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## On the Kneser–Tits problem

GOPAL PRASAD and M. S. RAGHUNATHAN

### Introduction

Let  $G$  be a semi-simple, simply connected algebraic group defined, isotropic and simple over a (commutative) field  $k$ . Let  $G(k)$  be the group of  $k$ -rational points of  $G$  and  $G(k)^+$  be the normal subgroup of  $G(k)$  generated by the  $k$ -rational points of the unipotent radicals of parabolic  $k$ -subgroups of  $G$ . The *Kneser–Tits problem* referred to in the title is the following: *Is  $G(k)^+ = G(k)$  for every  $G$  as above? The main object of this paper is to prove that for a field  $k$ , the Kneser–Tits problem has an affirmative solution iff  $G(k)^+ = G(k)$  for all simply connected,  $k$ -simple groups  $G$  of  $k$ -rank 1. This reduction of the Kneser–Tits problem is an immediate consequence of Theorem A proved below. After this work was complete, we learnt from Armand Borel that Theorem A was conjectured by Jacques Tits in a lecture at the Institute for Advanced Study (Princeton), and was proved by him for some fields by a method different from ours.*

The proof of Theorem A depends on a theorem on Galois cohomology (Theorem B) which may be of some independent interest.

In case  $k$  is a local field, the Kneser–Tits problem has an affirmative solution. This was essentially proved by V. P. Platonov [4] using the known results on classical groups and detailed knowledge of classification. He also gave the first examples of fields for which the Kneser–Tits problem has a negative answer (see Tits [8] for a survey). In §2 of this paper we use the reduction of the Kneser–Tits problem to rank 1 groups stated above to provide a simple proof of its affirmative solution for the local fields. This simple proof devised by the first-named author was the starting point of the present work. We hope to come back to the problem for global fields in the near future.

**1.1.** Let  $k$  be a (commutative) field,  $\mathcal{K}$  be a fixed separable closure of  $k$  and let  $\Gamma = \text{Gal}(\mathcal{K}/k)$ . Let  $G$  be a semi-simple, simply connected group defined over  $k$ . Let  $S$  be a maximal  $k$ -split torus of  $G$ . Let  $\dim S = r$  ( $:= k$ -rank  $G$ ). We assume that  $r > 0$  i.e.,  $G$  is isotropic over  $k$ ; we also assume that  $G$  is  $k$ -simple, i.e., it has no proper *connected* normal subgroup defined over  $k$ .

**1.2.** Let  $T$  be a maximal torus of  $G$  containing  $S$  and defined over  $k$ . Let  $\Phi$  be the set of roots of  $G$  relative to  $T$ . We fix a Borel subgroup  $B$  defined over  $\mathcal{K}$ ,  $B \supset T$ , and contained in a minimal parabolic  $k$ -subgroup of  $G$ . This induces an ordering on  $\Phi$ ; let  $\Delta$  be the set of all simple roots with respect to this ordering. Let  $\Delta_0$  be the subset of  $\Delta$  consisting of those roots which are trivial on  $S$ . There is an action of  $\Gamma$  on  $\Delta$  (the  $*$ -action) defined in Tits [7: §2.3]; both  $\Delta_0$  and  $\Delta - \Delta_0$  are stable under this action. Since  $k$ -rank  $G = r$ , there are  $r$   $\Gamma$ -orbits in  $\Delta - \Delta_0$ .

**1.3.** For a simple root  $a$ , let  $U_a$  and  $U_{-a}$  be the root subgroups associated with  $a$  and  $-a$  respectively;  $U_a$  and  $U_{-a}$  are connected unipotent  $\mathcal{K}$ -subgroups of  $G$ , of dimension 1, normalized by  $T$ . Since  $G$  is simply connected,  $\forall a \in \Delta$ , the subgroup generated by  $U_a$  and  $U_{-a}$  is  $\mathcal{K}$ -isomorphic to  $SL_2$ ; let  $T_a$  be its intersection with  $T$ , then  $T_a$  is a one dimensional torus defined over  $\mathcal{K}$ , and as  $G$  is simply connected,  $T$  is a direct product of the  $T_a$  ( $a \in \Delta$ ). For a subset  $\Theta$  of  $\Delta$ , let  $T_\Theta$  be the subtorus generated by the tori  $T_a$ ,  $a \in \Theta$ .

**1.4.** For a  $k$ -subgroup  $H$  of  $G$ , as usual,  $H(k)$  will denote the group of  $k$ -rational points of  $H$ , and  $H(k)^+$  will denote the normal subgroup of  $H(k)$  generated by the  $k$ -rational points of the unipotent radicals of the parabolic  $k$ -subgroups of  $H$ .

**1.5.** For a  $\Gamma$ -stable subset  $\Theta$  of  $\Delta - \Delta_0$ , let  $T^\Theta$  be the identity component of  $\bigcap_{\theta \in \Theta \cup \Delta_0} \text{Ker } \theta$ . Let  $\mathcal{M}_\Theta$  be the centralizer of  $T^\Theta$  in  $G$ . Then  $\mathcal{M}_\Theta$  is a connected reductive subgroup defined over  $k$ ; in fact it is a Levi  $k$ -subgroup of a parabolic  $k$ -subgroup of  $G$  (cf. Tits [7: §2.5.4]). Let  $\mathcal{G}_\Theta$  be the derived subgroup of  $\mathcal{M}_\Theta$ . Then  $\mathcal{G}_\Theta$  is a semi-simple, simply connected,  $k$ -subgroup of  $G$ , and hence it is a direct product of its connected  $k$ -simple normal subgroups. Let  $A_\Theta$  be the product of all connected  $k$ -simple normal subgroups of  $\mathcal{G}_\Theta$  which are anisotropic over  $k$ , and  $G_\Theta$  be the product of all connected  $k$ -simple  $k$ -isotropic subgroups. Then the  $k$ -rank of  $G_\Theta$  is equal to the number of  $\Gamma$ -orbits in  $\Theta$ , and  $\mathcal{G}_\Theta$  is a direct product (over  $k$ ) of  $A_\Theta$  and  $G_\Theta$ . It is easily seen that  $\mathcal{M}_\Theta$  is a semi-direct product of  $T_{\Theta'}$  and  $\mathcal{G}_\Theta$ ; where  $\Theta'$  is the complement of  $\Theta$  in  $\Delta - \Delta_0$ . Hence, the natural homomorphism:  $\mathcal{M}_\Theta(\mathcal{K}) \rightarrow (\mathcal{M}_\Theta/\mathcal{G}_\Theta)(\mathcal{K})$  is surjective.

