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On the Kneser–Tits problem

GOPAL PRASAD and M. S. RAGHUNATHAN

Introduction

Let G be a semi-simple, simply connected algebraic group defined, isotropic and simple over a (commutative) field k. Let $G(k)$ be the group of k-rational points of G and $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k-rational points of the unipotent radicals of parabolic k-subgroups of G. The *Kneser–Tits problem* referred to in the title is the following: *Is* $G(k)^+ = G(k)$ *for every* G *as above*? The main object of this paper is to prove that *for a field* k, *the Kneser–Tits problem has an affirmative solution iff* $G(k)^+ = G(k)$ *for all simply connected,* k*-simple groups* G *of* k*-rank* 1. This reduction of the Kneser–Tits problem is an immediate consequence of Theorem A proved below. After this work was complete, we learnt from Armand Borel that Theorem A was conjectured by Jacques Tits in a lecture at the Institute for Advanced Study (Princeton), and was proved by him for some fields by a method different from ours.

The proof of Theorem A depends on a theorem on Galois cohomology (Theorem B) which may be of some independent interest.

In case k is a local field, the Kneser–Tits problem has an affirmative solution. This was essentially proved by V. P. Platonov [4] using the known results on classical groups and detailed knowledge of classification. He also gave the first examples of fields for which the Kneser–Tits problem has a negative answer (see Tits [8] for a survey). In §2 of this paper we use the reduction of the Kneser–Tits problem to rank 1 groups stated above to provide a simple proof of its affirmative solution for the local fields. This simple proof devised by the first-named author was the starting point of the present work. We hope to come back to the problem for global fields in the near future.

1.1. Let k be a (commutative) field, $\mathcal{K}$ be a fixed separable closure of k and let $\Gamma = \text{Gal}(\mathcal{K}/k)$. Let G be a semi-simple, simply connected group defined over k. Let S be a maximal k-split torus of G. Let $\dim S = r$ $(:= k\text{-rank } G)$. We assume that $r > 0$ i.e., G is isotropic over k; we also assume that G is k-simple, i.e., it has no proper *connected* normal subgroup defined over k.

1.2. Let T be a maximal torus of G containing S and defined over k. Let Φ be the set of roots of G relative to T. We fix a Borel subgroup B defined over $\mathcal{K}$, $B \supset T$, and contained in a minimal parabolic k-subgroup of G. This induces an ordering on Φ; let Δ be the set of all simple roots with respect to this ordering. Let Δ_0 be the subset of Δ consisting of those roots which are trivial on S. There is an action of Γ on Δ (the $*$-action) defined in Tits [7: §2.3]; both Δ_0 and $\Delta - \Delta_0$ are stable under this action. Since k-rank $G = r$, there are r Γ-orbits in $\Delta - \Delta_0$.

1.3. For a simple root a, let U_a and U_{-a} be the root subgroups associated with a and $-a$ respectively; U_a and U_{-a} are connected unipotent $\mathcal{K}$-subgroups of G, of dimension 1, normalized by T. Since G is simply connected, $\forall a \in \Delta$, the subgroup generated by U_a and U_{-a} is $\mathcal{K}$-isomorphic to SL_2; let T_a be its intersection with T, then T_a is a one dimensional torus defined over $\mathcal{K}$, and as G is simply connected, T is a direct product of the $T_a (a \in \Delta)$. For a subset Θ of Δ, let T_Θ be the subtorus generated by the tori T_a, $a \in \Theta$.

1.4. For a k-subgroup H of G, as usual, $H(k)$ will denote the group of k-rational points of H, and $H(k)^+$ will denote the normal subgroup of $H(k)$ generated by the k-rational points of the unipotent radicals of the parabolic k-subgroups of H.

1.5. For a Γ-stable subset Θ of $\Delta - \Delta_0$, let T^Θ be the identity component of $\bigcap_{\theta \in \Theta \cup \Delta_0} \operatorname{Ker} \theta$. Let $\mathcal{M}_\Theta$ be the centralizer of T^Θ in G. Then $\mathcal{M}_\Theta$ is a connected reductive subgroup defined over k; in fact it is a Levi k-subgroup of a parabolic k-subgroup of G (cf. Tits [7: §2.5.4]). Let $\mathcal{G}_\Theta$ be the derived subgroup of $\mathcal{M}_\Theta$. Then $\mathcal{G}_\Theta$ is a semi-simple, simply connected, k-subgroup of G, and hence it is a direct product of its connected k-simple normal subgroups. Let A_Θ be the product of all connected k-simple normal subgroups of $\mathcal{G}_\Theta$ which are anisotropic over k, and G_Θ be the product of all connected k-simple k-isotropic subgroups. Then the k-rank of G_Θ is equal to the number of Γ-orbits in Θ, and $\mathcal{G}_\Theta$ is a direct product (over k) of A_Θ and G_Θ. It is easily seen that $\mathcal{M}_\Theta$ is a semi-direct product of $T_{\Theta'}$ and $\mathcal{G}_\Theta$; where Θ' is the complement of Θ in $\Delta - \Delta_0$. Hence, the natural homomorphism: $\mathcal{M}_\Theta(\mathcal{K}) \to (\mathcal{M}_\Theta/\mathcal{G}_\Theta)(\mathcal{K})$ is surjective.

We shall denote the centralizer of S in G by $\mathcal{M}$ and sometimes also by M. Let $\mathcal{G}$ be the derived group of $\mathcal{M}$. Then $\mathcal{M} = \mathcal{M}_\varnothing$; $\mathcal{G} = \mathcal{G}_\varnothing$ (where $\varnothing$ is the empty subset of $\Delta - \Delta_0$). $\mathcal{G}$ is anisotropic over k, and it is easy to see that A_Θ is a normal subgroup of $\mathcal{G}$ for every Γ-stable subset Θ of $\Delta - \Delta_0$.

For a Γ-stable subset Θ of $\Delta - \Delta_0$, let S_Θ be the maximal k-split torus of G_Θ contained in S, and let M_Θ denote the centralizer of S_Θ in G_Θ. Then M_Θ is a connected reductive k-subgroup. Moreover, since $\mathcal{G}_\Theta$ is a direct product of G_Θ

and A_Θ, the centralizer of S_Θ in $\mathscr{G}_\Theta$ is just $A_\Theta \cdot M_\Theta$ (direct product). It is easy to see, by considering the reductive groups $S \cdot G_\Theta$ and $S \cdot \mathscr{G}_\Theta$, that $M_\Theta = M \cap G_\Theta$ and $\mathscr{M} \cap \mathscr{G}_\Theta = A_\Theta \cdot M_\Theta$.

