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# Classification and stable classification of manifolds: some examples

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## I. Introduction

Two closed $n$-dimensional differentiable manifolds $M$ and $N$ are called stably diffeomorphic if there exists an integer $r \in \mathbb{N}$ such that

$$M \# rS \cong N \# rS$$

where

$$S = \begin{cases} S^k \times S^k, & n = 2k \\ S^k \times S^{k+1}, & n = 2k+1 \end{cases}.$$

In the literature there are some cases of manifolds which can be classified up to stable diffeomorphism, for instance 1-connected 4-manifolds [11] and manifolds of type B(SO) [4]. The problem we want to discuss in this paper is how much information about a differentiable manifold is lost if one passes to its stable diffeomorphism class. We begin with the following observation about the stable classification of even-dimensional manifolds. We use the notations of [14].

PROPOSITION I.1. $n \geqq 2$. *Let $M^{2n}$ and $N^{2n}$ be two normally bordant manifolds. Then $M$ and $N$ are stably diffeomorphic. More precisely if $n > 2$ and $W$ is a normal cobordism between $M$ and $N$ with surgery obstruction $\theta(W) \in L_{2n+1}$ $(\mathbb{Z}[\pi_1(M)], w_1)$ represented by a matrix $A \in SU_r$ $(\mathbb{Z}[\pi_1(M)])$ then*

$$M \# r(S^n \times S^n) \cong N \# r(S^n \times S^n).$$

*Proof.* By ([14], Theorem 6.5) for $n > 2$ we can assume that $W$ is the normal cobordism constructed in the proof of this theorem. That means: $W = M \times I \natural r(S^n \times D^{n+1}) \cup r$ handles of index $n+1$. If one considers the dual handle

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decomposition starting from $N$ we obtain the same picture. Thus

$$M \# r(S^n \times S^n) \cong N \# r(S^n \times S^n).$$

In dimension 4 the same proof works after additional stabilization by the methods of ([2], [8]).

*Remark.* The corresponding statement for odd-dimensional manifolds is false. For instance a homotopy sphere $\Sigma^{4k-1} \in bP_{4k}$ is normally cobordant to $S^{4k-1}$ but if $\Sigma \# r(S)$ was diffeomorphic to $S^{4k-1} \# r(S)$ then $\Sigma$ would be contained in the inertia group of $r(S^{2k} \times S^{2k-1})$ which is zero or $\mathbb{Z}_2$ by [3].

For some manifolds $M$ the set of diffeomorphism classes of manifolds which are normally cobordant to $M$ was computed and this leads to some non-cancellation results.

EXAMPLE. $n \geqq 5$. The set of diffeomorphism classes of manifolds which are normally cobordant to $T^n$ contains the non-trivial orbit space $H^3(T^n; \mathbb{Z}_2)/H^1(T^n; \mathbb{Z}_2)$ as a subset ([17], § 15 A). By the proposition above for $n$ even they are all stably diffeomorphic to $T^n$. Thus for even $n \geqq 6$ there exist many fake tori which are stably diffeomorphic to $T^n$ but not diffeomorphic.

*Remark.* For $n \geqq 5$ all fake tori are homeomorphic to $T^n$ ([17], p. 227).

In this example cancellation holds in the topological category. We will now give some examples which show that in general stably diffeomorphic manifolds are not even homotopy equivalent. The easiest examples of this type are obtained from stably parallelizable $(2n-1)$ connected $4n$-manifolds, $n > 1$. According to Wall [12] two such manifolds with equivalent intersection forms differ only by a homotopy sphere. Thus if $M^{4n}$ and $N^{4n}$ $(n > 1)$ are two such manifolds with non-equivalent intersection forms (which one can construct by plumbing, compare [1]) then $M \# S^{2n} \times S^{2n}$ and $N \# S^{2n} \times S^{2n}$ have equivalent intersection forms [6] and thus there exists a homotopy sphere $\Sigma$ such that $\Sigma \# M \# S$ and $\Sigma \# N \# S$ are diffeomorphic but $\Sigma \# M$ and $N$ are not even homotopy equivalent.

In dimension $8k$ one can use the invariant consisting of the isomorphism class of the triple $(H_{4k}(M; \mathbb{Z}), \circ, P_k)$, where $\circ$ is the intersection form and $P_k: H_{4k}(M; \mathbb{Z}) \to \mathbb{Z}$ is the Pontrjagin class, to detect $(4k-1)$ connected $8k$-manifolds with equivalent intersection forms which are stably diffeomorphic but not diffeomorphic.

*Remark.* In dimension 4 the cancellation problem for connected sum with $S^2 \times S^2$ is much more difficult. By recent results of S. K. Donaldson (compare

Atiyah's talk at the "Bonner Mathematische Arbeitstagung" 1982) the intersection form of a 1-connected differentiable 4-manifold is always indefinite or the standard definite form. Thus it is completely determined by its rank, signature and type. This implies the intersection form can not be used to distinguish stably diffeomorphic 1-connected differentiable 4-manifolds. Moreover it implies two 1-connected differentiable 4-manifolds are stably diffeomorphic if and only if they are homotopy equivalent. By Freedman's result two smooth homotopy equivalent 1-connected 4-manifolds are homeomorphic [18]. Thus the existence of stably diffeomorphic 1-connected 4-manifolds which are not diffeomorphic is equivalent to the existence of a 1-connected 4-manifold which has two different differentiable structures. It should be remarked here that Freedman's work implies that there are many 1-connected topological 4-manifolds which are stably homeomorphic but not homotopy equivalent.

The examples of non-cancellation results described above are of the following type: (a) stably diffeomorphic manifolds of dimension $2n>4$ which are normally bordant and modulo some indeterminacy are distinguished in $L^s_{2n+1}(\mathbb{Z}[\pi_1])$.

(b) manifolds of dimension $4k$, $k>1$, which are distinguished by the intersection form together with some stable tangent bundle information.

We will show in this paper that in odd dimensions also stably diffeomorphic doesn't imply diffeomorphic and in dimension $=0 \bmod 4$ there are stably diffeomorphic manifolds which have equivalent intersection forms and are stably parallelizable but are not even homotopy equivalent. In particular this seems to be the first non-cancellation result in dimension 4.

THEOREM I.1 (see also Theorem III.3). *For all* $n \geqq 4$ *with* $n \neq 2 \bmod 4$ *there exist stably diffeomorphic manifolds with trivial stable tangent bundle, and equivalent intersection forms if* $n=0 \bmod 4$, *which are not diffeomorphic. In fact these manifolds are not even homotopy equivalent.*

We will give an explicit construction of such manifolds in the following section. In it we will compute some basic invariants such as cohomology and intersection form and deduce the theorem in the odd-dimensional case from a non-cancellation result for some CW-complexes. The even-dimensional case is comparatively more difficult. In Section III we will introduce the basic invariant for this case and compute it for some examples in order to obtain the even-dimensional result.

*Remarks.* (1) Our examples are not 1-connected, the smallest fundamental group is $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. The existence of 1-connected examples in dimension 4 is equivalent to the existence of a closed 1-connected differentiable 4-manifold which has two different differentiable structures.

(2) It is perhaps interesting to note that even if our examples are not normally cobordant they are distinguished within a $L$-group.

(3) Our construction also works in dimension $4k+2$ but in this case the invariant is too weak to distinguish.

## II. Boundaries of thickenings

Our $k$-dimensional examples will be the boundary of a thickening of a finite $[k/2]$-dimensional CW-complex $X$. More precisely, let $X$ be a finite $n$-dimensional CW-complex. Then there exists a thickening of $X$ in $\mathbb{R}^{k+1}$ for $k \geqq 2n$ ([13]). This means that there exists a compact $k+1$-dimensional submanifold with boundary $N(X)$ in $\mathbb{R}^{k+1}$ which is simply homotopy equivalent to $X$. The boundary of $N(X)$ is unique up to $s$-cobordisms and we denote it by $M(X)^k$. Moreover Wall has proved that if $X$ and $Y$ are simply homotopy equivalent then $M(X)^k$ and $M(Y)^k$ differ by an $s$-cobordism. Thus we have a map from the set of simple homotopy equivalence classes of finite $n$-dimensional complexes into the set of diffeomorphism classes ($s$-cobordism classes if $k=4$) of $k$-dimensional manifolds.

*Remark.* There are several ways to visualize a thickening of $X$ in $\mathbb{R}^{k+1}$. One possibility is to consider $X$ as subpolyhedron of $\mathbb{R}^{k+1}$ and to take a smooth regular neighborhood of it in $\mathbb{R}^{k+1}$. Another possibility is to construct $N(X)$ as a handle body whose handles correspond to the cells of $X$ and the way the handles are attached is determined by the fact that the resulting manifold is to be stably parallelizable. We will discuss the consequences of this description of $N(X)$ later.

From all constructions one can easily see the following property of $M(X)^k$.