We shall denote the centralizer of  $S$  in  $G$  by  $\mathcal{M}$  and sometimes also by  $M$ . Let  $\mathcal{G}$  be the derived group of  $\mathcal{M}$ . Then  $\mathcal{M} = \mathcal{M}_\emptyset$ ;  $\mathcal{G} = \mathcal{G}_\emptyset$  (where  $\emptyset$  is the empty subset of  $\Delta - \Delta_0$ ).  $\mathcal{G}$  is anisotropic over  $k$ , and it is easy to see that  $A_\Theta$  is a normal subgroup of  $\mathcal{G}$  for every  $\Gamma$ -stable subset  $\Theta$  of  $\Delta - \Delta_0$ .

For a  $\Gamma$ -stable subset  $\Theta$  of  $\Delta - \Delta_0$ , let  $S_\Theta$  be the maximal  $k$ -split torus of  $G_\Theta$  contained in  $S$ , and let  $M_\Theta$  denote the centralizer of  $S_\Theta$  in  $G_\Theta$ . Then  $M_\Theta$  is a connected reductive  $k$ -subgroup. Moreover, since  $\mathcal{G}_\Theta$  is a direct product of  $G_\Theta$

and  $A_{\theta}$ , the centralizer of  $S_{\theta}$  in  $\mathcal{G}_{\theta}$  is just  $A_{\theta} \cdot M_{\theta}$  (direct product). It is easy to see, by considering the reductive groups  $S \cdot G_{\theta}$  and  $S \cdot \mathcal{G}_{\theta}$ , that  $M_{\theta} = M \cap G_{\theta}$  and  $\mathcal{M} \cap \mathcal{G}_{\theta} = A_{\theta} \cdot M_{\theta}$ .

**1.6.** Let  $\Theta_i, i = 1, \dots, r$ , be the  $\Gamma$ -orbits in  $\Delta - \Delta_0$ . Recall that  $G_{\theta_i}$  is a semi-simple simply connected  $k$ -subgroup of  $G$  of  $k$ -rank 1; it is  $k$ -simple since it does not contain any connected normal  $k$ -anisotropic subgroup. It follows from the Bruhat-decomposition that  $G(k) = M(k) \cdot G(k)^+$ . Thus  $G(k)^+ = G(k)$  if and only if  $G(k)^+ \supset M(k)$ . Similarly as  $G_{\theta}(k) = M_{\theta}(k) \cdot G_{\theta}(k)^+$ ,  $G_{\theta}(k)^+ = G_{\theta}(k)$  if and only if  $G_{\theta}(k)^+ \supset M_{\theta}(k)$ . In view of these observations, the following Theorem A implies that the Kneser–Tits problem for a field  $k$  has an affirmative solution if and only if for every  $k$ -simple simply connected group  $G$  of  $k$ -rank 1,  $G(k)^+ = G(k)$ .

**THEOREM A.** Assume that  $k$ -rank  $G \geq 2$ . Then  $M(k)$  is generated by the subgroups  $M_{\theta_i}(k)$  ( $1 \leq i \leq r$ ).

**1.7. Remark.** If  $k$  is an infinite field, then  $G(k)^+$  has no proper non-central normal subgroups (Tits [6: Main Theorem]), in particular it is *perfect* i.e.  $(G(k)^+, G(k)^+) = G(k)^+$ . Now Theorem A implies that to prove that  $G(k)$  is perfect for all  $k$ -simple, simply connected  $k$ -isotropic  $G$ , it suffices to prove that this is so for all  $k$ -simple, simply connected groups of  $k$ -rank 1.

We shall prove Theorem A using the following:

**THEOREM B.** For  $i \leq n$ , let  $\Delta_i$  be a  $\Gamma$  ( $= \text{Gal}(\mathcal{K}/k)$ )-stable subset of  $\Delta - \Delta_0$  such that  $\bigcap_{i=1}^n \Delta_i = \emptyset$ . Then the natural morphism:

$$H^1(k, \mathcal{G}) \rightarrow \prod_{i=1}^n H^1(k, \mathcal{G}_{\Delta_i}),$$

induced by the inclusion of  $\mathcal{G}$  in  $\mathcal{G}_{\Delta_i}$  ( $1 \leq i \leq n$ ), is injective (i.e., its kernel is trivial).

Now assuming Theorem B we shall prove Theorem A:

**NOTATION.** In the sequel we shall denote the complement of  $\Theta_i$  in  $\Delta - \Delta_0$  by  $\Theta'_i$  and  $A_{\theta_i}, \mathcal{G}_{\theta_i}, G_{\theta_i}, \mathcal{M}_{\theta_i}, M_{\theta_i}$  and  $T_{\theta_i}$  by  $A_i, \mathcal{G}_i, G_i, \mathcal{M}_i, M_i$  and  $T_i$  respectively.

*Proof of Theorem A.* It is obvious from the Tits index ([7]) of  $G/k$  that given a connected normal  $k$ -simple subgroup of the derived group  $\mathcal{G}$  of  $\mathcal{M}$ , there is an

$i(\leq r)$  such that  $G_{\Theta_i}$ , and therefore  $M_{\Theta_i}$ , contains it. Now since  $\mathcal{G}$  is a direct product of its connected normal  $k$ -simple subgroups, we conclude that the subgroup generated by the  $M_{\Theta_i}(k)$  ( $1 \leq i \leq r$ ) contains  $\mathcal{G}(k)$ .

The inclusion of  $\mathcal{M}$  in  $\mathcal{M}_i$  induces a  $k$ -rational homomorphism  $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$ , and also a homomorphism  $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$  of abstract groups. We now observe that the  $k$ -rational homomorphism  $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$  is an isomorphism. In fact, as  $\mathcal{M}_i$  is a semi-direct product of the torus  $T_i = T_{\Theta_i}$  and the normal semi-simple subgroup  $\mathcal{G}_i$ ,  $\mathcal{M}_i/\mathcal{G}_i$  is isomorphic to  $T_i (= T_{\Theta_i})$  and as  $\mathcal{M}$  is a semi-direct product of  $T_{\Delta-\Delta_0}$  and  $\mathcal{G}$ ,  $\mathcal{M}/\mathcal{G}$  is isomorphic to  $T_{\Delta-\Delta_0}$ . But  $T_{\Delta-\Delta_0}$  is a direct product of the tori  $T_i$  since  $\Delta - \Delta_0$  is a disjoint union of the  $\Theta_i$  ( $1 \leq i \leq r$ ). From this we conclude at once that the homomorphism  $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$  is an isomorphism.

The commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/\mathcal{G} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \prod_{i=1}^r \mathcal{G}_i & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i \longrightarrow 1, \end{array}$$

gives the following commutative diagram involving Galois cohomology:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{G}(k) & \longrightarrow & \mathcal{M}(k) & \longrightarrow & (\mathcal{M}/\mathcal{G})(k) \longrightarrow H^1(k, \mathcal{G}) \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \prod_{i=1}^r \mathcal{G}_i(k) & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i(k) & \longrightarrow & \prod_{i=1}^r (\mathcal{M}_i/\mathcal{G}_i)(k) \longrightarrow \prod_{i=1}^r H^1(k, \mathcal{G}_i), \end{array}$$

in which the horizontal rows are exact. Now since  $H^1(k, \mathcal{G}) \rightarrow \prod_{i=1}^r H^1(k, \mathcal{G}_i)$  is injective (Theorem B), we easily conclude from the second commutative diagram that the natural homomorphism  $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$  is surjective; now since  $\bigcap_{i=1}^r \mathcal{G}_i = \mathcal{G}$ , it follows that the induced homomorphism  $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$  is an isomorphism. It is evident from this that  $\mathcal{M}(k)$  is generated by the subgroups  $\mathcal{C}_i := \mathcal{M}(k) \cap \bigcap_{j \neq i} \mathcal{G}_j(k)$  ( $i \leq r$ ). But  $\bigcap_{j \neq i} \mathcal{G}_j = \bigcap_{j \neq i} \mathcal{G}_{\Theta_j} = \mathcal{G}_{\Theta_i}$ . Therefore

$$\mathcal{C}_i = \mathcal{M}(k) \cap \bigcap_{j \neq i} \mathcal{G}_j(k) = (\mathcal{M} \cap \mathcal{G}_{\Theta_i})(k) = A_{\Theta_i}(k) \cdot M_{\Theta_i}(k) \quad (\text{cf. 1.5}).$$

As the subgroup generated by the  $M_{\Theta_i}(k)$  ( $1 \leq i \leq r$ ) contains  $\mathcal{G}(k)$  and hence also  $A_{\Theta_c}(k)$  for  $1 \leq c \leq r$  (recall that  $A_{\Theta_c}$  is a normal subgroup of  $\mathcal{G}$ ), we conclude that  $\mathcal{M}(k) (= \mathcal{M}(k))$  is generated by the subgroups  $M_{\Theta_i}(k)$ ,  $1 \leq i \leq r$ . This proves Theorem A.

## §2. The Kneser–Tits problem for nonarchimedean local fields

We will now prove that the Kneser–Tits problem has an affirmative solution if  $k$  is a nonarchimedean local (i.e. locally compact, non-discrete, totally disconnected) field. For such a field it is known that  $H^1(k, \mathcal{G})$  is trivial (recall that  $\mathcal{G}$  is connected and simply connected): If  $k$  is a local field of characteristic zero, this was proved by M. Kneser ([3]) and then by Bruhat–Tits ([2]) for local fields of arbitrary characteristic. Thus, for a local field, Theorem B is an immediate consequence of this result. The first-named author originally proved Theorem A for local fields and deduced the Kneser–Tits conjecture in that case, the deduction is described below:

Let  $k$  be a nonarchimedean local field and let  $G$  be a  $k$ -simple, simply connected  $k$ -group of  $k$ -rank 1. Then ([1: 6.21(ii)]) there exists a finite separable extension  $K$  of  $k$  and an absolutely simple, simply connected group  $G$  defined over  $K$ , and of  $K$ -rank 1, such that  $G = R_{K/k}(G)$ ;  $K$  is again a nonarchimedean local field and from the classification (due to Kneser in characteristic zero and due to Bruhat–Tits in arbitrary characteristic) of absolutely simple groups over such a field we know that an absolutely simple, simply connected  $K$ -group of  $K$ -rank 1 is one of the following (note that there are no rank 1 forms of exceptional groups over a nonarchimedean local field):

- (i)  $SL_{2,D}$ , where  $D$  is a finite dimensional central division algebra over  $K$ .
- (ii)  $SU(f)$ , where  $f$  is a hermitian form, of Witt index 1, in 3 or 4 variables, defined in terms of a quadratic Galois extension  $K$  of  $K$ .
- (iii) The spin group of a  $\sigma$ -quadratic form of Witt index 1 and rank 4 or 5, or the symplectic group of a  $\sigma$ -antihermitian form of rank 2 or 3 and Witt index 1; where  $\sigma$  is an involution of the quaternion central division algebra  $D$  over  $K$  such that the dimension of  $D^\sigma$ , the space of symmetric elements, is 3.

For each of the above groups  $G$ , it is known that  $G(K)^+ = G(K)$ ; see, for example, [8].

## §3

We shall now begin our proof of Theorem B. A standard argument which uses the fact that there is a finite separable extension  $K$  of  $k$  and an absolutely simple, simply connected group defined over  $K$  such that  $G$  is obtained from it by restriction of scalars ([1: 6.21(ii)]), and Shapiro’s lemma in Galois cohomology (Serre [5: 5.8(b)]), allows us to assume that  $G$  is absolutely simple (and of  $k$ -rank  $\geq 2$ ). The proof (of Theorem B) uses the classification of absolutely simple groups in terms of Tits index (see Tits [7]); we shall assume familiarity with it.