1.6. Let Θ_i, $i = 1, \ldots, r$, be the Γ-orbits in $\Delta - \Delta_0$. Recall that G_{Θ_i} is a semi-simple simply connected k-subgroup of G of k-rank 1; it is k-simple since it does not contain any connected normal k-anisotropic subgroup. It follows from the Bruhat-decomposition that $G(k) = M(k) \cdot G(k)^+$. Thus $G(k)^+ = G(k)$ if and only if $G(k)^+ \supset M(k)$. Similarly as $G_\Theta(k) = M_\Theta(k) \cdot G_\Theta(k)^+$, $G_\Theta(k)^+ = G_\Theta(k)$ if and only if $G_\Theta(k)^+ \supset M_\Theta(k)$. In view of these observations, the following Theorem A implies that the Kneser–Tits problem for a field k has an affirmative solution if and only if for every k-simple simply connected group G of k-rank 1, $G(k)^+ = G(k)$.

THEOREM A. *Assume that k-rank $G \geq 2$. Then $M(k)$ is generated by the subgroups $M_{\Theta_i}(k)$ $(1 \leq i \leq r)$.*

1.7. *Remark.* If k is an infinite field, then $G(k)^+$ has no proper non-central normal subgroups (Tits [6: Main Theorem]), in particular it is *perfect* i.e. $(G(k)^+, G(k)^+) = G(k)^+$. Now Theorem A implies that to prove that $G(k)$ is perfect for all k-simple, simply connected k-isotropic G, it suffices to prove that this is so for all k-simple, simply connected groups of k-rank 1.

We shall prove Theorem A using the following:

THEOREM B. *For $i \leq n$, let Δ_i be a Γ $(= \mathrm{Gal}(\mathscr{K}/k))$-stable subset of $\Delta - \Delta_0$ such that $\bigcap_{i=1}^n \Delta_i = \varnothing$. Then the natural morphism:*

$$H^1(k, \mathscr{G}) \to \prod_{i=1}^n H^1(k, \mathscr{G}_{\Delta_i}),$$

induced by the inclusion of $\mathscr{G}$ in $\mathscr{G}_{\Delta_i}$ $(1 \leq i \leq n)$, is injective (i.e., its kernel is trivial).

Now assuming Theorem B we shall prove Theorem A:

NOTATION. In the sequel we shall denote the complement of Θ_i in $\Delta - \Delta_0$ by Θ_i' and $A_{\Theta_i'}$, $\mathscr{G}_{\Theta_i'}$, $G_{\Theta_i'}$, $\mathscr{M}_{\Theta_i'}$, $M_{\Theta_i'}$ and T_{Θ_i} by A_i, $\mathscr{G}_i$, G_i, $\mathscr{M}_i$, M_i and T_i respectively.

Proof of Theorem A. It is obvious from the Tits index ([7]) of G/k that given a connected normal k-simple subgroup of the derived group $\mathscr{G}$ of $\mathscr{M}$, there is an

$i(\leq r)$ such that G_{Θ_i}, and therefore M_{Θ_i}, contains it. Now since $\mathscr{G}$ is a direct product of its connected normal k-simple subgroups, we conclude that the subgroup generated by the $M_{\Theta_i}(k)$ $(1\leq i\leq r)$ contains $\mathscr{G}(k)$.

The inclusion of $\mathscr{M}$ in $\mathscr{M}_i$ induces a k-rational homomorphism $\mathscr{M}/\mathscr{G}\to\prod_{i=1}^r \mathscr{M}_i/\mathscr{G}_i$, and also a homomorphism $\mathscr{M}(k)/\mathscr{G}(k)\to\prod_{i=1}^r \mathscr{M}_i(k)/\mathscr{G}_i(k)$ of abstract groups. We now observe that the k-rational homomorphism $\mathscr{M}/\mathscr{G}\to\prod_{i=1}^r \mathscr{M}_i/\mathscr{G}_i$ is an isomorphism. In fact, as $\mathscr{M}_i$ is a semi-direct product of the torus $T_i = T_{\Theta_i}$ and the normal semi-simple subgroup $\mathscr{G}_i$, $\mathscr{M}_i/\mathscr{G}_i$ is isomorphic to $T_i(=T_{\Theta_i})$ and as $\mathscr{M}$ is a semi-direct product of $T_{\Delta-\Delta_0}$ and $\mathscr{G}$, $\mathscr{M}/\mathscr{G}$ is isomorphic to $T_{\Delta-\Delta_0}$. But $T_{\Delta-\Delta_0}$ is a direct product of the tori T_i since $\Delta-\Delta_0$ is a disjoint union of the Θ_i $(1\leq i\leq r)$. From this we conclude at once that the homomorphism $\mathscr{M}/\mathscr{G}\to\prod_{i=1}^r \mathscr{M}_i/\mathscr{G}_i$ is an isomorphism.

The commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{M}/\mathscr{G} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
1 & \longrightarrow & \prod_{i=1}^r \mathscr{G}_i & \longrightarrow & \prod_{i=1}^r \mathscr{M}_i & \longrightarrow & \prod_{i=1}^r \mathscr{M}_i/\mathscr{G}_i & \longrightarrow & 1,
\end{array}$$

gives the following commutative diagram involving Galois cohomology:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{G}(k) & \longrightarrow & \mathscr{M}(k) & \longrightarrow & (\mathscr{M}/\mathscr{G})(k) & \longrightarrow & H^1(k,\mathscr{G})\\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \downarrow\\
1 & \longrightarrow & \prod_{i=1}^r \mathscr{G}_i(k) & \longrightarrow & \prod_{i=1}^r \mathscr{M}_i(k) & \longrightarrow & \prod_{i=1}^r (\mathscr{M}_i/\mathscr{G}_i)(k) & \longrightarrow & \prod_{i=1}^r H^1(k,\mathscr{G}_i),
\end{array}$$