LEMMA II.1.

$$M(X \vee S^n)^k = \left\{\begin{matrix} M(X)^k \# S^n \times S^n, & k=2n \\ M(X)^k \# S^n \times S^{n+1}, & k=2n+1 \end{matrix}\right\}.$$

Thus the construction $X \mapsto M(X)^k$ gives a connection between our problem and the homotopy theoretical problem of the stable classification of $n$-dimensional CW-complexes ($X$ and $Y$ are *stably homotopy equivalent* if $X \underset{r}{\vee} S^n \simeq Y \underset{r}{\vee} S^n$) and the classification up to homotopy equivalence. The first homotopy theoretical results in this direction were obtained by J. H. C. Whitehead who proved that if $X$ and $Y$ are 2-dimensional CW-complexes with isomorphic fundamental groups and the same Euler characteristic then $X$ and $Y$ are stably homotopy equivalent. This result is also valid for $n$-dimensional complexes whose universal cover is $(n-1)$-connected.

It took rather a long time until one could show there exist complexes of this type which are stably simply homotopy equivalent but not homotopy equivalent.

THEOREM ([5], [9], [10]). *For all* $n \geqq 2$ *there exist finite* $n$*-dimensional CW-complexes* $X$ *and* $Y$ *such that* (i) *the universal covers are* $(n-1)$*-connected,* (ii) $X \vee S^n$ *and* $Y \vee S^n$ *are simply homotopy equivalent.* (iii) $X$ *and* $Y$ *are not homotopy equivalent.*

Thus if $X$ and $Y$ satisfy (i), (ii) and (iii) we know $M(X)^k$ and $M(Y)^k$ are stably diffeomorphic for $k = 2n$ or $2n+1$. Furthermore we know they are stably parallelizable and the intersection form of $M(X)^{2n}$ is completely determined by the Euler characteristic. This follows since the signature of $M(X)^{2n}$ is zero and the intersection form is of even type as $M(X)^{2n}$ is stably parallelizable, hence it is classified by its rank ([6]). Thus we would have examples of stably diffeomorphic but not diffeomorphic manifolds if $M(X)^k = M(Y)^k$ would imply $X$ and $Y$ are homotopy equivalent. This is easy to show for $k = 2n+1$. In fact we have a somewhat stronger result, which implies the odd-dimensional case of Theorem I.1.

PROPOSITION II.1. *Let* $X$ *and* $Y$ *be finite* $n$*-dimensional CW-complexes. If* $M(X)^{2n+1}$ *and* $M(Y)^{2n+1}$ *are homotopy equivalent then* $X$ *and* $Y$ *are homotopy equivalent.*

*Proof.* The first observation is that if we have a thickening $N$ of $X$ in $\mathbb{R}^k$ then $N \times I \subset \mathbb{R}^{k+1}$ is a thickening in $\mathbb{R}^{k+1}$. As $X$ has a thickening $N(X)$ in $\mathbb{R}^{2n+1}$ we know that $M(X)^{2n+1} = \partial(N(x) \times I) = N(X) \cup (-N(X))$, the double of $N(X)$. Since $N(X)$ has a handle decomposition whose handles correspond to the cells of $X$ we obtain a handle decomposition of $M(X)^{2n+1}$ whose handles of index $\leqq n$ form $N(X)$ and the handles of index $> n$ correspond to $-N(X)$. This implies $\pi_1(N(X)^{2n+1}) = \pi_1(N(X)) = \pi_1(X)$.

A general position argument implies $H_i\ (\partial N(X); \Lambda) \xrightarrow{\cong} H_i(N(X); \Lambda)$, where $\Lambda = \mathbb{Z}[\pi_1(x)]$ is the group ring, is an isomorphism for $i < n$ and surjective for $i = n$. This and the Mayer–Vietoris sequence imply

$$H_i(N(X); \Lambda) \xrightarrow{\cong} H_i(M(X)^{2n+1}; \Lambda) \quad \text{for} \quad i \leqq n.$$

Now given a homotopy equivalence $f: M(X)^{2n+1} \to M(Y)^{2n+1}$ one can deform it so that $N(X)$ is mapped into $N(Y)$. Assuming this, $f|_{N(X)}: N(X) \to N(Y)$ is an isomorphism on $\pi_1$ and on all homology groups with coefficients in $\Lambda$. Thus $N(X) \simeq N(Y)$ and the result follows as $N(X)$ and $N(Y)$ are homotopy equivalent to $X$ and $Y$ respectively.

In the proof of the preceding proposition we computed the homology of $M(X)^{2n+1}$. It is easy to make a similar computation for $M(X)^{2n}$ but we also must know the cellular chain complex with coefficients in $\Lambda$. For this we will show $M(X)^{2n}$ can also be written as a double $L(X) \cup -L(X)$ where $L(X)$ is also a thickening of $X$ but in general is not contained in $\mathbb{R}^{2n}$.

PROPOSITION II.2. *Let X be a finite n-dimensional CW-complex. Then there exists a* $2n$*-dimensional thickening* $L(X)$ (*which in general is not contained in* $\mathbb{R}^{2n}$ *and is not unique*) *of X such that*

(i) $M(X)^{2n} = L(X) \cup -L(X)$ *and*
(ii) *the intersection form* $H_n(L(X); \mathbb{Z}) \otimes H_n(L(X); \mathbb{Z}) \to \mathbb{Z}$ *is zero.*

*Proof.* We construct $L(X)$ as follows. Let $Y$ be the $(n-1)$-skeleton of $X$ and $N(Y)$ a thickening of $Y$ in $\mathbb{R}^{2n}$. Then $N(Y) \times I$ is a thickening of $Y$ in $\mathbb{R}^{2n+1}$ and we know that we can obtain the thickening $N(X)$ of $X$ in $\mathbb{R}^{2n+1}$ by adding $n$-handles to $N(Y) \times I$ which correspond to the $n$-cells of $X$. Given a characteristic embedding $f: S^{n-1} \times D^{n+1} \hookrightarrow \partial(N(Y) \times I)$ of such a handle we can isotope the embedding such that $g = f|_{S^{n-1} \times D^n}$ maps into $\partial(N(Y))$ and $f = g \times Id: S^{n-1} \times D^{n+1} \to \partial N \subset \partial(N(Y) \times I)$.

For by a general position argument we can find an embedding $S^{n-1} \subset \partial N(Y)$ which in $\partial(N(Y) \times I)$ is isotopic to $f|_{S^{n-1} \times \{0\}}$. Thus we can assume $f|_{S^{n-1} \times \{0\}}$ is contained in $\partial N(Y)$. This embedding has trivial normal bundle in $\partial N(Y)$ so we can extend it to an embedding $g: S^{n-1} \times D^n \hookrightarrow \partial(N(Y))$. As $\pi_{n-1}(\mathrm{SO}(n)) \to \pi_{n-1}(\mathrm{SO}(n+1))$ is surjective we can choose this embedding so that $g \times Id: S^{n-1} \times D^{n+1} \to \partial N(Y) \times I \subset \partial(N(Y) \times I)$ is isotopic to $f$.

If we choose characteristic embeddings with this property we see $N(X) = L(X) \times I$ where $L(X)$ is obtained from $N(Y)$ by adding handles with the g's. In particular it follows that $M(X)^{2n} = L(X) \cup -L(X)$.

For the proof of the second statement we use the fact that we are free to choose the embedding $S^{n-1} \subset \partial N(Y)$ which in $\partial(N(Y) \times I)$ is isotopic to $f|_{S^{n-1} \times \{0\}}$ arbitrarily within its isotopy class. Further, if we have chosen $g = S^{n-1} \times D^n \hookrightarrow \partial(N(Y))$ we can twist this by an arbitrary element of $\mathrm{Ker}\,(\pi_{n-1}(\mathrm{SO})(n)) \to \pi_{n-1}(\mathrm{SO})(n+1))$.

Now let $g_i : S^{n-1}\times D^n \hookrightarrow \partial N(Y)$ $1\leqq i\leqq r$ be the disjoint characteristic embeddings of $L(X)$. Then $H_n(L(X);\mathbb{Z})=\operatorname{Ker}(\mathbb{Z}^r\to H_{n-1}(N(Y);\mathbb{Z}))$ where $e_i\mapsto (g_i)_*[s^{n-1}]$. This kernel is a direct summand in $\mathbb{Z}^r=\pi_n(X, Y)$ as $H_{n-1}(Y;\mathbb{Z})$ is torsion free ($Y$ is a $(n-1)$-dimensional complex). Thus if we possibly change the presentation of $X$ as CW-complex we can assume that $e_1,\dots, e_j$ form a basis of $\operatorname{Ker}(\mathbb{Z}^r\to H_{n-1}(NY;\mathbb{Z}))$.

