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Commutators of diffeomorphisms, III: a group which is not perfect

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The group of C^r diffeomorphisms of the real line with compact support is perfect if $r \neq 2$ (cf. [1–3]). It is unknown whether this is the case if $r = 2$ (cf. [4]).

In this note, we will give a very simple proof that the group G of compactly supported C^1 diffeomorphisms of the real line whose first derivative has bounded variation is not perfect. For $f \in G$, $\log Df$ is a compactly supported function of bounded variation. Let $D \log Df$ denote the derivative of $\log Df$ in the sense of the theory of distributions. It is well known that $D \log Df$ is a compactly supported Radon measure. In other words, if we think of $D \log Df$ as a linear functional on the space of C^∞ functions on $\mathbb{R}$, then $D \log Df$ has a unique linear continuous extension to the space of continuous functions on $\mathbb{R}$, where we provide this last space with the C^0 topology.

A self homeomorphism f of $\mathbb{R}$ induces automorphism f^* of the continuous functions on $\mathbb{R}$, defined by $f^*u = u \circ f$. The dual of f^* is an automorphism f_* of the space of compactly supported Radon measures on $\mathbb{R}$. Another way of describing f_* is to observe that if X is a Borel subset of $\mathbb{R}$, then $(f_*\mu)(X) = \mu(f^{-1}X)$, for any Radon measure μ. If u is a compactly supported function of bounded variation, then

$$f_*^{-1}Du = D(u \circ f).$$

An easy way to see this is to use the fact that $Du(I_{a,b}) = u(b-0) - u(a+0)$, where $I_{a,b}$ is the open interval (a, b), and this uniquely specifies Du as a Radon measure. It follows that if $f, g \in G$, then

$$D \log D(f \circ g) = g_*^{-1} D \log Df + D \log Dg.$$

Any Radon measure μ uniquely decomposes as a sum $\mu = \mu_{\text{reg}} + \mu_{\text{sing}}$ where the regular part μ_{reg} vanishes on all Borel sets of zero Lebesque measure and the singular part μ_{sing} has support in some Borel set X of zero Lebesque measure, in the sense that $\mu_{\text{sing}}(Y) = \mu_{\text{sing}}(Y \cap X)$ for all Borel sets Y. Note that if $g \in G$, then g is Lipschitz, so it preserves the decomposition of a Radon measure into its

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regular and singular parts, i.e.

$$(g_*\mu)_{\mathrm{reg}} = g_*(\mu_{\mathrm{reg}}),\ (g_*\mu)_{\mathrm{sing}} = g_*(\mu_{\mathrm{sing}}).$$

We let $\int \mu$ denote the total mass of μ, i.e. $\mu(\mathbb{R})$. For $f \in G$, we define

$$\pi(f) = \int (D \log Df)_{\mathrm{reg}}.$$

Since $\int g_*\mu = \int \mu$, we have

$$\pi(fg) = \int (g_*^{-1} D \log Df + D \log Dg)_{\mathrm{reg}} = \pi(f) + \pi(g).$$

In other words $\pi : G \to \mathbb{R}$ is a homomorphism.

In fact, the homomorphism π is surjective. This is easy to prove: Consider $a \in \mathbb{R}$ and construct a compactly supported real valued function u of a real variable such that $u(0) = a$, $u(1) = 0$, u is C^1 outside of $[0, 1]$, and $u \mid [0, 1]$ is a monotone function whose derivative (as a Radon measure) is totally singular with respect to Lebesque measure. Such a function may be constructed, for example, by letting u be the primitive of an appropriate totally singular measure in $[0, 1]$ and extending u to be C^1 outside of $[0, 1]$. We further require that $\int_{-\infty}^{\infty} (e^{u(t)} - 1)\, dt = 0$. This may be arranged by altering u (if necessary) outside the interval $[0, 1]$. Let f be the primitive of e^u which is the identity near $-\infty$. Then $f \in G$ and

$$\pi(f) = \int_{-\infty}^{0} Du + \int_{1}^{\infty} Du = a,$$

since $(Du)_{\mathrm{reg}} \mid [0, 1] = 0$ and $Du_{\mathrm{reg}} = Du$, elsewhere. We have proved:

THEOREM. *$\pi : G \to \mathbb{R}$ is a surjective homomorphism.*

More generally, let R be a family of Borel subsets of $\mathbb{R}$ which is G-invariant, closed under countable unions, and satisfies the condition that if $X \in R$ and Y is a Borel subset of X, then $Y \in R$. Each compactly supported Radon measure μ has a unique decomposition

$$\mu = \mu_{R\text{-reg}} + \mu_{R\text{-sing}}$$

where the R-regular part of μ vanishes on all members of R and the R-singular part has support in a member of R.

For $f \in G$, let

$$\pi_R(f) = \int D(\log Df)_{R\text{-reg}}.$$

The same argument as before shows that π_R is a homomorphism.

We may obtain many different homomorphisms of G onto $\mathbb{R}$ this way, for example, by taking R to be the set of subsets of Hausdorff dimension $\leqslant\alpha$, for $0\leqslant\alpha<1$, or of vanishing α-dimensional Hausdorff measure, for $0\leqslant\alpha\leqslant 1$, or of Hausdorff dimension $<\alpha$, for $0<\alpha\leqslant 1$, or the family of subsets which for any $\varepsilon>0$ can be covered by a countable family of intervals whose lengths satisfy $\sum_{i=1}^{\infty}-(\log l_i)^{-1}<\varepsilon$, etc.

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