in which the horizontal rows are exact. Now since $H^1(k,\mathscr{G})\to\prod_{i=1}^r H^1(k,\mathscr{G}_i)$ is injective (Theorem B), we easily conclude from the second commutative diagram that the natural homomorphism $\mathscr{M}(k)/\mathscr{G}(k)\to\prod_{i=1}^r \mathscr{M}_i(k)/\mathscr{G}_i(k)$ is surjective; now since $\bigcap_{i=1}^r \mathscr{G}_i=\mathscr{G}$, it follows that the induced homomorphism $\mathscr{M}(k)/\mathscr{G}(k)\to\prod_{i=1}^r \mathscr{M}_i(k)/\mathscr{G}_i(k)$ is an isomorphism. It is evident from this that $\mathscr{M}(k)$ is generated by the subgroups $\mathscr{C}_i := \mathscr{M}(k)\cap\bigcap_{j\neq i}\mathscr{G}_j(k)$ $(i\leq r)$. But $\bigcap_{j\neq i}\mathscr{G}_j=\bigcap_{j\neq i}\mathscr{G}_{\Theta'_j}=\mathscr{G}_{\Theta_i}$. Therefore

$$\mathscr{C}_i=\mathscr{M}(k)\cap\bigcap_{j\neq i}\mathscr{G}_j(k)=(\mathscr{M}\cap\mathscr{G}_{\Theta_i})(k)=A_{\Theta_i}(k)\cdot M_{\Theta_i}(k)\quad \text{(cf. 1.5).}$$

As the subgroup generated by the $M_{\Theta_i}(k)$ $(1\leq i\leq r)$ contains $\mathscr{G}(k)$ and hence also $A_{\Theta_c}(k)$ for $1\leq c\leq r$ (recall that A_{Θ_c} is a normal subgroup of $\mathscr{G}$), we conclude that $M(k)(=\mathscr{M}(k))$ is generated by the subgroups $M_{\Theta_i}(k)$, $1\leq i\leq r$. This proves Theorem A.

§2. The Kneser–Tits problem for nonarchimedean local fields

We will now prove that the Kneser–Tits problem has an affirmative solution if k is a nonarchimedean local (i.e. locally compact, non-discrete, totally disconnected) field. For such a field it is known that $H^1(k, \mathcal{G})$ is trivial (recall that $\mathcal{G}$ is connected and simply connected): If k is a local field of characteristic zero, this was proved by M. Kneser ([3]) and then by Bruhat–Tits ([2]) for local fields of arbitrary characteristic. Thus, for a local field, Theorem B is an immediate consequence of this result. The first-named author originally proved Theorem A for local fields and deduced the Kneser–Tits conjecture in that case, the deduction is described below:

Let k be a nonarchimedean local field and let G be a k-simple, simply connected k-group of k-rank 1. Then ([1: 6.21(ii)]) there exists a finite separable extension K of k and an absolutely simple, simply connected group G defined over K, and of K-rank 1, such that $G = R_{\mathsf{K}/k}(\mathsf{G})$; K is again a nonarchimedean local field and from the classification (due to Kneser in characteristic zero and due to Bruhat–Tits in arbitrary characteristic) of absolutely simple groups over such a field we know that an absolutely simple, simply connected K-group of K-rank 1 is one of the following (note that there are no rank 1 forms of exceptional groups over a nonarchimedean local field):

(i) $SL_{2,D}$, where D is a finite dimensional central division algebra over K.

(ii) $SU(f)$, where f is a hermitian form, of Witt index 1, in 3 or 4 variables, defined in terms of a quadratic Galois extension K of K.

(iii) The spin group of a σ-quadratic form of Witt index 1 and rank 4 or 5, or the symplectic group of a σ-antihermitian form of rank 2 or 3 and Witt index 1; where σ is an involution of the quaternion central division algebra D over K such that the dimension of D^σ, the space of symmetric elements, is 3.

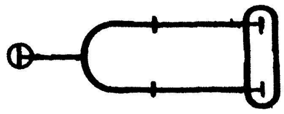
For each of the above groups G, it is known that $\mathsf{G}(\mathsf{K})^+ = \mathsf{G}(\mathsf{K})$; see, for example, [8].

§3

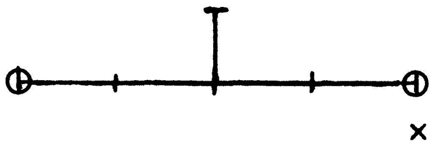
We shall now begin our proof of Theorem B. A standard argument which uses the fact that there is a finite separable extension K of k and an absolutely simple, simply connected group defined over K such that G is obtained from it by restriction of scalars ([1: 6.21(ii)]), and Shapiro's lemma in Galois cohomology (Serre [5: 5.8(b)]), allows us to assume that G is absolutely simple (and of k-rank ≥ 2). The proof (of Theorem B) uses the classification of absolutely simple groups in terms of Tits index (see Tits [7]); we shall assume familiarity with it.

From the Tits index of absolutely simple k-groups of k-rank ≥ 2 we see that if the Tits index is *not* one of the following six:

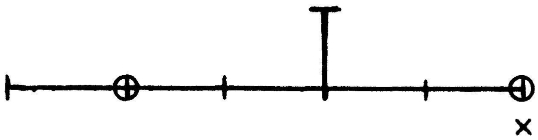
(i) ${}^2E_{6,2}$:



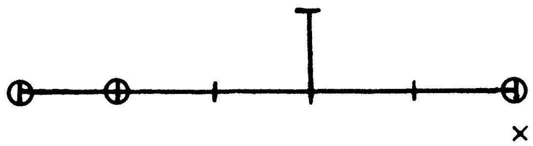
(ii) ${}^1E_{6,2}^{28}$:



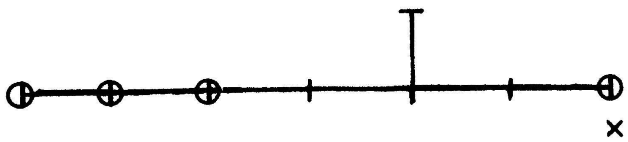
(iii) $E_{7,2}^{31}$:



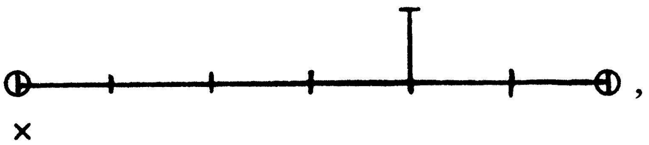
(iv) $E_{7,3}^{28}$:



(v) $E_{8,4}^{28}$:



(vi) $E_{8,2}^{66}$:



then there exists a Γ-orbit in $\Delta - \Delta_0$ such that if Θ is its complement in $\Delta - \Delta_0$, then, in the notation introduced in 1.5, G_Θ has at most one connected normal k-simple subgroup which meets $\mathcal{G}$ non-trivially and this connected normal k-simple subgroup is k-isomorphic to $R_{K/k}(\mathrm{G})$, where K is a Galois extension of k (of degree ≤ 2) and G is an absolutely simple K-isotropic group of inner type A. We know that $\mathcal{G}_\Theta$ is a direct product of A_Θ and G_Θ (and A_Θ is a factor of $\mathcal{G}$). Hence, the natural map $H^1(k, A_\Theta) \to H^1(k, \mathcal{G}_\Theta)$ is injective. Now it is not hard to see that to prove Theorem B for a group with Tits index different from the 6 indices listed above, it is enough to prove the following:

3.1. PROPOSITION. *Let* G *be an absolutely simple, simply connected group of inner type* A *which is defined and isotropic over a field* K. *Let* S *be a maximal* K-*split torus of* G *and* H *be a connected normal* K-*simple subgroup of the derived group of the centralizer of* S *in* G. *Then the natural map* $H^1(\mathsf{K}, \mathsf{H}) \to H^1(\mathsf{K}, \mathsf{G})$ *is injective.*

Proof. There exists a central division algebra D over K such that G is K-isomorphic to the group $SL_{m,\mathsf{D}}$, where $m = k$-rank G+1. We identify G with $SL_{m,\mathsf{D}}$ and for S take the K-split torus such that

$$\mathsf{S}(\mathsf{K}) = \left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} \;\middle|\; \lambda_i \in \mathsf{K}^\times, \Pi\lambda_i = 1 \right\}.$$

Then the centralizer of S is the diagonal subgroup of $SL_{m,\mathsf{D}}$, and there is a positive integer $i \le m$ such that H is the subgroup of the diagonal group consisting of the elements whose j-th diagonal entry is 1 for all $j \ne i$; H is clearly k-isomorphic to $SL_{1,\mathsf{D}}$. In the sequel we shall identify $SL_{1,\mathsf{D}}$ with H.

Now we consider the group $GL_{m,\mathsf{D}}$. We embedd $GL_{1,\mathsf{D}}$ in $GL_{m,\mathsf{D}}$ as the subgroup of the diagonal group consisting of the elements with the j-th diagonal entry 1 for all $j \ne i$. H is now the kernel of the reduced norm map $\mathrm{Nrd}: GL_{1,\mathsf{D}} \to$ Mult. The commutative diagram of K–groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SL_{m,\mathsf{D}} & \longrightarrow & GL_{m,\mathsf{D}} & \xrightarrow{\mathrm{Nrd}} & \mathrm{Mult} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & SL_{1,\mathsf{D}} & \longrightarrow & GL_{1,\mathsf{D}} & \xrightarrow{\mathrm{Nrd}} & \mathrm{Mult} & \longrightarrow & 1 \end{array}$$

gives the following commutative diagram in which the horizontal rows are exact in view of the vanishing[1] of $H^1(\mathsf{K}, GL_{n,\mathsf{D}})$ for all $n \ge 1$:

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & SL_m(\mathsf{D}) & \longrightarrow & GL_m(\mathsf{D}) & \xrightarrow{\mathrm{Nrd}} & \mathsf{K}^\times & \longrightarrow & H^1(\mathsf{K}, SL_{m,\mathsf{D}}) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \uparrow & & \\ 1 & \longrightarrow & SL_1(\mathsf{D}) & \longrightarrow & GL_1(\mathsf{D}) & \xrightarrow{\mathrm{Nrd}} & \mathsf{K}^\times & \longrightarrow & H^1(\mathsf{K}, SL_{1,\mathsf{D}}) & \longrightarrow & 1. \end{array}$$

From the theory of Dieudonné determinants it is obvious that the image of $GL_m(\mathsf{D})$ in $\mathsf{K}^\times$ equals that of $GL_1(\mathsf{D})$, from this and the above commutative diagram we conclude at once that $H^1(\mathsf{K}, SL_{1,\mathsf{D}}) \to H^1(\mathsf{K}, SL_{m,\mathsf{D}})$ is injective, i.e., in the notation of the proposition, the natural map $H^1(\mathsf{K}, \mathsf{H}) \to H^1(\mathsf{K}, \mathsf{G})$ is injective. This proves the proposition.

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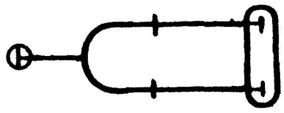
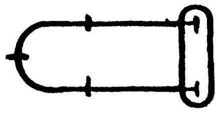
[1] This vanishing is a well-known theorem of Hilbert and Speiser.

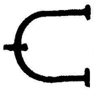
§4

We shall now prove Theorem B for groups with Tits index the first of the six exceptional ones listed in §3 i.e.,

$^2E_{6,2}$: 

Let Θ be the unique distinguished Γ-orbit consisting of 2 simple roots. Then the Tits index of $\mathscr{G}_\Theta (= \mathscr{G}_\Theta)$ is the following:



Moreover, the Tits index of $\mathscr{G}$ $(\subset \mathscr{G}_\Theta)$ is . Now let l be the quadratic Galois extension of k such that $\mathscr{G}_\Theta/l$ is an inner form of a split group. There is an anisotropic hermitian form f in 4 variables, defined in terms of the nontrivial automorphism σ of l/k, such that $\mathscr{G}$ is k-isomorphic to $SU(f)$, whereas $\mathscr{G}_\Theta$ is k-isomorphic to $SU(f \perp h)$, where h is the hyperbolic form in 2 variables. Now we consider the following commutative diagram in which the horizontal rows are exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SU(f \perp h) & \longrightarrow & U(f \perp h) & \longrightarrow & \mathscr{T} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ & \longrightarrow & SU(f) & \longrightarrow & U(f) & \longrightarrow & \mathscr{T} & \longrightarrow & 1; \end{array}$$

where $\mathscr{T}$ is the torus of dimension 1 defined and anisotropic over k which splits over l, (then $\mathscr{T}(k) = \{x \in l^\times \mid x\sigma(x) = 1\}$) and $U(f \perp h) \to \mathscr{T}$, as well as $U(f) \to \mathscr{T}$, are the determinant maps. It is obvious that both $U(f \perp h)(k) \to \mathscr{T}(k)$ and $U(f)(k) \to \mathscr{T}(k)$ are surjective. Therefore, the natural morphisms $H^1(k, SU(f \perp h)) \to H^1(k, U(f \perp h))$ and $H^1(k, SU(f)) \to H^1(k, U(f))$ are injective. On the other hand, Witt's cancellation theorem (for hermitian forms) implies at once that $H^1(k, U(f)) \to H^1(k, U(f \perp h))$ is injective. Now it is obvious that $H^1(k, SU(f)) \to H^1(k, SU(f \perp h))$ is injective, i.e., $H^1(k, \mathscr{G}) \to H^1(k, \mathscr{G}_\Theta)$ is injective. From this Theorem B follows for groups of type $^2E_{6,2}$.

