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On the Kneser–Tits problem

GOPAL PRASAD and M. S. RAGHUNATHAN

Introduction

Let G be a semi-simple, simply connected algebraic group defined, isotropic and simple over a (commutative) field k . Let $G(k)$ be the group of k -rational points of G and $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k -rational points of the unipotent radicals of parabolic k -subgroups of G . The *Kneser–Tits problem* referred to in the title is the following: *Is $G(k)^+ = G(k)$ for every G as above?* The main object of this paper is to prove that *for a field k , the Kneser–Tits problem has an affirmative solution iff $G(k)^+ = G(k)$ for all simply connected, k -simple groups G of k -rank 1.* This reduction of the Kneser–Tits problem is an immediate consequence of Theorem A proved below. After this work was complete, we learnt from Armand Borel that Theorem A was conjectured by Jacques Tits in a lecture at the Institute for Advanced Study (Princeton), and was proved by him for some fields by a method different from ours.

The proof of Theorem A depends on a theorem on Galois cohomology (Theorem B) which may be of some independent interest.

In case k is a local field, the Kneser–Tits problem has an affirmative solution. This was essentially proved by V. P. Platonov [4] using the known results on classical groups and detailed knowledge of classification. He also gave the first examples of fields for which the Kneser–Tits problem has a negative answer (see Tits [8] for a survey). In §2 of this paper we use the reduction of the Kneser–Tits problem to rank 1 groups stated above to provide a simple proof of its affirmative solution for the local fields. This simple proof devised by the first-named author was the starting point of the present work. We hope to come back to the problem for global fields in the near future.

1.1. Let k be a (commutative) field, \mathcal{K} be a fixed separable closure of k and let $\Gamma = \text{Gal}(\mathcal{K}/k)$. Let G be a semi-simple, simply connected group defined over k . Let S be a maximal k -split torus of G . Let $\dim S = r$ ($:= k$ -rank G). We assume that $r > 0$ i.e., G is isotropic over k ; we also assume that G is k -simple, i.e., it has no proper *connected* normal subgroup defined over k .

1.2. Let T be a maximal torus of G containing S and defined over k . Let Φ be the set of roots of G relative to T . We fix a Borel subgroup B defined over \mathcal{K} , $B \supset T$, and contained in a minimal parabolic k -subgroup of G . This induces an ordering on Φ ; let Δ be the set of all simple roots with respect to this ordering. Let Δ_0 be the subset of Δ consisting of those roots which are trivial on S . There is an action of Γ on Δ (the $*$ -action) defined in Tits [7: §2.3]; both Δ_0 and $\Delta - \Delta_0$ are stable under this action. Since k -rank $G = r$, there are r Γ -orbits in $\Delta - \Delta_0$.

1.3. For a simple root a , let U_a and U_{-a} be the root subgroups associated with a and $-a$ respectively; U_a and U_{-a} are connected unipotent \mathcal{K} -subgroups of G , of dimension 1, normalized by T . Since G is simply connected, $\forall a \in \Delta$, the subgroup generated by U_a and U_{-a} is \mathcal{K} -isomorphic to SL_2 ; let T_a be its intersection with T , then T_a is a one dimensional torus defined over \mathcal{K} , and as G is simply connected, T is a direct product of the $T_a (a \in \Delta)$. For a subset Θ of Δ , let T_Θ be the subtorus generated by the tori $T_a, a \in \Theta$.

1.4. For a k -subgroup H of G , as usual, $H(k)$ will denote the group of k -rational points of H , and $H(k)^+$ will denote the normal subgroup of $H(k)$ generated by the k -rational points of the unipotent radicals of the parabolic k -subgroups of H .

1.5. For a Γ -stable subset Θ of $\Delta - \Delta_0$, let T^Θ be the identity component of $\bigcap_{\theta \in \Theta \cup \Delta_0} \text{Ker } \theta$. Let \mathcal{M}_Θ be the centralizer of T^Θ in G . Then \mathcal{M}_Θ is a connected reductive subgroup defined over k ; in fact it is a Levi k -subgroup of a parabolic k -subgroup of G (cf. Tits [7: §2.5.4]). Let \mathcal{G}_Θ be the derived subgroup of \mathcal{M}_Θ . Then \mathcal{G}_Θ is a semi-simple, simply connected, k -subgroup of G , and hence it is a direct product of its connected k -simple normal subgroups. Let A_Θ be the product of all connected k -simple normal subgroups of \mathcal{G}_Θ which are anisotropic over k , and G_Θ be the product of all connected k -simple k -isotropic subgroups. Then the k -rank of G_Θ is equal to the number of Γ -orbits in Θ , and \mathcal{G}_Θ is a direct product (over k) of A_Θ and G_Θ . It is easily seen that \mathcal{M}_Θ is a semi-direct product of $T_{\Theta'}$ and \mathcal{G}_Θ ; where Θ' is the complement of Θ in $\Delta - \Delta_0$. Hence, the natural homomorphism: $\mathcal{M}_\Theta(\mathcal{K}) \rightarrow (\mathcal{M}_\Theta/\mathcal{G}_\Theta)(\mathcal{K})$ is surjective.

We shall denote the centralizer of S in G by \mathcal{M} and sometimes also by M . Let \mathcal{G} be the derived group of \mathcal{M} . Then $\mathcal{M} = \mathcal{M}_\emptyset$; $\mathcal{G} = \mathcal{G}_\emptyset$ (where \emptyset is the empty subset of $\Delta - \Delta_0$). \mathcal{G} is anisotropic over k , and it is easy to see that A_Θ is a normal subgroup of \mathcal{G} for every Γ -stable subset Θ of $\Delta - \Delta_0$.