Denote the corresponding elements in $(H_n(L(X);\mathbb{Z})$ by $\hat{e}_1,\dots,\hat{e}_s$. It is a well known fact that for $i\neq j$

$$\hat{e}_i\circ\hat{e}_j=\pm L(g_i(S^{n-1}\times\{0\}), g_j(S^{n-1}\times\{0\})),$$

where the linking numbers of the null-homologous *embeddings* $g_i(S^{n-1}\times\{0\})$ and $g_j(S^{n-1}\times\{0\})$ in $\partial N(Y)$ are given by $C\circ(g_j)[S^{n-1}]$ where $C$ is a chain bounding $g_i(S^{n-1}\times\{0\})$. But for $i<j$ we can change this linking number by an arbitrary integer if we add an appropriate multiple of $g_i(\{*\}\times S^{n-1})$ to $g_i(S^{n-1}\times\{*\})$ which is allowed as it does not change the isotopy class. Thus we can change our characteristic embeddings so that for $i\neq j$:

$$\hat{e}_i\circ\hat{e}_j=0.$$

For $n$ odd we are finished and for $k$ even we know $\hat{e}_i\circ\hat{e}_i$ is even as $L(X)$ is stably parallelizable. We are allowed to change $f_i : S^{n-1}\times D^n\to\partial N(Y)$ by twisting with any element $\alpha$ in $\operatorname{Ker}(\pi_{n-1}(\mathrm{SO}(n))\to\pi_{n-1}(\mathrm{SO}((n+1)))$. If we do this then only $\hat{e}_i\circ\hat{e}_i$ changes, namely by the Euler number of the bundle over $S^n$ corresponding to $\alpha$. Thus we also can achieve $\hat{e}_i\circ\hat{e}_i=0$ as there exist bundles over $S^n$ with arbitrary even Euler number.

Now we compute the cellular chain complex of $M(X)^{2n}$. For a left module $A$ we denote by $A^*=\operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ the $\Lambda$-module with $(g\cdot f)(x)=f(g^{-1}\cdot x)$.

PROPOSITION II.3. *Let* $M(X)^{2n}=L(X)\cup -L(X)$ *as in Proposition II.2. With respect to the corresponding handle decomposition the cellular chain complex of* $M(X)^{2n}$ *is given by*

$$0\to C_0(X;\Lambda)^*\xrightarrow{\pm\partial_1^*}C_1(X;\Lambda)^*\to\cdots\to C_{n-1}(X;\Lambda)^*\xrightarrow{(\pm\partial_n^*,\,0)}$$

$$C_n(X;\Lambda)^*\oplus C_n(X;\Lambda)\xrightarrow{\binom{0}{\partial_n}}C_{n-1}(X;\Lambda)\xrightarrow{\partial_{n-1}}\cdots\xrightarrow{\partial_1}C_0(X;\Lambda)\to 0,$$

(the sign of $\partial_k^*$ is $(-1)^k$) *where* $\Lambda$ *is the group ring* $\mathbb{Z}(\pi)$ *and* $\pi=\pi_1(X)\overset{\cong}{\Rightarrow}\pi_1(M(X)^{2n})$.

*Proof.* We recall that $L(X)$ is homotopy equivalent to $X$ and has a handle decomposition corresponding to the cell decomposition of $X$. Thus for $i \leqq n-1$ the left hand disks (notation as in [7]) of $M(X)^{2n}$ correspond to the $i$-cells of $X$ and up to dim $n-1$ the cellular chain complex of $M(X)^{2n}$ is given by

$$C_{n-1}(X; \Lambda) \to \cdots \to C_0(X;\Lambda) \to 0.$$

For $i > n$ the left hand $i$-disks of $M(X)^{2n}$ are all contained in $-L(X)$ and correspond to the right hand $i$-disks of $-L(X)$. From the duality between right hand and left hand disks we know by ([7]) that $C_i(M(X)^{2n}; \Lambda) \cong C_{2n-i}(-L(X); \Lambda)^* = C_{2n-i}(X; \Lambda)^*$ and that the cellular chain complex in dimensions $\geqq n+1$ of $M(X)^{2n}$ is given by

$$0 \to C_0(X; \Lambda)^* \xrightarrow{\pm\partial_1^*} C_1(X; \Lambda)^* \to \cdots \to C_{n-1}(X; \Lambda)^*.$$

There are two types of left hand $n$-disks in $M(X)^{2n}$, those sitting in $L(X)$ and those sitting in $-L(X)$ which again correspond to the right hand $n$-disks in $-L(X)$. Thus as above we have a splitting

$$C_n(M(X)^{2n}; \Lambda) = C_n(X; \Lambda)^* \oplus C_n(X; \Lambda)$$

For the boundary operator $C_{n+1}(M(X)^{2n}; \Lambda) = C_{n-1}(X; \Lambda)^* \to C_n(X; \Lambda)^* \oplus C_n(X; \Lambda)$ it is clear from ([7]) that it is of the form $(\pm\partial_n^*, ?)$. On the other hand the boundary of a left hand $(n+1)$-disk in $C_{n+1}(M(X)^{2n}; \Lambda)$ which corresponds to a right-hand $(n+1)$-disk of $-L(X)$ is contained in $-L(X)_*$ so it has no component in $L(X)$ and thus ? is 0.

For the boundary operator $C_n(M(X)^{2n}; \Lambda) = C_n(X; \Lambda)^* \oplus C_n(X; \Lambda) \to C_{n-1}(X; \Lambda)$ it is clear that it is of the form $\begin{pmatrix} ? \\ \partial_n \end{pmatrix}$. To show $? = 0$ in this case we consider a left hand $n$-disk in $M(X)^{2n}$ which corresponds to an element of $C_n(X; \Lambda)^*$. Geometrically it is given by a right hand $n$-disk of $-L(X)$. The boundary of it is zero in $C_{n-1}(L(X); \Lambda)$ as it bounds the same disk considered as sitting in $L(X)$.

Finally for $\pi_1(X)$ finite we need some information about the intersection form of the universal cover $\widetilde{M(X)}^{2n}$. This form is invariant under the action of $\pi_i$. If we consider $H_n(M(X)^{2n}; \Lambda)$ as a $\mathbb{Z}$-module then

$$H_n(M(X)^{2n}; \Lambda) = H_n(\widetilde{M(X)}^{2n}; \mathbb{Z}).$$

From Proposition II.3 we know that

$$H_n(M(X)^{2n}; \Lambda) = H^n(X; \Lambda) \oplus H_n(X; \Lambda)$$

or as $H^n(X; \Lambda) = H_n(X; \Lambda)^*$ ($\tilde{X}$ is $(n-1)$-connected) we have

$$H_n(M(X)^{2n}; \Lambda) = H_n(X; \Lambda)^* \oplus H_n(X; \Lambda).$$

Thus $H_n(\widetilde{M(X)}^{2n}; \mathbb{Z}) \cong H_n(\tilde{X}; \mathbb{Z})^* \oplus H_n(\tilde{X}; \mathbb{Z})$.

PROPOSITION II.4. *Let $\pi_1(X)$ be finite. If we write $M(X)^{2n}$ as $L(X) \cup -L(X)$ as in Proposition II. 2 then with respect to the splitting above:*

$$H_n(\widetilde{M(X)}^{2n}; \mathbb{Z}) = H_n(\tilde{X}; \mathbb{Z})^* \oplus H_n(\tilde{X}; \mathbb{Z})$$

the intersection form is equal to

$$\begin{pmatrix} 0 & \langle\, ,\rangle \\ (-1)^n \langle\, ,\rangle & ? \end{pmatrix}$$

*Proof.* The splitting $H_n(\widetilde{M(X)}^{2n}; \mathbb{Z}) = H_n(\tilde{X}; \mathbb{Z})^* \oplus H_n(\tilde{X}; \mathbb{Z})$ corresponds to the splitting of the exact sequence

$$\begin{array}{l} 0 \to H_{n+1}(\widetilde{L(X)} \times I, \partial(\widetilde{L(X)} \times I); \mathbb{Z}) \xrightarrow{\ \partial\ } H_n(\widetilde{L(X)} \cup -\widetilde{L(X)}; \mathbb{Z}) \\ \qquad\qquad \uparrow \cong \qquad\qquad\qquad\qquad\qquad\qquad \underset{p_*}{\overset{i_*}{\rightleftarrows}} H_n(\widetilde{L(X)} \times I; \mathbb{Z}) \to 0 \\ \quad H^n(\widetilde{L(X)}; \mathbb{Z}) \\ \qquad \wr\Vert \\ \quad H^n(\tilde{X}; \mathbb{Z}) \\ \qquad \surd \end{array}$$

where $p$ is the projection onto $\widetilde{L(X)} \times \{0\}$.

It is well known that for $W$ an oriented $2n$-manifold and $a \in H_{n+1}(W, \partial W; \mathbb{Z})$, $b \in H_n(\partial W; \mathbb{Z})$ the intersection number $\partial a \circ b$ is equal to $\pm\langle \Delta^{-1} a, i_* b\rangle$ where $\Delta$ is Poincaré duality.

Thus in our case the intersection form vanishes on *im* $(\partial)$ and corresponds to the Kronecker product for $a \in H^n(\tilde{X}; \mathbb{Z})$ and $b \in H_n(\tilde{X}; \mathbb{Z})$.

Some final remarks. From the results mentioned before we know there exist finite $n$-dimensional complexes $X$ and $Y$ such that $\tilde{X}$ and $\tilde{Y}$ are $(n-1)$-connected and $X$ and $Y$ stably simply homotopy-equivalent. Thus $M(X)^{2n}$ and $M(Y)^{2n}$ are stably diffeomorphic. We will construct in the next section an invariant which for $n$ even shows that for certain $X$ and $Y$, $M(X)^{2n}$ and $M(Y)^{2n}$ are not homotopy equivalent. We have already mentioned that $M(X)^{2n}$ and $M(Y)^{2n}$ have equivalent intersection forms. This will prove our Theorem I.1.

