Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 59 (1984)

Artikel: A curious remark concerning the geometric transfer map.
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DOI: https://doi.org/10.5169/seals-45384

Comment. Math. Helvetici 59 (1984) 86–110 0010-2571/84/010086-25$01.50 + 0.20/0

A curious remark concerning the geometric transfer map

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Let $\mathcal{D}_n^r$ denote the identity component of the group of C^r compactly supported diffeomorphisms of $\mathbb{R}^n$. In two papers [10], [11], I proved that $\mathcal{D}_n^r$ is perfect, i.e. equal to its own commutator subgroup, provided $r \neq n+1$. However, the case $r = n+1$ is still open; it is not known whether $\mathcal{D}_n^{n+1}$ is perfect.

In this paper, I will give an elementary example which shows where the proof breaks down when $r = n+1$. The example is given in §2. A more general form of this example is given in §5. One has an assertion which is *true* when $r \neq n+1$ and *false* when $r = n+1$. In the version given in §2, the assertion is that a certain geometric transfer map is surjective. The proof in the case $r \neq n+1$ is analogous to part of the proof which I gave in [10] and [11]. The fact that this result is false for $r = n+1$ shows why my method cannot work in that case. However, the examples given in §2 and §5 of this paper are too special to suggest a proof that $\mathcal{D}_n^{n+1}$ is not perfect.

In §1, I define the group $\mathcal{D}_n^r$ and some related groups in detail. In §3, I briefly outline how M. Herman and Thurston proved that $\mathcal{D}_n^\infty$ is perfect, using K.A.M. theory. In §4, I discuss the connection of the result of this note with the method of [10] and [11]. In §5, I give an example of a linear mapping between spaces of C^r vector fields which is surjective when $r \neq n+1$ and is not surjective when $r = n+1$. This example generalizes the example in §2. The proof of non-surjectivity in the case $r = n+1$ is much more difficult than it was for the geometric transfer map considered in §2. It is given in §6.

In the appendix, I give one of Thurston's proofs of his result leading to the perfectness of $\mathcal{D}_n^\infty$. I believe this proof has not previously been published.

I would like to thank Jürgen Moser for encouraging me to write up these results.

§1. Definitions

If r is a positive integer of ∞, a mapping will be said to be C^r if it is r times continuously differentiable. If r is a real number >1, and not an integer, a mapping will be said to be C^r if it is $C^{[r]}$ (where $[r]$ denotes the greatest integer $\leq r$), and its $[r]^{\text{th}}$ derivative satisfies a Hölder condition of order $r-[r]$. A C^r diffeomorphism is a C^r mapping with a C^r inverse. The *support* of a diffeomorphism φ of a manifold M onto itself is $\{x \in M : \varphi(x) \neq x\}$. The support of φ will be denoted supp φ.

If K is a subset of a manifold M, we let $\mathcal{D}^r_K(M)$ denote the group of all C^r diffeomorphisms φ of M onto itself such that supp $\varphi \subset K$. We let $\mathcal{D}^r(M)$ denote the group of all C^r diffeomorphisms of M with compact support. We have

$$\mathcal{D}^r(M) = \bigcup_K \mathcal{D}^r_K(M) = \varinjlim_K \mathcal{D}^r_K(M),$$

where K runs over all compact subsets of M. We provide $\mathcal{D}^r_K(M)$ with the C^1 topology and $\mathcal{D}^r(M)$ with the direct limit topology. We let $\mathcal{D}^r(M)^0$ denote the component of the identity in $\mathcal{D}^r(M)$.

It is not difficult to show that an element of $\mathcal{D}^r(M)$ is in $\mathcal{D}^r(M)^0$ if and only if it is C^r isotopic to the identity by an isotopy with compact support.

From Smale's h-cobordism theorem [18], [20] and Cerf's theorem "pseudo-isotopy implies isotopy" [2], it follows that $\mathcal{D}^r(\mathbb{R}^n)/\mathcal{D}^r(\mathbb{R}^n)^0$ is isomorphic to the Kervaire-Milnor group Γ_{n+1} of homotopy $(n+1)$-spheres [8], when $n \geq 5$. For example [8], $\mathcal{D}^r(\mathbb{R}^6)/\mathcal{D}^r(\mathbb{R}^6)^0 \approx \mathbb{Z}/28\mathbb{Z}$.

The group $\mathcal{D}^r_n$, mentioned in the introduction, is defined to be $\mathcal{D}^r(\mathbb{R}^n)^0$.

§2. The geometric transfer map

Let $T^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| = \cdots = |z_n| = 1\}$. Let $A > 1$ be an integer. Let $\Gamma : T^n \to T^n$ be the covering mapping defined by

$$\Gamma(z_1, \ldots, z_n) = (z_1^A, \ldots, z_n^A).$$

Let θ^r denote the $\mathbb{R}$ vector space of C^r vector fields on T_n. *The geometric transfer map (associated to Γ)*

$$\mathrm{Tr} : \theta^r \to \theta^r$$

is defined by

$$\mathrm{Tr}\,(X)(x) = \sum_{y \in \Gamma^{-1}(x)} \Gamma_*(X(y)),$$

for $X \in \theta^r$, $x \in T^n$. Note that $\Gamma_*(X(y))$ is well defined because Γ is a local diffeomorphism. The method of [10] and [11] for proving that $\mathcal{D}_n^r$ is perfect when $r = n+1$ suggests the following:

Question. Is $\mathrm{Tr} - \mathrm{id}$ surjective?

How the method of [10] and [11] suggests this question will be explained in §4. The answer to it is *yes* if $r \neq n+1$ and *no* if $r = n+1$. This shows clearly that the method of [10] and [11] cannot succeed if $r = n+1$, although it gives no information on whether the result might still be true. More precise information is given by the following result:

PROPOSITION 1. *If* $r > n+1$, *then* $\mathrm{Tr} - \mathrm{id}$ *is an isomorphism. If* $r < n+1$, *then* $\mathrm{Tr} - \mathrm{id}$ *is surjective with infinite dimensional kernel. If* $r = n+1$, *then* $\mathrm{Tr} - \mathrm{id}$ *is not surjective, but is injective and has dense image.*

Proof. Introduce angular coordinates $\varphi_1, \ldots, \varphi_n$ on T^n, defined by the formula $z_j = \exp(2\pi i \varphi_j)$, where $\exp t = e^t$. Using the trivialization of the tangent bundle of T^n associated to this coordinate system, we may think of a vector field on T^n as a mapping of T^n into $\mathbb{R}^n$. We provide T^n with Haar measure. We let θ_0^r denote the set of vector fields on T^n whose integral over T^n vanishes. We identify $\mathbb{R}^n$ with the constant vector fields. Then

$$\theta^r = \theta_0^r \oplus \mathbb{R}^n$$

and Tr preserves each summand. It is easily seen that $\mathrm{Tr} \mid \mathbb{R}^n$ is multiplication by A^{n+1}. Hence, $(\mathrm{Tr} - \mathrm{id}) \mid \mathbb{R}^n$ is an isomorphism, and the assertions of Proposition 1 are equivalent to the corresponding assertions for θ_0^r in place of θ^r. Some of the assertions for θ_0^r are consequences of the following result:

LEMMA. *Let T and U be bounded linear mappings of a Banach space E into itself. Suppose* $TU = \mathrm{id}$, $\|T\|\,\|U\| = 1$, *and T has infinite dimensional kernel. Then the spectrum of T is the closed ball of radius* $\|T\|$, *the mapping* $T - \lambda$ *is surjective with infinite dimensional kernel for* $|\lambda| < \|T\|$, *and is not surjective for* $|\lambda| = \|T\|$.

Proof. For $|\lambda| < \|T\|$, the mapping $\mathrm{id} - \lambda U$ is invertible, since $\|\lambda U\| < 1$, by the hypothesis that $\|T\|\,\|U\| = 1$. Clearly, $(T - \lambda)U(\mathrm{id} - \lambda U)^{-1} = \mathrm{id}$. This shows that $T - \lambda$ is surjective, when $|\lambda| < \|T\|$. It is easily verified that

$$\ker(T - \lambda) = (\mathrm{id} - \lambda U)^{-1}(\ker T).$$

Since $\ker T$ is infinite dimensional and $\mathrm{id} - \lambda U$ is an isomorphism when $|\lambda| < \|T\|$, it follows that $\ker(T - \lambda)$ is infinite dimensional, when $|\lambda| < \|T\|$.