§5

In this section we shall complete the proof of Theorem B by proving it for the groups of the remaining five exceptional types. We begin with the following two lemmas.

5.1. LEMMA. *Let* P *be a parabolic* k*-subgroup of a connected reductive* k*-group* G, *and* M *be a maximal reductive* k*-subgroup of* P. *Then the natural morphism*

$$H^1(k, \mathrm{M}) \to H^1(k, \mathrm{G})$$

is injective.

Proof. Since the natural map $\mathrm{G}(k) \to (\mathrm{G/P})(k)$ is surjective (Botel–Tits [1: 4.13(a)]), the morphism

$$H^1(k, \mathrm{P}) \to H^1(k, \mathrm{G})$$

is injective. Therefore, to prove the lemma, it suffices to observe that if U is the unipotent radical of P, then U is defined over k and $\mathrm{P} = \mathrm{M} \ltimes \mathrm{U}$ (a semi-direct product), and hence the natural morphism

$$H^1(k, \mathrm{M}) \to H^1(k, \mathrm{P})$$

is injective.

5.2. LEMMA. *Let* G *and* M *be as in the preceding lemma. Let* $\mathscr{G}$ *be the derived subgroup of* M *and* S *be the central torus of* M. *Let* $\mathscr{G}_0$ *and* $\mathscr{G}_*$ *be two connected normal* k*–subgroups of* $\mathscr{G}$ *such that* $\mathscr{G}$ *is an almost direct product of* $\mathscr{G}_0$ *and* $\mathscr{G}_*$. *Let* $\mathscr{C}$ *be the finite group scheme* $\mathscr{G}_0 \cap \mathrm{S}\mathscr{G}_*$. *Then the kernel of the natural morphism*

$$H^1(k, \mathscr{G}_0) \to H^1(k, \mathrm{G})$$

is contained in the image of

$$H^1(k, \mathscr{C}) \to H^1(k, \mathscr{G}_0).$$

Proof. Since the morphism $H^1(k, \mathrm{M}) \to H^1(k, \mathrm{G})$ is injective (Lemma 5.1), the kernel of $H^1(k, \mathscr{G}_0) \to H^1(k, \mathrm{G})$ coincides with the kernel of $H^1(k, \mathscr{G}_0) \to H^1(k, \mathrm{M})$. But $C := \mathrm{Ker}\,(H^1(k, \mathscr{G}_0) \to H^1(k, \mathrm{M}))$ is clearly contained in the kernel of the morphism $H^1(k, \mathscr{G}_0) \to H^1(k, \mathrm{M/S}\mathscr{G}_*)$ induced by the k-homomorphism $\mathscr{G}_0 \to \mathrm{M/S}\mathscr{G}_*$. Now as the natural homomorphism $\mathscr{G}_0/\mathscr{C} \to \mathrm{M/S}\mathscr{G}_*$ is a k-isomorphism, we conclude that C is contained in the kernel of $H^1(k, \mathscr{G}_0) \to H^1(k, \mathscr{G}_0/\mathscr{C})$, and from this the lemma is obvious.

Before proceeding further with the proof of Theorem B in the remaining exceptional cases, we shall recall some of the basic notions of the theory of quadratic forms.

5.3. Let p be the characteristic of k. If $p = 2$, let $\wp(k) = \{x + x^2 \mid x \in k\}$; $\wp(k)$ is a subgroup of k.

A quadratic form is said to be *nondefective* if the associated bilinear form is nondegenerate.

The *rank* (or the *dimension*) of a nondefective quadratic form is by definition the dimension of the underlying k-vector space, and the *Witt index* (over k) is the dimension of a maximal isotropic k-vector subspace.

For a quadratic form $\mathfrak{f}/k$, the *discriminant* (when $p=2$, it is also called the *Arf invariant*) $d(\mathfrak{f})$ will have the usual meaning. We recall that if $p\neq 2$, $d(\mathfrak{f})$ is an element of $k^\times/k^{\times 2}$, and if $p=2$, $d(\mathfrak{f})$ is an element of $k/\wp(k)$. We shall say that a quadratic form $\mathfrak{f}$ of rank $2n$ has *trivial signed discriminant* if its discriminant equals that of the hyperbolic form of rank $2n$, or, equivalently, if the special orthogonal group $SO(\mathfrak{f})$ is of *inner* type over k.

Let $\mathfrak{q}$ be a nondefective anisotropic quadratic form over k, of rank 2, and K be the quadratic Galois extension of k over which it is hyperbolic, then $d(\mathfrak{q})$ is the image (in $k^\times/k^{\times 2}$ if $p\neq 2$ and in $k/\wp(k)$ if $p=2$) of the norm of any element of $K^\times$ of trace zero if $p\neq 2$ and of trace 1 if $p=2$. Since $\mathfrak{q}$ is a multiple of the norm-form of K/k, we conclude that the discriminant $d(\mathfrak{q})$ determines $\mathfrak{q}$ up to a scalar multiple.

If over k, $\mathfrak{f}$ is an orthogonal direct sum of the nondefective quadratic forms $\mathfrak{q}_i$, $1\leq i\leq n$, of rank 2, then $d(\mathfrak{f})$ is the product of the $d(\mathfrak{q}_i)$ $(1\leq i\leq n)$ if $p\neq 2$, and it is the sum of the $d(\mathfrak{q}_i)$'s if $p=2$.