There is an obvious question to be asked. Namely in the non-simply connected case there is another invariant which one might use to distinguish stably diffeomorphic manifolds: the intersection form of the universal cover considered as a $\mathbb{Z}$-valued form of $\Lambda$-modules. In our case it is not difficult to show that if $|\pi_1(X)|$ is odd then the intersection form on $H_n(\widetilde{M(X)^{2n}};\mathbb{Z})=H^n_*(X;\Lambda)^*\oplus H_n(X;\Lambda)$ is equivalent to $\begin{pmatrix}0 & \langle,\rangle\\ \langle,\rangle & 0\end{pmatrix}$ under a transformation of the form $\begin{pmatrix}1 & 0\\ Y & 1\end{pmatrix}$. Thus also these invariants agree in our examples.

## III.1. The 4$n$-dimensional case

In this section we will summarize the geometric results of section II, translate to algebra, and outline the procedure we will use to produce the examples of dimension $4n$. $G$ will always denote a finite group and $\Lambda$ its integral group ring. All modules will be finitely generated.

Recall that if $A$ is a $\Lambda$-lattice (i.e. free as an abelian group), $A^*=\text{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ is the $\Lambda$-module with $(g\cdot f)(x)=f(g^{-1}x)$. The function $A\to A^*$ is exact on $\Lambda$-lattices and maps projectives to projectives since $\Lambda\simeq\Lambda^*$ via the $\Lambda$-map which sends $1\in A$ to $\pi_1$, projection on the identity component.

If $A$ is a $\Lambda$-module $\hat{H}^0(G,A)=A^G/NA$ will denote the 0th Tate cohomology group of $G$ with coefficients in $A$. Since it is used often in this paper we will sometimes denote this group by $\hat{A}$.

Let $n\geqq 1$ and consider the following categories.

(i) $\mathcal{FC}_n(\mathcal{C}_n)$; the category whose objects are $\Lambda$-free (projective) chain complexes $C_*$ of length $n$, exact except at $C_0$ and $C_n$, with $H_0(C_*)=\mathbb{Z}$. (i.e. free (projective) syzygies of $\mathbb{Z}$). The maps are chain maps inducing the identity on $H_0$. We will denote $H_n(C_*)$ by $\pi_n(C_*)$ or $\pi_n$ or sometimes even $\pi$ if $C_*$ and $n$ are clear.

(ii) $\mathcal{FM}_{2n}(\mathcal{M}_{2n})$: the category whose objects are pairs $(D_*,\Phi_D)$ where $D_*$ is a free (projective) chain complex of length $2n$ exact except at $D_0, D_n, D_{2n}$ with $H_0(D_*)\simeq H_{2n}(D_*)=\mathbb{Z}$. $\Phi_D: H_n(D_*)\to H_n(D_*)^*$ is a $\Lambda$-isomorphism i.e. is the adjoint of a nonsingular $\Lambda$-equivariant bilinear form on $H_n(D_*)$. If $A$ is any lattice and $(A,b)$ is a $\mathbb{Z}$-valued $\Lambda$-form on $A$, then $b$ induces forms $(A^G, b^G)$ on the fixed point set and $(\hat{A}, \hat{b})$ on the Tate group. The form $\hat{b}$ is a $\mathbb{Z}/|G|$-valued form.

The maps in the category $\mathcal{FM}_{2n}(\mathcal{M}_{2n})$ will be chain maps inducing the identity on $H_0$ and $H_{2n}$ and an isometry of $(\hat{H}^0(G, H_n(C_*)), \hat{b}_C)$ with $(\hat{H}^0(G, H_n(D_*)), \hat{b}_D)$.

(iii) $\widetilde{\mathcal{FC}}_n(\tilde{\mathcal{C}}_n)$; the category whose objects are the objects of $\mathcal{FC}_n(\mathcal{C}_n)$. A morphism $C \to C'$ consists of a pair of homomorphisms $f: C \to C'$ and $g: C' \to C$.

Consider the functor $M: \widetilde{\mathcal{FC}_n(\mathcal{C}_n)} \to \mathcal{FM}_{2n}(\mathcal{M}_{2n})$ defined as follows. $M(C_*) = (D_*, \Phi_D)$ where

$$D_*: C_0^* \xrightarrow{\pm\partial_1^*} C_1^* \longrightarrow \cdots \xrightarrow{\pm\partial_{n-1}^*} C_{n-1}^* \xrightarrow{(\pm\partial_n^*, 0)} C_n^* \oplus C_n$$

$$\xrightarrow{\binom{0}{\partial_n}} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0, \ \Phi_{D*} = id, \qquad \text{sign of } \partial_k^*: (-1)^k$$

If $f: C_* \to C'_*$ and $g: C'_* \to C_*$ then $M(f, g) = h$ where $h_i = f_i$, $0 \leqq i < n$; $h_i = g_i^*$, $n+1 \leqq i \leqq 2n$ and $h_n = g_n^* \oplus f_n$. We will see in Section 5 that $M(f, g)$ is a map in $\mathcal{FM}_{2n}(\mathcal{M}_{2n})$.

Finally, let $M$ be a $\Lambda$-module and $\alpha \in \operatorname{Aut} G$. Denote by ${}_\alpha M$ the $\Lambda$-module where $g \cdot {}_\alpha m = \alpha(g) \cdot m$. Note that if $f: M \to N$ is a $\Lambda$-map, then $f$ is also a $\Lambda$-map $f: {}_\alpha M \to {}_\alpha N$. Hence if $C_*$ is a $\Lambda$-chain complex then so is ${}_\alpha C_*$ where $({}_\alpha C_*)_n = {}_\alpha C_n$ and ${}_\alpha \partial_n = \partial_n$.

In Section II we saw that the manifold $MX^{2n}$ had the following properties. (i) $\pi_1 MX \simeq \pi_1 X$, (ii) the chain complex $C(MX; \Lambda)$ is chain homotopy equivalent to $M(C(X^n; \Lambda))$ (Proposition II.3) (iii) under the chain homotopy equivalence in (ii) the intersection form $b_x$ on $H_n(MX: \Lambda)$ corresponds to the form on $H_n(M(C(X^n; \Lambda))) = H_n(X, \Lambda)^* \oplus H_n(X, \Lambda)$ given by the matrix $\begin{pmatrix} 0 & \langle , \rangle \\ \langle , \rangle & ? \end{pmatrix}$ where $\langle , \rangle$ is the Kronecker product on $H_n(X, \Lambda)$ (Proposition II.4), (iv) the form $b_x$ when restricted to $(H_n(X, \Lambda)^* \oplus H_n(X, \Lambda))^G$ is given by $\begin{pmatrix} 0 & \langle , \rangle \\ (-1)^n \langle , \rangle & 0 \end{pmatrix} = e_x$. The zeros follow from Proposition II.2 as over the rationals $(H_n(X; \Lambda)^* \oplus H_n(X; \Lambda))^G \otimes \mathbb{Q} \cong H_n(\widetilde{MX}; \mathbb{Q})^G \cong H_n(MX; \mathbb{Q}) \cong H_n(LX: \mathbb{Q}) \oplus H_n(-LX; \mathbb{Q})$, the isomorphisms respect the splittings and the forms vanish on $H_n(LX; \mathbb{Q})$ and $H_n(-LX; \mathbb{Q})$.

Our procedure is as follows. Let $X$ and $Y$ be two finite $n$-complexes with the same Euler characteristic and $(n-1)$-connected universal cover together with isomorphisms of the fundamental groups to a given group $G$ (polarized complexes). We denote the cellular chain complexes of $X$ and $Y$ with $\Lambda$-coefficients for a moment by $C$ and $D$. The cellular chain complexes of $MX$ and $MY$ are then given by $M(C)$ and $M(D)$.

Let $h:M(C)\to M(D)$ be any chain map which induces the identity in dim 0 and $2n$. The map $H_n(h):H_n(M(C))\to H_n(M(D))$ induces a map of Tate cohomology groups, $H_n(h)^{\wedge}:\hat{H}^0H_n(M(C))\to\hat{H}^0H_n(M(D))$. We show that this map is unique moduo composition with

$\begin{pmatrix}\mathrm{Id} & \alpha\\ 0 & \mathrm{Id}\end{pmatrix}$, where we split $H_n(M(C))$ as $H_n(C)^*\oplus H_n(C)$. (Proposition III.4).

If $MX$ and $MY$ are polarized orientation preserving homotopy equivalent then we can take the induced cellular map for the $h$. It induces a map $H_n(M(C))^G\to H_n(M(D))^G$ which is an isometry of the restriction of the intersection form to the fixed point sets. If $|G|$ is odd then all maps $\begin{pmatrix}\mathrm{Id} & \alpha\\ 0 & \mathrm{Id}\end{pmatrix}$ on Tate groups as above are induced by isometries of the fixed point sets (Lemma III.5). Thus if $MX$ and $MY$ are polarized orientation preserving homotopy equivalent then for every chain map $h$ the map $H_n(h)^{\wedge}$ is induced by an isometry of the fixed point sets.

