardinal κ, a subset $X \subseteq \kappa$ is *closed* if for each $Y \subseteq X$, $\sup(Y) < \kappa$ implies $\sup(Y) \in X$ (where $\sup(Y)$ is the supremum of Y). A subset X is *unbounded* (or *cofinal*) in κ if $\sup(X) = \kappa$. The cofinality, $\mathrm{cf}(\kappa)$, of an infinite cardinal κ is the cardinality of the smallest $X \subseteq \kappa$ such that X is cofinal in κ. An infinite cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$.

Suppose that κ is a regular uncountable cardinal. A subset $C \subseteq \kappa$ is called a *cub* if it is closed and unbounded; $I \subseteq \kappa$ is *thin* if $\kappa - I$ contains a cub. The thin subsets of κ form an ideal $\mathcal{J}(\kappa)$ of the Boolean algebra $\mathcal{P}(\kappa)$ of all subsets of κ. Let $D(\kappa)$ denote the quotient algebra $\mathcal{P}(\kappa)/\mathcal{J}(\kappa)$; denote the image of $I \subseteq \kappa$ in $D(\kappa)$ by $\tilde{I}$. We have $\tilde{I} = \tilde{J}$ if and only if there is a cub C such that $I \cap C = J \cap C$. The least element of $D(\kappa)$ is $0 \stackrel{\text{def}}{=} \tilde{\varnothing} = \mathcal{J}(\kappa)$, and the largest element is $1 \stackrel{\text{def}}{=} \tilde{\kappa}$ which is just the filter dual to $\mathcal{J}(\kappa)$. A subset $I \subseteq \kappa$ is called *stationary* if $\tilde{I} \neq 0$; that is, for all cubs C, $I \cap C \neq \tilde{\varnothing}$. It can be proved that for every element $e \in D(\kappa) - \{0\}$, the interval $[0, e] = \{f \in D(\kappa) \mid f \leq e\}$ has cardinality 2^κ; in particular, we have $|D(\kappa)| = 2^\kappa$.

Now let X be an ω-filtered K-vector space of dimension κ. A subspace Y of X is said to be *small* if $\dim_K(Y) < \kappa$. An (ascending) *κ-filtration* of X is a family of subspaces $\{X_\nu \mid \nu < \kappa\}$ of X such that

(0) $X_0 = 0$;

(i) if $\mu < \nu$ then $X_\mu \subseteq X_\nu$ (i.e., it is a *chain*);

(ii) X_ν is a small subspace of X for all ν;

(iii) if ν is a limit ordinal, $X_\nu = \bigcup_{\mu<\nu} X_\mu$ (the chain is *smooth*); and

(iv) $X = \bigcup_{\nu<\kappa} X_\nu$.

To indicate that $\{X_\nu \mid \nu < \kappa\}$ is a κ-filtration of X we will simply write $X = \bigcup_{\nu<\kappa} X_\nu$. The following observation is crucial.

LEMMA 2.1. *If $X = \bigcup_{\nu<\kappa} X_\nu$ and $X = \bigcup_{\nu<\kappa} X'_\nu$ are two κ-filtrations of X, then the set $C = \{\nu < \kappa \mid X_\nu = X'_\nu\}$ is a cub.*

The proof of this is essentially the same as for abelian groups (cf. [E1, p. 260]) and is therefore omitted. $\square$

A filtered subspace Y of X is called a *κ-summand* of X if Y is a direct summand of every intermediate subspace Z such that $\dim_K(Z/Y) < \kappa$. Given a κ-filtration $X = \bigcup_{\nu<\kappa} X_\nu$, we consider the set

$$E = \{\nu < \kappa \mid X_\nu \text{ is not a } \kappa\text{-summand of } X\}.$$

If we choose another filtration $X=\bigcup_{\nu<\kappa} X'_\nu$ and let

$$E'=\{\nu<\kappa \mid X'_\nu \text{ is not a } \kappa\text{-summand of } X\},$$

then by 2.1 there is a cub C such that $E\cap C=E'\cap C$. Thus $\tilde{E}\in D(\kappa)$ is an invariant of X which will be denoted by $\Gamma(X)$.

Analogously to [E1, Thm. 2.5] one proves

THEOREM 2.2. *Let X be an ω-filtered vector space of regular uncountable dimension κ. Then*

(1) *$\Gamma(X)=0$ if and only if X is a direct sum of small subspaces.*

(2) *If $\Gamma(X)\neq 1$ then every small subspace Y of X is contained in a small κ-summand Y' of X.* □

From now on we mainly concentrate on the case that $\dim(X)=\aleph_1$. Since countable dimensional separated spaces are projective, 2.2(1) has the following consequence.

COROLLARY 2.3. *A separated ω-filtered vector space X of dimension $\aleph_1$ satisfies $\Gamma(X)=0$ if and only if it is projective.* □

We note that by 1.2 and 1.3 a subspace Y of a separated space X is an ω_1-summand if and only if Y is closed in X. Therefore, if $\dim(X)=\omega_1$, $X=\bigcup_{\nu<\omega_1} X_\nu$ is any ω_1-filtration of X, and $E=\{\nu<\omega_1 \mid X_\nu \text{ is not closed in } X\}$, then $\Gamma(X)=\tilde{E}$.

An ω-filtered vector space X will be called *ω_1-separable* if every countable subset of X is contained in a projective direct summand (hence in a countable dimensional such summand). X will be termed *weakly ω_1-separable* if it is separated and every countable subset of X is contained in a closed countable dimensional subspace. Clearly, every ω_1-separable space is weakly ω_1-separable.

COROLLARY 2.4. *If X is a separated ω-filtered vector space of dimension $\aleph_1$ such that $\Gamma(X)\neq 1$, then X is weakly ω_1-separable.* □

Let X be a separated space with $\dim(X)=\aleph_1$, and let Y be a subspace of X with the same dimension. Given an ω_1-filtration $X=\bigcup_{\nu<\omega_1} X_\nu$, we let $Y_\nu=Y\cap X_\nu$ for all $\nu<\omega_1$. This defines an ω_1-filtration $Y=\bigcup_{\nu<\omega_1} Y_\nu$ such that if X_ν is closed in X then so is Y_ν in Y. We conclude that $\Gamma(Y)\leq\Gamma(X)$.

THEOREM 2.5. *If X has dimension $\aleph_1$ and Y is a subspace with* $\dim(X/Y) \leq \aleph_0$, *then* $\Gamma(Y) = \Gamma(X)$.

Before we can prove this we need to rephrase in the language of ω-filtered vector spaces a significant relation which has been introduced by Hill in the setting of valuated vector spaces [Hi]. Given an ω-filtered vector space X and subspaces Y, Z of X, Y is said to be *compatible* with Z if for all $n < \omega$, $(Y + X^n) \cap Z \subseteq (Y \cap Z) + X^n$. We note that this relation is symmetric, and we write $Y \| Z$ if Y is compatible with Z. The following facts are straightforward to see.

LEMMA 2.6. (a) *If $\{Z_\nu \mid \nu < \rho\}$ is an ascending chain of subspaces of X with $Y \| Z_\nu$ for all $\nu < \rho$, then $Y \| \bigcup_{\nu < \rho} Z_\nu$.*

(b) *If $Y \| Z$ then the natural map $Y/Y \cap Z \to (Y+Z)/Z$ is an isomorphism in $\mathscr{F}\mathscr{V}$.* $\square$

PROPOSITION 2.7. *Let X be an ω-filtered vector space of regular uncountable dimension κ. Then for any subspace Y of X there is a κ-filtration $X = \bigcup_{\nu < \kappa} X_\nu$ such that for all $\nu < \kappa$, $Y \| X_\nu$.*

This is a consequence of [Hi; Lemma 3]. In this particular case, however, we are in a position to offer a much simpler proof.