From the Tits index of absolutely simple  $k$ -groups of  $k$ -rank  $\geq 2$  we see that if

the Tits index is *not* one of the following six:

- (i)  ${}^2E_{6,2}$ :
- (ii)  ${}^1E_{6,2}^{28}$ :
- (iii)  $E_{7,2}^{31}$ :
- (iv)  $E_{7,3}^{28}$ :
- (v)  $E_{8,4}^{28}$ :
- (vi)  $E_{8,2}^{66}$ :

then there exists a  $\Gamma$ -orbit in  $\Delta - \Delta_0$  such that if  $\Theta$  is its complement in  $\Delta - \Delta_0$ , then, in the notation introduced in 1.5,  $G_\Theta$  has at most one connected normal  $k$ -simple subgroup which meets  $\mathcal{G}$  non-trivially and this connected normal  $k$ -simple subgroup is  $k$ -isomorphic to  $R_{K/k}(\mathbf{G})$ , where  $K$  is a Galois extension of  $k$  (of degree  $\leq 2$ ) and  $\mathbf{G}$  is an absolutely simple  $K$ -isotropic group of inner type  $A$ . We know that  $\mathcal{G}_\Theta$  is a direct product of  $A_\Theta$  and  $G_\Theta$  (and  $A_\Theta$  is a factor of  $\mathcal{G}$ ). Hence, the natural map  $H^1(k, A_\Theta) \rightarrow H^1(k, \mathcal{G}_\Theta)$  is injective. Now it is not hard to see that to prove Theorem B for a group with Tits index different from the 6 indices listed above, it is enough to prove the following:

**3.1. PROPOSITION.** *Let  $G$  be an absolutely simple, simply connected group of inner type  $A$  which is defined and isotropic over a field  $K$ . Let  $S$  be a maximal  $K$ -split torus of  $G$  and  $H$  be a connected normal  $K$ -simple subgroup of the derived group of the centralizer of  $S$  in  $G$ . Then the natural map  $H^1(K, H) \rightarrow H^1(K, G)$  is injective.*

*Proof.* There exists a central division algebra  $D$  over  $K$  such that  $G$  is  $K$ -isomorphic to the group  $SL_{m,D}$ , where  $m = k\text{-rank } G + 1$ . We identify  $G$  with  $SL_{m,D}$  and for  $S$  take the  $K$ -split torus such that

$$S(K) = \left\{ \left[ \begin{array}{cccc} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \lambda_m \end{array} \right] \mid \lambda_i \in K^\times, \prod \lambda_i = 1 \right\}.$$

Then the centralizer of  $S$  is the diagonal subgroup of  $SL_{m,D}$ , and there is a positive integer  $i \leq m$  such that  $H$  is the subgroup of the diagonal group consisting of the elements whose  $j$ -th diagonal entry is 1 for all  $j \neq i$ ;  $H$  is clearly  $k$ -isomorphic to  $SL_{1,D}$ . In the sequel we shall identify  $SL_{1,D}$  with  $H$ .

Now we consider the group  $GL_{m,D}$ . We embed  $GL_{1,D}$  in  $GL_{m,D}$  as the subgroup of the diagonal group consisting of the elements with the  $j$ -th diagonal entry 1 for all  $j \neq i$ .  $H$  is now the kernel of the reduced norm map  $\text{Nrd}: GL_{1,D} \rightarrow \text{Mult}$ . The commutative diagram of  $K$ -groups:

$$\begin{array}{ccccccc} 1 & \longrightarrow & SL_{m,D} & \longrightarrow & GL_{m,D} & \xrightarrow{\text{Nrd}} & \text{Mult} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & SL_{1,D} & \longrightarrow & GL_{1,D} & \xrightarrow{\text{Nrd}} & \text{Mult} \longrightarrow 1 \end{array}$$

gives the following commutative diagram in which the horizontal rows are exact in view of the vanishing<sup>(1)</sup> of  $H^1(K, GL_{n,D})$  for all  $n \geq 1$ :

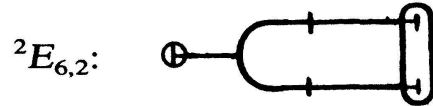
$$\begin{array}{ccccccc} 1 & \longrightarrow & SL_m(D) & \longrightarrow & GL_m(D) & \xrightarrow{\text{Nrd}} & K^\times \longrightarrow H^1(K, SL_{m,D}) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & SL_1(D) & \longrightarrow & GL_1(D) & \xrightarrow{\text{Nrd}} & K^\times \longrightarrow H^1(K, SL_{1,D}) \longrightarrow 1. \end{array}$$

From the theory of Dieudonné determinants it is obvious that the image of  $GL_m(D)$  in  $K^\times$  equals that of  $GL_1(D)$ , from this and the above commutative diagram we conclude at once that  $H^1(K, SL_{1,D}) \rightarrow H^1(K, SL_{m,D})$  is injective, i.e., in the notation of the proposition, the natural map  $H^1(K, H) \rightarrow H^1(K, G)$  is injective. This proves the proposition.

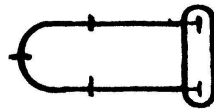
<sup>1</sup>This vanishing is a well-known theorem of Hilbert and Speiser.

§4

We shall now prove Theorem B for groups with Tits index the first of the six exceptional ones listed in §3 i.e.,



Let  $\Theta$  be the unique distinguished  $\Gamma$ -orbit consisting of 2 simple roots. Then the Tits index of  $\mathcal{G}_\Theta (= \mathcal{G}_\Theta)$  is the following:



Moreover, the Tits index of  $\mathcal{G} (\subset \mathcal{G}_\Theta)$  is . Now let  $l$  be the quadratic Galois extension of  $k$  such that  $\mathcal{G}_\Theta/l$  is an inner form of a split group. There is an anisotropic hermitian form  $f$  in 4 variables, defined in terms of the nontrivial automorphism  $\sigma$  of  $l/k$ , such that  $\mathcal{G}$  is  $k$ -isomorphic to  $SU(f)$ , whereas  $\mathcal{G}_\Theta$  is  $k$ -isomorphic to  $SU(f \perp h)$ , where  $h$  is the hyperbolic form in 2 variables. Now we consider the following commutative diagram in which the horizontal rows are exact:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & SU(f \perp h) & \longrightarrow & U(f \perp h) & \longrightarrow & \mathcal{T} \longrightarrow 1 \\
 & & \uparrow & & \uparrow & & \parallel \\
 & & SU(f) & \longrightarrow & U(f) & \longrightarrow & \mathcal{T} \longrightarrow 1;
 \end{array}$$

where  $\mathcal{T}$  is the torus of dimension 1 defined and anisotropic over  $k$  which splits over  $l$ , (then  $\mathcal{T}(k) = \{x \in l^\times \mid x\sigma(x) = 1\}$ ) and  $U(f \perp h) \rightarrow \mathcal{T}$ , as well as  $U(f) \rightarrow \mathcal{T}$ , are the determinant maps. It is obvious that both  $U(f \perp h)(k) \rightarrow \mathcal{T}(k)$  and  $U(f)(k) \rightarrow \mathcal{T}(k)$  are surjective. Therefore, the natural morphisms  $H^1(k, SU(f \perp h)) \rightarrow H^1(k, U(f \perp h))$  and  $H^1(k, SU(f)) \rightarrow H^1(k, U(f))$  are injective. On the other hand, Witt's cancellation theorem (for hermitian forms) implies at once that  $H^1(k, U(f)) \rightarrow H^1(k, U(f \perp h))$  is injective. Now it is obvious that  $H^1(k, SU(f)) \rightarrow H^1(k, SU(f \perp h))$  is injective, i.e.,  $H^1(k, \mathcal{G}) \rightarrow H^1(k, \mathcal{G}_\Theta)$  is injective. From this Theorem B follows for groups of type  ${}^2E_{6,2}$ .