We will use this information to distinguish certain $MX$ and $MY$. For this we study the maps and how they change if we choose different polarizations more systematically in the rest of part III.3.

To study the isometries of the fixed point set $H_n\ (M(C))^G$ we have to determine the restriction of the intersection form to them. As the forms are given by evaluation this is a purely algebraic problem which we investigate in part III.2.

With this information we construct in III.4 some non-cancellation examples. It is easy to see that if the complex is of the form $X\vee S^n$ then for every $h$ as above $H_n(h)^{\wedge}$ is induced by an isometry. Thus we only get informations if the Euler characteristics of $X$ and $Y$ are minimal.

In some cases, for instance if $G=(\mathbb{Z}/p)^s$, the isometries of the fixed point sets determine elements in the Wall group $L_1^0(\mathbb{Z})$ (for the notation compare [15]). If in addition $X$ and $Y$ have minimal Euler characteristic the Tate groups are the reduction of the fixed point sets mod $p$, thus $H_n(h)^{\wedge}$ defines an element in $L_1^0(\mathbb{Z}/p)$. The knowledge of these Wall groups and of the map between them leads to non-cancellation results for some oriented manifolds $MX$ and $MY$ in every dimension $=0 \bmod 4$ (Corollary after Theorem III.3). As $MX$ is diffeomorphic to $-MX$ (by § 2 $MX$ is a double) we can forget the orientations.

**III.2. The forms $e^G$ and $\hat{e}$ on $(\pi_n^* \oplus \pi_n)^G$ and $(\pi_n^* \oplus \pi)\hat{}$**

The evaluation form $e$ on $H_n(M(C)) = \pi^* \oplus \pi$ ($\pi = H_n(C_*)$) gives rise to forms $e^G$ and $\hat{e}$ on $(\pi^* \oplus \pi)^G$ and $(\pi^* \oplus \pi)\hat{}$ respectively. In this section we wish to evaluate these forms. More generally we wish to determine the forms $(H_n(D_*)^G, \Phi^G)$ and $(\widehat{H_n(D_*)}, \hat{\Phi})$ induced by $(D_*, \Phi) \in \mathcal{M}_{2n}$. The form $e^G$ is not quite the evaluation form since as we shall see below $(\pi^*)^G \neq (\pi^G)^*$ in general. However there is a close relationship between them.

PROPOSITION III.1. *If $A$ is an $\Lambda$-module, there exists an exact sequence*

$$0 \to (A^*)^G \xrightarrow{\text{rest}} (A^G)^* \to L_A \to 0 \tag{1}$$

*where $L_A$ is finite.*

*Proof.* From the exact sequence $0 \to A^G \to A \to A/A^G \to 0$ we obtain the sequence

$$0 \to \mathrm{Hom}_G(A/A^G, \mathbb{Z}) \to \mathrm{Hom}_G(A, \mathbb{Z}) \xrightarrow{\text{rest}} \mathrm{Hom}_G(A^G, \mathbb{Z}) \to \mathrm{Ext}^1_G(A/A^G, \mathbb{Z})$$

$$\to \mathrm{Ext}^1_G(A, \mathbb{Z}) \to \mathrm{Ext}^1_G(A^G, \mathbb{Z})$$

Since $\mathrm{Hom}_G(A, \mathbb{Z}) = (A^*)^G$ and obviously $\mathrm{Hom}_G(A^G, \mathbb{Z}) = \mathrm{Hom}_\mathbb{Z}(A^G, \mathbb{Z}) = (A^G)^*$. It is enough to show $\mathrm{Hom}_G(A/A^G, \mathbb{Z}) = (0)$.

But this is obvious for if $\Phi: A/A^G \to \mathbb{Z}$ is a $G$-map, then $0 = \Phi(\sum gx) = |G|\,\Phi(x)$ for all $x \in A$. Note that $L_A$ is finite since $\mathrm{Ext}^1_G(A/A^G, \mathbb{Z})$ is.

The usefulness of this proposition is due to the fact that the form $e^G$ on $(\pi^* \oplus \pi)^G$ is obviously the restriction of the evaluation form on $(\pi^G)^* \oplus \pi^G$ and hence if we know the torsion coefficients of $L_\pi$, say $\beta_1, \ldots, \beta_s$, then we may choose a basis $\{f_j\}$, of $(\pi^G)^*$ so that $\{\beta_j f_j\}$ is a basis of $(\pi^*)^G$. If $\{x_j\}$ is the dual basis to $\{f_j\}$, then with respect to the basis $\{\beta_j f_j, x_j\}$ of $(\pi^*)^G \oplus \pi^G$ the form $e^G$ has matrix

$$\left(\begin{array}{c|c} O & \begin{matrix}\beta_1 & & \\ & \ddots & \\ & & \beta_s\end{matrix} \\ \hline \begin{matrix}\beta_1 & & \\ & \ddots & \\ & & \beta_s\end{matrix} & O \end{array}\right)$$

In order to compute $L_A$ we have the following,

LEMMA III.1. *If $A$ is a $\Lambda$-lattice, then in* (1) $\operatorname{rest}(N(A^*)) = |G|\,(A^G)^*$ *where* $NA = \{\sum ga \mid a \in A\}$ *for any $G$ module $A$.*

*Proof.* We first note that for any finitely generated $\Lambda$-lattice $A$, $A/A^G$ is a $\Lambda$-lattice since it is clear that $na \in A^G$, $n \in \mathbb{Z}$ only if $a \in A^G$.

If $h \in N(A^*)$ then for $x \in A$, $h(x) = \sum_g h^1(g^{-1}x)$ for some $\mathbb{Z}$-map $h^1 : A \to \mathbb{Z}$. If $x \in A^G$, then $h(x) = |G|\, h^1(x)$ and so $\operatorname{rest}(h) \in |G|\,(A^G)^*$. Conversely if $h = |G|\, h^1$ for some $h^1 : A^G \to \mathbb{Z}$ then since $A/A^G$ is a lattice, $A \simeq A^G \oplus A/A^G$ as abelian groups. We may therefore define $\tilde{h} : A \to \mathbb{Z}$ by $\tilde{h} \mid A^G = \tilde{h}^1$, $\tilde{h} \mid A/A^G = 0$. $\tilde{h} \in A^*$ and for $x \in A^G$, $N\tilde{h}(x) = \sum_g \tilde{h}(g^{-1}x) = |G|\, h^1(x) = h(x)$.

Therefore we have the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & NA^* & \xrightarrow[\cong]{\text{rest}} & |G|\,(A^G)^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (A^*)^G & \xrightarrow{\text{rest}} & (A^G)^* & \longrightarrow & L_A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\simeq & & \\
0 & \longrightarrow & \hat{H}^0(A^*) & \longrightarrow & (A^G)^*/|G|\,(A^G)^* & \longrightarrow & \text{coker} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{2}$$

Hence to compute $L_A$ it is enough to compute the cokernel of the map $\hat{H}^0(A^*) \to (A^G)^*/|G|\,(A^G)^*$.

Since $|G|$ annihilates all Tate cohomology groups of $G$, the torsion coefficients of $\hat{H}^0(G, A)$ must divide $|G|$.

PROPOSITION III.2. *If the torsion coefficients of $\hat{H}^0(G, A)$ are $\tau_1 | \tau_2 | \cdots | \tau_k$ where $k = \operatorname{rank} A^G$ and we allow $\tau_i = 1$ then $L_A$ has torsion coefficients* $|G|/\tau_k \mid |G|/\tau_{k-1} \mid \cdots \mid |G|/\tau_1$.

The proposition is an immediate consequence of the following lemma.

LEMMA III.2. If $\varphi : \bigoplus_1^k \mathbb{Z}/s_i \to (\mathbb{Z}/s)^k$ *is an embedding where $s_i \mid s$ for all $i$, then cokernel* $\varphi \simeq \bigoplus_i \mathbb{Z}/(s/s_i)$.

*Proof.* By considering the $p$-torison it is enough to assume $s = p^l$, $s_i = p^{l_i}$, $l_i \leq l$. Consider the generator $e_k$ of $\mathbb{Z}/p^{l_k} \cdot p^{l_k} e_k = 0$ implies $\varphi e_k$ has order $p^{l_k}$ and therefore $\varphi e_k = p^{l-l_k}$ for some $y \in (\mathbb{Z}/p^l)^k$ of order exactly $p^l$. But any element of order $p^l$ is part of a basis of $(\mathbb{Z}/p^l)^k$ since it must contain a component relatively prime to $p$. Therefore there exists an automorphism $f: (\mathbb{Z}/p^l)^k \to (\mathbb{Z}/p^l)^k$ such that $f(y) = \bar{e}_k$ a generator of the last factor of $(\mathbb{Z}/p^l)^k$. Since $f$ is an isomorphism, coker $f \circ \varphi \simeq$ coker $\varphi$, $f \circ \varphi(e_k) = p^{l-l_k} \bar{e}_k$ and coker $f \circ \varphi \simeq \mathbb{Z}/p^{l-l_k} \oplus$ coker $\bar{\varphi}$

$$
\begin{array}{ccccc}
\mathbb{Z}/p^{l_k} & \longrightarrow & \mathbb{Z}/p^{l} & \longrightarrow & \mathbb{Z}/p^{l-l_k} \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\
\bigoplus_{1}^{k} \mathbb{Z}/p^{l_i} & \xrightarrow{f \circ \varphi} & (\mathbb{Z}/p^{l})^{k} & \longrightarrow & \operatorname{coker} f \circ \varphi \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\
\bigoplus_{1}^{k-1} \mathbb{Z}/p^{l_i} & \xrightarrow{\bar{\varphi}} & (\mathbb{Z}/p^{l})^{k-1} & \longrightarrow & \operatorname{coker} \bar{\varphi}
\end{array}
$$

since all the vertical maps split. Hence we are done by induction.