Proof. Let $\{x_\nu \mid \nu < \kappa\}$ be a basis of X. By induction on $\nu < \kappa$ we define subspaces X_ν of dimension $< \kappa$ such that $Y \| X_\nu$ and for all $\mu < \nu$, $x_\mu \in X_\nu$. Suppose that X_μ has been defined for all $\mu < \nu$. If ν is a limit ordinal, we let $X_\nu = \bigcup_{\mu < \nu} X_\mu$. Clearly $x_\mu \in X_\nu$ for all $\mu < \nu$, and $Y \| X_\nu$ by 2.6(a). If ν is a successor, say $\nu = \mu + 1$, we define an ascending sequence $\{Z_k \mid k < \omega\}$ of subspaces of dimension $< \kappa$ such that $Z_0 = X_\mu + Kx_\mu$, and for all k, $n < \omega$,

$$(Y + X^n) \cap Z_k \subseteq (Y \cap Z_{k+1}) + X^n.$$

This is possible because $\dim(Z_k) < \kappa$. Now let $X_\nu = \bigcup_{k < \omega} Z_k$. Then of course $x_\alpha \in X_\nu$ for all $\alpha < \nu$ and $Y \| X_\nu$. Thus by construction $X = \bigcup_{\nu < \kappa} X_\nu$ is a κ-filtration with the desired property. $\square$

Proof of 2.5. By hypothesis and 2.7 there is an ω_1-filtration $X = \bigcup_{\nu < \omega_1} X_\nu$ such that $Y + X_1 = X$, and $Y \| X_\nu$ for all $\nu < \omega_1$. Therefore for all $\nu < \omega_1$, $Y/Y \cap X_\nu \cong (Y + X_\nu)/X_\nu$ (in $\mathscr{F}\mathscr{V}$) by 2.6(b). But this means that for all $\nu < \omega_1$, $Y \cap X_\nu$ is closed in Y if and only if X_ν is closed in X. Consequently, $\Gamma(Y) = \Gamma(X)$. $\square$

The *final dimension* of an ω-filtered vector space X is given by

$$\operatorname{findim}(X) = \inf\{\dim(X^n) \mid n<\omega\}.$$

We note that if $\operatorname{findim}(X)=\aleph_0$ then $X=X_0\oplus X_1$ where X_0 is discrete and $\dim(X_1)=\aleph_0$. Therefore, if X is weakly ω_1-separable but not projective we have $\operatorname{findim}(B)\geq\omega_1$ for any basic subspace B of X.

We next prove that for any stationary subset $E\subseteq\omega_1$, $\Gamma^{-1}(\tilde{E})$ has cardinality $\geq 2^{\aleph_1}$. In fact, our construction will provide ω_1-separable spaces, so that the assumption on the given basic subspace is inevitable. Analogous results hold in the case of ω_1-free groups [E2; Thm. 11.2] and ω_1-separable p-groups [Hu; Thm. 2.7]; in fact, for the scalar field $\mathbb{Z}(p)$ our theorem follows from that in [Hu].

THEOREM 2.8. *Let B be a projective ω-filtered vector space with $\dim(B)=\operatorname{findim}(B)=\aleph_1$, and let E be a stationary subset of ω_1. Then there exist $2^{\aleph_1}$ mutually nonisomorphic ω_1-separable ω-filtered vector spaces X_i $(i<2^{\aleph_1})$ of dimension $\aleph_1$ such that for all i, $\Gamma(X_i)=\tilde{E}$ and B is isomorphic to a basic subspace of X_i.*

Before beginning the proof of 2.8, we prove a lemma which will also be used in later sections.

LEMMA 2.9. *Let Y, Z, S be subspaces of the separated space X such that Y is dense in Z and $Y+S=Y\oplus S$. Then we also have $Z+S=Z\oplus S$.*

Proof. We first note that, since X is separated, for any pair (U, V) of subspaces of X the statement $U+V=U\oplus V$ is equivalent to

$$\text{for all } u\in U,\ v\in V \text{ and } n<\omega,\qquad u+v\in X^n \text{ implies } v\in X^n. \tag{*}$$

Assuming (*) for the pair (Y, S), we show that (*) also holds for (Z, S). So let $z\in Z$, $s\in S$ and $n<\omega$ such that $z+s\in X^n$. By hypothesis there exists $y\in Y$ such that $z-y\in X^n$. But then $y+s=(z+s)-(z-y)\in X^n$. It follows that $s\in X^n$, as desired. $\square$

Proof of 2.8. Write $B=\bigoplus_{n<\omega}B_n$ where B_n is homogeneous of value n. Since countable dimensional summands do not change the Γ-invariant, we may assume that for all n, either $B_n=0$ or $\dim(B_n)=\omega_1$. Furthermore, we may assume that E consists of limit ordinals. Recall that a *ladder* on a limit ordinal $\delta<\omega_1$ is a strictly increasing function $\eta_\delta:\omega\to\delta$ whose range is cofinal in δ. A *ladder system* on E is an indexed family $\eta=\{\eta_\delta \mid \delta\in E\}$ such that η_δ is a ladder on δ. We shall first construct for any ladder system η an ω_1-separable space $X=X(\eta)$ as a subspace of $\hat{B}=\prod_{n<\omega} B_n$.

If $\dim(B_n)=\omega_1$ let $\{x_\nu^{(n)} \mid \nu<\omega_1\}$ be a basis of B_n; otherwise let $x_\nu^{(n)}=0$ for all $\nu<\omega_1$. For any $\nu<\omega_1$ let

$$S_\nu=\bigoplus_{n<\omega} Kx_\nu^{(n)}$$

$$B(\nu)=\bigoplus_{n<\omega}\langle x_\mu^{(n)} \mid \mu<\nu\rangle, \qquad B(\nu)'=\bigoplus_{n<\omega}\langle x_\mu^{(n)} \mid \mu\geq\nu\rangle,$$

$$\hat{B}(\nu)=\prod_{n<\omega}\langle x_\mu^{(n)} \mid \mu<\nu\rangle, \qquad \hat{B}(\nu)'=\prod_{n<\omega}\langle x_\mu^{(n)} \mid \mu\geq\nu\rangle.$$

We observe that for all ν, $B=B(\nu)\oplus B(\nu)'$ and $\hat{B}=\hat{B}(\nu)\oplus\hat{B}(\nu)'$. For each $\delta\in E$ we define $y_\delta=(y_\delta(n))_{n<\omega}\in\hat{B}$ by $y_\delta(n)=x_{\eta_\delta(n)}^{(n)}$; note that $y_\delta\in\hat{B}(\delta)-\bigcup_{\nu<\delta}\hat{B}(\nu)$.

Now define $X=X(\eta)$ as the union of a smooth chain $\{X_\nu \mid \nu<\omega_1\}$ of countable dimensional subspaces of $\hat{B}$ where $X_0=0$ and

$$X_{\nu+1}=\begin{cases} X_\nu+S_\nu & \text{if } \nu\notin E;\\ X_\nu+S_\nu+Ky_\nu & \text{if } \nu\in E.\end{cases}$$

The following statements are easily checked:

(a) For all $\nu<\omega_1$, $B(\nu)\subseteq X_\nu\subseteq\hat{B}(\nu)$;

(b) if $\nu\in E$ then X_ν is not closed in $X_{\nu+1}$.