Obviously, spec $T \subset$ the closed ball of radius $\|T\|$. We have just shown that $\ker(T-\lambda) \neq 0$, when $|\lambda| < \|T\|$. Hence, spec $T =$ the closed ball of radius $\|T\|$. It is easy to see that for any bounded linear operator T of a Banach space into itself, if λ is in the boundary of spec T, then $T-\lambda$ is not surjective. (The idea of the proof is that if $T-\lambda$ is surjective, then it has non-trivial kernel, since $\lambda \in$ spec T. But, then any small perturbation of $T-\lambda$ is surjective with non-trivial kernel, and it follows that λ is in the interior of spec T, contrary to hypothesis.) For the operator T which we are considering here, we have shown that the boundary of spec T is the set of λ satisfying $|\lambda| = \|T\|$. Hence, $T-\lambda$ is not surjective for such λ. □

Proof of Proposition 1 (cont.). Let r be an integer and let $X \in \theta^r$. By the r^{th} total derivative $D^rX(x)$ of X at x, we mean the collection of all partial derivatives of order r at x, i.e. the numbers

$$\frac{\partial^{|\alpha|} X^i}{\partial \varphi^\alpha}(x), \qquad i = 1, \ldots, n; |\alpha| = r,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $X = (X^1, \ldots, X^n)$. We set

$$\|D^rX(x)\|^2 = \sum_{i=1}^{n} \sum_{|\alpha|=r} \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!} \left| \frac{\partial^{|\alpha|} X^i}{\partial \varphi^\alpha} \right|^2.$$

In the case that r is an integer, we provide θ_0^r with the norm

$$\|X\|_r = \sup_{x \in T^n} \|D^rX(x)\|.$$

When r is not an integer, we use the norm

$$\|X\|_r = \sup_{\substack{x, y \in T^n \\ x \neq y}} \frac{\|D^{[r]}X(x) - D^{[r]}X(y)\|}{d(x, y)^{r-[r]}}$$

where $d(x, y)$ is the distance between x and y associated to the Riemannian metric $d\varphi_1^2 + \cdots + d\varphi_n^2$ on T^n. In either case θ_0^r is a Banach space, with respect to the norm $\| \ \|_r$.

For a bounded linear operator L of θ_0^r into itself, we will use $\|L\|_r$ to denote the operator norm of L, i.e.

$$\|L\|_r = \sup_{\|X\|_r = 1} \|L(X)\|_r.$$

Let $\mathrm{Tr}_0 = \mathrm{Tr} \mid \theta_0^r$. We have

$$D^{[r]}(\mathrm{Tr}_0 X)(x) = \sum_{y \in \Gamma^{-1}(x)} A^{1-[r]} D^{[r]} X(y),$$

which implies

$$\|\mathrm{Tr}_0\|_r = A^{n+1-r}.$$

Let $U: \theta_0^r \to \theta_0^r$ be defined by

$$U(X)(y) = A^{-n} \Gamma_*^{-1}(X(\Gamma(y))),$$

for $X \in \theta_0^r$, $y \in T^n$. Then $\mathrm{Tr}_0 \circ U = \mathrm{id}$. We have

$$D^{[r]}(UX)(y) = A^{[r]-1-n} D^{[r]} X(\Gamma y),$$

which implies

$$\|U\|_r = A^{r-1-n}.$$

It is easily seen that $\ker \mathrm{Tr}_0$ is infinite dimensional.

We have just shown that all hypotheses of the lemma are satisfied, where $E = \theta_0^r$ and $T = \mathrm{Tr}_0$. From the lemma and our formula for $\|\mathrm{Tr}_0\|_r$, we obtain the conclusion of Proposition 1 for the case $r \neq n+1$ and we obtain that $\mathrm{Tr} - \mathrm{id}$ is not surjective in the case $r = n+1$. Since θ_0^r is dense in θ_0^{n+1} for $r > n+1$, and $\theta^r = (\mathrm{Tr}_0 - \mathrm{id})(\theta_0^r) \subset (\mathrm{Tr}_0 - \mathrm{id})(\theta_0^{n+1})$, we see that $\mathrm{Tr}_0 - \mathrm{id}$ has dense image, for the case $r = n+1$.

If $X \in \theta_0^{n+1}$ and $x \in T^n$, then $D^{n+1}(\mathrm{Tr}_0^k X)(x)$ is the average of $D^{n+1}X(y)$, where y ranges over $\Gamma^{-k}(x)$. Since the integral of X over T^n is zero, it follows from the fundamental theorem of Riemann integration that this average tends to 0, uniformly in x, so that $\|\mathrm{Tr}_0^k(X)\|_{n+1} \to 0$ as $k \to \infty$. Hence

$$(\mathrm{id} + \mathrm{Tr}_0 + \cdots + \mathrm{Tr}_0^N)(\mathrm{id} - \mathrm{Tr}_0)(X) = (\mathrm{id} - \mathrm{Tr}_0^{N+1})(X) \to X, \quad \text{as} \quad N \to \infty$$

and it follows that $\mathrm{Tr}_0 - \mathrm{id}$ is injective. □

§3. Related results

In the next section, I will explain how my result about the geometric transfer map is related to the proof that $\mathcal{D}_n^r$ is perfect when $r \neq n+1$. In this section, I will discuss how M. Herman and Thurston proved that $\mathscr{D}_n^\infty$ is perfect, using K.A.M. theory.

Let G be a connected topological group. One way to prove that G is perfect would be to find a single element $f \in G$ such that f could be written as a product of commutators

$$f = [g_1, h_1] \cdots [g_m, h_m],$$

where $[g, h] = ghg^{-1}h^{-1}$, and such that for every f' near f, we have that the functional equation

$$f' = [g_1', h_1'] \cdots [g_m', h_m'] \tag{1}$$

has a solution, where g_i' and h_i' are elements of the group. For, then $f'f^{-1}$ could be written as a product of commutators, so every element in a sufficiently small neighborhood of the identity could be written as a product of commutators. Since G is a connected group, it is generated by any neighborhood of the identity, so the solvability of (1) for every f' in a neighborhood of f implies that G is perfect.

The obvious way to try to solve (1) for every f' in a neighborhood of f is to prove an appropriate implicit function theorem. One case in which this can be done is when $G = \mathscr{D}^\infty(T^n)^0$. Let $R_\alpha : T^n \to T^n$ be defined by

$$R_\alpha(z) = (e^{2\pi i\alpha(1)}z_1, \ldots, e^{2\pi i\alpha(n)}z_n),$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $z = (z_1, \ldots, z_n) \in T^n$. We will suppose that α is *Diophantine*, i.e. that there exists $\delta > 0$ and $N > 0$ such that

$$|q_0 + q_1\alpha_1 + \cdots + q_n\alpha_n| > \delta(|q_0| + \cdots + |q_n|)^{-n},$$

for all $(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}$. It is a consequence of K.A.M. theory that for $f \in G$ near the identity, there exists $\beta \in \mathbb{R}^n$ near α and $g \in G$ near the identity such that

$$R_\beta f = gR_\alpha g^{-1}. \tag{2}$$

Writing this in the form $f = R_{\alpha-\beta}R_\alpha^{-1}gR_\alpha g^{-1}$, we reduce our problem to the problem of expressing $R_{\alpha-\beta}$ as a product of commutators. Let $\gamma_i = (0, \ldots, 0, \alpha_i - \beta_i, 0, \ldots, 0)$, where the non-zero term appears in the i^{th} place. Then $R_{\alpha-\beta} = R_{\gamma(1)} \cdots R_{\gamma(n)}$, and it is enough to express $R_{\gamma(i)}$ as a product of commutators. For this, it is enought to express the rotation of the circle through $\alpha_i - \beta_i$ as a product of commutators in the group of orientation preserving diffeomorphisms of the circle. But this is easily done: rotations of the circle are in PSL(2, $\mathbb{R}$), and is easily seen that PSL(2, $\mathbb{R}$) is perfect.

The K.A.M. method of solving (2), for f near the identity, uses an implicit function theorem. We use "+" for the standard group operation on T^n, i.e. the group operation defined by coordinatwise multiplication. For f and g near the identity in $G=\mathcal{D}^\infty(T^n)^0$, the mappings $f-\text{id}$, $g-\text{id}: T^n \to T^n$ are homotopic to the constant mapping and so can be lifted to mappings $f, g: T^n \to \mathbb{R}^n$, which are near zero. Let us write $\mathcal{C}$ for the vector space of C^∞ mappings of T^n into $\mathbb{R}^N$. The linearized form of (2) is

$$\beta+\hat{f}=\hat{g}R_\alpha+\alpha-\hat{g}. \tag{3}$$

Given $\hat{f}\in\mathcal{C}$, one wishes to solve (3) for $\beta\in\mathbb{R}^n$ and $\hat{g}\in\mathcal{C}$. This is easily done, by expanding everything in Fourier series. Here, the fact that α is Diophantine is crucial. Starting from the solvability of the linearized equation (3), Moser proved [19] an implicit function theorem which shows that (2) has a solution (β, g), where g is C^r, provided that f is in a sufficiently small neighborhood of the identity. Here, r may be taken to be an arbitrarily large integer, but the neighborhood depends on r. M. Herman improved this by showing [5], [6], [7] that g could actually be taken to be C^∞, for f in an appropriate neighborhood of the identity. This result showed that $\mathcal{D}^\infty(T^n)^0$ is perfect.