5.4. The *Witt invariant* $w(\mathfrak{f})$ of a nondefective quadratic form $\mathfrak{f}/k$ of *even* rank is by definition the class of the Clifford algebra of $\mathfrak{f}$ in the Brauer group of k; it is an element of order 2 in the Brauer group. We recall that if $\mathfrak{f}$ is a quadratic form of rank $2n$, with trivial signed discriminant, then the Witt invariant of $\mathfrak{f}$ has the following useful description: Let h be the hyperbolic form of rank $2n$ and let Spin (h) and $SO(h)$ be respectively the spin group and the special orthogonal group of h. Then since the discriminant of $\mathfrak{f}$ equals that of h, the quadratic form $\mathfrak{f}$ is obtained from h by twisting by a Galois cocycle with values in $SO(h)$. Let c denote the cohomology class in $H^1(k, SO(h))$ determined by the cocycle. Now consider the natural central isogeny:

$$1\to \mu_2\to \mathrm{Spin}\,(h)\to SO(h)\to 1,$$

where μ_2 is the kernel of the endomorphism $x\mapsto x^2$ of GL_1 (it is a finite group scheme defined over k). It gives rise to the following exact sequence:

$$H^1(k, \mathrm{Spin}\,(h))\to H^1(k, SO(h))\xrightarrow{\delta} H^2(k, \mu_2),$$

then $w(\mathfrak{f})=\delta(c)$ in the natural identification of $H^2(k, \mu_2)$ with the subgroup of the Brauer group of k consisting of the elements of order 2.

Now we observe that if $\mathfrak{f}$ is an *anisotropic* quadratic form of rank 6 which has trivial signed discriminant, then its Witt invariant is the class of a division algebra of degree 4 (i.e. of dimension 16). This follows immediately from the fact that Spin (h), where h is the hyperbolic form of rank 6, is isomorphic to SL_4 over the base field, and the only anisotropic *inner* twists of SL_4 are of the form $SL_{1,D}$, D a central division algebra of dimension 16 over the base field.

5.5. Now we assume that G is an absolutely simple, simply connected algebraic group of type one of the remaining five: ${}^1E^{28}_{6,2}$, $E^{31}_{7,2}$, $E^{28}_{7,3}$, $E^{28}_{8,4}$, $E^{66}_{8,2}$. Let $\mathcal{G}$ be (as in §1) the semi-simple anistropic kernel of G. Let $\mathcal{G}_0$ be the unique connected normal k-subgroup of $\mathcal{G}$ of type D_n $(n=4$ or $6)$ and in case G is of type $E^{31}_{7,2}$, let $\mathcal{G}_*$ be the connected normal k-subgroup of $\mathcal{G}$ of type A_1, in all the other cases let $\mathcal{G}_*$ be trivial. Then $\mathcal{G}$ is a direct product of $\mathcal{G}_0$ and $\mathcal{G}_*$.

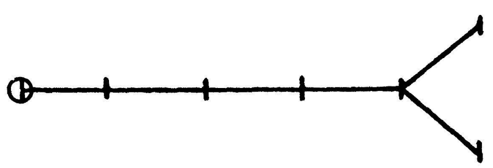
5.6. Let a be the simple root corresponding to the vertex in the Tits index marked with a cross (in §3) and let Θ be the set of distinguished simple roots $\neq a$. To establish Theorem B in the cases under consideration, it clearly suffices to prove that the natural morphism

$$H^1(k, \mathcal{G}_0) \to H^1(k, G_{\{a\}}) \times H^1(k, G_\Theta)$$

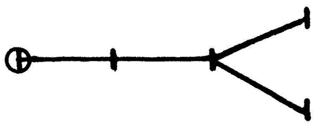
is injective.

Let $S_{\{a\}}$ (resp. S_Θ) be the maximal k-split torus of $G_{\{a\}}$ (resp. G_Θ) contained in S, and let $\mathbf{Z} = \mathcal{G}_0 \cap S_{\{a\}}$, $\mathcal{X} = \mathcal{G}_0 \cap S_\Theta \mathcal{G}_*$. Then it is easily seen, using the Tits indices, that both $\mathbf{Z}$ and $\mathcal{X}$ are k-isomorphic to the group scheme $\boldsymbol{\mu}_2$. Moreover, the center of $\mathcal{G}_0$ is a direct sum of $\mathbf{Z}$ and $\mathcal{X}$.

Now we observe that there is a nondefective, anisotropic quadratic form $\mathfrak{f}/k$ with trivial discriminant, $\mathfrak{f}$ of rank 12 in case G is of type $E^{66}_{8,2}$, and of rank 8 in all the other cases, such that $\mathcal{G}_0$ is k-isomorphic to Spin $(\mathfrak{f})$ and the kernel of the natural central isogeny π : Spin $(\mathfrak{f}) \to SO(\mathfrak{f})$ is $\mathbf{Z}$ $(= \mathcal{G}_0 \cap S_{\{a\}})$. This follows from the fact that $\mathcal{G}_0$ is the semi-simple anisotropic kernel of the simply connected, absolutely simple group $G_{\{a\}}$, and $G_{\{a\}}$ is the spin group of a nondefective quadratic form, of Witt index 1, which has trivial signed discriminant, since its Tits index is



in case G is of type $E^{66}_{8,2}$, and



in all the other cases. We shall identify $\mathcal{G}_0$ with Spin $(\mathfrak{f})$ in the sequel.

5.7. LEMMA. *If G is of type $E^{66}_{8,2}$, then the Witt invariant of $\mathfrak{f}$ over k is trivial.*

Proof. Any connected absolutely simple algebraic group of type E_8 is simply connected and is isomorphic to its automorphism group. Therefore, as the semi-simple anisotropic kernel of a k-form of type $E^{66}_{8,2}$ is an absolutely simple, simply connected group of type D_6, it is obtained from the split group of type E_8 by twisting by a Galois cocycle with values in the spin group of the hyperbolic form h of rank 12 (the spin group embedded as a maximal semi-simple k-subgroup of a parabolic k-subgroup of the split group of type E_8). Hence, $\mathfrak{f}$ is obtained from h by twisting by a cocycle whose cohomology class lies in the image of the natural morphism.

$$H^1(k, \mathrm{Spin}\,(h)) \to H^1(k, SO(h)).$$

This implies the lemma (see 5.4).

5.8. We now note, for future use, that the Witt index of the quadratic form $\mathfrak{f}$ is *even* over any extension of k: this is seen easily from the classification of inner k-forms of types E_6, E_7 and E_8 in terms of the Tits indices given in Tits [7].

5.9. Now let c be an element of the kernel of the natural morphism

$$H^1(k, \mathcal{G}_0) \to H^1(k, G_{\{a\}}) \times H^1(k, G_{\Theta}).$$

We shall prove that c is trivial, this will establish Theorem B (see 5.6).