Furthermore, we claim that

(c) if $\nu\notin E$ then $X=X_\nu\oplus(X\cap\hat{B}(\nu)')$.

To prove this we note that if $\delta\in E$, $\delta>\nu$, y_δ may be written $y_\delta=z+w$ where $z\in B(\nu)$ and $w\in\hat{B}(\nu)'$. Therefore $X=X_\nu+(X\cap\hat{B}(\nu)')$, and the claim follows by 2.9 since $B(\nu)$ is dense in X_ν. Now (a) implies that B is a basic subspace of X, and from (b) and (c) we infer that X is ω_1-separable and $\Gamma(X)=\tilde{E}$.

Our next aim is to prove that if η and η^1 are sufficiently different ladder systems then $X(\eta)$ and $X(\eta^1)$ are not isomorphic. (Here the argument is simpler than in the group theory case). Let $h:\omega\to\omega$ be a strictly increasing function such that for all n, $\dim(B_{h(n)})=\omega_1$. Let $\eta=\{\eta_\delta \mid \delta\in E\}$ and $\eta^1=\{\eta_\delta^1 \mid \delta\in E\}$ be ladder systems on E such that for each $\delta\in E$ the following condition holds:

$$\eta_\delta^1(h(n))\geq\eta_\delta(h(n+1)) \quad \text{for all} \quad n<\omega. \tag{*}$$

We claim that $X\not\cong Y$ where $X=X(\eta)$ and $Y=X(\eta^1)$. Suppose to the contrary that there exists an isomorphism $f:X\xrightarrow{\sim}Y$. Then there is $\delta\in E$ and a strictly increasing sequence of ordinals $\{\nu(n) \mid n<\omega\}$ with $\sup\{\nu(n)\}=\delta$ such that

$f(X_{\nu(n)}) = Y_{\nu(n)}$ for all n and $f(X_\delta) = Y_\delta$. Now by definition we have

$$X_{\delta+1} = X_\delta + S_\delta + Kz, \qquad Y_{\delta+1} = Y_\delta + S_\delta + Kw$$

where $z = (x^{(n)}_{\eta_\delta(n)})_{n<\omega}$ and $w = (x^{(n)}_{\eta^1_\delta(n)})_{n<\omega}$. Since $z \in \bar{X}^X_\delta$ (the closure of X_δ in X), $f(z)$ belongs to $\bar{Y}^Y_\delta$. But $\bar{Y}^Y_\delta = Y_\delta + Kw$, hence $f(z) = y + \lambda w$ for some $y \in Y_\delta$ and $\lambda \in K - \{0\}$.

Now choose n large enough so that $y \in Y_{\nu(n)}$ and let $d = \max\{i < \omega \mid \eta_\delta(h(i)) < \nu(n)\}$. Define $u = (u_k)_{k<\omega} \in \hat{B}$ by

$$u_k = \begin{cases} x^{(k)}_{\eta_\delta(k)} & \text{if } k \le h(d) \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have $u \in X_{\nu(n)}$ and $z - u \in X^{h(d)+1}$. It follows that $y - f(u) \in Y_{\nu(n)}$ and $f(z) - f(u) \in Y^{h(d)+1}$. But this is impossible for $\eta^1_\delta(h(d)) \ge \nu(n)$ by (*) and thus the $h(d)$th component of $f(z) - f(u)$ cannot be zero because $\lambda x^{(h(d))}_{\eta^1_\delta(h(d))} \ne 0$.

This constructs two non-isomorphic spaces X and Y with the desired properties. To obtain $2^{\aleph_1}$ different ones we proceed as in [E2; pp. 111–112]. □

We wish to apply Theorem 2.8 to p-groups. For a *separable* p-group G (i.e., $p^\omega G = 0$) of cardinality ω_1 the Γ-invariant can be defined by $\Gamma(G) = \tilde{E}$ where

$$E = \{\nu < \omega_1 \mid G_\nu \text{ is not closed in } G\}$$

and $G = \bigcup_{\nu<\omega_1} G_\nu$ is any ω_1-filtration (cf. [Hu], remark after Cor. 1.3). Of course, we may assume that each G_ν is a pure subgroup of G. But then for all $\nu < \omega_1$, G_ν is closed in G if and only if $G_\nu[p]$ is closed in $G[p]$. Thus we obtain

PROPOSITION 2.10. *For any separable p-group G of cardinality $\aleph_1$ we have* $\Gamma(G) = \Gamma(G[p])$. □

Recall that a p-group is termed ω_1-*separable* if every countable subset of G is contained in a (countable) Σ-cyclic direct summand of G, whereas G is *weakly* ω_1-*separable* if it is separable and every countable subset is contained in a countable closed pure subgroup of G. We have not been able to derive [Hu; Thm. 2.7] from 2.8 above, because we do not know whether every ω_1-separable ω-filtered $\mathbb{Z}(p)$-vector space is the socle of an ω_1-separable p-group. However, we obtain the following slightly weaker result.

COROLLARY 2.11. *Let B be a Σ-cyclic p-group of cardinality $\aleph_1$ and final* *rank $\aleph_1$, and let E be a stationary subset of ω_1. Then there exist $2^{\aleph_1}$ mutually nonisomorphic weakly ω_1-separable p-groups G_i $(i<2^{\aleph_1})$ such that for all i, B is a basic subgroup of G_i and $\Gamma(G_i)=\tilde{E}$.*

Proof. This is a consequence of 1.12, 2.8 and the following theorem of Megibben's which we quote explicitly since it will be applied again several times. □

THEOREM 2.12 [Meg2; Thm. 1.1]. *Let G be a separable p-group. Then G is weakly ω_1-separable if and only if $G[p]$ is weakly ω_1-separable as an ω-filtered $\mathbb{Z}(p)$-vector space.* □

3. The realization of Γ-invariants by subspaces

In this section we attempt to imitate inside a given ω-filtered vector space X the construction of the proof of Theorem 2.8 in order to find a dense subspace with a prescribed value of Γ. This leads us to introduce another set-theoretic invariant, $\Sigma(X)$, taking its values in $D(\omega_1)$. It will turn out that any value of Γ between 0 and $\Sigma(X)$ can be realized (Theorem 3.4).

In the sequel, X will always denote a separated ω-filtered vector space of dimension ω_1. Given an ω_1-filtration $X=\bigcup_{\nu<\omega_1} X_\nu$, we let

$$E=\left\{\nu\in\operatorname{Lim}(\omega_1)\mid \bar{X}_\nu\neq\bigcup_{\mu<\nu}\bar{X}_\mu\right\}$$

where $\operatorname{Lim}(\omega_1)$ is the set of all limit ordinals $<\omega_1$, and $\bar{X}_\nu$ denotes the closure of X_ν in X. Let $X=\bigcup_{\nu<\omega_1} X'_\nu$ be another ω_1-filtration and let $E'=\{\nu\in\operatorname{Lim}(\omega_1)\mid \bar{X}'_\nu\neq\bigcup_{\mu<\nu}\bar{X}'_\mu\}$. From 2.1 we know that $C=\{\nu<\omega_1\mid X_\nu=X'_\nu\}$ is a cub, and thus so is C', the set of limit points of C. But $E\cap C'=E'\cap C'$, so that $\tilde{E}\in D(\omega_1)$ is an invariant of X which will be denoted by $\Sigma(X)$.

We note that $\Sigma(X)=0$ if and only if X admits an ω_1-filtration $X=\bigcup_{\mu<\nu}X_\nu$ such that for all $\nu\in\operatorname{Lim}(\omega_1)$, $\bar{X}_\nu=\bigcup_{\mu<\nu}\bar{X}_\mu$. If this holds we shall say that X satisfies SCC (the *smooth chain of closures* condition). For example, every projective space satisfies SCC; on the other hand, any space X containing a countable dimensional basic subspace trivially satisfies SCC.