This computes the form $e^G$ on $(\pi^* \oplus \pi)^G$. The form $\hat{e}$ on $(\pi^* \oplus \pi)\hat{}$ is induced from $e^G$ and so has the same matrix but considered as a $\mathbb{Z}/|G|$-valued form.

COROLLARY. *Suppose $H_n(G)$ is elementary abelian (i.e. all $\tau_i = \tau_j = \tau$) and $d(H_n(G)) = \chi(C_*) + (-1)^{n+1}$. Then $e^G(\hat{e})$ is a multiple of the hyperbolic form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ on $(\pi^* \oplus \pi)^G$ $((\pi^* \oplus \pi)\hat{})$. More precisely there exists a basis $\{x_i, y_i\}$ such that $x_i y_i = b \in \mathbb{Z}(\mathbb{Z}/|G|)$ for $b = |G|/\tau$. Hence by dividing through by $b$ we may consider $e^G$ to be the $\mathbb{Z}$-valued hyperbolic form and $\hat{e}$ to be the $\mathbb{Z}/\tau$-valued hyperbolic form.*

*Proof.* Since $\mathbb{Z} \to C_0^* \to \cdots \to C_n^* \to \pi^* \to 0$ is exact we have by degree shifting $\hat{H}^0(\pi^*) \simeq H^{n+1}(G, \mathbb{Z})$ which is in turn isomorphic to $H_n(G, \mathbb{Z})$ since $G$ is finite. Moreover the same sequence (or its dual) shows rank $(\pi^*)^G =$ rank $\pi^G = \chi(C_*) + (-1)^{n+1}$. Since $d(H_n)(G)) = \chi(C_*) + (-1)^{n+1}$ all the torsion coefficients of $L_\pi$ equal $b = |G|/\tau$ by the proposition and the result follows.

We shall now proceed to show that if $(D, \phi)$ is an arbitrary element of $\mathcal{M}_{2n}$ then the induced form on $H_n(D)\hat{}$ is non-singular.

LEMMA III.3. *Suppose $h: F_1 \to F_2$ is an embedding of a free abelian group in another of the same rank. Suppose $F_0 \subseteq F_1$ is a subgroup such that $h(F_0) = kF_2$ for some $k \in \mathbb{Z}^+$. Let $\pi_1: F_1 \to F_1/F_0$ and $\pi_2: F_2 \to F_2/F_1$ be the natural projections. Suppose $\{f_j\}, \{g_j\}$ are bases of $F_1, F_2$ resp. such that $h(f_j) = \beta_j g_j$. Then $\{\pi_1 f_j\}$ is a basis of $F_1/F_0$.*

*Proof.* Clearly $\{\pi_2 g_j\}$ is a basis of $F_2/kF_2$. Since $\pi_1(kg_j) = 0$, $\beta_j \mid k$.

(i) $\pi_1 f_j$ has order $k/\beta_j$ since $(k/\beta_j)\pi_1 f_j$ maps to the image of $kg_j = 0$ in $F_2/kF_2$. Since $F_1/F_0 \hookrightarrow F_2/kF_2$, $(k/\beta_j)\pi_1 f_j = 0$. If $\alpha\pi_1 f_j = 0$, then $\alpha\beta_j g_j$ maps to zero in $F_2/kF_2$ so $k \mid \alpha\beta_j$ which is equivalent to saying $\dfrac{k}{\beta_j} \mid \alpha$.

(ii) If $\sum m_j \pi_1 f_j = 0$ then $\sum m_j \beta_j g_j$ maps to zero in $F_2/kF_2$ so as above, $\dfrac{k}{\beta_j} \Bigm| m_j$.

LEMMA III.4. *Let $A$ be a $\Lambda$-lattice and consider the dual of* (1) *from Proposition III.*1,

$$0 \to A^G \xrightarrow{(\text{rest})^*} (A^{*G})^* \to \text{Ext}\,(L_A, \mathbb{Z}) \simeq L_A \to 0.$$

*Then* $\text{rest}^*((N_A)) = |G|\,(A^{*G})^*$, *where we have identified $A$ and $A^{**}$ by means of the natural isomorphism.*

*Proof.* Let $x \in A^{*G}$ and $y \in NA$, then $(\text{rest})^*(y)(x) = x(y)$. If $y = \sum h\bar{y}$ for $\bar{y} \in A$, then $x(y) = |G|\,x(\bar{y})$. If $i: A^{*G} \to A^*$ is the inclusion, $i^*(\bar{y}) \in (A^{*G})^*$ and $|G|\,i(\bar{y})(x) = |G|\,\bar{y}(ix) = |G|\,x(\bar{y})$. Hence $\text{rest}^*(y) \in |G|\,(A^{*G})^*$. Conversely if $g = |G|\,f \in |G|\,(A^{*G})^*(ix)$, then since $A/A^G$ is a $\Lambda$-lattice, $i^*: A \to (A^{*G})^*(ix)$ is onto. If $i^*(z) = f$ and $y = gz \in NA$, $\text{rest}^*(y)(x) = |G|\,x(z) = g(x)$ for $x \in A^{*G}$.

THEOREM III.1. *If $(A, \phi)$ is a non singular $\Lambda$-form, then the induced form $(\hat{H}^0(A), \hat{\phi})$ is non-singular.*

*Proof.* Since $\phi: A \to A^*$ is an isomorphism, $A$ and hence $A/A^G$ are $\Lambda$-lattices. Consider the following diagram.

$$\begin{array}{ccccc}
NA & \xrightarrow[\simeq]{N\phi} & N(A^*) & \xrightarrow{\simeq} & |G|A^{G*} \\
\downarrow & & \downarrow & & \downarrow \\
A^G & \xrightarrow[\simeq]{\phi^G} & A^{*G} & \xrightarrow{r} & A^{G*} \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(A) & \xrightarrow{\simeq} & \hat{H}^0(A^*) & \longrightarrow & A^{G*}/|G|\,A^{G*}
\end{array}$$

By Lemmas III.1 and III.3 we may choose bases $\{f_j\}$, $\{g_j^*\}$ of $A^G$, $A^{G*}$ respectively so that (i) $r \circ \phi^G(f_j) = \beta_j g_j^*$ (ii) $\{\pi_1 f_j\}$ is a basis of $\hat{H}^0(A)$. If we dualize $(r \circ \phi^G)^*(g_j) = \beta_j f_j^*$ and use Lemma III.1 and III.4 we have $\{\pi_1 g_j\}$ is also a basis of $\hat{H}^0(A)$. Moreover $\hat{\phi}(\pi_1 f_j, \pi_1 g_k) = \delta_{jk}\beta_k \bmod |G|$ since $\hat{\phi}$ is induced by the form on $A^G$ whose adjoint map is $r \circ \phi^G$. Hence if $x = \sum m_i \pi_i f_i \in \hat{H}^0(A)$ has $\hat{\phi}(x, y) = 0$ for all $y$ we must have $\beta_k m_k = 0 \bmod |G|$ for all $k$ which is equivalent to $\frac{|G|}{\beta_k} \mid m_k \Leftrightarrow x = 0$ since $\pi_1 f_k$ has order $\frac{|G|}{\beta_k}$. Hence the adjoint map $\beta\hat{\phi} : \hat{H}^0(A) \to \mathrm{Hom}_{\mathbb{Z}}(\hat{H}^0(A), \mathbb{Z}/|G|)$ is a monomorphism and hence an isomorphism since both groups are finite and abstractly isomorphic.

## III.3. The invariant

Let $\mathbb{D}_* \in \mathcal{M}_{2n}$, consider the exact sequence $0 \to \operatorname{Im} \partial_{n+1} \xrightarrow{i} \ker \partial_n \to H_n \to 0$ and let

(a) $K(\mathbb{D}_*) = \operatorname{coker}\{\hat{H}^0(i) : \hat{H}^0(\operatorname{im} \partial_{n+1}) \to \hat{H}^0(\ker \partial_n)\}$

(b) $L(\mathbb{D}_*) = \ker\{\hat{H}^1(i) : \hat{H}^1(\operatorname{im} \partial_{n+1}) \to \hat{H}^1(\ker \partial_n)\}$

so that one has the exact sequence

$$0 \to K(\mathbb{D}_*) \to \hat{H}^0(H_n) \to L(\mathbb{D}_*) \to 0 \tag{3}$$

PROPOSITION III.3. *Suppose* $\mathbb{D}_*$, $\mathbb{E}_* \in \mathcal{M}_{2n}$ *and* $h : \mathbb{D}_* \to \mathbb{E}_*$ *is a chain map inducing the identity on* $H_0$ *and* $H_{2n}$. *Then* (i) $\hat{H}^0(h_*) : \hat{H}^0(H_n(D_*)) \to \hat{H}^0(H_n(E_*))$ *is an isomorphism.* (ii) *If* $\bar{h} : \mathbb{D}_* \to \mathbb{E}_*$ *is another chain map (inducing identity on* $H_0$ *and* $H_{2n}$) *then* $\hat{H}^0(\bar{h}_*)^{-1}\hat{H}^0(h_*) = \mathrm{Id} + j_D\alpha \cdot \pi_D$ *for some map* $\alpha : L(\mathbb{D}_*) \to K(\mathbb{D}_*)$.