Herman's result actually proves a little more: the universal covering group $\tilde{\mathcal{D}}^\infty(T^n)^0$ of $\mathcal{D}^\infty(T^n)^0$ is perfect. More generally, consider a connected topological group G which admits a universal covering group $\tilde{G}$. Suppose that for any neighborhood U of the identity in G there is a neighborhood V of the identity in G such that whenever $f'f^{-1}\in V$, the equation (1) has a solution with $g_i'g_i^{-1}\in U$ and $h_i'h_i^{-1}\in U$, for $i=1,\dots,n$. Then, it is easy to see that $\tilde{G}$ is perfect. Herman's theorem shows that this condition is satisfied for $G=\mathcal{D}^\infty(T^n)^0$.

It is easily seen that if $\tilde{\mathcal{D}}^r(\mathbb{R}^n)^0$ is perfect, then $\tilde{\mathcal{D}}^r(M)^0$ is perfect for every n-manifold M, since $\tilde{\mathcal{D}}^r(M)^0$ is generated by elements having support in open balls. More generally, this argument shows that when M is connected, the natural homomorphism

$$\iota: H_1(\tilde{\mathcal{D}}^r(\mathbb{R}^n)^0)\to H_1(\tilde{\mathcal{D}}^r(M)^0),$$

induced by any C^r embedding of $\mathbb{R}^n$ in M, is surjective. Here $H_1(G)$ denotes the first homology group of G in the sense of Eilenberg and Maclane; this is the same as the commutator quotient group $G/[G, G]$. Note that ι is independent of the embedding of $\mathbb{R}^n$ in M, since any two such embeddings are isotopic. Thurston showed that ι is an isomorphism. (We give a proof in the appendix). In particular, since $H_1(\tilde{\mathcal{D}}^\infty(T^n)^0)=0$, by M. Herman's theorem, it follows that $H_1(\tilde{\mathcal{D}}^\infty(\mathbb{R}^n)^0)=0$, i.e. $\tilde{\mathcal{D}}^\infty(\mathbb{R}^n)^0$ is perfect. This obviously implies that $\tilde{\mathcal{D}}_n^\infty=\tilde{\mathcal{D}}^\infty(\mathbb{R}^n)^0$ is perfect.

These results were extended to the volume perserving case by Thurston [22] and to the symplectic case by Banyaga [1].

§4. Commutators of C^r diffeomorphisms

For finite r, a different argument is needed: For $f\in C^r$, it is not generally possible to solve (3) with g in C^r. This is the case no matter what α is. If α is Diophantine and r is large enough, then it is possible to solve (3) with $g\in C^{r-d(\alpha)}$, where $d(\alpha)$ depends only on α. But, $d(\alpha)$ is always positive. This state of affairs is often expressed by saying that the solution of (3) involves loss of derivatives. Similarly, the solution of (2) involves loss of derivatives. Consequently, it is not possible to prove that $\mathcal{D}^r(T^n)^0$ is perfect by using equation (2). In [10] and [11], I found another method which works when $r\neq n+1$, and is independent of the very difficult K.A.M. theory.

Following the method which I used in [10] and [11], we consider $f\in\mathcal{D}_n^r$. It is enough to show that if f is sufficiently close to the identity, then it is in the commutator subgroup. There is no loss of generality in assuming that the support of f is in the interior of the cube

$$D_n=\{x\in\mathbb{R}^n: -2\le x_j\le 2 \text{ for } 1\le j\le n\}.$$

What I did in [10] can be expressed in terms of solving the following functional equation:

$$f=\tilde{A}^{-1}(\tau_1^{-1}\lambda_1\tau_1)\cdots(\tau_n^{-1}\lambda_n\tau_n)u\lambda_n^{-1}\cdots\lambda_1^{-1}\tilde{A}u^{-1}, \tag{4}$$

for f in a sufficiently small neighborhood of the identity. Here, $\tilde{A}$ and the τ_i are fixed elements of $\mathcal{D}_n^r$ which will be defined below. Likewise, what I did in [11] can be expressed in terms of solving a slightly different functional equation:

$$f=\tilde{A}(\tau_n^{-1}\lambda_n\tau_n)\cdots(\tau_1^{-1}\lambda_1\tau_1)u\lambda_1^{-1}\cdots\lambda_n^{-1}\tilde{A}^{-1}u^{-1} \tag{5}$$

for f in a sufficiently small neighborhood of the identity.

We recall from [10] that $\tilde{A}$ was an element of $\mathcal{D}_n^r$ whose restriction to D_n was a multiplication by some large number A. Also, τ_i was the time-one mapping associated to the vector field $\rho\partial_i$, where ∂_i denoted the unit vector field on $\mathbb{R}^n$ in the direction of the i^{th} coordinate and $\rho(x_1,\ldots,x_n)=\rho_1(x_1)\cdots\rho_1(x_n)$, where ρ_1 was a C^∞ non-negative function on $\mathbb{R}$, which was identically one on $[-2A,2A]$, and which had support in a finite interval.

The arguments given in [10] show that when $r > n+1$, the equation (4) can be solved for $\lambda_1, \ldots, \lambda_n$, and u with support in a fixed compact set, provided that A is large enough. Moreover, these could be chosen to be in prescribed C^r neighborhood of the identity, provided that f had support in the interior of D_n and was in an appropriately small neighborhood of the identity to begin with. The arguments given in [11] show the same result for $r < n+1$, where the equation (4) is replaced the equation (5).

An idea of why the results of [10] and [11] might be true may be suggested by the following considerations. The linearized form of the equation (4) is

$$\tilde{A}_* \hat{f} = (\tau_{1*}^{-1}\hat{\lambda}_1 - \hat{\lambda}_1) + \cdots + (\tau_{n*}^{-1}\hat{\lambda}_n - \hat{\lambda}_n) + (\hat{u} - \hat{A}_* \hat{u}). \tag{6}$$

Here, we write $f = \mathrm{id} + \hat{f}$, etc., and think of $\hat{f}$, etc. as vector fields on $\mathbb{R}^n$, so $\tilde{A}_*$ denotes the action of $\tilde{A}$ on vector fields. Of course, (6) is obtained from (4) by considering one parameter families $\lambda_{1s}, \ldots, \lambda_{ns}, u_s$ satisfying $\lambda_{10} = \cdots = \lambda_{n0} = u_0 = \mathrm{id}$, defining f_s by (4), differentiating with respect to s, and evaluating at $s = 0$. The equation (6) is linear, so it is easier to study than (4). We wish to solve (6) for every C^r vector field $\hat{f}$ with support in the interior of D_n. Moreover, the solution $(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$ is required to be an $(n+1)$-tuple of C^r vector fields with compact support.

The linearized form of equation (5) is

$$\hat{f} = (\tau_{1*}^{-1}\hat{\lambda}_1 - \hat{\lambda}_1) + \cdots + (\tau_{n*}^{-1}\hat{\lambda}_n - \hat{\lambda}_n) + (\tilde{A}_* \hat{u} - \hat{u}). \tag{7}$$

Since we may obviously write $\hat{f} = \hat{A}_* \hat{f}_1$, with $\operatorname{supp}(\hat{f}_1) \in D_n$, the solvability of (6), in the sense we have just discussed, implies the solvability of (7). The solvability of (6), for $r \neq n+1$ and A sufficiently large, will be proved in the next section.

If one had an appropriate implicit function theorem, then the solvability of (6) would imply the solvability of (4) and the solvability of (7) would imply the solvability of (5). No implicit function theorem which permits one to make such deductions is known. Instead, in [10] and [11], I was able to find arguments in the nonlinear case, analogous to those we use here in the linear case, to prove the solvability of (4) and (5).

§5. The question of solvability of (6) and a generalization of proposition 1

We begin by considering the following simplified form of equation (6):

$$\hat{f} = (T_{1*}\hat{\lambda}_1 - \hat{\lambda}_1) + \cdots + (T_{n*}\hat{\lambda}_n - \hat{\lambda}_n) + (\hat{u} - A_* \hat{u}). \tag{8}$$

Here, $T_i : \mathbb{R}^n \to \mathbb{R}^n$ is the unit translation in the i^{th} coordinate, i.e. $T_i(x_1, \ldots x_n) = (x_1, \ldots, x_i + 1, \ldots, x_n)$. Also, A denotes a positive number and $A : \mathbb{R}^n \to \mathbb{R}^n$

denotes multiplication by A. We may pose the question: given a C^r vector field f on $\mathbb{R}^n$ with compact support, do there exist C^r vector fields $\hat{\lambda}_1, \ldots, \hat{\lambda}_n, \hat{u}$ on $\mathbb{R}^n$ with compact support which satisfy equation (8)?

This is a simplified version of the question we posed in §4 concerning the solvability of (6), simplified in the sense that τ_i and $\tilde{A}$ are replaced by the *affine* transformations T_i and A. Note that since we are considering arbitrary vector fields with compact support, replacing $\tilde{A}_*\hat{f}$ (of equation (6)) with $\hat{f}$ makes no difference. Also, we can replace T_i by T_i^{-1} without changing anything. Replacing τ_i and $\tilde{A}$ with the affine transformations T_i and A provides the simplification: it makes the analysis easier.