If ρ is an ordinal $\leq\omega_1$, the *diagonal intersection* of a family $\{I_\alpha\mid\alpha<\rho\}$ of subsets of ω_1 is defined by $\Delta_{\alpha<\rho}I_\alpha=\{\nu<\omega_1\mid \nu\in\bigcap_{\alpha<\min(\nu,\rho)}I_\alpha\}$, whereas the *diagonal union* is $\nabla_{\alpha<\rho}I_\alpha=\{\nu<\omega_1\mid\nu\in\bigcup_{\alpha<\min(\nu,\rho)}I_\alpha\}$. From the fact that the diagonal intersection of ρ cubs is again a cub [J; Lemma 7.5], it follows that

$(\nabla_{\alpha<\rho} I_\alpha)^\sim$ is the supremum in $D(\omega_1)$ of the family $\{\tilde{I}_\alpha : \alpha<\rho\}$; we denote it by $\bigvee_{\alpha<\rho} \tilde{I}_\alpha$. Of course $\bigvee_{\alpha<\rho} \tilde{I}_\alpha = (\bigcup_{\alpha<\rho} I_\alpha)^\sim$ if $\rho<\omega_1$.

PROPOSITION 3.1. *Let ρ be an ordinal $\leq\omega_1$, and let $X=\bigoplus_{\alpha<\rho} X^{(\alpha)}$ where for each $\alpha<\rho$, $X^{(\alpha)}$ has dimension ω_1. Then $\Sigma(X)=\bigvee_{\alpha<\rho}\Sigma(X^{(\alpha)})$.*

Proof. For each $\alpha<\rho$ choose an ω_1-filtration $X^{(\alpha)}=\bigcup_{\nu<\omega_1} X^{(\alpha)}_\nu$ and let $E_\alpha = \{\nu\in \mathrm{Lim}\,(\omega_1) \mid \overline{X^{(\alpha)}_\nu} \neq \bigcup_{\mu<\nu} \overline{X^{(\alpha)}_\mu}\}$. Let $X=\bigcup_{\nu<\omega_1} X_\nu$ be the ω_1-filtration given by $X_\nu = \bigoplus_{\alpha<\min(\nu,\rho)} X^{(\alpha)}_\nu$, and let $E=\{\nu\in\mathrm{Lim}\,(\omega_1) \mid \bar{X}_\nu \neq \bigcup_{\mu<\nu}\bar{X}_\mu\}$. Now for each limit ordinal ν we have

$$\bigcup_{\mu<\nu} \bar{X}_\mu = \bigcup_{\mu<\nu}\ \bigoplus_{\alpha<\min(\mu,\rho)} \overline{X^{(\alpha)}_\mu} = \bigoplus_{\alpha<\min(\nu,\rho)}\ \bigcup_{\mu<\nu} \overline{X^{(\alpha)}_\mu}.$$

We infer that $E=\nabla_{\alpha<\rho} E_\alpha$ and hence $\Sigma(X)=(\nabla_{\alpha<\rho} E_\alpha)^\sim = \bigvee_{\alpha<\rho}\Sigma(X^{(\alpha)})$. □

Thus, in particular, $X=\bigoplus_{\alpha<\rho} X^{(\alpha)}$ satisfies SCC if and only if so does $X^{(\alpha)}$ for each $\alpha<\rho$. This implies, for instance, that any space X with a basic subspace of countable final dimension satisfies SCC.

PROPOSITION 3.2. *For any subspace Y of X we have $\Sigma(Y)\leq\Sigma(X)$. In particular, if X satisfies SCC, then so does every subspace of X.*

Proof. By 2.7 there is an ω_1-filtration $X=\bigcup_{\nu<\omega_1} X_\nu$ such that for all ν, $Y \parallel X_\nu$. For all $\nu<\omega_1$, let $Y_\nu = Y\cap X_\nu$. Clearly, $\bar{Y}^Y_\nu$ is contained in $Y\cap\bar{X}_\nu$. On the other hand, by the choice of the X_ν's

$$Y\cap\bar{X}_\nu = \bigcap_{n<\omega} [Y\cap(X_\nu+X^n)] \subseteq \bigcap_{n<\omega} [(Y_\nu+X^n)\cap Y] = \bar{Y}^Y_\nu.$$

Therefore, if ν is a limit ordinal such that $\bar{X}_\nu = \bigcup_{\mu<\nu}\bar{X}_\mu$, we obtain

$$\bar{Y}^Y_\nu = Y\cap\bar{X}_\nu = \bigcup_{\mu<\nu}(Y\cap\bar{X}_\mu) = \bigcup_{\mu<\nu} \bar{Y}^Y_\mu.$$

We conclude that $\Sigma(Y)\leq\Sigma(X)$, as desired. □

A (weakly) ω_1-separable space will be called *proper* if it is not projective.

PROPOSITION 3.3. (a) *For each space X we have $\Sigma(X)\leq\Gamma(X)$.*

(b) *If X is weakly ω_1-separable, then $\Sigma(X)=\Gamma(X)$. Thus no proper weakly ω_1-separable space satisfies* SCC.

Proof. Statement (a) is clear from the definitions. To prove (b) we let $X = \bigcup_{\nu<\omega_1} X_\nu$ be an ω_1-filtration such that for all $\nu<\omega_1$, $X_{\nu+1}$ is closed in X. Then for each $\nu \in \operatorname{Lim}(\omega_1)$ we have $X_\nu = \bigcup_{\mu<\nu} \bar{X}_\mu$. Therefore $\bar{X}_\nu \neq \bigcup_{\mu<\nu} \bar{X}_\mu$ if and only if X_ν is not closed in X; hence $\Sigma(X) = \Gamma(X)$. □

Remark. Let X be a space satisfying $\Sigma(X) < \Gamma(X)$. Then X is not weakly ω_1-separable and hence $\Gamma(X) = 1$. Actually such spaces are in abundance: To any subset E of ω_1 there exists a space X such that $\Sigma(X) = \tilde{E}$ and $\Gamma(X) = 1$: take $X = X_0 \oplus X_1$ where $\dim(X_0) = \dim(X_1) = \omega_1$, X_0 has a countable dimensional basic subspace, and X_1 is weakly ω_1-separable with $\Gamma(X_1) = \tilde{E}$.

Let X be a space of dimension $\aleph_1$, and let Y be a subspace of X with $\Gamma(Y) \neq \Gamma(X)$. Then Y is weakly ω_1-separable, hence $\Gamma(Y) = \Sigma(Y) \leq \Sigma(X)$ (by 3.3(b) and 3.2). Therefore the next theorem (which is the main result of this section) is best possible.

THEOREM 3.4. *Let X be a separated ω-filtered vector space of dimension $\aleph_1$, and let E be a subset of ω_1 such that $\Sigma(X) = \tilde{E}$. Then for every subset E' of E there exists a weakly ω_1-separable dense subspace Y of X of dimension $\aleph_1$ such that $\Gamma(Y) = \Sigma(Y) = \tilde{E}'$.*

COROLLARY 3.5. *If X is weakly ω_1-separable with $\Gamma(X) = \tilde{E}$, then for every subset E' of E there exists a dense subspace Y of X satisfying $\Gamma(Y) = E'$.* □

Remark. In [EMS] a result similar to 3.5 has been proved for strongly κ-free groups of regular uncountable cardinality κ.