Let Z and $\mathscr{Z}$ be as in 5.6. From Lemma 5.2 applied in turn to $\mathrm{G} = G_{\{a\}}$ and $\mathrm{G} = G_{\Theta}$, we conclude that c lies in the intersection of the images of the following natural morphisms:

$$H^1(k, \mathscr{Z}) \to H^1(k, \mathcal{G}_0)$$

and

$$H^1(k, \mathrm{Z}) \to H^1(k, \mathcal{G}_0).$$

Hence, in particular c is mapped onto the trivial element of $H^1(k, SO(\mathfrak{f}))$ under the central isogeny $(\mathcal{G}_0 =)\,\mathrm{Spin}\,(\mathfrak{f}) \to SO(\mathfrak{f})$ (whose kernel is Z).

We fix an element $\mathfrak{c} \in H^1(k, \mathscr{Z})$ which is mapped onto $c \in H^1(k, \mathcal{G}_0)$. Since $\mathscr{Z}$ is k-isomorphic to $\boldsymbol{\mu}_2$, there is a natural identification of $H^1(k, \mathscr{Z})$ with $k^\times/k^{\times 2}$. Let

$s \in k^\times$ be such that, in the identification of $H^1(k, \mathscr{Z})$ with $k^\times/k^{\times 2}$, $\mathfrak{c}$ corresponds to s. Now we observe that under the central isogeny $\mathrm{Spin}(\mathfrak{f}) \to SO(\mathfrak{f})$, $\mathscr{Z}$ is mapped onto the center of $\mathrm{SO}(\mathfrak{f})$ and from this we conclude that the image of the cohomology class c in $H^1(k, SO(\mathfrak{f}))$ corresponds to the quadratic form $s\mathfrak{f}$. But since the image of c in $H^1(k, SO(\mathfrak{f}))$ is trivial, $s\mathfrak{f}$ is equivalent to $\mathfrak{f}$ over k.

5.10. LEMMA. *Let φ be a nondefective anisotropic quadratic form such that φ is equivalent to $s\varphi$ $(s \in k^\times)$. Then there is a nondefective subform $\mathfrak{q}$ of φ of rank 2 such that $\mathfrak{q}$ is equivalent to $s\mathfrak{q}$.*

Proof. If s is a square in $k^\times$, the lemma is obvious, so we shall assume that s is not a square.

Let V be the k-vector space underlying φ and $\langle\, ,\rangle$ be the bilinear form associated with φ. We fix a $v \in V$ such that $\varphi(v) \neq 0$. Then since $\varphi \simeq s\varphi$, there is a $v' \in V$ such that $\varphi(v') = s\varphi(v)$. Now if $\langle v, v'\rangle \neq 0$, let $w = v'$; if $\langle v, v'\rangle = 0$, choose a $v_0 \in V$ such that $\langle v_0, v\rangle \cdot \langle v_0, v'\rangle \neq 0$, and let

$$w = v' - \frac{\langle v_0, v'\rangle}{\varphi(v_0)} v_0.$$

Then $\varphi(w) = \varphi(v') = s\varphi(v)$ and $\langle v, w\rangle \neq 0$. Also since s is not a square, w is not a scalar multiple of v. Let $\mathfrak{q}$ be the restriction of the quadratic form φ to the 2-dimensional subspace X spanned by v and w. It is easily seen that $\mathfrak{q}$ is a nondefective quadratic form. The k-linear automorphism of the vector subspace X defined by $v \mapsto w$, $w \mapsto sv$ provides an equivalence of the quadratic form $s\mathfrak{q}$ with $\mathfrak{q}$.

5.11. PROPOSITION. *There exist nondefective subforms $\mathfrak{q}_i, \mathfrak{q}_i'$ ($i = 1, 2$ if G is not of type $E_{8,2}^{66}$ and $i = 1, 2, 3$ if G is of type $E_{8,2}^{66}$) of $\mathfrak{f}$, of rank 2, such that $\mathfrak{f}$ is the orthogonal direct sum of the $\mathfrak{q}_i$'s and $\mathfrak{q}_i'$'s, and for each i*

(1) $\mathfrak{q}_i \simeq s\mathfrak{q}_i$, $\mathfrak{q}_i' \simeq s\mathfrak{q}_i'$

(2) *$\mathfrak{q}_i'$ is a scalar multiple of $\mathfrak{q}_i$; in particular $SO(\mathfrak{q}_i)$ is k-isomorphic to $SO(\mathfrak{q}_i')$.*

Proof. According to the preceding lemma, there is a nondefective subform $\mathfrak{q}_1$ of $\mathfrak{f}$ of rank 2 such that $\mathfrak{q}_1 \simeq s\mathfrak{q}_1$. Now let K be the quadratic Galois extension of k over which $\mathfrak{q}_1$ is hyperbolic, then $\mathfrak{q}_1$ is a multiple of the norm-form of K/k. Let $\mathfrak{q}_1^\perp$ be the orthogonal complement of $\mathfrak{q}_1$ in $\mathfrak{f}$. Then since the Witt index of $\mathfrak{f}$ over K is even (5.8), $\mathfrak{q}_1^\perp$ is isotropic over K. Therefore, there exist vectors v, w in the subspace corresponding to $\mathfrak{q}_1^\perp$ and $\alpha \in K - k$ such that

$$\mathfrak{q}_1^\perp(v + \alpha w) = \mathfrak{f}(v + \alpha w) = \mathfrak{f}(v) + \alpha\langle v, w\rangle + \alpha^2\mathfrak{f}(w) = 0.$$

Now since α is separable, we easily conclude that the restriction $\mathfrak{q}_1'$ of $\mathfrak{q}_1^\perp$ to the

2-dimensional subspace spanned by v and w is a nondefective quadratic form of rank 2 which is isotropic (and hence hyperbolic) over K. Therefore, $\mathfrak{q}_1'$ is a multiple of the norm-form of K/k. As $\mathfrak{q}_1$ is also a multiple of the norm-form of K/k and $\mathfrak{q}_1 \simeq s\mathfrak{q}_1$, we conclude that $\mathfrak{q}_1'$ is a multiple of $\mathfrak{q}_1$ and $\mathfrak{q}_1' \simeq s\mathfrak{q}_1'$.