Let $\tilde{\theta}^r$ denote the $\mathbb{R}$ vector space of C^r compactly supported vector fields on $\mathbb{R}^n$. Let

$$\Lambda : \bigoplus_{n+1} \tilde{\theta}^r \to \tilde{\theta}^r$$

be defined by

$$\Lambda(\hat{\lambda}_1, \ldots, \hat{\lambda}_n, \hat{u}) = (T_{1*}\hat{\lambda}_1 - \hat{\lambda}_1) + \cdots + (T_{n*}\hat{\lambda}_n - \hat{\lambda}_n) + (\hat{u} - A_*\hat{u}).$$

The question we posed in this section is equivalent to asking whether Λ is surjective. We will prove the following result:

PROPOSITION 2. *Λ is surjective if and only if $A \neq 1$ and $r \neq n+1$.*

The proof of surjectivity when $A \neq 1$ and $r \neq n+1$ will be given later in this section. The fact that Λ is not surjective when $A = 1$ is obvious. The fact that it is not surjective when $r \neq n+1$ is difficult; this will be proved in the next section. It depends on the fact that A and the T_i are affine; we do not know the answer to the question as to whether (6) is solvable when $r = n+1$ (and A is large). Quite possibly, the answer depends on how $\tilde{A}$ and τ_i are chosen, since there is some arbitrariness in the choice. We will prove the solvability of (6) when $r \neq n+1$ and A is large at the end of this section.

First, however, we show how Proposition 2 generalizes the part of Proposition 1 concerning surjectivity. Let

$$\Lambda_0 : \bigoplus_{n} \tilde{\theta}^r \to \tilde{\theta}^r$$

be defined by

$$\Lambda_0(\hat{\lambda}_1, \ldots, \hat{\lambda}_n) = (T_{1*}\hat{\lambda}_1 - \hat{\lambda}_1) + \cdots + (T_{n*}\hat{\lambda}_n - \hat{\lambda}_n)$$

Let $e:\mathbb{R}^n \to T^n$ be the covering mapping defined by $e(x_1, \ldots, x_n) = (\exp 2\pi i x_1, \ldots, \exp 2\pi i x_n)$. *The geometric transfer map* (*associated to e*)

$$\mathrm{Tr}_e: \tilde{\theta}^r \to \theta^r$$

is defined by

$$\mathrm{Tr}_e\,(X)(y) = \sum_{x \in e^{-1}(y)} e_*(X(x)),$$

for $X \in \tilde{\theta}^r$, $y \in T^n$.

The following sequence is exact:

$$\bigoplus_n \tilde{\theta}^r \xrightarrow{\Lambda_0} \theta^r \xrightarrow{\mathrm{Tr}_e} \theta^r \longrightarrow 0. \tag{$*$}$$

It is obvious that $\mathrm{Tr}_e\,\Lambda_0 = 0$. To prove that $\ker \mathrm{Tr}_e = \mathrm{im}\,\Lambda_0$, we consider the $\mathbb{R}$-vector space $\theta^{r,i}$ of C^r compactly supported vector fields on $T^i \times \mathbb{R}^{n-i}$, and let $\mathrm{Tr}_e^{i+1}: \theta^{r,i} \to \theta^{r,i+1}$ be the geometric tranfer map associated to the map

$$(z_1, \ldots, z_j, x_{j+1}, x_{j+2}, \ldots, x_n) \to (z_1, \ldots, z_i, \exp(2\pi i x_{j+1}), x_{j+2}, \ldots, x_n).$$

Then

$$\mathrm{Tr}_e = \mathrm{Tr}_e^n \circ \cdots \circ \mathrm{Tr}_e^1 : \tilde{\theta}^r = \theta^{r,0} \to \theta^r = \theta^{r,n}.$$

Let $X \in \ker \mathrm{Tr}_e$ and set $X' = \mathrm{Tr}_e^{n-1} \circ \cdots \circ \mathrm{Tr}_e^1(X)$. We have $X' = (T_{n*} - \mathrm{id})Y'$, where $Y'(x) = \sum_{k<0} T_{n*}^k X'(x)$. Note that for each $x \in T^{n-1} \times \mathbb{R}$ this is a finite sum, since X' has compact support. Since $X' \in \ker \mathrm{Tr}_e^n$, we have that Y' has compact support, i.e. it is in $\tilde{\theta}^{r,n-1}$. It follows that there exists $Y_1 \in \tilde{\theta}^r$ such that $Y' = \mathrm{Tr}_e^{n-1} \circ \cdots \circ \mathrm{Tr}_e^1(Y_1)$. Then

$$X - (T_{n*} - \mathrm{id})Y_1 \in \ker(\mathrm{Tr}_e^{n-1} \circ \cdots \circ \mathrm{Tr}_e^1).$$

In a similar way, we may prove that there exists $Y_2 \in \tilde{\theta}^r$ such that

$$X - (T_{n*} - \mathrm{id})Y_1 - (T_{n-1*} - \mathrm{id})Y_2 \in \ker(\mathrm{Tr}_e^{n-2} \circ \cdots \circ \mathrm{Tr}_e^1).$$

Continuing in this way, we obtain that $X \in \mathrm{im}\,\Lambda_0$. This proves exactness of $(*)$.

Now, suppose $A \in \mathbb{Z}$. Then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\bigoplus_n \tilde{\theta}^r & \xrightarrow{\Lambda_0} & \theta^r & \xrightarrow{\mathrm{Tr}_e} & \theta^r & \longrightarrow & 0 \\
\downarrow{\scriptstyle \oplus A_*} & & \downarrow{\scriptstyle A_*} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \\
\bigoplus_n \tilde{\theta}^r & \xrightarrow{\Lambda_0} & \tilde{\theta}^r & \xrightarrow{\mathrm{Tr}_e} & \theta^r & \longrightarrow & 0
\end{array}$$

where Tr is the geometric transfer map associated to Γ, which was defined in §2. It follows that $\mathrm{Tr}-\mathrm{id}$ is surjective if and only if $\tilde{\theta}^r = \mathrm{im}\,(A_* - \mathrm{id}) + \mathrm{im}\,\Lambda_0$, i.e. if and only if Λ is surjective. In other words, for the case $A \in \mathbb{Z}$, Proposition 2 is equivalent to the assertions in Proposition 1 which concern surjectivity.

Proof of Proposition 2 *in the case* $r \neq n+1$. We identify vector fields on $\mathbb{R}^n$ with mappings of $\mathbb{R}^n$ into itself in the standard way. We let $\tilde{\theta}_0^r$ denote the set of $X \in \tilde{\theta}^r$ whose integral over $\mathbb{R}^n$ vanishes. Then $\tilde{\theta}_0^r$ is a vector subspace of $\tilde{\theta}^r$, whose codimension in $\tilde{\theta}^r$ is n. We have the following commutative diagram, with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_n \tilde{\theta}^r \oplus \tilde{\theta}_0^r & \longrightarrow & \bigoplus_{n+1} \tilde{\theta}^r & \longrightarrow & \mathbb{R}^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{\Lambda}_0} & & \downarrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle A-1} & & \\
0 & \longrightarrow & \tilde{\theta}_0^r & \longrightarrow & \tilde{\theta}^r & \longrightarrow & \mathbb{R}^n & \longrightarrow & 0
\end{array}$$

where $\bar{\Lambda}_0$ denotes the restriction of Λ. Since $A \neq 1$, multiplication by $A-1$ is an isomorphism, and it follows that Λ is surjective if and only if $\bar{\Lambda}_0$ is surjective.

We let u be a C^∞ non-negative function on $\mathbb{R}$ with compact support such that

$$\sum_{n \in \mathbb{Z}} u(x+n) = 1,$$

for any $x \in \mathbb{R}$. For any positive number ω, we consider the function u_ω defined by

$$u_\omega(x) = \omega^{-1} u(\omega^{-1} x).$$

Then u_ω is a C^∞ non-negative function on $\mathbb{R}$ with compact support. If λ is a positive integer, we have

$$\sum_{n \in \mathbb{Z}} u_\lambda(x+n) = 1,$$

for any $x\in\mathbb{R}$. We let $U_\lambda:\theta_0^r\to\tilde{\theta}_0^r$ be defined by

$$U_\lambda(X)(x_1,\ldots,x_n)=u_\lambda(x_1)\cdots u_\lambda(x_n)Xe(x_1,\ldots,x_n),$$

for $X\in\theta_0^r$ and $(x_1,\ldots,x_n)\in\mathbb{R}^n$. If λ is a positive integer, then $\mathrm{Tr}_e\,U_\lambda=\mathrm{id}$.

To prove surjectivity when $r\neq n+1$, we will need:

LEMMA. *There exists a constant C, which depends only on n, r and u, such that*

$$\|\mathrm{Tr}_e\,A_*U_\lambda\|_r\leq(1+C\lambda^{-1})(A+C\lambda^{-1})^nA^{1-r}$$

for all positive numbers A and all positive integers λ.