COROLLARY 3.6. *Every separated ω-filtered vector space X of dimension $\aleph_1$ which fails to have SCC contains $2^{\aleph_1}$ mutually nonisomorphic dense subspaces which are weakly ω_1-separable and of codimension $\aleph_1$.*

Proof. This follows from 3.4, 2.5 and the fact that for each $e \in D(\omega_1) - \{0\}$ the interval $[0, e]$ has cardinality $2^{\aleph_1}$. □

Proof of Theorem 3.4: Let B be a basic subspace of X, let $X = \bigcup_{\nu<\omega_1} X_\nu$ be an ω_1-filtration, and let $B_\nu = B \cap X_\nu$ for all $\nu < \omega_1$. Since B is dense in X, there is for each $n<\omega$ a cub C_n such that for all $\nu \in C_n$, $B_\nu + (X^n \cap X_\nu) = X_\nu$. Now $C = \bigcap_{n<\omega} C_n$ is again a cub, hence we may assume that

(i) for all $\nu<\omega_1$, B_ν is dense in X_ν.

Furthermore, w.l.o.g. we may assume that E equals $\{\nu \in \operatorname{Lim}(\omega_1) \mid \bar{X}_\nu \neq \bigcup_{\mu<\nu} \bar{X}_\mu\}$ and that the following conditions hold:

(ii) for all $\nu \in E$, $(\bar{X}_\nu \cap X_{\nu+1}) - \bigcup_{\mu<\nu} \bar{X}_\mu \neq \varnothing$;

(iii) for all $\nu<\omega_1$, B_ν is a summand of B.

For each $\nu<\omega_1$, let $B_{\nu+1} = B_\nu \oplus S_\nu$.

Now if E' is a subset of E we define Y as the union of an ω_1-filtration $\{Y_\nu \mid \nu<\omega_1\}$ of subspaces of X such that

$$Y_{\nu+1}=\begin{cases} Y_\nu+S_\nu & \text{if} \quad \nu\notin E', \\ Y_\nu+S_\nu+Kx_\nu & \text{if} \quad \nu\in E', \end{cases}$$

where $x_\nu\in(\bar{X}_\nu\cap X_{\nu+1})-\bigcup_{\mu<\nu}\bar{X}_\mu$. By (ii) such an x_ν exists. Obviously, we have $B_\nu\subseteq Y_\nu\subseteq X_\nu$ for all $\nu<\omega_1$; thus by (i) Y_ν is dense in X_ν. If $\nu\in E'$ then Y_ν is not closed in Y because $x_\nu\in\bar{X}_\nu\cap Y=\bar{Y}_\nu^Y$ but $x_\nu\notin X_\nu$, hence $x_\nu\in\bar{Y}_\nu^Y-Y_\nu$.

It remains to show that Y_ν is closed in Y for all $\nu\in\omega_1-E'$. Given $\nu\notin E'$, we shall prove by induction on τ that Y_ν is closed in Y_τ for all $\tau>\nu$. There is no problem at limit stages. So suppose that $\tau=\mu+1$ where $\mu\geq\nu$. We wish to prove that $\bar{Y}_\nu\cap Y_{\mu+1}=\bar{Y}_\nu\cap Y_\mu$ and thus $\bar{Y}_\nu\cap Y_{\mu+1}=Y_\nu$ by induction hypothesis. So let $a\in\bar{Y}_\nu\cap Y_{\mu+1}$. Then we have

$$a=\begin{cases} y+s+\lambda x_\mu & \text{where} \quad y\in Y_\mu,\, s\in S_\mu,\, \lambda\in K \quad \text{if} \quad \mu\in E', \\ y+s & \text{where} \quad y\in Y_\mu,\, s\in S_\mu \quad \text{otherwise.} \end{cases}$$

If $\mu\in E'$ there is a σ, $\nu\leq\sigma<\mu$, such that $y\in Y_\sigma$, and thus $\lambda x_\mu=a-y-s\in\bar{X}_\sigma+S_\mu$. On the other hand, $\lambda x_\mu\in\bar{X}_\mu$ and hence $\lambda x_\mu\in\bar{X}_\mu\cap(\bar{X}_\sigma+S_\mu)$. Since B_μ is dense in X_μ, we have $\bar{X}_\mu\cap S_\mu=0$ by 2.9. Therefore $\lambda x_\mu\in\bar{X}_\sigma$ which implies $\lambda=0$, and thus in both cases $a=y+s$ for some $y\in Y_\mu$ and $s\in S_\mu$. But then $a\in\bar{X}_\mu\cap(Y_\mu+S_\mu)$, and the argument just used shows that indeed $a\in\bar{Y}_\nu\cap Y_\mu$.

Consequently, Y_ν is closed in Y for all $\nu\in\omega_1-E'$ and thus $\Gamma(Y)=\tilde{E}'$. Finally, since E' consists of limit ordinals, we infer that Y is weakly ω_1-separable, and hence $\Sigma(Y)=\Gamma(Y)$. □

As a further consequence, we obtain a characterization of spaces satisfying SCC.

COROLLARY 3.7. *For a separated ω-filtered vector space X of dimension $\aleph_1$ the following statements are equivalent*:

(a) *X satisfies* SCC;

(b) *every weakly ω_1-separable subspace of X is projective*;

(c) *every weakly ω_1-separable dense subspace of X is projective.*

Proof. This follows readily from 3.2, 3.3 and 3.4. □

On the other hand, assuming the Continuum Hypothesis (CH) we can construct an X which fails to have SCC yet does not have a *closed* subspace which is proper weakly ω_1-separable:

THEOREM 3.8. *Assume* CH. *Given a countable field* K, *there exists an* ω-*filtered* K-*vector space* X *of dimension* $\aleph_1$ *which fails to have* SCC, *but such that every closed subspace of* X *which is weakly* ω_1-*separable is discrete* (*hence projective*).

Proof. Let $B = \bigoplus_{n<\omega} B_n$ where for each n, B_n is homogeneous of value n, and $\dim(B_n) = \omega_1$. We construct X as a dense subspace of $\hat{B}$ as in the proof of 2.8, with the following changes.

Let E be a stationary set of limit ordinals $<\omega_1$. By CH the set of all countable subsets of $\hat{B}$ has cardinality ω_1. Fix an enumeration $\{Y_\nu \mid \nu \in \omega_1 - E\}$ of all nondiscrete countable (dimensional) subspaces of $\hat{B}$ such that each of them occurs ω_1 times.

Now define X_ν by induction on $\nu < \omega_1$ such that $B(\nu) \subseteq X_\nu \subseteq \hat{B}(\nu)$ (notation as in 2.8). The crucial cases are when $\nu \in E$, resp. when $\nu \in \omega_1 - E$ and $Y_\nu \subseteq X_\nu$. If $\nu \in E$ we let

$$X_{\nu+1} = X_\nu + S_\nu + K y_\nu$$

where S_ν is as in 2.8 and $y_\nu \in \hat{B}(\nu) - \bigcup_{\mu<\nu} \hat{B}(\mu)$. If $\nu \in \omega_1 - E$ and $Y_\nu \subseteq X_\nu$ we let

$$X_{\nu+1} = X_\nu + S_\nu + K z_\nu$$

where $z_\nu \in \ddot{Y}_\nu - Y_\nu$. Such a z_ν exists because every closed subspace of $\hat{B}$ is Π-homogeneous, hence Y_ν cannot be closed.