For a Γ -stable subset Θ of $\Delta - \Delta_0$, let S_Θ be the maximal k -split torus of G_Θ contained in S , and let M_Θ denote the centralizer of S_Θ in G_Θ . Then M_Θ is a connected reductive k -subgroup. Moreover, since \mathcal{G}_Θ is a direct product of G_Θ

and A_{θ} , the centralizer of S_{θ} in \mathcal{G}_{θ} is just $A_{\theta} \cdot M_{\theta}$ (direct product). It is easy to see, by considering the reductive groups $S \cdot G_{\theta}$ and $S \cdot \mathcal{G}_{\theta}$, that $M_{\theta} = M \cap G_{\theta}$ and $\mathcal{M} \cap \mathcal{G}_{\theta} = A_{\theta} \cdot M_{\theta}$.

1.6. Let θ_i , $i = 1, \dots, r$, be the Γ -orbits in $\Delta - \Delta_0$. Recall that G_{θ_i} is a semi-simple simply connected k -subgroup of G of k -rank 1; it is k -simple since it does not contain any connected normal k -anisotropic subgroup. It follows from the Bruhat-decomposition that $G(k) = M(k) \cdot G(k)^+$. Thus $G(k)^+ = G(k)$ if and only if $G(k)^+ \supset M(k)$. Similarly as $G_{\theta}(k) = M_{\theta}(k) \cdot G_{\theta}(k)^+$, $G_{\theta}(k)^+ = G_{\theta}(k)$ if and only if $G_{\theta}(k)^+ \supset M_{\theta}(k)$. In view of these observations, the following Theorem A implies that the Kneser–Tits problem for a field k has an affirmative solution if and only if for every k -simple simply connected group G of k -rank 1, $G(k)^+ = G(k)$.

THEOREM A. Assume that k -rank $G \geq 2$. Then $M(k)$ is generated by the subgroups $M_{\theta_i}(k)$ ($1 \leq i \leq r$).

1.7. Remark. If k is an infinite field, then $G(k)^+$ has no proper non-central normal subgroups (Tits [6: Main Theorem]), in particular it is *perfect* i.e. $(G(k)^+, G(k)^+) = G(k)^+$. Now Theorem A implies that to prove that $G(k)$ is perfect for all k -simple, simply connected k -isotropic G , it suffices to prove that this is so for all k -simple, simply connected groups of k -rank 1.

We shall prove Theorem A using the following:

THEOREM B. For $i \leq n$, let Δ_i be a Γ ($= \text{Gal}(\mathcal{K}/k)$)-stable subset of $\Delta - \Delta_0$ such that $\bigcap_{i=1}^n \Delta_i = \emptyset$. Then the natural morphism:

$$H^1(k, \mathcal{G}) \rightarrow \prod_{i=1}^n H^1(k, \mathcal{G}_{\Delta_i}),$$

induced by the inclusion of \mathcal{G} in \mathcal{G}_{Δ_i} ($1 \leq i \leq n$), is injective (i.e., its kernel is trivial).

Now assuming Theorem B we shall prove Theorem A:

NOTATION. In the sequel we shall denote the complement of θ_i in $\Delta - \Delta_0$ by θ'_i and $A_{\theta'_i}$, $\mathcal{G}_{\theta'_i}$, $G_{\theta'_i}$, $\mathcal{M}_{\theta'_i}$, $M_{\theta'_i}$ and $T_{\theta'_i}$ by A_i , \mathcal{G}_i , G_i , \mathcal{M}_i , M_i and T_i respectively.

Proof of Theorem A. It is obvious from the Tits index ([7]) of G/k that given a connected normal k -simple subgroup of the derived group \mathcal{G} of \mathcal{M} , there is an

$i(\leq r)$ such that G_{Θ_i} , and therefore M_{Θ_i} , contains it. Now since \mathcal{G} is a direct product of its connected normal k -simple subgroups, we conclude that the subgroup generated by the $M_{\Theta_i}(k)$ ($1 \leq i \leq r$) contains $\mathcal{G}(k)$.

The inclusion of \mathcal{M} in \mathcal{M}_i induces a k -rational homomorphism $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$, and also a homomorphism $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$ of abstract groups. We now observe that the k -rational homomorphism $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$ is an isomorphism. In fact, as \mathcal{M}_i is a semi-direct product of the torus $T_i = T_{\Theta_i}$ and the normal semi-simple subgroup \mathcal{G}_i , $\mathcal{M}_i/\mathcal{G}_i$ is isomorphic to $T_i (= T_{\Theta_i})$ and as \mathcal{M} is a semi-direct product of $T_{\Delta-\Delta_0}$ and \mathcal{G} , \mathcal{M}/\mathcal{G} is isomorphic to $T_{\Delta-\Delta_0}$. But $T_{\Delta-\Delta_0}$ is a direct product of the tori T_i since $\Delta - \Delta_0$ is a disjoint union of the Θ_i ($1 \leq i \leq r$). From this we conclude at once that the homomorphism $\mathcal{M}/\mathcal{G} \rightarrow \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i$ is an isomorphism.

The commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}/\mathcal{G} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \prod_{i=1}^r \mathcal{G}_i & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i/\mathcal{G}_i \longrightarrow 1, \end{array}$$

gives the following commutative diagram involving Galois cohomology:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{G}(k) & \longrightarrow & \mathcal{M}(k) & \longrightarrow & (\mathcal{M}/\mathcal{G})(k) \longrightarrow H^1(k, \mathcal{G}) \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 1 & \longrightarrow & \prod_{i=1}^r \mathcal{G}_i(k) & \longrightarrow & \prod_{i=1}^r \mathcal{M}_i(k) & \longrightarrow & \prod_{i=1}^r (\mathcal{M}_i/\mathcal{G}_i)(k) \longrightarrow \prod_{i=1}^r H^1(k, \mathcal{G}_i), \end{array}$$