Here, we continue to use the notation introduced in §2: $\|\ \|_r$ denotes both the norm on θ_0^r which was defined there and the operator norm on the vector space of bounded linear mappings of θ_0^r into itself. In addition, we let $\|\ \|_r$ denote the norm on $\tilde{\theta}^r$ which is defined in the same way as the norm on θ_0^r was defined, with the obvious change: T^n should be replaced by $\mathbb{R}^n$ in the definition. Note that $\mathrm{Tr}_e\,A_*U_\lambda$ is a bounded linear mapping of θ_0^r into itself.

Proof of the lemma. First, we consider the case when r is an integer. We set $v_\omega(x)=u_\omega(x_1)\cdots u_\omega(x_n)$ for $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$. For $X\in\theta_0^r$ and $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, the Leibniz formula for the derivative of a product gives

$$\|D^rU_\lambda(X)(x)\|\leq v_\lambda(x)\,\|D^rX(y)\|+C_1\sum_{0<\alpha\leq r}\|D^\alpha v_\lambda(x)\|\,\|D^{r-\alpha}X(y)\|,$$

where $y=e(x)$ and C_1 is a constant which depends only on n and r. We have

$$\|X\|_k\leq C_2\,\|X\|_r$$

for any integer $0\leq k\leq r$ and any $X\in\theta_0^r$, where C_2 is a constant which depends only on n and r. The fact that this is true for $k=0$ is a consequence of the fact that the integral of X over T^n vanishes. The fact that this is true for other k is a consequence of the fact that the integral of any derivative of X vanishes. Combining the two previous inequalities, we obtain

$$\begin{aligned}\|D^rU_\lambda(X)(x)\|&\leq\Big(v_\lambda(x)+C_3\sum_{0<\alpha\leq r}\|D^\alpha v_\lambda(x)\|\Big)\,\|X\|_r\\&\leq(v_\lambda(x)+C_4\lambda^{-n-1})\,\|X\|_r,\end{aligned}$$

where C_3 is a constant which depends only on n and r and C_4 is a constant which depends only on n, r and u. Then

$$\|D^r A_* U_\lambda(X)(x)\| \leq A^{1-r} \|D^r U_\lambda(X)(A^{-1}x)\| \\ \leq A^{1-r}(v_\lambda(A^{-1}x) + C_4\lambda^{-n-1}) \|X\|_r.$$

For $y \in T^n$, we have

$$D^r \operatorname{Tr}_e A_* U_\lambda(X)(y) = \sum_{k\in\mathbb{Z}^n} D^r A_* U_\lambda(X)(x+k),$$

where x is any element of $e^{-1}(y)$. Note that the support of $A_*U_\lambda(X)$ is contained in $A \operatorname{supp} v_\lambda = A\lambda(\operatorname{supp} u)^n$. Consequently, the sum on the right has at most $(A\lambda L+1)^n \leq (A\lambda+1)^n L^n$ non-vanishing terms, where L is the length of the shortest interval which contains supp u. (Note that $L \geq 1$.) It follows that

$$\|D^r \operatorname{Tr}_e A_* U_\lambda(X)(y)\| \leq A^{1-r}\Big(\sum_{k\in\mathbb{Z}^n} v_\lambda(A^{-1}(x+k)) + C_5(A\lambda+1)^n\lambda^{-n-1}\Big)\|X\|_r,$$

where C_5 is a constant which depends only on n, r, and u.

Moreover,

$$\sum_{k\in\mathbb{Z}^n} v_\lambda(A^{-1}(x+k)) = \prod_{i=1}^n \sum_{k\in\mathbb{Z}} u_\lambda(A^{-1}(x_i+k)) \\ = \lambda^{-n} \prod_{i=1}^n \sum_{k\in\mathbb{Z}} u(\lambda^{-1}A^{-1}(x_i+k))$$

It is easily verified that there exists a constant C_6, which depends only on u, such that

$$\sum_{k\in\mathbb{Z}} u_\omega(x+k) \leq 1 + C_6\omega^{-1}$$

for all positive numbers ω and all $x \in \mathbb{R}$. Here is the verification: For $\omega \leq L^{-1}$, the sum on the left has only one term, and we obtain that it is $\leq (\max u)\omega^{-1}$. Moreover, we have

$$\Big|\sum_{k\in\mathbb{Z}} u_\omega(x+k) - 1\Big| = \Big|\sum_{k\in\mathbb{Z}} u_\omega(x+k) - \int_{-\infty}^{\infty} u_\omega(x+y)\,dy\Big| \\ = \Big|\sum_{k\in\mathbb{Z}} \Big[u_\omega(x+k) - \int_{-1/2}^{1/2} u_\omega(x+k+y)\,dy\Big]\Big| \leq \tfrac{1}{2}(L\omega+2)\max u'_\omega \\ = \tfrac{1}{2}(L\omega+2)(\max u')\omega^{-2} \leq C_6\omega^{-1},$$

where the last inequality is valid for $\omega \geq L^{-1}$, if C_6 is large enough. Hence,

$$\left|\sum_{k\in\mathbb{Z}^n} v_\lambda(A^{-1}(x+k))\right| = A^n \prod_{i=1}^{n}\left|\sum_{k\in\mathbb{Z}} u_{\lambda A}(x_i+k)\right| \leq A^n(1+C_6\lambda^{-1}A^{-1})^n.$$

It follows that

$$\|D^r \operatorname{Tr}_e A_* U_\lambda(X)(y)\| \leq A^{1-r}(A^n(1+C_6\lambda^{-1}A^{-1})^n + C_5(A\lambda+1)^n\lambda^{-n-1})\,\|X\|_r.$$

Every term in the coefficient of $\|X\|_r$ has the form const. $A^a\lambda^b$, where $n+2-r\geq a-b\geq n+1-r$, and $b\leq 0$. Moreover, the coefficient of A^{n+1-r} is 1 and the coefficients of A^{n+2-r} vanishes. Since a positive integer is ≥ 1, we obtain that there exists a constant $C>0$, which depends only on n, r, and u, such that

$$\|D^r \operatorname{Tr}_e A_* U_\lambda(X)(y)\| \leq (1+C\lambda^{-1})(A+C\lambda^{-1})^n A^{1-r}\,\|X\|_r,$$

for all positive numbers A and all positive integers λ.

Now we consider the case when r is not an integer. For $X\in\theta_0^r$, $x=(x_1,\ldots,x_n)$ and $x'=(x_1',\ldots,x_n')\in\mathbb{R}^n$, we have

$$\begin{aligned}
&\|D^{[r]}U_\lambda(X)(x) - D^{[r]}U_\lambda(X)(x')\| \\
&\quad\leq |v_\lambda(x)-v_\lambda(x')|\,\|D^{[r]}X(y)\| + v_\lambda(x')\,\|D^{[r]}X(y)-D^{[r]}X(y')\| \\
&\quad + C_1 \sum_{0<\alpha\leq[r]} \|D^\alpha v_\lambda(x) - D^\alpha v_\lambda(x')\|\,\|D^{[r]-\alpha}X(y)\| \\
&\quad + \|D^\alpha v_\lambda(x')\|\,\|D^{[r]-\alpha}X(y)-D^{[r]-\alpha}X(y')\|,
\end{aligned}$$

where $y=e(x)$ and $y'=e(x')$. Moreover, $\|X\|_{[r]}\leq C_5\,\|X\|_r$,,where $C_5=\pi\sqrt{n}$, so $\|X\|_k\leq C_2C_5\,\|X\|_r$, for any integer $0\leq k\leq[r]$, and

$$\|D^{[r]-\alpha}X(y) - D^{[r]-\alpha}X(y')\| \leq \|X\|_{[r]-\alpha+1}\,\|x-x'\|,$$

so we obtain

$$\|D^{[r]}U_\lambda(X)(x) - D^{[r]}U_\lambda(X)(x')\| \leq (v_\lambda(x') + C_7\lambda^{-n-1})\,\|X\|_r\,\|x-x'\|^{r-[r]},$$

where C_7 is a constant which depends only on n, r and u. Then

$$\begin{aligned}
&\|D^{[r]}A_*U_\lambda(X)(x) - D^{[r]}A_*U_\lambda(X)(x')\| \\
&\quad = A^{1-[r]}\,\|D^{[r]}U_\lambda(X)(A^{-1}x) - D^{[r]}U_\lambda(X)(A^{-1}x')\| \\
&\quad \leq A^{1-[r]}(v_\lambda(A^{-1}x') + C_7\lambda^{-n-1})\,\|X\|_r\,\|A^{-1}x - A^{-1}x'\|^{r-[r]} \\
&\quad \leq A^{1-r}(v_\lambda(A^{-1}x') + C_7\lambda^{-n-1})\,\|X\|_r\,\|x-x'\|^{r-[r]}.
\end{aligned}$$

The rest of the proof may be done in exactly the same way as in the case when r is an integer. □

Proof of Proposition 2 *in the case* $r \neq n+1$ (*conclusion*). In either the case $r > n+1$, $A > 1$ or the case $r < n+1$, $A < 1$, the lemma shows that there exists an integer λ so large that

$$\|\mathrm{Tr}_e\, A_* U_\lambda\|_r < 1.$$

Since θ_0^r is a Banach space, it follows that $\mathrm{id} - \mathrm{Tr}_e\, A_* U_\lambda$ is an invertible linear mapping of θ_0^r onto itself. Let $X \in \tilde{\theta}_0^r$. There exists $Y \in \theta_0^r$ such that

$$\mathrm{Tr}_e\,(X) = (\mathrm{id} - \mathrm{Tr}_e\, A_* U_\lambda)(Y) = \mathrm{Tr}_e(\mathrm{id} - A_*)\, U_\lambda(Y).$$

Hence, $X - (\mathrm{id} - A_*)U_\lambda(Y) \in \ker \mathrm{Tr}_e = \mathrm{im}\,(T_{1*} - \mathrm{id}) + \cdots + \mathrm{im}\,(T_{n*} - \mathrm{id})$, by the exactness of $(*)$. It follows that $X \in \mathrm{im}\,(\mathrm{id} - A_*) + \mathrm{im}\,(T_{1*} - \mathrm{id}) + \cdots + \mathrm{im}\,(T_{n*} - \mathrm{id}) = \mathrm{im}\,\bar{\Lambda}_0$. Since X is an arbitrary element of $\tilde{\theta}_0^r$, this proves that $\bar{\Lambda}_0$ is surjective.