Finally, let $X = \bigcup_{\nu<\omega_1} X_\nu$. By construction we have $\ddot{X}_\nu^X \neq \bigcup_{\mu<\nu} \ddot{X}_\mu^X$ for each $\nu \in E$, hence $\Sigma(X) \geq \tilde{E} \neq 0$. On the other hand, for any nondiscrete countable subspace C of X there is a $\nu \in \omega_1 - E$ such that $C = Y_\nu \subseteq X_\nu$, hence C is not closed in X. We infer that X has the desired properties. $\square$

Theorem 3.4 and its corollaries have interesting applications to p-groups. The first of them is immediate from 3.7. (Following the terminology for spaces a (weakly) ω_1-separable p-group is said to be *proper* if it is not Σ-cyclic).

COROLLARY 3.9. *If* G *is a separable* p-*group of cardinality* $\aleph_1$ *whose socle satisfies* SCC, *then* G *does not contain any pure subgroup that is proper weakly* ω_1-*separable. Thus every pure subgroup* H *of* G *satisfies either* $\Gamma(H) = 0$ *or* $\Gamma(H) = 1$. $\square$

The next result follows from 3.4, 1.12 and 2.10.

COROLLARY 3.10. *Suppose that G is separable and of cardinality $\aleph_1$ such that $G[p]$ does not satisfy* SCC. *Then there is a stationary subset E of ω_1 such that for each $E' \subseteq E$ there is a pure dense subgroup H of G satisfying $\Gamma(H) = \tilde{E}'$. If G is weakly ω_1-separable, then E can be chosen so that $\Gamma(G) = \tilde{E}$.* □

As an immediate consequence of 3.9 and 3.10 we obtain the following

COROLLARY 3.11. *If a separable p-group G of cardinality $\aleph_1$ contains any pure subgroup which is proper weakly ω_1-separable, then G contains $2^{\aleph_1}$ pairwise nonisomorphic pure dense subgroups which are weakly ω_1-separable.* □

For the next application we need to recall some facts about ω-elongations. Given p-groups G and B, an *ω-elongation* of G by B is an exact sequence

$$(\varphi) \quad 0 \longrightarrow B \longrightarrow A \xrightarrow{\varphi} G \longrightarrow 0$$

such that $\mathrm{Ker}(\varphi) \subseteq p^{\omega}A$. Given an ω-elongation (φ), we define

$$P(\varphi) = \mathrm{Im}\,(\varphi_* : A[p] \to G[p]).$$

(If $G = A/p^{\omega}A$ and $\varphi : A \to A/p^{\omega}A$ is the natural map, we shall write $P(A)$ instead of $P(\varphi)$.) It is readily seen that $P(\varphi)$ is a dense subspace of $G[p]$. Conversely, to every dense subspace P of $G[p]$ there exists an ω-elongation (φ) of G by an elementary p-group B such that $P(\varphi) = P$. (Here a p-group B is called *elementary* if $pB = 0$; in this case B is a $\mathbb{Z}(p)$-vector space). More precisely, we have the following criterion which is essentially due to Richman [R].

CRITERION 3.12. *Let G and B be p-groups where B is elementary of dimension d and assume that ω-elongations of G by B exist. Then the map which associates to each ω-*elongation *(φ) of G by B the ω-filtered vector space $P(\varphi)$ gives a one-to-one correspondence between isomorphism classes of ω-elongations of G by B and equivalence classes of dense subspaces of $G[p]$ of codimension d.* □

Here two subspaces P, Q of $G[p]$ are called *equivalent* if there is an automorphism γ of G such that $\gamma(P) = Q$. Two ω-elongations $0 \to B \to A \xrightarrow{\varphi} G \to 0$ and $0 \to B \to A' \xrightarrow{\psi} G \to 0$ are *isomorphic* if there is an isomorphism $\alpha : A \xrightarrow{\sim} A'$ and an automorphism γ of G such that $\gamma \circ \varphi = \psi \circ \alpha$. By [N; Thm. 1.6] ω-elongations of G by B exist if and only if $\dim(B) \leq \mathrm{finrk}(G)$ where $\mathrm{finrk}(G) \overset{\mathrm{def}}{=} \inf\{\dim(p^n G[p]) \mid n < \omega\}$, the *final rank* of G.

The final result of this section strengthens Theorem 3.1 of [W] for a certain class of separable p-groups. Recall that, in contrast to this, if G is Σ-cyclic and B is any p-group, then every two ω-elongations of G by B are isomorphic. (This follows from [F1; Thm. 83.4].)

THEOREM 3.13. *Let G be a separable p-group of cardinality $\aleph_1$ containing a pure subgroup which is proper weakly ω_1-separable, and let B be an elementary p-group of dimension $\aleph_1$. Then there exist $2^{\aleph_1}$ mutually nonisomorphic ω-elongations of G by B.*

Proof. The hypothesis on G implies that $G[p]$ does not have SCC (by 3.9). Thus by 3.6 $G[p]$ contains $2^{\aleph_1}$ mutually non-isomorphic dense subspaces of codimension ω_1, and the theorem follows by 3.12. □

4. On $p^{\omega+1}$-injective p-groups

For any ordinal α, a reduced p-group A is called *p^α-injective* if $p^\alpha \operatorname{Ext}(C, A) = 0$ for all p-groups C. The group A is $p^{\omega+1}$-injective if and only if $p^\omega A \subseteq A[p]$ and $A/p^\omega A$ is torsion-complete (cf. [S; Thm. 54.5]). Such groups have been studied by Richman; as a consequence of the main theorem of [R] (cf. 3.12 above) he obtained:

CRITERION 4.1 [R; Corollary 1]. *Let A and B be $p^{\omega+1}$-injective p-groups. Then $A \cong B$ if and only if $P(A) \cong P(B)$ as ω-filtered vector spaces. Moreover, for any separated ω-filtered $\mathbb{Z}(p)$-vector space X there is a $p^{\omega+1}$-injective p-group A such that $P(A) \cong X$.* □

(The definition of $P(A)$ has been given before 3.12.) The following characterization of $p^{\omega+1}$-injectives is of quite some interest to us. This is essentially contained in [S; section 55]; for the sake of completeness we have chosen to include a proof. Let $\bar{G}$ denote the torsion completion of G, and let the $\mathbb{Z}(p)$-vector space $G[p^2]/G[p]$ always be equipped with the filtration from $G/G[p]$.

THEOREM 4.2. (a) *Let G be a separable p-group. Then $A = \bar{G}/G[p]$ is $p^{\omega+1}$-injective and $P(A) \cong G[p^2]/G[p]$ (as ω-filtered $\mathbb{Z}(p)$-vector spaces).*

(b) *If A is a $p^{\omega+1}$-injective p-group, then there exists a separable p-group G such that $A \cong \bar{G}/G[p]$.*

Proof. (a) If $A = \bar{G}/G[p]$ then clearly $p^\omega A = \bar{G}[p]/G[p]$. We infer that $A/p^\omega A \cong \bar{G}/\bar{G}[p] \cong p\bar{G}$, and the latter obviously is torsion-complete. Hence A is

$p^{\omega+1}$-injective. To verify that $P(A) \cong G[p^2]/G[p]$ (in $\mathscr{F}\mathscr{V}$) we consider the composite map $\psi: G/G[p] \hookrightarrow \bar{G}/G[p] \to \bar{G}/\bar{G}[p]$. Clearly ψ is injective and $\psi(G[p^2]/G[p]) \subseteq P(A)$. Furthermore, Im (ψ) is pure in $\bar{G}/\bar{G}[p]$, since G is pure in $\bar{G}$. So it remains to show that $P(A)$ is contained in the socle of Im (ψ). Let $x \in \bar{G}$ be such that $px \in G[p]$. Then by purity of G in $\bar{G}$ there is $y \in G$ such that $py = px$. It follows that $y \in G[p^2]$, $y - x \in \bar{G}[p]$, and hence $x + \bar{G}[p] = \psi(y + G[p])$.