in which the horizontal rows are exact. Now since $H^1(k, \mathcal{G}) \rightarrow \prod_{i=1}^r H^1(k, \mathcal{G}_i)$ is injective (Theorem B), we easily conclude from the second commutative diagram that the natural homomorphism $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$ is surjective; now since $\bigcap_{i=1}^r \mathcal{G}_i = \mathcal{G}$, it follows that the induced homomorphism $\mathcal{M}(k)/\mathcal{G}(k) \rightarrow \prod_{i=1}^r \mathcal{M}_i(k)/\mathcal{G}_i(k)$ is an isomorphism. It is evident from this that $\mathcal{M}(k)$ is generated by the subgroups $\mathcal{C}_i := \mathcal{M}(k) \cap \bigcap_{j \neq i} \mathcal{G}_j(k)$ ($i \leq r$). But $\bigcap_{j \neq i} \mathcal{G}_j = \bigcap_{j \neq i} \mathcal{G}_{\Theta_j} = \mathcal{G}_{\Theta_i}$. Therefore

$$\mathcal{C}_i = \mathcal{M}(k) \cap \bigcap_{j \neq i} \mathcal{G}_j(k) = (\mathcal{M} \cap \mathcal{G}_{\Theta_i})(k) = A_{\Theta_i}(k) \cdot M_{\Theta_i}(k) \quad (\text{cf. 1.5}).$$

As the subgroup generated by the $M_{\Theta_i}(k)$ ($1 \leq i \leq r$) contains $\mathcal{G}(k)$ and hence also $A_{\Theta_c}(k)$ for $1 \leq c \leq r$ (recall that A_{Θ_c} is a normal subgroup of \mathcal{G}), we conclude that $\mathcal{M}(k) (= \mathcal{M}(k))$ is generated by the subgroups $M_{\Theta_i}(k)$, $1 \leq i \leq r$. This proves Theorem A.

§2. The Kneser–Tits problem for nonarchimedean local fields

We will now prove that the Kneser–Tits problem has an affirmative solution if k is a nonarchimedean local (i.e. locally compact, non-discrete, totally disconnected) field. For such a field it is known that $H^1(k, \mathcal{G})$ is trivial (recall that \mathcal{G} is connected and simply connected): If k is a local field of characteristic zero, this was proved by M. Kneser ([3]) and then by Bruhat–Tits ([2]) for local fields of arbitrary characteristic. Thus, for a local field, Theorem B is an immediate consequence of this result. The first-named author originally proved Theorem A for local fields and deduced the Kneser–Tits conjecture in that case, the deduction is described below:

Let k be a nonarchimedean local field and let G be a k -simple, simply connected k -group of k -rank 1. Then ([1: 6.21(ii)]) there exists a finite separable extension K of k and an absolutely simple, simply connected group G defined over K , and of K -rank 1, such that $G = R_{K/k}(G)$; K is again a nonarchimedean local field and from the classification (due to Kneser in characteristic zero and due to Bruhat–Tits in arbitrary characteristic) of absolutely simple groups over such a field we know that an absolutely simple, simply connected K -group of K -rank 1 is one of the following (note that there are no rank 1 forms of exceptional groups over a nonarchimedean local field):

- (i) $SL_{2,D}$, where D is a finite dimensional central division algebra over K .
- (ii) $SU(f)$, where f is a hermitian form, of Witt index 1, in 3 or 4 variables, defined in terms of a quadratic Galois extension K of k .
- (iii) The spin group of a σ -quadratic form of Witt index 1 and rank 4 or 5, or the symplectic group of a σ -antihermitian form of rank 2 or 3 and Witt index 1; where σ is an involution of the quaternion central division algebra D over K such that the dimension of D^σ , the space of symmetric elements, is 3.

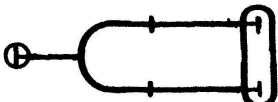
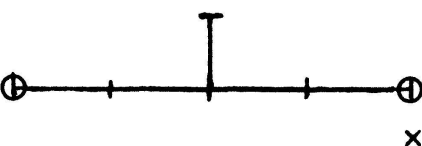
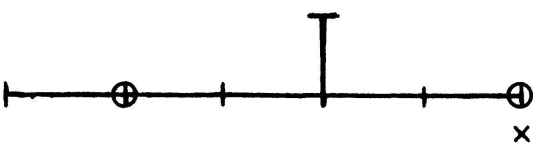
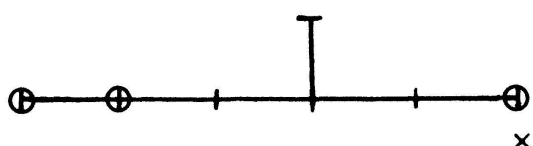
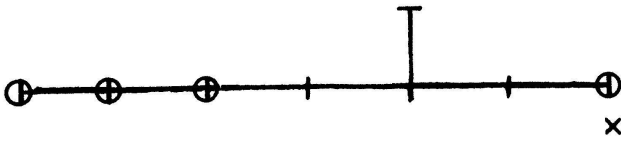
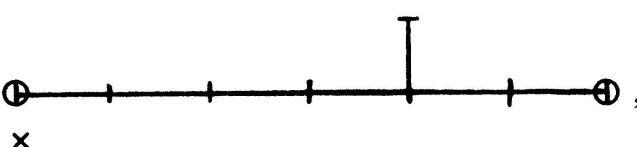
For each of the above groups G , it is known that $G(K)^+ = G(K)$; see, for example, [8].

§3

We shall now begin our proof of Theorem B. A standard argument which uses the fact that there is a finite separable extension K of k and an absolutely simple, simply connected group defined over K such that G is obtained from it by restriction of scalars ([1: 6.21(ii)]), and Shapiro’s lemma in Galois cohomology (Serre [5: 5.8(b)]), allows us to assume that G is absolutely simple (and of k -rank ≥ 2). The proof (of Theorem B) uses the classification of absolutely simple groups in terms of Tits index (see Tits [7]); we shall assume familiarity with it.