*Proof.* Since $0 \to \ker \partial_n \to D_n \to \cdots \to D_0 \to \mathbb{Z} \to 0$ is exact and the $D_i$ are projective, the chain map $h$ induces a map from $\hat{H}^i(\ker \partial_n^D) \to \hat{H}^i(\ker \partial_n^E)$ which is independent of $h$ (since any chain map lifts the identity) and hence must be an isomorphism for all $i$. Similarly the exact sequence $0 \to \mathbb{Z} \to D_{2n} \to \cdots \to D_{n+1} \to \operatorname{im} \partial_{n+1} \to 0$ implies that $h$ induces a map from $\hat{H}^i(\operatorname{im} \partial_{n+1}^{\mathbb{D}}) \to \hat{H}^i(\operatorname{im} \partial_{n+1}^{\mathbb{E}})$ which is also independent of $h$ and hence also an isomorphism. From the definitions of $K$ and $L$ we see $h$ induces unique isomorphisms $\mu(\mathbb{D}_*, \mathbb{E}_*) : K(\mathbb{D}_*) \to K(\mathbb{E}_*)$ and $\nu(\mathbb{D}_*, \mathbb{E}_*) : L(\mathbb{D}_*) \to L(\mathbb{E}_*)$. (i) and (ii) are now immediate.

Now suppose $\mathbb{D}_*$ and $\mathbb{E}_*$ are in the image of the functor $M : \tilde{\mathscr{C}}_n \to \mathcal{M}_{2n}$ (see Section III.1), say $\mathbb{D}_* = M(B_*)$, $\mathbb{E}_* = M(C_*)$.

The definition of $M(\mathbb{D}_*)$ shows immediately that the map $H^i(i): H^i(\operatorname{im}\partial_{n+1}) \to H^i(\operatorname{Ker}\partial_{n+1})$ is zero and we have $K(\mathbb{D}_*) = \hat{H}^0(\ker\partial_n)$ and $L(\mathbb{D}_*) = \hat{H}^1(\operatorname{im}\partial_{n+1})$. Since $0 \to \operatorname{im}\partial_{n+1} \to D_n \to \operatorname{coker}\partial_{n+1} \to 0$ is exact we have $L(\mathbb{D}_*) \simeq \hat{H}^0(\operatorname{coker}\partial_{n+1})$ and we will think of $\nu$ as being induced by the chain map on $\operatorname{coker}\partial^{\mathbb{D}}_{n+1} \to \operatorname{coker}\partial^{\mathbb{E}}_{n+1}$. As $\operatorname{Hom}_{\mathscr{C}_n}(B_*, C_*) \neq \phi$, there exists chain maps $h: M(f, g): M(B_*) \to M(C_*)$. Now we have

(i) $\operatorname{coker}\partial^D_{n+1} = \operatorname{coker}((\partial^B_n)^*, 0) = \pi_n(B_*)^* \oplus B_n$.

(ii) $\ker\partial^D_n = \ker\begin{pmatrix}0\\ \partial^B_n\end{pmatrix} = B^*_n \oplus \pi_n(B_*)$.

(iii) $H_n(M(B_*)) = \pi^*_n(B_*) \oplus \pi_n(B_*)$.

and similarly for $\mathbb{E}_*$. From the definition of $\mu$ and $\nu$ and the form of $M(f, g)$ we see $\mu$ is induced by $H_n(g)^* \oplus f_n$, $\nu$ by $g^*_n \oplus H_n(f)$ and $H_n(h) = H_n(g)^* \oplus H_n(f)$. Since $B^*_n(C^*_n)$ and $B_n(C_n)$ are $\Lambda$-projective $\hat{H}^0(B_n) = \hat{H}^0(B^*_n) = 0$ and we may identify $\hat{H}^0(\operatorname{coker}\partial^D_{n+1})$ (resp $\hat{H}^0(\ker\partial^0_n)$) with $\hat{H}^0(\pi_n(B_*)^*)$ (resp. $\hat{H}^0(\pi_n(B_*))$) via the projection and injection maps. Moreover using this identification the maps $\mu$, $\nu$ are given by $\hat{H}^0(H_n(g)^*)$ and $\hat{H}^0(H_n(f))$ respectively.

PROPOSITION III.4. *If $\mathbb{D}_*, \mathbb{E}_* \in \mathscr{M}_{2n}$ are in the image of the functor $M: \mathscr{C}_n \to \tilde{\mathscr{M}}_{2n}$, then (1) there exists a chain map $h: \mathbb{D}_* \to \mathbb{E}_*$ such that $\hat{H}^0(H_n(h)) = \mu(\mathbb{D}_*, \mathbb{E}_*) \oplus \nu(\mathbb{D}_*, \mathbb{E}_*)$. (2) The map is an isometry of the form $\hat{e}_{\mathbb{D}_*}$ with $\hat{e}_{\mathbb{E}_*}$. (3) If $\bar{h}: \mathbb{D}_* \to \mathbb{E}_*$ is any other map in $\mathscr{M}_{2n}$ then $\hat{H}^0(H_n(h)) = \begin{pmatrix}\mu & 0\\ 0 & \nu\end{pmatrix}\begin{pmatrix}\mathrm{Id} & \alpha\\ 0 & \mathrm{Id}\end{pmatrix}$ where $\begin{pmatrix}\mathrm{Id} & \alpha\\ 0 & \mathrm{Id}\end{pmatrix}$ is an isometry of $(\pi^*_n(B) \oplus \pi_n(B))\hat{}, \hat{e}_{\mathbb{D}_*})$.*

*Proof.* We have already verified everything but (2). We have seen from the above that the map $\mu \oplus \nu = \sigma$ is given by $\hat{H}^0(H_n(g)^*) \oplus \hat{H}^0(H_n(f))$ where $f: B_* \to \mathbb{C}_*$ and $g: \mathbb{C}_* \to B_*$ are chain lifts of $\mathrm{id}: \mathbb{Z} \to \mathbb{Z}$. Since $f \circ g: \mathbb{C}_* \to \mathbb{C}_*$ is a chain lift of the $\mathrm{id}: \mathbb{Z} \to \mathbb{Z}$ we have $\hat{H}^0(H_n(g)^*) = \hat{H}^0(H_n(f)^*)^{-1}$ (the map induced on $\hat{H}(\pi^*_n)$ is independent of the chain map). The form $\hat{e}_B(\hat{e}_C)$ is induced by $e^G_B(e^G_C)$ which is in turn the restriction of the evaluation form $e_B(e_C)$ on $\pi^*_n \oplus \pi_n$. Hence if $[\zeta] \in \hat{H}^0(\pi_n(B_*)^*)$, $\zeta \in (\pi_n(B_*)^*)^G$ and $[u] \in \hat{H}^0(\pi_n(B_*))$, $u \in \pi_n(B_*)^G$ then $\hat{e}_B([\zeta], [u]) = \zeta(u) \bmod \mathbb{Z}/|G|$. (Recall $\pi^{*G}_n \hookrightarrow (\pi^G_n)^*$.) On the other hand $\hat{H}^0(H_n(g)^*)([\zeta]) = [\zeta']$ where $H_n(f)^*(\zeta') = \zeta$ with $\zeta' \in (\pi_n(C_*)^*)^G$ and $\hat{H}^0(H_n(f))[u] = [H_n(f)(u)] = [u']$. Therefore $\hat{e}_C([\zeta'], [u']) = \zeta'(u') \bmod \mathbb{Z}/|G| = \zeta'(H_n(f)u) \bmod \mathbb{Z}/|G| = H_n(f)^*(\zeta')(u) \bmod \mathbb{Z}/|G| = \hat{e}_B[\zeta], [u])$. Therefore $\sigma$ is an isometry. (3) now follows since by definition any map $\bar{h} \in M_{2n}$ induces an isometry on $\hat{H}^0(H_n)$.

Suppose one can show the isometry $\sigma = \mu \oplus \nu$ of $\hat{e}_{\mathbb{D}_*}$ with $\hat{e}_{\mathbb{E}_*}$

($\mathbb{D}_* = MB_*, \mathbb{E}_* = M\mathbb{C}_*$) cannot be lifted to an isometry of $e^G_{\mathbb{D}_*}$ with $e^G_{\mathbb{E}_*}$; then the following lemma and Proposition III.4 (3) will show that there cannot be any chain homotopy equivalence of $MB_*$ with $M\mathbb{C}_*$ and which preserves the forms on $H_n(MB_*)$ and $H_n(MC_*)$ providing $|G|$ is odd.