Since $A_* - \mathrm{id} = -(A_*^{-1} - \mathrm{id})A_*$ and A_* is an isomorphism, we have $\mathrm{im}\,(A_* - \mathrm{id}) = \mathrm{im}\,(A_*^{-1} - \mathrm{id})$. Consequently, the case $r > n+1$, $A < 1$ and the case $r < n+1$, $A > 1$ may be reduced to the case already considered. □

Solvability of (6).

We begin with the following observation. Let $Y \in \tilde{\theta}^r$ and suppose Y has support in $A.D_n$. Then

$$\begin{aligned} Y \in \ker \mathrm{Tr}_e &\Leftrightarrow Y \in \mathrm{im}\,(T_{1*} - \mathrm{id}) + \cdots + \mathrm{im}\,(T_{n*} - \mathrm{id}) \\ &\Rightarrow Y \in \mathrm{im}\,(\tau_{1*} - \mathrm{id}) + \cdots + \mathrm{im}\,(\tau_{n*} - \mathrm{id}) \end{aligned}$$

The equivalence of the first two conditions is the same as the exactness of the sequence $(*)$. The fact that the second condition implies the third may be seen in the same was as the exactness of $(*)$ was proved, since that proof showed the existence of solutions having support in $[-2A, 2A]^n$ and $\tau_i = T_i$ on $[-2A, 2A-1]^n$.

In what follows, we let u be as in the preceding discussion, but suppose in addition that $\mathrm{supp}\, u \subset (-2, 2)$.

Previously, we remarked that the solvability of (6) implies the solvability of (7). But also, the solvability of (7) implies the solvability of (6). For, let $\hat{f} \in \tilde{\theta}^r$ and suppose that the support of $\hat{f}$ is in the interior of D_n. Then the support of

$Y = \tilde{A}\hat{f} - U_1 \operatorname{Tr}_e \tilde{A}\hat{f}$ is in the interior of $A \cdot D_n$ and $\operatorname{Tr}_e Y = 0$, since $\operatorname{Tr}_e U_1 = \mathrm{id}$. Hence, $Y \in \operatorname{im}(\tau_{1*} - \mathrm{id}) + \cdots + \operatorname{im}(\tau_{n*} - \mathrm{id})$. But the support of $U_1 \operatorname{Tr}_e \tilde{A}\hat{f}$ is in the interior of D_n, so in discussing the solvability of (6), we may replace $\tilde{A}\hat{f}$ with something having support in the interior of D_n, which is what we mean by saying that the solvability of (6) follows from the solvability of (7).

Next, we show that it is enough to consider the case when the integral of $\hat{f}$ vanishes. For, consider an arbitrary $\hat{f}$. Let $\hat{g} \in \theta^r$ satisfy $\int \hat{g} = (A^{n+1} - 1)^{-1} \int \hat{f}$ and $\operatorname{supp} \hat{g} \subset A^{-1} D_n$. Then the integral of $\hat{f} - (A_* - \mathrm{id})\hat{g}$ vanishes, and it is enough to solve (7) for $\hat{f}$ replaced by this element.

So, consider $\hat{f} \in \tilde{\theta}_0^r$ with support in the interior of D_n. Suppose $r > n+1$. By the lemma, we may choose A so large that $\|\operatorname{Tr}_e A_* U_1\| < 1$. Then $\mathrm{id} - \operatorname{Tr}_e A_* U_1$ is an isomorphism of θ_0^r onto itself, so there exists $Y \in \theta_0^r$ such that

$$\operatorname{Tr}_e(\hat{f}) = (\mathrm{id} - \operatorname{Tr}_e A_* U_1)(Y) = \operatorname{Tr}_e(\mathrm{id} - A_*) U_1(Y).$$

Let $\hat{u} = -U_1(Y)$. Then $\hat{f} - \tilde{A}_* \hat{u} + \hat{u} \in \operatorname{Ker} \operatorname{Tr}_e$. Since $\hat{f} - \tilde{A}_* \hat{u} + \hat{u}$ has support in $A \cdot D_n$, it follows that $\hat{f} - \tilde{A}_* \hat{u} + \hat{u} \in \operatorname{im}(\tau_{1*} - \mathrm{id}) + \cdots + \operatorname{im}(\tau_{n*} - \mathrm{id})$. This proves the solvability of (7) when $r > n+1$.

Now suppose $r < n+1$. Let $\lambda = [A^{-1}]$ and choose $A > 0$ so small that $\|\operatorname{Tr}_e A_* U_\lambda\| < 1$. This is possible by the lemma. Now we use the same reasoning as in the case $r > n+1$, using the fact that $\mathrm{id} - \operatorname{Tr}_e A_* U_\lambda$ is an isomorphism. □

§6. Proof of proposition 2 when $r = n+1$.

In the case that $r = n+1$, Proposition 2 says that Λ is not surjective. The proof of non-surjectivity of Λ is based on an idea different from that in §2. I have been unable to find a functional analytic proof, analogous to that in §2. Instead, I will give a proof based on the Fourier transform.

We will suppose that Λ is surjective and show that this leads to a contradiction. It will be convenient to use complex valued vector fields and the corresponding complex vector spaces, in order to make the application of the Fourier transform easier. In this section, θ^r will mean *complex valued* C^r vector fields on T^n, $\tilde{\theta}^r$ will mean *complex valued*, compactly supported C^r vector fields on $\mathbb{R}^n$, the mapping $\Lambda : \bigoplus_{n+1} \tilde{\theta}^r \to \tilde{\theta}^r$ will mean the complex linear mapping of complex vector spaces, defined by the same formula as previously, and so on. Obviously, these are harmless changes, since the old Λ is surjective if and only if the new Λ is surjective. We will suppose $A > 1$, since the case $A < 1$ reduces to this case.

From the hypothesis that Λ is surjective, it follows that

$$\operatorname{Tr}_e(A_* - \mathrm{id}) : \tilde{\theta}^r \to \theta^r$$

is surjective. It follows that for some $m \in \mathbb{Z}$, the image of

$$\mathrm{Tr}_e\,(A_* - \mathrm{id}) : \tilde{\theta}^r_m \to \theta^r$$

has a second category in θ^r, where θ^r_m denotes the space of C^r vector fields on $\mathbb{R}^n$ having support in a ball of radius m. This is because

$$\tilde{\theta}^r = \bigcup_{m \in \mathbb{Z}} \tilde{\theta}^r_m$$

and the Baire category theorems apply to θ^r, since it is a complete metric space. From now on, we fix a value of m for which this image has second category in θ^r.

LEMMA 1. *There exists $C > 0$, such that the following holds. For any $Y \in \theta^r$, there exists $X \in \tilde{\theta}^r_m$ such that*

$$\mathrm{Tr}_e\,(A_* - \mathrm{id})(X) = Y,$$

and

$$\|X\|_r < C\,\|Y\|_r.$$

Proof. $\tilde{\theta}^r_m$ is a Banach space with respect to the norm $\|\ \|_r$ and the usual proof of the open mapping theorem only requires the hypothesis that the image be of the second category, nor surjectivity. □

We will construct a $Y \in \theta^r$ for which there is no $X \in \tilde{\theta}^r_m$ satisfying the conclusions of Lemma 1. This will give a contradiction. For the construction of Y, we need the following result.

LEMMA 2. *Let $\varepsilon > 0$ and let N be a positive integer. Then there exist positive integers $p_{-N}, \ldots, p_N$ such that*

$$|Ap_i - p_{i+1}| < \varepsilon$$

for $i = -N, \ldots, N-1$.