From the Tits index of absolutely simple k -groups of k -rank ≥ 2 we see that if

the Tits index is *not* one of the following six:

- (i) ${}^2E_{6,2}$: 
- (ii) ${}^1E_{6,2}^{28}$: 
- (iii) $E_{7,2}^{31}$: 
- (iv) $E_{7,3}^{28}$: 
- (v) $E_{8,4}^{28}$: 
- (vi) $E_{8,2}^{66}$: 

then there exists a Γ -orbit in $\Delta - \Delta_0$ such that if Θ is its complement in $\Delta - \Delta_0$, then, in the notation introduced in 1.5, G_Θ has at most one connected normal k -simple subgroup which meets \mathcal{G} non-trivially and this connected normal k -simple subgroup is k -isomorphic to $R_{K/k}(G)$, where K is a Galois extension of k (of degree ≤ 2) and G is an absolutely simple K -isotropic group of inner type A . We know that \mathcal{G}_Θ is a direct product of A_Θ and G_Θ (and A_Θ is a factor of \mathcal{G}). Hence, the natural map $H^1(k, A_\Theta) \rightarrow H^1(k, \mathcal{G}_\Theta)$ is injective. Now it is not hard to see that to prove Theorem B for a group with Tits index different from the 6 indices listed above, it is enough to prove the following:

3.1. PROPOSITION. *Let G be an absolutely simple, simply connected group of inner type A which is defined and isotropic over a field K . Let S be a maximal K -split torus of G and H be a connected normal K -simple subgroup of the derived group of the centralizer of S in G . Then the natural map $H^1(K, H) \rightarrow H^1(K, G)$ is injective.*

Proof. There exists a central division algebra D over K such that G is K -isomorphic to the group $SL_{m,D}$, where $m = k\text{-rank } G + 1$. We identify G with $SL_{m,D}$ and for S take the K -split torus such that

$$S(K) = \left\{ \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ 0 & & & \lambda_m \end{bmatrix} \mid \lambda_i \in K^\times, \prod \lambda_i = 1 \right\}.$$

Then the centralizer of S is the diagonal subgroup of $SL_{m,D}$, and there is a positive integer $i \leq m$ such that H is the subgroup of the diagonal group consisting of the elements whose j -th diagonal entry is 1 for all $j \neq i$; H is clearly k -isomorphic to $SL_{1,D}$. In the sequel we shall identify $SL_{1,D}$ with H .

Now we consider the group $GL_{m,D}$. We embed $GL_{1,D}$ in $GL_{m,D}$ as the subgroup of the diagonal group consisting of the elements with the j -th diagonal entry 1 for all $j \neq i$. H is now the kernel of the reduced norm map $\text{Nrd}: GL_{1,D} \rightarrow \text{Mult}$. The commutative diagram of K -groups:

$$\begin{array}{ccccccc} 1 & \longrightarrow & SL_{m,D} & \longrightarrow & GL_{m,D} & \xrightarrow{\text{Nrd}} & \text{Mult} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & SL_{1,D} & \longrightarrow & GL_{1,D} & \xrightarrow{\text{Nrd}} & \text{Mult} \longrightarrow 1 \end{array}$$

gives the following commutative diagram in which the horizontal rows are exact in view of the vanishing⁽¹⁾ of $H^1(K, GL_{n,D})$ for all $n \geq 1$:

$$\begin{array}{ccccccc} 1 & \longrightarrow & SL_m(D) & \longrightarrow & GL_m(D) & \xrightarrow{\text{Nrd}} & K^\times \longrightarrow H^1(K, SL_{m,D}) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & SL_1(D) & \longrightarrow & GL_1(D) & \xrightarrow{\text{Nrd}} & K^\times \longrightarrow H^1(K, SL_{1,D}) \longrightarrow 1. \end{array}$$

From the theory of Dieudonné determinants it is obvious that the image of $GL_m(D)$ in K^\times equals that of $GL_1(D)$, from this and the above commutative diagram we conclude at once that $H^1(K, SL_{1,D}) \rightarrow H^1(K, SL_{m,D})$ is injective, i.e., in the notation of the proposition, the natural map $H^1(K, H) \rightarrow H^1(K, G)$ is injective. This proves the proposition.

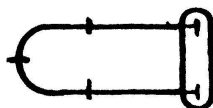
¹ This vanishing is a well-known theorem of Hilbert and Speiser.


§4

We shall now prove Theorem B for groups with Tits index the first of the six exceptional ones listed in §3 i.e.,

$${}^2E_{6,2}: \quad \text{Diagram: A circle with a dot inside, connected to a horizontal tube with two vertical lines on each side, and a vertical line on the right side of the tube.$$

Let Θ be the unique distinguished Γ -orbit consisting of 2 simple roots. Then the Tits index of $\mathcal{G}_\Theta (= \mathcal{G}_\Theta)$ is the following:



Moreover, the Tits index of $\mathcal{G} (\subset \mathcal{G}_\Theta)$ is . Now let l be the quadratic Galois extension of k such that \mathcal{G}_Θ/l is an inner form of a split group. There is an anisotropic hermitian form f in 4 variables, defined in terms of the nontrivial automorphism σ of l/k , such that \mathcal{G} is k -isomorphic to $SU(f)$, whereas \mathcal{G}_Θ is k -isomorphic to $SU(f \perp h)$, where h is the hyperbolic form in 2 variables. Now we consider the following commutative diagram in which the horizontal rows are exact:

$$\begin{array}{ccccccc} 1 & \longrightarrow & SU(f \perp h) & \longrightarrow & U(f \perp h) & \longrightarrow & \mathcal{T} \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ & \longrightarrow & SU(f) & \longrightarrow & U(f) & \longrightarrow & \mathcal{T} \longrightarrow 1; \end{array}$$