§5

In this section we shall complete the proof of Theorem B by proving it for the groups of the remaining five exceptional types. We begin with the following two lemmas.

**5.1. LEMMA.** *Let  $P$  be a parabolic  $k$ -subgroup of a connected reductive  $k$ -group  $G$ , and  $M$  be a maximal reductive  $k$ -subgroup of  $P$ . Then the natural morphism*

$$H^1(k, M) \rightarrow H^1(k, G)$$

*is injective.*

*Proof.* Since the natural map  $G(k) \rightarrow (G/P)(k)$  is surjective (Borel–Tits [1: 4.13(a)]), the morphism

$$H^1(k, P) \rightarrow H^1(k, G)$$

is injective. Therefore, to prove the lemma, it suffices to observe that if  $U$  is the unipotent radical of  $P$ , then  $U$  is defined over  $k$  and  $P = M \ltimes U$  (a semi-direct product), and hence the natural morphism

$$H^1(k, M) \rightarrow H^1(k, P)$$

is injective.

**5.2. LEMMA.** *Let  $G$  and  $M$  be as in the preceding lemma. Let  $\mathcal{G}$  be the derived subgroup of  $M$  and  $S$  be the central torus of  $M$ . Let  $\mathcal{G}_0$  and  $\mathcal{G}_*$  be two connected normal  $k$ -subgroups of  $\mathcal{G}$  such that  $\mathcal{G}$  is an almost direct product of  $\mathcal{G}_0$  and  $\mathcal{G}_*$ . Let  $\mathcal{C}$  be the finite group scheme  $\mathcal{G}_0 \cap S\mathcal{G}_*$ . Then the kernel of the natural morphism*

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G)$$

*is contained in the image of*

$$H^1(k, \mathcal{C}) \rightarrow H^1(k, \mathcal{G}_0).$$

*Proof.* Since the morphism  $H^1(k, M) \rightarrow H^1(k, G)$  is injective (Lemma 5.1), the kernel of  $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G)$  coincides with the kernel of  $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M)$ . But  $C := \text{Ker}(H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M))$  is clearly contained in the kernel of the morphism  $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M/S\mathcal{G}_*)$  induced by the  $k$ -homomorphism  $\mathcal{G}_0 \rightarrow M/S\mathcal{G}_*$ . Now as the natural homomorphism  $\mathcal{G}_0/\mathcal{C} \rightarrow M/S\mathcal{G}_*$  is a  $k$ -isomorphism, we conclude that  $C$  is contained in the kernel of  $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, \mathcal{G}_0/\mathcal{C})$ , and from this the lemma is obvious.

Before proceeding further with the proof of Theorem B in the remaining exceptional cases, we shall recall some of the basic notions of the theory of quadratic forms.

**5.3.** Let  $p$  be the characteristic of  $k$ . If  $p = 2$ , let  $\wp(k) = \{x + x^2 \mid x \in k\}$ ;  $\wp(k)$  is a subgroup of  $k$ .

A quadratic form is said to be *nondefective* if the associated bilinear form is nondegenerate.

The *rank* (or the *dimension*) of a nondefective quadratic form is by definition the dimension of the underlying  $k$ -vector space, and the *Witt index* (over  $k$ ) is the dimension of a maximal isotropic  $k$ -vector subspace.

For a quadratic form  $f/k$ , the *discriminant* (when  $p = 2$ , it is also called the *Arf invariant*)  $d(f)$  will have the usual meaning. We recall that if  $p \neq 2$ ,  $d(f)$  is an element of  $k^\times/k^{\times 2}$ , and if  $p = 2$ ,  $d(f)$  is an element of  $k/\wp(k)$ . We shall say that a quadratic form  $f$  of rank  $2n$  has *trivial signed discriminant* if its discriminant equals that of the hyperbolic form of rank  $2n$ , or, equivalently, if the special orthogonal group  $SO(f)$  is of *inner type* over  $k$ .

Let  $q$  be a nondefective anisotropic quadratic form over  $k$ , of rank 2, and  $K$  be the quadratic Galois extension of  $k$  over which it is hyperbolic, then  $d(q)$  is the image (in  $k^\times/k^{\times 2}$  if  $p \neq 2$  and in  $k/\wp(k)$  if  $p = 2$ ) of the norm of any element of  $K^\times$  of trace zero if  $p \neq 2$  and of trace 1 if  $p = 2$ . Since  $q$  is a multiple of the norm-form of  $K/k$ , we conclude that the discriminant  $d(q)$  determines  $q$  up to a scalar multiple.

If over  $k$ ,  $f$  is an orthogonal direct sum of the nondefective quadratic forms  $q_i$ ,  $1 \leq i \leq n$ , of rank 2, then  $d(f)$  is the product of the  $d(q_i)$  ( $1 \leq i \leq n$ ) if  $p \neq 2$ , and it is the sum of the  $d(q_i)$ 's if  $p = 2$ .