Proof. Let $0 < \delta < \varepsilon/(A+1)$. Let $x_p = e(p, Ap, \ldots, A^{2N+1}p)$, where $e : \mathbb{R}^{2N+1} \to T^{2N+1}$ is the standard covering map of the $(2N+1)$-torus. Then $x_1, x_2, \ldots$ is an infinite sequence in the compact space T^n, so we can find positive integers $q < q'$

such that $d(x_q, x_{q'}) < \delta$. Let $p_{-N} = q' - q$ and let p_i denote a closest integral approximation to $A^{N+i}p_{-N}$ for $i = -N, \ldots, N$. Then $|A^{N+i}p_{-N} - p_i| < \delta$ and

$$|Ap_i - p_{i+1}| < A\,|A^{N+i}p_{-N} - p_i| + |A^{N+i+1}p_{-N} - p_{i+1}| < (A+1)\delta < \varepsilon \quad \square$$

For $X \in \tilde{\theta}^r$, let $\hat{X}$ denote the Fourier transform of $\hat{X}$, i.e.

$$\hat{X}(\xi) = \int_{\mathbb{R}^n} X(x) e^{-2\pi i x \cdot \xi}\, dx,$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x \cdot \xi = \sum x_i \xi_i$, and $dx = dx_1 \cdots dx_n$. Here, we identify complex valued vector fields with $\mathbb{C}^n$ valued functions, so the above integral makes sense.

LEMMA 3. *There exists $C_1 > 0$ which depends only on m, n, and r, such that*

$$\|\hat{X}(\xi)\| < C_1 \|X\|_r (1 + \|\xi\|^r)^{-1}$$
$$\|D\hat{X}(\xi)\| < C_1 \|X\|_r (1 + \|\xi\|^r)^{-1}$$

for all $\xi \in \mathbb{R}^n$ and all $X \in \tilde{\theta}^r_m$.

This is a standard estimate in the theory of the Fourier transform. One has this estimate for the total derivative of any order, but the above is all we need. $\square$

For $Y \in \theta^r$, let $\hat{Y}$ denote the Fourier transform of Y, i.e.

$$\hat{Y}(\xi) = \int_{T^n} Y(z) z^{-\xi}\, dz,$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n$, $z = (z_1, \ldots, z_n) \in T^n$, $z^{-\xi} = z_1^{-\xi(1)} \cdots z_n^{-\xi(n)}$, and dz denotes Haar measure on T^n, normalized so that the total measure of T^n is one. The equation $\mathrm{Tr}_e\,(A_* - \mathrm{id})(X) = Y$ is equivalent to

$$A^{n+1}\hat{X}(A\xi) - \hat{X}(\xi) = \hat{Y}(\xi), \tag{9}$$

for all $\xi \in \mathbb{Z}^n$.

LEMMA 4. *Let $\varepsilon > 0$ and let N be a positive integer. Then there exist $\xi_{-N}, \ldots, \xi_N \in \mathbb{Z}^n \setminus \{0\}$ such that*

$$\|\xi_i - A^i \xi_0\| < \varepsilon, \qquad -N \le i \le N$$

and such that if $Y=\hat{Y}(\xi_0)z^{\xi(0)}$ *and* $X\in\tilde{\theta}^r_m$ *satisfies the conclusions of Lemma* 1 *in relation to* Y, *then*

$$\|B_i^{n+1}\hat{X}(\xi_i)-A^{n+1}\hat{X}(A\xi_0)\|<\varepsilon\,\|\hat{Y}(\xi_0)\|,\qquad 1\le i\le N,$$
$$\|B_i^{n+1}\hat{X}(\xi_i)-\hat{X}(\xi_0)\|<\varepsilon\,\|\hat{Y}(\xi_0)\|,\qquad -N\le i\le 0,$$

where $B_i=\|\xi_i\|/\|\xi_0\|$.

Proof. We will suppose that $\varepsilon<A^{-N}/4$. Obviously, there is no loss of generality in supposing this. Let δ be a positive number which satisfies

$$\delta<\min\,(C_2^{-1}N^{-1}[(3A^N/2)^{n+1}+(n+1)(3A^N/2)^n]^{-1},(A-1)A^{-N-1})\varepsilon,$$

where $C_2=CC_1A^{Nr}(4\pi)^r$. Let $p_{-N},\ldots,p_N$ be positive integers which satisfy the conclusion of Lemma 2, with ε replaced by δ. We set $\xi_i=(p_i,0,\ldots,0)$. Since $|p_{i+1}-Ap_i|<\delta<(A-1)A^{-N-1}\varepsilon$, we obtain the first inequality of Lemma 4.

Since $Y(z)=\hat{Y}(\xi_0)z^{\xi(0)}$, we obtain $\|Y\|_r=(2\pi)^r\,\|\hat{Y}(\xi_0)\|\,\|\xi_0\|^r$. Since X satisfies the conclusions of Lemma 1 with relation to Y, we then obtain

$$\|X\|_r\le C(2\pi)^r\,\|\hat{Y}(\xi_0)\|\,\|\xi_0\|^r$$

From Lemma 3, we obtain that if $\|\xi\|>A^{-N}\,\|\xi_0\|/2$ then

$$\|\hat{X}(\xi)\|<C_2\,\|\hat{Y}(\xi_0)\|,\ \|D\hat{X}(\xi)\|<C_2\,\|\hat{Y}(\xi_0)\|.$$

By the first inequality of Lemma 4 and the fact that $\varepsilon<A^{-N}/4$, the condition $\|\xi\|>A^{-N}\,\|\xi_0\|/2$ is satisfied on the line segment joining $A\xi_i$ and ξ_{i+1}, for $i=-N,\ldots,N-1$. By the mean value theorem, we then obtain

$$\|\hat{X}(\xi_{i+1})-\hat{X}(A\xi_i)\|\le C_2\,\|\xi_{i+1}-A\xi_i\|\,\|\hat{Y}(\xi_0)\|\le C_2\delta\,\|\hat{Y}(\xi_0)\|.$$

Clearly, $B_i\le A^i+\varepsilon\le 3A^i/2$, for $i=-N,\ldots,N$ and $B_{i+1}-AB_i<\delta$, so

$$B_{i+1}^{n+1}-A^{n+1}B_i^{n+1}\le(n+1)(3A^{i+1}/2)^n\delta,\qquad i=-N,\ldots,N-1.$$

We have, by (9):

$$A^{n+1}\hat{X}(A\xi_i)-\hat{X}(\xi_i)=\hat{Y}(\xi_i)=0,\qquad i\ne 0.$$

Hence

$$\begin{aligned}\|B^{n+1}_{i+1}\hat{X}(\xi_{i+1})-B^{n+1}_{i}\hat{X}(\xi_i)\| &= \|B^{n+1}_{i+1}\hat{X}(\xi_{i+1})-B^{n+1}_{i}A^{n+1}\hat{X}(A\xi_i)\|\\ &\le B^{n+1}_{i+1}\|\hat{X}(\xi_{i+1})-\hat{X}(A\xi_i)\|+|B^{n+1}_{i+1}-B^{n+1}_{i}A^{n+1}|\,\|\hat{X}(A\xi_i)\|\\ &\le C_2(B^{n+1}_{i+1}+(n+1)(3A^{i+1}/2)^n\delta\,\|\hat{Y}(\xi_0)\|\\ &\le C_2((3A^N/2)^{n+1}+(n+1)(3A^N/2)^n\,\delta\,\|\hat{Y}(\xi_0)\|\\ &\le \varepsilon\,\|\hat{Y}(\xi_0)\|/N,\end{aligned}$$

if $i\neq 0$ and $-N\le i\le N-1$. A similar argument implies

$$\|B^{n+1}_1\hat{X}(\xi_1)-A^{n+1}\hat{X}(A\xi_0)\|<\varepsilon\,\|\hat{Y}(\xi_0)\|/N$$

Combining these inequalities, we get the second and third inequalities of Lemma 4. □

End of the proof that Λ is not surjective.

We define a norm $\|\ \|_{r,2}$ on θ^r_m by

$$\|X\|^2_{r,2}=\int_{\mathbb{R}^n}\|D^rX(x)\|^2\,dx$$

and a pseudonorm $\|\ \|_{r,2}$ on θ^r by the same formula with the domain of integration changed to T^n from $\mathbb{R}^n$. We have

$$\|X\|_{r,2}\le C_3\,\|X\|_r$$
$$\|\mathrm{Tr}_e\,X\|_{r,2}\le C_3\,\|X\|_{r,2},$$

for $X\in\tilde{\theta}^r_m$, where C_3 is a constant which depends only on n and m. Moreover, for $Z\in\theta^r$, we have

$$\|Z\|^2_{r,2}=\sum_{\xi}\|\xi\|^{2r}\,\|\hat{Z}(\xi)\|^2,$$

by Parseval's equation.