where \mathcal{T} is the torus of dimension 1 defined and anisotropic over k which splits over l , (then $\mathcal{T}(k) = \{x \in l^\times \mid x\sigma(x) = 1\}$) and $U(f \perp h) \rightarrow \mathcal{T}$, as well as $U(f) \rightarrow \mathcal{T}$, are the determinant maps. It is obvious that both $U(f \perp h)(k) \rightarrow \mathcal{T}(k)$ and $U(f)(k) \rightarrow \mathcal{T}(k)$ are surjective. Therefore, the natural morphisms $H^1(k, SU(f \perp h)) \rightarrow H^1(k, U(f \perp h))$ and $H^1(k, SU(f)) \rightarrow H^1(k, U(f))$ are injective. On the other hand, Witt's cancellation theorem (for hermitian forms) implies at once that $H^1(k, U(f)) \rightarrow H^1(k, U(f \perp h))$ is injective. Now it is obvious that $H^1(k, SU(f)) \rightarrow H^1(k, SU(f \perp h))$ is injective, i.e., $H^1(k, \mathcal{G}) \rightarrow H^1(k, \mathcal{G}_\Theta)$ is injective. From this Theorem B follows for groups of type ${}^2E_{6,2}$.

§5

In this section we shall complete the proof of Theorem B by proving it for the groups of the remaining five exceptional types. We begin with the following two lemmas.

5.1. LEMMA. *Let P be a parabolic k -subgroup of a connected reductive k -group G , and M be a maximal reductive k -subgroup of P . Then the natural morphism*

$$H^1(k, M) \rightarrow H^1(k, G)$$

is injective.

Proof. Since the natural map $G(k) \rightarrow (G/P)(k)$ is surjective (Borel–Tits [1: 4.13(a)]), the morphism

$$H^1(k, P) \rightarrow H^1(k, G)$$

is injective. Therefore, to prove the lemma, it suffices to observe that if U is the unipotent radical of P , then U is defined over k and $P = M \ltimes U$ (a semi-direct product), and hence the natural morphism

$$H^1(k, M) \rightarrow H^1(k, P)$$

is injective.

5.2. LEMMA. *Let G and M be as in the preceding lemma. Let \mathcal{G} be the derived subgroup of M and S be the central torus of M . Let \mathcal{G}_0 and \mathcal{G}_* be two connected normal k -subgroups of \mathcal{G} such that \mathcal{G} is an almost direct product of \mathcal{G}_0 and \mathcal{G}_* . Let \mathcal{C} be the finite group scheme $\mathcal{G}_0 \cap S\mathcal{G}_*$. Then the kernel of the natural morphism*

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G)$$

is contained in the image of

$$H^1(k, \mathcal{C}) \rightarrow H^1(k, \mathcal{G}_0).$$

Proof. Since the morphism $H^1(k, M) \rightarrow H^1(k, G)$ is injective (Lemma 5.1), the kernel of $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G)$ coincides with the kernel of $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M)$. But $C := \text{Ker}(H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M))$ is clearly contained in the kernel of the morphism $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, M/S\mathcal{G}_*)$ induced by the k -homomorphism $\mathcal{G}_0 \rightarrow M/S\mathcal{G}_*$. Now as the natural homomorphism $\mathcal{G}_0/\mathcal{C} \rightarrow M/S\mathcal{G}_*$ is a k -isomorphism, we conclude that C is contained in the kernel of $H^1(k, \mathcal{G}_0) \rightarrow H^1(k, \mathcal{G}_0/\mathcal{C})$, and from this the lemma is obvious.

Before proceeding further with the proof of Theorem B in the remaining exceptional cases, we shall recall some of the basic notions of the theory of quadratic forms.

5.3. Let p be the characteristic of k . If $p = 2$, let $\wp(k) = \{x + x^2 \mid x \in k\}$; $\wp(k)$ is a subgroup of k .

A quadratic form is said to be *nondefective* if the associated bilinear form is nondegenerate.

The *rank* (or the *dimension*) of a nondefective quadratic form is by definition the dimension of the underlying k -vector space, and the *Witt index* (over k) is the dimension of a maximal isotropic k -vector subspace.

For a quadratic form f/k , the *discriminant* (when $p = 2$, it is also called the *Arf invariant*) $d(f)$ will have the usual meaning. We recall that if $p \neq 2$, $d(f)$ is an element of $k^\times/k^{\times 2}$, and if $p = 2$, $d(f)$ is an element of $k/\wp(k)$. We shall say that a quadratic form f of rank $2n$ has *trivial signed discriminant* if its discriminant equals that of the hyperbolic form of rank $2n$, or, equivalently, if the special orthogonal group $SO(f)$ is of *inner type* over k .

Let q be a nondefective anisotropic quadratic form over k , of rank 2, and K be the quadratic Galois extension of k over which it is hyperbolic, then $d(q)$ is the image (in $k^\times/k^{\times 2}$ if $p \neq 2$ and in $k/\wp(k)$ if $p = 2$) of the norm of any element of K^\times of trace zero if $p \neq 2$ and of trace 1 if $p = 2$. Since q is a multiple of the norm-form of K/k , we conclude that the discriminant $d(q)$ determines q up to a scalar multiple.

If over k , f is an orthogonal direct sum of the nondefective quadratic forms q_i , $1 \leq i \leq n$, of rank 2, then $d(f)$ is the product of the $d(q_i)$ ($1 \leq i \leq n$) if $p \neq 2$, and it is the sum of the $d(q_i)$'s if $p = 2$.