**5.4.** The *Witt invariant*  $w(f)$  of a nondefective quadratic form  $f/k$  of *even* rank is by definition the class of the Clifford algebra of  $f$  in the Brauer group of  $k$ ; it is an element of order 2 in the Brauer group. We recall that if  $f$  is a quadratic form of rank  $2n$ , with trivial signed discriminant, then the Witt invariant of  $f$  has the following useful description: Let  $h$  be the hyperbolic form of rank  $2n$  and let  $\text{Spin}(h)$  and  $SO(h)$  be respectively the spin group and the special orthogonal group of  $h$ . Then since the discriminant of  $f$  equals that of  $h$ , the quadratic form  $f$  is obtained from  $h$  by twisting by a Galois cocycle with values in  $SO(h)$ . Let  $c$  denote the cohomology class in  $H^1(k, SO(h))$  determined by the cocycle. Now consider the natural central isogeny:

$$1 \rightarrow \mu_2 \rightarrow \text{Spin}(h) \rightarrow SO(h) \rightarrow 1,$$

where  $\mu_2$  is the kernel of the endomorphism  $x \mapsto x^2$  of  $GL_1$  (it is a finite group scheme defined over  $k$ ). It gives rise to the following exact sequence:

$$H^1(k, \text{Spin}(h)) \rightarrow H^1(k, SO(h)) \xrightarrow{\delta} H^2(k, \mu_2),$$

then  $w(f) = \delta(c)$  in the natural identification of  $H^2(k, \mu_2)$  with the subgroup of the Brauer group of  $k$  consisting of the elements of order 2.

Now we observe that if  $f$  is an *anisotropic* quadratic form of rank 6 which has trivial signed discriminant, then its Witt invariant is the class of a division algebra of degree 4 (i.e. of dimension 16). This follows immediately from the fact that  $\text{Spin}(h)$ , where  $h$  is the hyperbolic form of rank 6, is isomorphic to  $SL_4$  over the base field, and the only anisotropic *inner* twists of  $SL_4$  are of the form  $SL_{1,D}$ ,  $D$  a central division algebra of dimension 16 over the base field.

**5.5.** Now we assume that  $G$  is an absolutely simple, simply connected algebraic group of type one of the remaining five:  ${}^1E_{6,2}^{28}$ ,  $E_{7,2}^{31}$ ,  $E_{7,3}^{28}$ ,  $E_{8,4}^{28}$ ,  $E_{8,2}^{66}$ . Let  $\mathcal{G}$  be (as in §1) the semi-simple anisotropic kernel of  $G$ . Let  $\mathcal{G}_0$  be the unique connected normal  $k$ -subgroup of  $\mathcal{G}$  of type  $D_n$  ( $n = 4$  or  $6$ ) and in case  $G$  is of type  $E_{7,2}^{31}$ , let  $\mathcal{G}_*$  be the connected normal  $k$ -subgroup of  $\mathcal{G}$  of type  $A_1$ , in all the other cases let  $\mathcal{G}_*$  be trivial. Then  $\mathcal{G}$  is a direct product of  $\mathcal{G}_0$  and  $\mathcal{G}_*$ .

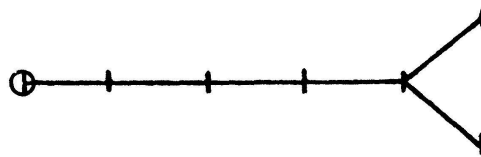
**5.6.** Let  $a$  be the simple root corresponding to the vertex in the Tits index marked with a cross (in §3) and let  $\Theta$  be the set of distinguished simple roots  $\neq a$ . To establish Theorem B in the cases under consideration, it clearly suffices to prove that the natural morphism

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G_{\{a\}}) \times H^1(k, G_\Theta)$$

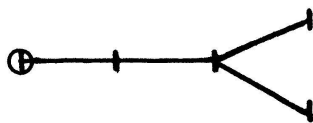
is injective.

Let  $S_{\{a\}}$  (resp.  $S_\Theta$ ) be the maximal  $k$ -split torus of  $G_{\{a\}}$  (resp.  $G_\Theta$ ) contained in  $S$ , and let  $Z = \mathcal{G}_0 \cap S_{\{a\}}$ ,  $\mathcal{Z} = \mathcal{G}_0 \cap S_\Theta \mathcal{G}_*$ . Then it is easily seen, using the Tits indices, that both  $Z$  and  $\mathcal{Z}$  are  $k$ -isomorphic to the group scheme  $\mu_2$ . Moreover, the center of  $\mathcal{G}_0$  is a direct sum of  $Z$  and  $\mathcal{Z}$ .

Now we observe that there is a nondefective, anisotropic quadratic form  $f/k$  with trivial discriminant,  $f$  of rank 12 in case  $G$  is of type  $E_{8,2}^{66}$ , and of rank 8 in all the other cases, such that  $\mathcal{G}_0$  is  $k$ -isomorphic to  $\text{Spin}(f)$  and the kernel of the natural central isogeny  $\pi : \text{Spin}(f) \rightarrow SO(f)$  is  $Z (= \mathcal{G}_0 \cap S_{\{a\}})$ . This follows from the fact that  $\mathcal{G}_0$  is the semi-simple anisotropic kernel of the simply connected, absolutely simple group  $G_{\{a\}}$ , and  $G_{\{a\}}$  is the spin group of a nondefective quadratic form, of Witt index 1, which has trivial signed discriminant, since its Tits index is



in case  $G$  is of type  $E_{8,2}^{66}$ , and



in all the other cases. We shall identify  $\mathcal{G}_0$  with  $\text{Spin}(f)$  in the sequel.

**5.7. LEMMA.** *If  $G$  is of type  $E_{8,2}^{66}$ , then the Witt invariant of  $f$  over  $k$  is trivial.*

*Proof.* Any connected absolutely simple algebraic group of type  $E_8$  is simply connected and is isomorphic to its automorphism group. Therefore, as the semi-simple anisotropic kernel of a  $k$ -form of type  $E_{8,2}^{66}$  is an absolutely simple, simply connected group of type  $D_6$ , it is obtained from the split group of type  $E_8$  by twisting by a Galois cocycle with values in the spin group of the hyperbolic form  $h$  of rank 12 (the spin group embedded as a maximal semi-simple  $k$ -subgroup of a parabolic  $k$ -subgroup of the split group of type  $E_8$ ). Hence,  $f$  is obtained from  $h$  by twisting by a cocycle whose cohomology class lies in the image of the natural morphism.

$$H^1(k, \text{Spin}(h)) \rightarrow H^1(k, \text{SO}(h)).$$

This implies the lemma (see 5.4).

**5.8.** We now note, for future use, that the Witt index of the quadratic form  $f$  is *even* over any extension of  $k$ : this is seen easily from the classification of inner  $k$ -forms of types  $E_6$ ,  $E_7$  and  $E_8$  in terms of the Tits indices given in Tits [7].

