

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 60 (1985)

Artikel: Commutators of diffeomorphisms, III: a group which is not perfect.
Autor: Mather, John N.
DOI: <https://doi.org/10.5169/seals-46303>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 13.10.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Commutators of diffeomorphisms, III: a group which is not perfectJOHN N. MATHER⁽¹⁾

The group of C^r diffeomorphisms of the real line with compact support is perfect if $r \neq 2$ (cf. [1–3]). It is unknown whether this is the case if $r = 2$ (cf. [4]).

In this note, we will give a very simple proof that the group G of compactly supported C^1 diffeomorphisms of the real line whose first derivative has bounded variation is not perfect. For $f \in G$, $\log Df$ is a compactly supported function of bounded variation. Let $D \log Df$ denote the derivative of $\log Df$ in the sense of the theory of distributions. It is well known that $D \log Df$ is a compactly supported Radon measure. In other words, if we think of $D \log Df$ as a linear functional on the space of C^∞ functions on \mathbb{R} , then $D \log Df$ has a unique linear continuous extension to the space of continuous functions on \mathbb{R} , where we provide this last space with the C^0 topology.

A self homeomorphism f of \mathbb{R} induces automorphism f^* of the continuous functions on \mathbb{R} , defined by $f^*u = u \circ f$. The dual of f^* is an automorphism f_* of the space of compactly supported Radon measures on \mathbb{R} . Another way of describing f_* is to observe that if X is a Borel subset of \mathbb{R} , then $(f_*\mu)(X) = \mu(f^{-1}X)$, for any Radon measure μ . If u is a compactly supported function of bounded variation, then

$$f_*^{-1}Du = D(u \circ f).$$

An easy way to see this is to use the fact that $Du(I_{a,b}) = u(b-0) - u(a+0)$, where $I_{a,b}$ is the open interval (a, b) , and this uniquely specifies Du as a Radon measure. It follows that if $f, g \in G$, then

$$D \log D(f \circ g) = g_*^{-1}D \log Df + D \log Dg.$$

Any Radon measure μ uniquely decomposes as a sum $\mu = \mu_{\text{reg}} + \mu_{\text{sing}}$ where the regular part μ_{reg} vanishes on all Borel sets of zero Lebesgue measure and the singular part μ_{sing} has support in some Borel set X of zero Lebesgue measure, in the sense that $\mu_{\text{sing}}(Y) = \mu_{\text{sing}}(Y \cap X)$ for all Borel sets Y . Note that if $g \in G$, then g is Lipschitz, so it preserves the decomposition of a Radon measure into its

¹ Partially supported by an NSF grant.

regular and singular parts, i.e.

$$(g_*\mu)_{\text{reg}} = g_*(\mu_{\text{reg}}), (g_*\mu)_{\text{sing}} = g_*(\mu_{\text{sing}}).$$

We let $\int \mu$ denote the total mass of μ , i.e. $\mu(\mathbb{R})$. For $f \in G$, we define

$$\pi(f) = \int (D \log Df)_{\text{reg}}.$$

Since $\int g_*\mu = \int \mu$, we have

$$\pi(fg) = \int (g_*^{-1} D \log Df + D \log Dg)_{\text{reg}} = \pi(f) + \pi(g).$$

In other words $\pi : G \rightarrow \mathbb{R}$ is a homomorphism.

In fact, the homomorphism π is surjective. This is easy to prove: Consider $a \in \mathbb{R}$ and construct a compactly supported real valued function u of a real variable such that $u(0) = a$, $u(1) = 0$, u is C^1 outside of $[0, 1]$, and $u|_{[0, 1]}$ is a monotone function whose derivative (as a Radon measure) is totally singular with respect to Lebesgue measure. Such a function may be constructed, for example, by letting u be the primitive of an appropriate totally singular measure in $[0, 1]$ and extending u to be C^1 outside of $[0, 1]$. We further require that $\int_{-\infty}^{\infty} (e^{u(t)} - 1) dt = 0$. This may be arranged by altering u (if necessary) outside the interval $[0, 1]$. Let f be the primitive of e^u which is the identity near $-\infty$. Then $f \in G$ and

$$\pi(f) = \int_{-\infty}^0 Du + \int_1^{\infty} Du = a,$$

since $(Du)_{\text{reg}}|_{[0, 1]} = 0$ and $Du_{\text{reg}} = Du$, elsewhere. We have proved:

THEOREM. $\pi : G \rightarrow \mathbb{R}$ is a surjective homomorphism.

More generally, let R be a family of Borel subsets of \mathbb{R} which is G -invariant, closed under countable unions, and satisfies the condition that if $X \in R$ and Y is a Borel subset of X , then $Y \in R$. Each compactly supported Radon measure μ has a unique decomposition

$$\mu = \mu_{R\text{-reg}} + \mu_{R\text{-sing}}$$

where the R -regular part of μ vanishes on all members of R and the R -singular part has support in a member of R .

For $f \in G$, let

$$\pi_R(f) = \int D(\log Df)_{R\text{-reg}}.$$

The same argument as before shows that π_R is a homomorphism.

We may obtain many different homomorphisms of G onto \mathbb{R} this way, for example, by taking R to be the set of subsets of Hausdorff dimension $\leq \alpha$, for $0 \leq \alpha < 1$, or of vanishing α -dimensional Hausdorff measure, for $0 \leq \alpha \leq 1$, or of Hausdorff dimension $< \alpha$, for $0 < \alpha \leq 1$, or the family of subsets which for any $\varepsilon > 0$ can be covered by a countable family of intervals whose lengths satisfy $\sum_{i=1}^{\infty} -(\log l_i)^{-1} < \varepsilon$, etc.

Acknowledgement

I wrote this note while visiting ETH, Zürich, (Forschungsinstitut für Mathematik). I would like to thank ETH for its hospitality.

REFERENCES

- [1] D. B. A. EPSTEIN, *Commutators of C^∞ Diffeomorphisms*, Comment. Math. Helv. 59 (1984), 111–122.
- [2] J. N. MATHER, *Commutators of Diffeomorphisms*, Comment. Math. Helv. 49 (1974), 512–528, MR 50 No. 8600.
- [3] J. N. MATHER, *Commutators of Diffeomorphisms, II*, Comment. Math. Helv. 50 (1975), 33–40, MR 51, No. 11576.
- [4] J. N. MATHER, *A Curious Remark Concerning the Geometric Transfer Map*, Comment. Math. Helv. 59 (1984), 86–110.

Princeton University
Princeton N1 08544
USA

Received July 9, 1984