5.4. The *Witt invariant* $w(f)$ of a nondefective quadratic form f/k of *even* rank is by definition the class of the Clifford algebra of f in the Brauer group of k ; it is an element of order 2 in the Brauer group. We recall that if f is a quadratic form of rank $2n$, with trivial signed discriminant, then the Witt invariant of f has the following useful description: Let h be the hyperbolic form of rank $2n$ and let $\text{Spin}(h)$ and $SO(h)$ be respectively the spin group and the special orthogonal group of h . Then since the discriminant of f equals that of h , the quadratic form f is obtained from h by twisting by a Galois cocycle with values in $SO(h)$. Let c denote the cohomology class in $H^1(k, SO(h))$ determined by the cocycle. Now consider the natural central isogeny:

$$1 \rightarrow \mu_2 \rightarrow \text{Spin}(h) \rightarrow SO(h) \rightarrow 1,$$

where μ_2 is the kernel of the endomorphism $x \mapsto x^2$ of GL_1 (it is a finite group scheme defined over k). It gives rise to the following exact sequence:

$$H^1(k, \text{Spin}(h)) \rightarrow H^1(k, SO(h)) \xrightarrow{\delta} H^2(k, \mu_2),$$

then $w(f) = \delta(c)$ in the natural identification of $H^2(k, \mu_2)$ with the subgroup of the Brauer group of k consisting of the elements of order 2.

Now we observe that if f is an *anisotropic* quadratic form of rank 6 which has trivial signed discriminant, then its Witt invariant is the class of a division algebra of degree 4 (i.e. of dimension 16). This follows immediately from the fact that $\text{Spin}(h)$, where h is the hyperbolic form of rank 6, is isomorphic to SL_4 over the base field, and the only anisotropic *inner* twists of SL_4 are of the form $\text{SL}_{1,D}$, D a central division algebra of dimension 16 over the base field.

5.5. Now we assume that G is an absolutely simple, simply connected algebraic group of type one of the remaining five: ${}^1E_{6,2}^{28}$, $E_{7,2}^{31}$, $E_{7,3}^{28}$, $E_{8,4}^{28}$, $E_{8,2}^{66}$. Let \mathcal{G} be (as in §1) the semi-simple anisotropic kernel of G . Let \mathcal{G}_0 be the unique connected normal k -subgroup of \mathcal{G} of type D_n ($n = 4$ or 6) and in case G is of type $E_{7,2}^{31}$, let \mathcal{G}_* be the connected normal k -subgroup of \mathcal{G} of type A_1 , in all the other cases let \mathcal{G}_* be trivial. Then \mathcal{G} is a direct product of \mathcal{G}_0 and \mathcal{G}_* .

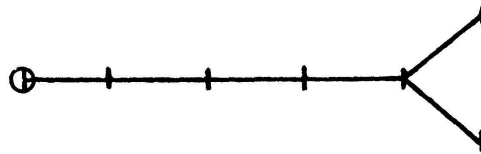
5.6. Let a be the simple root corresponding to the vertex in the Tits index marked with a cross (in §3) and let Θ be the set of distinguished simple roots $\neq a$. To establish Theorem B in the cases under consideration, it clearly suffices to prove that the natural morphism

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G_{\{a\}}) \times H^1(k, G_{\Theta})$$

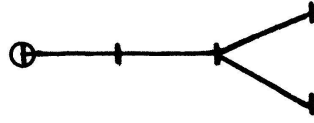
is injective.

Let $S_{\{a\}}$ (resp. S_{Θ}) be the maximal k -split torus of $G_{\{a\}}$ (resp. G_{Θ}) contained in S , and let $Z = \mathcal{G}_0 \cap S_{\{a\}}$, $\mathcal{Z} = \mathcal{G}_0 \cap S_{\Theta} \mathcal{G}_*$. Then it is easily seen, using the Tits indices, that both Z and \mathcal{Z} are k -isomorphic to the group scheme μ_2 . Moreover, the center of \mathcal{G}_0 is a direct sum of Z and \mathcal{Z} .

Now we observe that there is a nondefective, anisotropic quadratic form f/k with trivial discriminant, f of rank 12 in case G is of type $E_{8,2}^{66}$, and of rank 8 in all the other cases, such that \mathcal{G}_0 is k -isomorphic to $\text{Spin}(f)$ and the kernel of the natural central isogeny $\pi : \text{Spin}(f) \rightarrow \text{SO}(f)$ is $Z (= \mathcal{G}_0 \cap S_{\{a\}})$. This follows from the fact that \mathcal{G}_0 is the semi-simple anisotropic kernel of the simply connected, absolutely simple group $G_{\{a\}}$, and $G_{\{a\}}$ is the spin group of a nondefective quadratic form, of Witt index 1, which has trivial signed discriminant, since its Tits index is



in case G is of type $E_{8,2}^{66}$, and



in all the other cases. We shall identify \mathcal{G}_0 with $\text{Spin}(\mathfrak{f})$ in the sequel.

5.7. LEMMA. *If G is of type $E_{8,2}^{66}$, then the Witt invariant of \mathfrak{f} over k is trivial.*

Proof. Any connected absolutely simple algebraic group of type E_8 is simply connected and is isomorphic to its automorphism group. Therefore, as the semi-simple anisotropic kernel of a k -form of type $E_{8,2}^{66}$ is an absolutely simple, simply connected group of type D_6 , it is obtained from the split group of type E_8 by twisting by a Galois cocycle with values in the spin group of the hyperbolic form h of rank 12 (the spin group embedded as a maximal semi-simple k -subgroup of a parabolic k -subgroup of the split group of type E_8). Hence, \mathfrak{f} is obtained from h by twisting by a cocycle whose cohomology class lies in the image of the natural morphism.

$$H^1(k, \text{Spin}(h)) \rightarrow H^1(k, \text{SO}(h)).$$

This implies the lemma (see 5.4).

5.8. We now note, for future use, that the Witt index of the quadratic form \mathfrak{f} is *even* over any extension of k : this is seen easily from the classification of inner k -forms of types E_6 , E_7 and E_8 in terms of the Tits indices given in Tits [7].

5.9. Now let c be an element of the kernel of the natural morphism

$$H^1(k, \mathcal{G}_0) \rightarrow H^1(k, G_{\{a\}}) \times H^1(k, G_{\theta}).$$

We shall prove that c is trivial, this will establish Theorem B (see 5.6).