Now let $\mathfrak{f}_1 = \mathfrak{q}_1 \perp \mathfrak{q}_1'$. Then the discriminant of $\mathfrak{f}_1$ is trivial. Let $\mathfrak{f}_2$ be the orthogonal complement of $\mathfrak{f}_1$ in $\mathfrak{f}$. Then the discriminant of $\mathfrak{f}_2$ is trivial and as $\mathfrak{f}_1 \simeq s\mathfrak{f}_1$, by Witt's cancellation theorem $\mathfrak{f}_2 \simeq s\mathfrak{f}_2$. We shall now consider the cases where $\mathfrak{f}$ is of rank 8. Let $\mathfrak{q}_2$ be a nondefective subform of $\mathfrak{f}_2$ of rank 2 such that $\mathfrak{q}_2 \simeq s\mathfrak{q}_2$ (Lemma 5.10) and $\mathfrak{q}_2'$ be its orthogonal complement in $\mathfrak{f}_2$. Then the discriminant of $\mathfrak{q}_2$ equals that of $\mathfrak{q}_2'$ and hence $\mathfrak{q}_2'$ is a scalar multiple of $\mathfrak{q}_2$ (5.3), in particular $\mathfrak{q}_2' \simeq s\mathfrak{q}_2'$.

Now we consider the case where $\mathfrak{f}$ is of rank 12, then G is of type $E_{8,2}^{66}$, $\mathfrak{f}_2$ is an anisotropic form of rank 8 and trivial discriminant. We claim that the Witt index of $\mathfrak{f}_2$ over any quadratic Galois extension of k is even. To prove this we consider a quadratic Galois extension l of k such that $\mathfrak{f}_2$ is isotropic over l. Then as the discriminant of $\mathfrak{f}_2$ is trivial, the Witt index of $\mathfrak{f}_2$ over l can not be 3; assume, if possible, that it is 1. Then since the Witt invariant of $\mathfrak{f}/k$ is zero (Lemma 5.7), the Witt invariant of $\mathfrak{f}_1/l$ equals that of $\mathfrak{f}_2/l$. Now since by hypothesis $\mathfrak{f}_2/l$ is of Witt index 1, over l it is an orthogonal direct sum of the hyperbolic form of rank 2 and an anisotropic form of rank 6. Therefore, the Witt invariant of $\mathfrak{f}_2/l$ is the class of a division algebra of degree 4 in the Brauer group of l (5.4). But since $\mathfrak{f}_1/k$ is an anisotropic form of rank 4 of trivial discriminant, it is a multiple of the norm-form of a *quaternion* division algebra D, and its Witt invariant is the class of D in the Brauer group of k. Therefore, the Witt invariant of $\mathfrak{f}_1/l$ is the class of $D \otimes_k l$. We conclude thus that the class of a division algebra of degree 4 (in the Brauer group of l) contains $D \otimes_k l$. This is absurd, and hence the Witt index of $\mathfrak{f}_2$ over l can not be 1. This proves that the Witt index of $\mathfrak{f}_2$ over l is even. Now since $\mathfrak{f}_2$ is of rank 8, we can prove, as before, that there exist 4 nondefective quadratic forms $\mathfrak{q}_2$, $\mathfrak{q}_2'$, $\mathfrak{q}_3$ and $\mathfrak{q}_3'$ of rank 2 such that $\mathfrak{f}_2$ is an orthogonal direct sum of these; $\mathfrak{q}_i \simeq s\mathfrak{q}_i$, $\mathfrak{q}_i' \simeq s\mathfrak{q}_i'$ and $\mathfrak{q}_i'$ is a scalar multiple of $\mathfrak{q}_i$ $(i = 2, 3)$. This proves the proposition.

5.12. We fix a set of nondefective subforms $\mathfrak{q}_i$, $\mathfrak{q}_i'$, of $\mathfrak{f}$, of rank 2, as in the preceding proposition. Let $\bar{T}_i = SO(\mathfrak{q}_i)(\subset SO(\mathfrak{f}))$ and $\bar{T}_i' = SO(\mathfrak{q}_i')(\subset SO(\mathfrak{f}))$. Then (for all i) $\bar{T}_i$ and $\bar{T}_i'$ are isomorphic k-tori of dimension 1. Let $\pi : \mathrm{Spin}(\mathfrak{f}) \to SO(\mathfrak{f})$ be the usual central isogeny and let $T_i = \pi^{-1}(\bar{T}_i)$ and $T_i' = \pi^{-1}(\bar{T}_i')$. Then for all i, T_i and T_i' are isomorphic k-tori.

$\prod_i (\bar{T}_i \times \bar{T}_i')$ is a maximal torus of $SO(\mathfrak{f})$ and there is a unique k-embedding of μ_2 into $\bar{T}_i$, as well as in $\bar{T}_i'$. The center C of $SO(\mathfrak{f})$ is the "diagonally" embedded μ_2 in $\prod_i (\bar{T}_i \times \bar{T}_i')$.

Let θ_i be a fixed k-isomorphism of $\bar{T}_i$ onto $\bar{T}_i'$ (note that there are exactly two distinct k-isomorphisms of $\bar{T}_i$ onto $\bar{T}'_i$), we shall let θ_i also denote the induced k-isomorphism of T_i onto T'_i. Let $\bar{\mathcal{T}}_i = \{x \cdot \theta_i(x) \mid x \in \bar{T}_i\}$, $\mathcal{T}_i = \{x \cdot \theta_i(x) \mid x \in T_i\}$, and let $\bar{\mathcal{T}} = \prod_i \bar{\mathcal{T}}_i$, $\mathcal{T} = \prod_i \mathcal{T}_i$. It is easily seen that $\forall i$, the restriction of π to $\mathcal{T}_i$ is an isomorphism onto $\bar{\mathcal{T}}_i$ and hence the restriction of π to $\mathcal{T}$ is an isomorphism onto $\bar{\mathcal{T}}$. Also, if necessary after changing the isomorphism θ_i for any one i, we can ensure that $\mathcal{Z} \subset \mathcal{T}$. We shall assume in the sequel that this is the case. Now we assert that $\mathfrak{c}(\in H^1(k, \mathcal{Z}))$ is mapped onto the trivial element of $H^1(k, \mathcal{T})$ under the morphism induced by the inclusion $\mathcal{Z} \to \mathcal{T}$. To see this, we observe that the image of $\mathfrak{c}$ in $H^1(k, \bar{\mathcal{T}})$ is trivial: this is a simple consequence of the fact that for $\forall i$, $\mathfrak{q}_i \simeq s\mathfrak{q}_i$. Now since $\pi|_{\mathcal{T}} : \mathcal{T} \to \bar{\mathcal{T}}$ is a k-isomorphism which maps $\mathcal{Z}$ onto C, our assertion follows. It is obvious now that c, being the image of $\mathfrak{c}$ in $H^1(k, \mathcal{G}_0)$, is trivial because $\mathcal{Z} \subset \mathcal{T} \subset \mathcal{G}_0$. This completes the proof of Theorem B.

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