**5.9.** Now let  $c$  be an element of the kernel of the natural morphism

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G_{\{a\}}) \times H^1(k, G_{\theta}).$$

We shall prove that  $c$  is trivial, this will establish Theorem B (see 5.6).

Let  $Z$  and  $\mathcal{Z}$  be as in 5.6. From Lemma 5.2 applied in turn to  $G = G_{\{a\}}$  and  $G = G_{\theta}$ , we conclude that  $c$  lies in the intersection of the images of the following natural morphisms:

$$H^1(k, \mathcal{Z}) \rightarrow H^1(k, \mathcal{G}_0)$$

and

$$H^1(k, Z) \rightarrow H^1(k, \mathcal{G}_0).$$

Hence, in particular  $c$  is mapped onto the trivial element of  $H^1(k, \text{SO}(f))$  under the central isogeny  $(\mathcal{G}_0 =) \text{Spin}(f) \rightarrow \text{SO}(f)$  (whose kernel is  $Z$ ).

We fix an element  $c \in H^1(k, \mathcal{Z})$  which is mapped onto  $c \in H^1(k, \mathcal{G}_0)$ . Since  $\mathcal{Z}$  is  $k$ -isomorphic to  $\mu_2$ , there is a natural identification of  $H^1(k, \mathcal{Z})$  with  $k^\times/k^{\times 2}$ . Let

$s \in k^\times$  be such that, in the identification of  $H^1(k, \mathcal{X})$  with  $k^\times/k^{\times 2}$ ,  $c$  corresponds to  $s$ . Now we observe that under the central isogeny  $\text{Spin}(\mathfrak{f}) \rightarrow \text{SO}(\mathfrak{f})$ ,  $\mathcal{X}$  is mapped onto the center of  $\text{SO}(\mathfrak{f})$  and from this we conclude that the image of the cohomology class  $c$  in  $H^1(k, \text{SO}(\mathfrak{f}))$  corresponds to the quadratic form  $s\mathfrak{f}$ . But since the image of  $c$  in  $H^1(k, \text{SO}(\mathfrak{f}))$  is trivial,  $s\mathfrak{f}$  is equivalent to  $\mathfrak{f}$  over  $k$ .

**5.10. LEMMA.** *Let  $\varphi$  be a nondefective anisotropic quadratic form such that  $\varphi$  is equivalent to  $s\varphi$  ( $s \in k^\times$ ). Then there is a nondefective subform  $q$  of  $\varphi$  of rank 2 such that  $q$  is equivalent to  $sq$ .*

*Proof.* If  $s$  is a square in  $k^\times$ , the lemma is obvious, so we shall assume that  $s$  is not a square.

Let  $V$  be the  $k$ -vector space underlying  $\varphi$  and  $\langle, \rangle$  be the bilinear form associated with  $\varphi$ . We fix a  $v \in V$  such that  $\varphi(v) \neq 0$ . Then since  $\varphi \simeq s\varphi$ , there is a  $v' \in V$  such that  $\varphi(v') = s\varphi(v)$ . Now if  $\langle v, v' \rangle \neq 0$ , let  $w = v'$ ; if  $\langle v, v' \rangle = 0$ , choose a  $v_0 \in V$  such that  $\langle v_0, v \rangle \cdot \langle v_0, v' \rangle \neq 0$ , and let

$$w = v' - \frac{\langle v_0, v' \rangle}{\varphi(v_0)} v_0.$$

Then  $\varphi(w) = \varphi(v') = s\varphi(v)$  and  $\langle v, w \rangle \neq 0$ . Also since  $s$  is not a square,  $w$  is not a scalar multiple of  $v$ . Let  $q$  be the restriction of the quadratic form  $\varphi$  to the 2-dimensional subspace  $X$  spanned by  $v$  and  $w$ . It is easily seen that  $q$  is a nondefective quadratic form. The  $k$ -linear automorphism of the vector subspace  $X$  defined by  $v \mapsto w, w \mapsto sv$  provides an equivalence of the quadratic form  $sq$  with  $q$ .

**5.11. PROPOSITION.** *There exist nondefective subforms  $q_i, q'_i$  ( $i = 1, 2$  if  $G$  is not of type  $E_{8,2}^{66}$  and  $i = 1, 2, 3$  if  $G$  is of type  $E_{8,2}^{66}$ ) of  $\mathfrak{f}$ , of rank 2, such that  $\mathfrak{f}$  is the orthogonal direct sum of the  $q_i$ 's and  $q'_i$ 's, and for each  $i$*

- (1)  $q_i \simeq sq_i, q'_i \simeq sq'_i$
- (2)  $q'_i$  is a scalar multiple of  $q_i$ ; in particular  $\text{SO}(q_i)$  is  $k$ -isomorphic to  $\text{SO}(q'_i)$ .

*Proof.* According to the preceding lemma, there is a nondefective subform  $q_1$  of  $\mathfrak{f}$  of rank 2 such that  $q_1 \simeq sq_1$ . Now let  $K$  be the quadratic Galois extension of  $k$  over which  $q_1$  is hyperbolic, then  $q_1$  is a multiple of the norm-form of  $K/k$ . Let  $q_1^\perp$  be the orthogonal complement of  $q_1$  in  $\mathfrak{f}$ . Then since the Witt index of  $\mathfrak{f}$  over  $K$  is even (5.8),  $q_1^\perp$  is isotropic over  $K$ . Therefore, there exist vectors  $v, w$  in the subspace corresponding to  $q_1^\perp$  and  $\alpha \in K - k$  such that

$$q_1^\perp(v + \alpha w) = \mathfrak{f}(v + \alpha w) = \mathfrak{f}(v) + \alpha \langle v, w \rangle + \alpha^2 \mathfrak{f}(w) = 0.$$

Now since  $\alpha$  is separable, we easily conclude that the restriction  $q'_1$  of  $q_1^\perp$  to the

2-dimensional subspace spanned by  $v$  and  $w$  is a nondefective quadratic form of rank 2 which is isotropic (and hence hyperbolic) over  $K$ . Therefore,  $q'_1$  is a multiple of the norm-form of  $K/k$ . As  $q_1$  is also a multiple of the norm-form of  $K/k$  and  $q_1 \simeq sq_1$ , we conclude that  $q'_1$  is a multiple of  $q_1$  and  $q'_1 \simeq sq'_1$ .