Let Z and \mathcal{Z} be as in 5.6. From Lemma 5.2 applied in turn to $G = G_{\{a\}}$ and $G = G_{\theta}$, we conclude that c lies in the intersection of the images of the following natural morphisms:

$$H^1(k, \mathcal{Z}) \rightarrow H^1(k, \mathcal{G}_0)$$

and

$$H^1(k, Z) \rightarrow H^1(k, \mathcal{G}_0).$$

Hence, in particular c is mapped onto the trivial element of $H^1(k, \text{SO}(\mathfrak{f}))$ under the central isogeny $(\mathcal{G}_0 =) \text{Spin}(\mathfrak{f}) \rightarrow \text{SO}(\mathfrak{f})$ (whose kernel is Z).

We fix an element $c \in H^1(k, \mathcal{Z})$ which is mapped onto $c \in H^1(k, \mathcal{G}_0)$. Since \mathcal{Z} is k -isomorphic to μ_2 , there is a natural identification of $H^1(k, \mathcal{Z})$ with $k^\times/k^{\times 2}$. Let

$s \in k^\times$ be such that, in the identification of $H^1(k, \mathcal{X})$ with $k^\times/k^{\times 2}$, c corresponds to s . Now we observe that under the central isogeny $\text{Spin}(\mathfrak{f}) \rightarrow \text{SO}(\mathfrak{f})$, \mathcal{X} is mapped onto the center of $\text{SO}(\mathfrak{f})$ and from this we conclude that the image of the cohomology class c in $H^1(k, \text{SO}(\mathfrak{f}))$ corresponds to the quadratic form $s\mathfrak{f}$. But since the image of c in $H^1(k, \text{SO}(\mathfrak{f}))$ is trivial, $s\mathfrak{f}$ is equivalent to \mathfrak{f} over k .

5.10. LEMMA. *Let φ be a nondefective anisotropic quadratic form such that φ is equivalent to $s\varphi$ ($s \in k^\times$). Then there is a nondefective subform q of φ of rank 2 such that q is equivalent to sq .*

Proof. If s is a square in k^\times , the lemma is obvious, so we shall assume that s is not a square.

Let V be the k -vector space underlying φ and \langle, \rangle be the bilinear form associated with φ . We fix a $v \in V$ such that $\varphi(v) \neq 0$. Then since $\varphi \simeq s\varphi$, there is a $v' \in V$ such that $\varphi(v') = s\varphi(v)$. Now if $\langle v, v' \rangle \neq 0$, let $w = v'$; if $\langle v, v' \rangle = 0$, choose a $v_0 \in V$ such that $\langle v_0, v \rangle \cdot \langle v_0, v' \rangle \neq 0$, and let

$$w = v' - \frac{\langle v_0, v' \rangle}{\varphi(v_0)} v_0.$$

Then $\varphi(w) = \varphi(v') = s\varphi(v)$ and $\langle v, w \rangle \neq 0$. Also since s is not a square, w is not a scalar multiple of v . Let q be the restriction of the quadratic form φ to the 2-dimensional subspace X spanned by v and w . It is easily seen that q is a nondefective quadratic form. The k -linear automorphism of the vector subspace X defined by $v \mapsto w$, $w \mapsto sv$ provides an equivalence of the quadratic form sq with q .

5.11. PROPOSITION. *There exist nondefective subforms q_i, q'_i ($i = 1, 2$ if G is not of type $E_{8,2}^{66}$ and $i = 1, 2, 3$ if G is of type $E_{8,2}^{66}$) of \mathfrak{f} , of rank 2, such that \mathfrak{f} is the orthogonal direct sum of the q_i 's and q'_i 's, and for each i*

- (1) $q_i \simeq sq_i$, $q'_i \simeq sq'_i$
- (2) q'_i is a scalar multiple of q_i ; in particular $\text{SO}(q_i)$ is k -isomorphic to $\text{SO}(q'_i)$.

Proof. According to the preceding lemma, there is a nondefective subform q_1 of \mathfrak{f} of rank 2 such that $q_1 \simeq sq_1$. Now let K be the quadratic Galois extension of k over which q_1 is hyperbolic, then q_1 is a multiple of the norm-form of K/k . Let q_1^\perp be the orthogonal complement of q_1 in \mathfrak{f} . Then since the Witt index of \mathfrak{f} over K is even (5.8), q_1^\perp is isotropic over K . Therefore, there exist vectors v, w in the subspace corresponding to q_1^\perp and $\alpha \in K - k$ such that

$$q_1^\perp(v + \alpha w) = \mathfrak{f}(v + \alpha w) = \mathfrak{f}(v) + \alpha \langle v, w \rangle + \alpha^2 \mathfrak{f}(w) = 0.$$

Now since α is separable, we easily conclude that the restriction q'_1 of q_1^\perp to the

2-dimensional subspace spanned by v and w is a nondefective quadratic form of rank 2 which is isotropic (and hence hyperbolic) over K . Therefore, q'_1 is a multiple of the norm-form of K/k . As q_1 is also a multiple of the norm-form of K/k and $q_1 \simeq sq_1$, we conclude that q'_1 is a multiple of q_1 and $q'_1 \simeq sq'_1$.

Now let $f_1 = q_1 \perp q'_1$. Then the discriminant of f_1 is trivial. Let f_2 be the orthogonal complement of f_1 in f . Then the discriminant of f_2 is trivial and as $f_1 \simeq sf_1$, by Witt's cancellation theorem $f_2 \simeq sf_2$. We shall now consider the cases where f is of rank 8. Let q_2 be a nondefective subform of f_2 of rank 2 such that $q_2 \simeq sq_2$ (Lemma 5.10) and q'_2 be its orthogonal complement in f_2 . Then the discriminant of q_2 equals that of q'_2 and hence q'_2 is a scalar multiple of q_2 (5.3), in particular $q'_2 \simeq sq'_2$.