(b) By Corollary 1.10 there is a p-group H such that $H[p] \cong P(A)$. Now if G is a p-group such that $pG \cong H$, then $P(A) \cong G[p^2]/G[p]$, and the result follows from 4.1 and 4.2(a). □

For a separable p-group G, we say that $A = \bar{G}/G[p]$ is the $p^{\omega+1}$-injective p-group *associated* with G. Clearly G is Σ-cyclic if and only if $P(A)$ is projective, and if G, H are Σ-cyclic and $B = \bar{H}/H[p]$, then $A \cong B$ if and only if $pG \cong pH$. Furthermore, we have the following.

PROPOSITION 4.3. *Let G be a separable p-group, and let A be its associated $p^{\omega+1}$-injective group. Then*

(a) *G is weakly ω_1-separable if and only if so is $P(A)$;*

(b) *If $|G| = \omega_1$, then $\Gamma(G) = \Gamma(P(A))$.*

Proof. Because of 4.1, 2.12 and 2.10 it suffices to show that $G[p]$ is weakly ω_1-separable if and only if so is $G[p^2]/G[p]$, resp. $\Gamma(G[p]) = \Gamma(G[p^2]/G[p])$. But this will follow from the subsequent lemma. □

Given any space $X \in \mathscr{F}\mathscr{V}$, for each $n < \omega$ define spaces $X^{(-n)}$ and $X^{(n)}$ in $\mathscr{F}\mathscr{V}$ by

$$(X^{(-n)})^k = X^{n+k}, \qquad (X^{(n)})^k = X^{\max\{0,k-n\}}, \qquad k < \omega.$$

Note that if $X = G[p]$ then $X^{(-1)} \cong G[p^2]/G[p]$.

LEMMA 4.4. *For all $n < \omega$ we have*

(a) *X is weakly ω_1-separable if and only if so is $X^{(-n)}$;*

(b) *If $\dim(X) = \omega_1$ then $\Gamma(X) = \Gamma(X^{(-n)})$.*

Proof. (a) The "only if" part is straightforward to prove. Suppose now that $X^{(-n)}$ is weakly ω_1-separable, and let Y be a countable dimensional subspace of X. Let $Y = Y_n \oplus Y^n$, and let $X = X_n \oplus X^n$ such that $Y_n \subseteq X_n$. Since $Y_n \cap X^n = 0$, such an X_n exists, and Y_n is a direct summand of X_n because Y_n is discrete. By hypothesis, $Y^{(-n)}$ is contained in a closed countable dimensional subspace W of $X^{(-n)}$. Then we have $Y^n \subseteq W^{(n)} \subseteq X^n$ and $W^{(n)}$ is closed in X^n. Therefore, $Y_n \oplus W^{(n)}$ is a closed countable dimensional subspace of X containing Y.

(b) Let $X = X_n \oplus X^n$. Choose ω_1-filtrations $X_n = \bigcup_{\nu<\omega_1} Y_\nu$ and $X^n = \bigcup_{\nu<\omega_1} Z_\nu$. Then $X_\nu = Y_\nu \oplus Z_\nu$ defines an ω_1-filtration of X. But by definition X_ν is closed in X if and only if Z_ν is closed in X^n if and only if $(X_\nu)^{(-n)}$ is closed in $X^{(-n)}$. Hence $\Gamma(X) = \Gamma(X^{(-n)})$, as desired. □

Remarks. (1) Similarly one proves that X is ω_1-separable if and only if $X^{(-n)}$ is ω_1-separable.

(2) Note that if G is ω_1-separable and A is its associated $p^{\omega+1}$-injective, then every countable subset of A is contained in a direct summand with countable basic subgroup.

COROLLARY 4.5. *A p-group G is weakly ω_1-separable if and only if so is p^nG for some n.*

Proof. This is immediate from 4.4(a) and 2.12. □

For ω_1-separable p-groups the corresponding result is contained in [Cu]. We conclude this section with an abundance result which is a consequence of 2.8, 4.1, 4.2 and 4.3.

COROLLARY 4.6. *For any stationary subset E of ω_1 there exist $2^{\aleph_1}$ weakly ω_1-separable p-groups G_i $(i<2^{\omega_1})$ of cardinality $\aleph_1$ such that their associated $p^{\omega+1}$-injective groups $A_i = \bar{G}_i/G_i[p]$ are mutually nonisomorphic yet for all i, $\Gamma(G_i) = \Gamma(P(A_i)) = \tilde{E}$.* □

5. Projective resolutions and socles of $p^{\omega+1}$-projective p-groups

We begin this section by considering the following question: Given $X \in \mathscr{F}\mathscr{V}$, for which projective spaces K does there exist a projective resolution

$$0 \to K \to P \to X \to 0$$

in $\mathscr{F}\mathscr{V}$? The result of this investigation (Theorem 5.3) will be applied to characterize the socles of $p^{\omega+1}$-projective p-groups.

PROPOSITION 5.1. *Let X be an ω-filtered vector space of countable dimension, and let S be a coinfinite subset of ω. Then there is a projective resolution*

$$0 \to K \to P \to X \to 0$$

such that for all $n \in \omega$,

$$f_n(K) = \begin{cases} 0 & \text{if } n \in S, \\ 1 & \text{otherwise.} \end{cases}$$

(For the definition of $f_n(K)$ see Section 1.)

Proof. Since dim (X) is countable, we have $X = X^\infty \oplus Q$ where Q is projective. Thus we may assume that $X = X^\infty$; say $X = \bigoplus_{i \in I} Kx_i$, where $|I| \leq \omega$. Let $\bar{S} = \omega - S$, and decompose $\bar{S} = \coprod_{k<\rho} S_k$ (disjoint union) such that each S_k is infinite and $\rho = |I|$. Define the projective space P by $P^m = \bigoplus_{\substack{n \in \bar{S} \\ n \geq m}} Ka_n$, and let $\pi : P \to X$ be given by $\pi(a_n) = x_k$, where k is the unique number such that $n \in S_k$. Clearly, π is a cokernel, and if $K = \text{Ker}(\pi)$ we obtain a projective resolution

$$0 \to K \to P \xrightarrow{\pi} X \to 0.$$

Clearly, for $n \in \omega$,

$$f_n(P) = \begin{cases} 0 & \text{if } n \in S \\ 1 & \text{otherwise.} \end{cases}$$

Since $f_n(P) = f_n(K) + f_n(X)$, and $f_n(X) = 0$, the result follows. $\square$

COROLLARY 5.2. *If X is a countable dimensional ω-filtered vector space and K is any nondiscrete projective space, then there exists a projective resolution*

$$0 \to K \to P \to X \to 0.$$

Proof. Since K is nondiscrete, we may write $K = K_0 \oplus K_1$ such that, for some coinfinite subset S of ω,

$$f_n(K_0) = \begin{cases} 0 & \text{if } n \in S; \\ 1 & \text{otherwise.} \end{cases}$$

Now the result follows from 5.1. $\square$

THEOREM 5.3. *Let X be an ω-filtered vector space of dimension κ. Then for any projective ω-filtered vector space K with* fin dim $(K) \geq \kappa$ *there exists a projective resolution*

$$0 \to K \to P \to X \to 0.$$

Proof. The proof is by induction on κ. For $\kappa \leq \omega$ this is precisely Corollary 5.2. Now suppose that κ is uncountable. In this case, it suffices to consider spaces K such that for some coinfinite subset S of ω,

$$f_n(K) = \begin{cases} 0 & \text{if } n \in S; \\ \kappa & \text{otherwise.} \end{cases}$$