Let $\xi_{-N},\dots,\xi_N$, X, and Y be as in Lemma 4. Let $Z=\mathrm{Tr}_e\,X$. We have that $\hat{Z}(\xi)=\hat{X}(\xi)$, for $\xi\in\mathbb{Z}^n$. Then

$$\begin{aligned}\|Z\|^2_{r,2} &= \sum_{\xi}\|\xi\|^{2r}\,\|\hat{X}(\xi)\|^2\ge\sum_{i=-N}^{N}\|\xi_i\|^{2r}\,\|\hat{X}(\xi_i)\|^2\\ &\ge[(N+1)(\|\hat{X}(\xi_0)\|-\varepsilon\,\|\hat{Y}(\xi_0)\|)^2+N(\|A^{n+1}\hat{X}(A\xi_0)\|-\varepsilon\,\|\hat{Y}(\xi_0)\|)^2]\,\|\xi_0\|^{2r}\\ &\ge 2N(\tfrac{1}{2}-\varepsilon)^2\,\|\hat{Y}(\xi_0)\|^2\,\|\xi_0\|^{2r}=N(1-2\varepsilon)^2\,\|Y\|^2_r/2(2\pi)^{2r},\end{aligned}$$

where the second inequality is a consequence of Lemma 4, the fact that $r=n+1$, and the definition of B_i; the third inequality is a consequence of (9); and the last equation is a consequence of $Y(z)=\hat{Y}(\xi_0)z^{\xi(0)}$. Taking $\varepsilon<\frac{1}{8}$, we then have

$$\|X\|_r \geq C_3^{-2}\|Z\|_{r,2} \geq \sqrt{N}C_3^{-2}\|Y\|_r/2(2\pi)^r.$$

Since C_3 is a constant which depends only on n and m, and N may be taken to be arbitrarily large, this contradicts Lemma 1. This contradiction proves that Λ is not surjective when $r=n+1$. □

Appendix: Proof that ι is an isomorphism

This refers to the natural homomorphism

$$\iota: H_1(\tilde{\mathscr{D}}^r(\mathbb{R}^n)^0) \to H_1(\tilde{\mathscr{D}}^r(M)^0),$$

defined in §3, where H_* denotes Eilenberg–MacLane homology. Thurston gave two proofs that ι is an isomorphism. One relies on the theory of Haefliger's classifying space. The other, which is elementary, but very clever, was generalized to the case of symplectic diffeomorphisms by Banyaga [1]. Here, we explain the proof which relies on the theory of Haefliger's classifying space. This proof generalizes my earlier proof, valid for the case $n=1$ [cf. 12, Corollary 4].

The homomorphism ι is surjective since $\tilde{\mathscr{D}}^r(M)^0$ is generated by elements having support in open balls.

We may use the theory of Haefliger's classifying space to construct a homomorphism $j: H_1(\tilde{\mathscr{D}}^r(M)^0) \to H_1(\tilde{\mathscr{D}}^r(\mathbb{R}^n)^0)$, such that $j\iota=$ identity, as follows. An element of $\tilde{\mathscr{D}}^r(M)^0$ consists of a pair (f,γ), where $f\in\mathscr{D}^r(M)^0$ and γ is a homotopy class (rel. endpoints) of curves connecting the identity to f in $\mathscr{D}^r(M)^0$. Let (f,γ) be such a pair and let $\{f_t\}\in\gamma$, so $f_t\in\tilde{\mathscr{D}}^r(M)$ for each $t\in[0,1]$, $f_0=$ identity, and $f_1=f$. Since any such curve may be smoothed we may suppose that f_t is a C^r function on $M\in[0,1]$, that $f_t=$ identity for t near 0 and that $f_1=f$ for t near 1.

For each point $x\in M$, we have the curve $\{(f_t(x),t): t\in[0,1]\}$. This defines a family of curves in $M\times[0,1]$, which may be pushed down into $M\times T^1$ by means of the mapping $t\mapsto e^{2\pi it}$ of $[0,1]$ onto T^1. The resulting family of curves in $M\times T^1$ is a C^r foliation of $M\times T^1$ by curves. Since this is a codimension n foliation ($n=\dim M$), the theory of Haefliger's classifying space [3], [4] associates to it a fiber homotopy class of mappings $\Gamma=\Gamma_{(f,\gamma)}: M\times T^1\to B\Gamma_n^r$ such that the diagram

$$\begin{array}{ccc} M \times T^1 & \xrightarrow{\Gamma} & B\Gamma^r_n \\ \Big\downarrow{\scriptstyle \text{proj.}} & & \Big\downarrow{\scriptstyle \nu} \\ M & \xrightarrow{\tau_M} & BGl(n) \end{array}$$

commutes. Here τ_M is a mapping associated to the tangent bundle of M, according to the classical theory of classifying spaces [21], and ν is a mapping associated to the normal bundle of the canonical Haefliger structure on $B\Gamma^r_n$. We may think of Γ as a homotopy class of sections of

$$(\tau_M \circ \text{proj.})^* B\Gamma^r_n,$$

which is a bundle over $M \times T^n$. The fiber of this bundle is the same as the fiber of ν, i.e. $B\bar{\Gamma}^r_n$. According to obstruction theory [21], homotopy classes of sections of this bundle are in $1-1$ correspondence with elements of $H^{n+1}(M \times T^1, \pi_{n+1}(B\bar{\Gamma}^r_n))$, since $B\bar{\Gamma}^r_n$ is n-connected, by Haefliger's theory [3], [4]. (Here, H^* means singular cohomology). In order to define this $1-1$ correspondence, it is necessary to choose a basepoint in the space of sections which corresponds to the zero element of $H^{n+1}(M \times T^1, \pi_{n+1}(B\bar{\Gamma}^r_n))$; we let the basepoint be $\Gamma_{(\text{id},\text{id})}$, where (id, id) denotes the identity element of $\tilde{\bar{\mathcal{D}}}^r(M)^0$. Since $H^{n+1}(M \times T^1, \pi_{n+1}(B\bar{\Gamma}^r_n)) = \pi_{n+1}(B\bar{\Gamma}^r_n)$, we have defined a mapping

$$\tilde{\bar{\mathcal{D}}}^r(M)^0 \to \pi_{n+1}(B\bar{\Gamma}^r_n).$$

It is easily verified that this mapping is a homomorphism of groups.

Thurston proved that there is a mapping

$$B\bar{\mathcal{D}}^r_n \to \Omega^n B\bar{\Gamma}^r_n$$

which induces isomorphism in (singular) homology. Here, $\Omega^n X$ denotes the n^{th} loop space of X and $B\bar{\mathcal{D}}^r_n$ denotes the homotopy theoretic fiber of the identity mapping

$$B\mathcal{D}^{r,\delta}_n \to B\mathcal{D}^r_n$$

where $\mathcal{D}^{r,\delta}_n$ denotes $\mathcal{D}^r_n$ provided with the discrete topology. Peviously, I had proved this theorem for the case $n=1$ ([12], [13]). A proof of the result for general n (due to Thurston) may be found in [16]. A short outline of another proof (also due to Thurston) may be found in [14]. (See also [15]). A later proof by another method may be found in [17]. The latter paper contains the first published proof for the case $r=0$.

Obviously, we have

$$\pi_{n+1}(B\bar{\Gamma}^r_n)=\pi_1(\Omega^n B\bar{\Gamma}^r_n)=H_1(\Omega^n B\bar{\Gamma}^r_n),$$

and, by Thurston's theorem, this is the same as $H_1(B\bar{\mathscr{D}}^r_n)$. It is easily seen that $B\bar{\mathscr{D}}^r_n$ is the homotopy theoretic fiber of the identity mapping $B\tilde{\mathscr{D}}^{r,\delta}_n \to B\tilde{\mathscr{D}}^r_n$, where $\tilde{\mathscr{D}}^{r,\delta}_n$ denotes $\tilde{\mathscr{D}}^r_n$ with the discrete topology. Since $\tilde{\mathscr{D}}^r_n$ is simply connected, the first and second homology groups of $B\tilde{\mathscr{D}}^r_n$ vanish, and it follows that $H_1(B\bar{\mathscr{D}}^r_n)=H_1(B\tilde{\mathscr{D}}^{r,\delta}_n)$. Hence

$$\pi_{n+1}(B\bar{\Gamma}^r_n)=H_1(B\tilde{\mathscr{D}}^{r,\delta}_n)=\frac{\tilde{\mathscr{D}}^r_n}{[\tilde{\mathscr{D}}^r_n,\tilde{\mathscr{D}}^r_n]}$$

and the homomorphism we constructed above has the form

$$\tilde{\mathscr{D}}^r(M)^0\to\frac{\tilde{\mathscr{D}}^r_n}{[\tilde{\mathscr{D}}^r_n,\tilde{\mathscr{D}}^r_n]}.$$

Obviously, the commutator subgroup of $\tilde{\mathscr{D}}^r(M)^0$ goes to zero under this homomorphism. Consequently, we have an induced homomorphism

$$j: H_1(\tilde{\mathscr{D}}^r(M)^0)=\frac{\tilde{\mathscr{D}}^r(M)^0}{[\tilde{\mathscr{D}}^r(M)^0,\tilde{\mathscr{D}}^r(M)^0]}\to H_1(\tilde{\mathscr{D}}^r_n)$$

where, now, H_* means Eilenberg–MacLane homology. It is easily checked that $j\iota=$ identity. Consequently ι is injective. □

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Received March 21, 1983/September 5, 1983