Now we consider the case where f is of rank 12, then G is of type $E_{8,2}^{66}$, f_2 is an anisotropic form of rank 8 and trivial discriminant. We claim that the Witt index of f_2 over any quadratic Galois extension of k is even. To prove this we consider a quadratic Galois extension l of k such that f_2 is isotropic over l . Then as the discriminant of f_2 is trivial, the Witt index of f_2 over l can not be 3; assume, if possible, that it is 1. Then since the Witt invariant of f/k is zero (Lemma 5.7), the Witt invariant of f_1/l equals that of f_2/l . Now since by hypothesis f_2/l is of Witt index 1, over l it is an orthogonal direct sum of the hyperbolic form of rank 2 and an anisotropic form of rank 6. Therefore, the Witt invariant of f_2/l is the class of a division algebra of degree 4 in the Brauer group of l (5.4). But since f_1/k is an anisotropic form of rank 4 of trivial discriminant, it is a multiple of the norm-form of a *quaternion* division algebra D , and its Witt invariant is the class of D in the Brauer group of k . Therefore, the Witt invariant of f_1/l is the class of $D \otimes_k l$. We conclude thus that the class of a division algebra of degree 4 (in the Brauer group of l) contains $D \otimes_k l$. This is absurd, and hence the Witt index of f_2 over l can not be 1. This proves that the Witt index of f_2 over l is even. Now since f_2 is of rank 8, we can prove, as before, that there exist 4 nondefective quadratic forms q_2, q'_2, q_3 and q'_3 of rank 2 such that f_2 is an orthogonal direct sum of these; $q_i \simeq sq_i$, $q'_i \simeq sq'_i$ and q'_i is a scalar multiple of q_i ($i = 2, 3$). This proves the proposition.

5.12. We fix a set of nondefective subforms q_i, q'_i , of f , of rank 2, as in the preceding proposition. Let $\bar{T}_i = SO(q_i) (\subset SO(f))$ and $\bar{T}'_i = SO(q'_i) (\subset SO(f))$. Then (for all i) \bar{T}_i and \bar{T}'_i are isomorphic k -tori of dimension 1. Let $\pi : \text{Spin}(f) \rightarrow SO(f)$ be the usual central isogeny and let $T_i = \pi^{-1}(\bar{T}_i)$ and $T'_i = \pi^{-1}(\bar{T}'_i)$. Then for all i , T_i and T'_i are isomorphic k -tori.

$\prod_i (\bar{T}_i \times \bar{T}'_i)$ is a maximal torus of $SO(f)$ and there is a unique k -embedding of μ_2 into \bar{T}_i , as well as in \bar{T}'_i . The center C of $SO(f)$ is the “diagonally” embedded μ_2 in $\prod_i (\bar{T}_i \times \bar{T}'_i)$.

Let θ_i be a fixed k -isomorphism of \bar{T}_i onto \bar{T}'_i (note that there are exactly two

distinct k -isomorphisms of \bar{T}_i onto \bar{T}'_i), we shall let θ_i also denote the induced k -isomorphism of T_i onto T'_i . Let $\bar{\mathcal{T}}_i = \{x \cdot \theta_i(x) \mid x \in \bar{T}_i\}$, $\mathcal{T}_i = \{x \cdot \theta_i(x) \mid x \in T_i\}$, and let $\bar{\mathcal{T}} = \prod_i \bar{\mathcal{T}}_i$, $\mathcal{T} = \prod_i \mathcal{T}_i$. It is easily seen that $\forall i$, the restriction of π to \mathcal{T}_i is an isomorphism onto $\bar{\mathcal{T}}_i$ and hence the restriction of π to \mathcal{T} is an isomorphism onto $\bar{\mathcal{T}}$. Also, if necessary after changing the isomorphism θ_i for any one i , we can ensure that $\mathcal{Z} \subset \mathcal{T}$. We shall assume in the sequel that this is the case. Now we assert that $c(\epsilon H^1(k, \mathcal{Z}))$ is mapped onto the trivial element of $H^1(k, \mathcal{T})$ under the morphism induced by the inclusion $\mathcal{Z} \rightarrow \mathcal{T}$. To see this, we observe that the image of c in $H^1(k, \bar{\mathcal{T}})$ is trivial: this is a simple consequence of the fact that for $\forall i$, $q_i \simeq sq_i$. Now since $\pi|_{\mathcal{T}} : \mathcal{T} \rightarrow \bar{\mathcal{T}}$ is a k -isomorphism which maps \mathcal{Z} onto C , our assertion follows. It is obvious now that c , being the image of c in $H^1(k, \mathcal{G}_0)$, is trivial because $\mathcal{Z} \subset \mathcal{T} \subset \mathcal{G}_0$. This completes the proof of Theorem B.

REFERENCES

- [1] A. BOREL and J. TITS, *Groupes réductifs*, Publ. Math. I.H.E.S. #27 (1965), 55–150.
- [2] F. BRUHAT and J. TITS, *Groupes algébriques simples sur un corps local*, in Proc. Conf. on Local fields (Driebergen 1966, Ed. T.A. Springer). Springer-Verlag, Berlin 1967, pp. 23–36.
- [3] M. KNESER, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über p -adischen Körpern I, II*, Math. Z. 88 (1965), 40–47, 89 (1965), 250–272.
- [4] V. P. PLATONOV, *The problem of strong approximation and the Kneser-Tits conjecture*, Math. USSR Izv. 3 (1969), 1139–1147; Addendum: *ibid* 4 (1970), 784–786.
- [5] J-P. SERRE, *Cohomologie Galoisienne*, Lecture notes #5, Springer-Verlag, Heidelberg (1965).
- [6] J. TITS, *Algebraic and abstract simple groups*, Ann. of Math. 80 (1964), 313–329.
- [7] J. TITS, *Classification of algebraic semisimple groups*, Proc. Symp. in pure Math., AMS #9 (1966), 33–62.
- [8] J. TITS, *Groupes de Whitehead de groupes algébriques simples sur un corps*, Séminaire Bourbaki #505 (June 1977).

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