We represent X as the union $\bigcup_{\nu<\kappa} X_\nu$ of a smooth chain of subspaces such that $\dim(X_\nu) < \kappa$. For each $\nu < \kappa$ we choose a projective resolution

$$0 \longrightarrow K_\nu \longrightarrow P_\nu \xrightarrow{\varphi_\nu} X_{\nu+1}/X_\nu \longrightarrow 0$$

such that

$$f_n(K_\nu) = \begin{cases} 0 & \text{if } n \in S; \\ \dim(X_{\nu+1}/X_\nu) & \text{otherwise.} \end{cases}$$

By induction hypothesis such a resolution exists. Now by induction on ν we define projective resolutions

$$P(\nu): 0 \longrightarrow \bigoplus_{\mu<\nu} K_\mu \longrightarrow \bigoplus_{\mu<\nu} P_\mu \xrightarrow{\Psi_\nu} X_\nu \longrightarrow 0$$

such that for all $\rho < \nu$ the diagram

$$D(\rho, \nu): \quad \begin{tikzcd} 0 \arrow[r] & \bigoplus_{\mu<\rho} K_\mu \arrow[r] \arrow[d, hook] & \bigoplus_{\mu<\rho} P_\mu \arrow[r, "\Psi_\rho"] \arrow[d, hook] & X_\rho \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \bigoplus_{\mu<\nu} K_\mu \arrow[r] & \bigoplus_{\mu<\nu} P_\mu \arrow[r, "\Psi_\nu"] & X_\nu \arrow[r] & 0 \end{tikzcd}$$

commutes. Suppose that $P(\mu)$ has been defined for all $\mu < \nu$. If ν is a limit ordinal, we take unions. If ν is a successor, say $\nu = \rho + 1$, we shall construct $P(\nu)$ such that the diagram $D(\rho, \nu)$ commutes. Then by induction hypothesis $D(\mu, \nu)$ will commute for all $\mu < \nu$.

Since P_ρ is projective, there exists a map $\theta: P_\rho \to X_{\rho+1}$ making the diagram

$$\begin{tikzcd} & P_\rho \arrow[dl, "\theta"'] \arrow[d, "\varphi_\rho"] \\ X_{\rho+1} \arrow[r] & X_{\rho+1}/X_\rho \end{tikzcd}$$

commute. Let $\tilde{P}_\rho = \bigoplus_{\mu<\rho} P_\rho$, and let $\chi = \iota \circ \Psi_\rho : \tilde{P}_\rho \to X_{\rho+1}$, where $\iota : X_\rho \to X_{\rho+1}$ denotes inclusion. Then there is a unique morphism $\Psi_{\rho+1} : \tilde{P}_\rho \oplus P_\rho \to X_{\rho+1}$ satisfying $\Psi_{\rho+1} \restriction \tilde{P}_\rho = \chi$ and $\Psi_{\rho+1} \restriction P_\rho = \theta$; an easy computation shows that $\Psi_{\rho+1}$ is a cokernel. Now let $N = \operatorname{Ker}(\Psi_{\rho+1})$. It is easy to see that the exact sequence of vector spaces

$$0 \to \bigoplus_{\mu<\rho} K_\mu \to N \to K_\rho \to 0$$

is in fact exact in $\mathcal{FV}$. Therefore $N \cong \bigoplus_{\mu\le\rho} K_\mu$, and the construction of $P(\nu)$ is accomplished.

Finally, taking limits, we obtain a projective resolution

$$0 \to \bigoplus_{\mu<\kappa} K_\mu \to \bigoplus_{\mu<\kappa} P_\mu \to X \to 0$$

and by construction we have $K \cong \bigoplus_{\mu<\kappa} K_\mu$. □

We are now going to apply Theorem 5.3 to p-groups. For any ordinal α, a p-group G is called *p^α-projective* if for all p-groups A, $p^\alpha \operatorname{Ext}(G, A) = 0$. The group G is $p^{\omega+1}$-projective if and only if $G[p]$ contains a subgroup K such that G/K is Σ-cyclic if and only if G is the quotient of a Σ-cyclic p-group modulo an elementary p-group (cf. e.g., [S; Thm. 38.1]). Fuchs and Irwin have shown that $p^{\omega+1}$-projective p-groups are determined by their socles regarded as valuated vector spaces [FI; Thm. 3]. (Notice that since such groups have length $\le\omega+1$, we may as well regard their socles as objects in $\mathcal{FV}$.)

THEOREM 5.4. *Suppose that X is an ω-filtered $\mathbb{Z}(p)$-vector space such that $X = Y \oplus P$, where P is projective and* $\operatorname{fin\,dim}(X) = \operatorname{fin\,dim}(P)$. *Then there is a unique $p^{\omega+1}$-projective p-group G such that $G[p] \cong X$ (in $\mathcal{FV}$).*

Proof. Let $K = P^{(1)}$ (cf. Section 4). Clearly $\operatorname{fin\,dim}(K) = \operatorname{fin\,dim}(P)$. Therefore by 5.3 there exists a projective resolution

$$0 \to K \to Q \to Y \to 0.$$

At this point we could appeal to [FI; Thm. 4], but instead we give a self-contained construction. Now let F be the Σ-cyclic p-group determined by $F[p] = Q$ and let $G = F/K$ which is $p^{\omega+1}$-projective. Since K is a projective subsocle of F, it supports a pure (Σ-cyclic) subgroup U of F. Let $i : U \hookrightarrow F$ denote inclusion, let

$\pi: U \to U/K$ be the natural map, and define

$$\Delta : U \to F \oplus U/K$$

to be the unique map with components i and $-\pi$. It is readily seen that $\Delta(U)$ is pure in $F \oplus U/K$ and that $\operatorname{Coker}(\Delta) \cong G$. Hence we have obtained a pure-projective resolution

$$0 \to U \xrightarrow{\Delta} F \oplus U/K \to G \to 0.$$

Passing to the socles, we infer from 1.8 that $G[p] \cong (F/U)[p] \oplus (U/K)[p]$ (in $\mathcal{FV}$). But $(F/U)[p] \cong Y$ and $(U/K)[p] \cong K^{(-1)} \cong P$. Hence $G[p] \cong Y \oplus P = X$, as desired. Finally, uniqueness of G follows from Theorem 3 of [FI] quoted above. □

A p-group G is called *C-decomposable* if $G = H \oplus C$ where C is Σ-cyclic and has the same final rank as G. Analogously, we call an ω-filtered vector space X *C-decomposable* if $X = Y \oplus P$ where P is projective and $\operatorname{fin\,dim}(P) = \operatorname{fin\,dim}(X)$. Clearly, if G is C-decomposable, then so is $G[p]$. It follows from [FI; p. 466] that every $p^{\omega+1}$-projective p-group is C-decomposable. Consequently, together with 5.4 we obtain the following.

COROLLARY 5.5. *Let X be an ω-filtered $\mathbb{Z}(p)$-vector space. Then X is the socle of a $p^{\omega+1}$-projective p-group if and only if X is C-decomposable.* □

Remark. Theorem 5.4 is implicit in [CuM]; in there, a different means of proof is employed.

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Dept. of Mathematics
University of California
Irvine, Calif. 92717/USA

Mathematisches Institut
der Universität
Alberstr. 23b
D-7800 Freiburg i.Br./BRD

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