Now let  $f_1 = q_1 \perp q'_1$ . Then the discriminant of  $f_1$  is trivial. Let  $f_2$  be the orthogonal complement of  $f_1$  in  $f$ . Then the discriminant of  $f_2$  is trivial and as  $f_1 \simeq sf_1$ , by Witt's cancellation theorem  $f_2 \simeq sf_2$ . We shall now consider the cases where  $f$  is of rank 8. Let  $q_2$  be a nondefective subform of  $f_2$  of rank 2 such that  $q_2 \simeq sq_2$  (Lemma 5.10) and  $q'_2$  be its orthogonal complement in  $f_2$ . Then the discriminant of  $q_2$  equals that of  $q'_2$  and hence  $q'_2$  is a scalar multiple of  $q_2$  (5.3), in particular  $q'_2 \simeq sq'_2$ .

Now we consider the case where  $f$  is of rank 12, then  $G$  is of type  $E_{8,2}^{66}$ ,  $f_2$  is an anisotropic form of rank 8 and trivial discriminant. We claim that the Witt index of  $f_2$  over any quadratic Galois extension of  $k$  is even. To prove this we consider a quadratic Galois extension  $l$  of  $k$  such that  $f_2$  is isotropic over  $l$ . Then as the discriminant of  $f_2$  is trivial, the Witt index of  $f_2$  over  $l$  can not be 3; assume, if possible, that it is 1. Then since the Witt invariant of  $f/k$  is zero (Lemma 5.7), the Witt invariant of  $f_1/l$  equals that of  $f_2/l$ . Now since by hypothesis  $f_2/l$  is of Witt index 1, over  $l$  it is an orthogonal direct sum of the hyperbolic form of rank 2 and an anisotropic form of rank 6. Therefore, the Witt invariant of  $f_2/l$  is the class of a division algebra of degree 4 in the Brauer group of  $l$  (5.4). But since  $f_1/k$  is an anisotropic form of rank 4 of trivial discriminant, it is a multiple of the norm-form of a *quaternion* division algebra  $D$ , and its Witt invariant is the class of  $D$  in the Brauer group of  $k$ . Therefore, the Witt invariant of  $f_1/l$  is the class of  $D \otimes_k l$ . We conclude thus that the class of a division algebra of degree 4 (in the Brauer group of  $l$ ) contains  $D \otimes_k l$ . This is absurd, and hence the Witt index of  $f_2$  over  $l$  can not be 1. This proves that the Witt index of  $f_2$  over  $l$  is even. Now since  $f_2$  is of rank 8, we can prove, as before, that there exist 4 nondefective quadratic forms  $q_2, q'_2, q_3$  and  $q'_3$  of rank 2 such that  $f_2$  is an orthogonal direct sum of these;  $q_i \simeq sq_i, q'_i \simeq sq'_i$  and  $q'_i$  is a scalar multiple of  $q_i$  ( $i = 2, 3$ ). This proves the proposition.

**5.12.** We fix a set of nondefective subforms  $q_i, q'_i$ , of  $f$ , of rank 2, as in the preceding proposition. Let  $\bar{T}_i = SO(q_i) (\subset SO(f))$  and  $\bar{T}'_i = SO(q'_i) (\subset SO(f))$ . Then (for all  $i$ )  $\bar{T}_i$  and  $\bar{T}'_i$  are isomorphic  $k$ -tori of dimension 1. Let  $\pi : \text{Spin}(f) \rightarrow SO(f)$  be the usual central isogeny and let  $T_i = \pi^{-1}(\bar{T}_i)$  and  $T'_i = \pi^{-1}(\bar{T}'_i)$ . Then for all  $i$ ,  $T_i$  and  $T'_i$  are isomorphic  $k$ -tori.

$\prod_i (\bar{T}_i \times \bar{T}'_i)$  is a maximal torus of  $SO(f)$  and there is a unique  $k$ -embedding of  $\mu_2$  into  $\bar{T}_i$ , as well as in  $\bar{T}'_i$ . The center  $C$  of  $SO(f)$  is the "diagonally" embedded  $\mu_2$  in  $\prod_i (\bar{T}_i \times \bar{T}'_i)$ .

Let  $\theta_i$  be a fixed  $k$ -isomorphism of  $\bar{T}_i$  onto  $\bar{T}'_i$  (note that there are exactly two

distinct  $k$ -isomorphisms of  $\bar{T}_i$  onto  $\bar{T}'_i$ ), we shall let  $\theta_i$  also denote the induced  $k$ -isomorphism of  $T_i$  onto  $T'_i$ . Let  $\bar{\mathcal{T}}_i = \{x \cdot \theta_i(x) \mid x \in \bar{T}_i\}$ ,  $\mathcal{T}_i = \{x \cdot \theta_i(x) \mid x \in T_i\}$ , and let  $\bar{\mathcal{T}} = \prod_i \bar{\mathcal{T}}_i$ ,  $\mathcal{T} = \prod_i \mathcal{T}_i$ . It is easily seen that  $\forall i$ , the restriction of  $\pi$  to  $\mathcal{T}_i$  is an isomorphism onto  $\bar{\mathcal{T}}_i$  and hence the restriction of  $\pi$  to  $\mathcal{T}$  is an isomorphism onto  $\bar{\mathcal{T}}$ . Also, if necessary after changing the isomorphism  $\theta_i$  for any one  $i$ , we can ensure that  $\mathcal{Z} \subset \mathcal{T}$ . We shall assume in the sequel that this is the case. Now we assert that  $c(\epsilon H^1(k, \mathcal{Z}))$  is mapped onto the trivial element of  $H^1(k, \mathcal{T})$  under the morphism induced by the inclusion  $\mathcal{Z} \rightarrow \mathcal{T}$ . To see this, we observe that the image of  $c$  in  $H^1(k, \bar{\mathcal{T}})$  is trivial: this is a simple consequence of the fact that for  $\forall i$ ,  $q_i \simeq \text{sq}_i$ . Now since  $\pi|_{\mathcal{T}} : \mathcal{T} \rightarrow \bar{\mathcal{T}}$  is a  $k$ -isomorphism which maps  $\mathcal{Z}$  onto  $C$ , our assertion follows. It is obvious now that  $c$ , being the image of  $c$  in  $H^1(k, \mathcal{G}_0)$ , is trivial because  $\mathcal{Z} \subset \mathcal{T} \subset \mathcal{G}_0$ . This completes the proof of Theorem B.

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