LEMMA III.5. *If $|G|$ is odd, every isometry of $\hat{e}$ of the form $\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$ lifts to an isometry of $e^G$.*

*Proof.* It is sufficient to prove it for $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$. Let $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_n \end{pmatrix}$ $b_n \mid \cdots \mid b_1$ and $H = \hat{H}^0(\pi_n) = \mathbb{Z}/t_1 \times \cdots \times \mathbb{Z}/t_n$ where $t_1 \mid \cdots \mid t_n$. By Proposition III.2, $t_i b_i = |G|$. The form $e^G$ is $\left(\mathbb{Z}^n \times \mathbb{Z}^n, \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}\right)$ and the form $\hat{e}$ is $\left(H \times H, \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}\right)$ the last being considered as a $\mathbb{Z}/|G|$-valued form. Denote the basis of the first copy of $H$ by $\{x_i\}$ and the second copy by $\{x_i^*\}$ so that the order of $x_i$ and $x_i^*$ is $t_i$ and $\hat{e}$ $(x_i, x_i^*) = b_i \bmod |G|$; all other products are zero. Let $A: H \to H$ be given by $Ax_i = \sum a_{ij} x_j^*$ where $a_{ij}$ are only defined mod $t_j$. Since the order of $x_i$ is $t_i$ we see that for $i<j$ we may write $a_{ij} = a'_{ij} t_j t_i^{-1}$ where $a'_{ij}$ is defined mod $t_i$. The fact that $\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$ is an isometry of $\hat{e}$ is equivalent to

(i) $2a_{ii} b_i \equiv 0 (\bmod |G|)$

(ii) for $i<j$, $a_{ji} b_i + b_j a'_{ij} t_j t_i^{-1} \equiv 0 \ (\bmod |G|)$.

Since $(2, |G|) = 1$, (i) says $a_{ii} \equiv 0 \bmod \dfrac{|G|}{b_i} \equiv 0 \bmod t_i$. Since $b_j \mid b_i$, (ii) says $a_{ji} b_i b_j^{-1} + a'_{ij} t_j t_i^{-1} \equiv 0 \bmod \dfrac{|G|}{b_j} \equiv 0 \bmod t_j$. But $b_i b_j^{-1} = t_j t_i^{-1}$. So (ii) implies $a_{ji} + a'_{ij} \equiv 0 \bmod t_i$. Now define $C: \mathbb{Z}^n \to \mathbb{Z}^n$ by $c_{ii} = 0$ and for $j>i$ let $c_{ji}$ be any integer congruent to $a_{ji} \bmod t_i$. Let $c_{ij} = -b_j^{-1} b_i c_{ji}$ for $i<j$ then clearly $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$ is an isometry of $\begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$ on $\mathbb{Z}^n \times \mathbb{Z}^n$. But $c_{ij} = -b_j^{-1} b_i c_{ji} = -t_i^{-1} t_j a_{ji} \bmod b_i = a'_{ij} t_j t_i^{-1} \bmod t_i$, i.e. $C$ reduces to $A$.

In order to show $\sigma$ does not lift to an isometry of $e^G_{\mathbb{D}}$ with $e^G_{\mathbb{E}}$ it is easier to collect the $\sigma(D, \mathbb{E})$ into a group valued invariant. Let $\mathbb{C}_1, \mathbb{C}_2 \in \mathscr{C}_n$.

DEFINITION. (i) An isometry of $((\pi_1^* \oplus \pi_1)\hat{}, \hat{e}_1) \to ((\pi_2^* \oplus \pi_2)\hat{}, \hat{e}_2)$ is called diagonal if there exists maps $f: \pi_1^G \to \pi_2^G$ and $g: \pi_2^G \to \pi_1^G$ inducing isomorphisms $\hat{f}: \hat{\pi}_1 \to \hat{\pi}_2$ and $g^* \mid (\pi_1^*)^G : \hat{\pi}_1^* \to \hat{\pi}_2^*$ such that $\rho = (g^* \mid)\hat{} \oplus \hat{f}$.

(ii) An isometry $\lambda$ of $((\pi_1^* \oplus \pi_1)^G, e_1^G) \to ((\pi_2^* \oplus \pi_2)^G, e_2^G)$ is called diagonal if it is of the form $f^{*-1} \mid \pi_1^{*G} \oplus f$ for some isomorphism $f: \pi_1^G \to \pi_2^G$.

Note that the isometry $\sigma$ of $\hat{e}_1$ to $\hat{e}_2$ is a diagonal isometry.

We will denote by $\operatorname{Isom}(\hat{e}_1, \hat{e}_2)(\operatorname{Diag}\operatorname{Isom}(\hat{e}_1, \hat{e}_2))$ the set of isometries (diagonal isometries) of $((\pi_1^* \oplus \pi_1)^\wedge, \hat{e}_1)$ with $((\pi_2^* \oplus \pi_2)^\wedge, \hat{e}_2)$ and by $\operatorname{Isom}(e_1^G, e_2^G)(\operatorname{Diag}\operatorname{Isom}(e_1^G, e_2^G))$ the subset of $\operatorname{Isom}(\hat{e}_1, \hat{e}_2)(\operatorname{Diag}\operatorname{Isom}(\hat{e}_1, \hat{e}_2))$ induced by an isometry (diagonal isometry) of $((\pi_1^* \oplus \pi_1)^G, e_1^G)$ with $((\pi_2^* \oplus \pi_2)^G, e_2^G)$.

PROPOSITION III.5. Given $\mathbb{C}_*, \mathbb{C}'_* \in \mathscr{C}_n$ such that $\chi(\mathbb{C}_*) = \chi(\mathbb{C}'_*)$ then there exists a diagonal isometry $\varphi: ((\pi_n'^* \oplus \pi_n')^G, e'^G) \to ((\pi_n^* \oplus \pi_n)^G, e^G)$ inducing a diagonal isometry $\hat{\varphi}: ((\pi_n'^* \oplus \pi_n')^\wedge, e') \to ((\pi_n^* \oplus \pi_n)^\wedge, \hat{e})$.

*Proof.* Recall from Lemmas III.1 and III.4 we have commutative diagrams.

$$\begin{array}{ccccccc}
\hat{H}^0(\pi) & \xrightarrow{\simeq} & (\pi^{G^*})^*/(N\pi)^{**} & \rightarrowtail & (\pi^{*G})^*/|G|(\pi^{*G})^* & \rightarrowtail\!\!\twoheadrightarrow & \text{coker} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \simeq \\
\pi^G & \xrightarrow{j} & (\pi^{G^*})^* & \xrightarrow{\text{rest}^*} & (\pi^{*G})^* & \twoheadrightarrow & \operatorname{Ext}(L_\pi, \mathbb{Z}) \\
\uparrow & & \uparrow & & \uparrow & & \\
N\pi & \xrightarrow{j} & (N\pi)^{**} & \longrightarrow & |G|(\pi^{*G})^* & &
\end{array} \qquad (2')$$

$$\begin{array}{ccccc}
\hat{H}^0(\pi^*) & \rightarrowtail & \pi^{G^*}/|G|\cdot \pi^{G^*} & \twoheadrightarrow & \text{coker} \\
\uparrow & & \uparrow & & \uparrow \simeq \\
\pi^{*G} & \xrightarrow{\text{rest}} & \pi^{G^*} & \twoheadrightarrow & L_\pi \\
\uparrow & & \uparrow & & \\
N\pi^* & \xrightarrow{\simeq} & |G|\pi^{G^*} & &
\end{array} \qquad (2)$$

There are similar diagrams for $\pi'$. Now since $\chi(C_*) = \chi(C'_*)$ there exists an isomorphism $\rho: L_\pi \to L_{\pi'}$. This follows since $\chi(C_*) = \chi(C'_*)$ implies by Schanuels lemma and semi-local cancellation that $\pi_{(G)} \approx \pi'_{(G)}$ ($A_{(G)}$ denotes $A$ localized at those primes dividing the order of $G$). Since localization commutes with Ext and $|G| \operatorname{Ext} = 0$ for lattices we have

$$L_\pi \simeq \ker\{\operatorname{Ext}_G^1(\pi/\pi^G, \mathbb{Z}) \to \operatorname{Ext}_G^1(\pi, \mathbb{Z})\} \simeq L_{\pi'}.$$

This isomorphism can be lifted to an isomorphism $\beta^*:\pi^{G^*}\to\pi'^{G^*}$. From (2) we see that $\beta^*\mid\pi^{*^G}$ induces an isomorphism $\hat{H}^0(\pi^*)\to\hat{H}^0(\pi'^*)$. Dualizing we see that $\beta:\pi'^G\to\pi^G$ induces an isomorphism $\hat{H}^0(\pi')\to\hat{H}^0(\pi)$. It is obvious that $(\beta^*\mid\pi^{*^G})^{-1}\oplus\beta:(\pi'^*\oplus\pi')^G\to(\pi^*\oplus\pi)^G$ is a diagonal isometry of $e'^G$ to $e^G$ and induces a diagonal isometry of $\hat{e}'$ to $\hat{e}$.

This proposition allows one to collect the invariants $\sigma(\mathbb{D}_*,\mathbb{E}_*)$ of Proposition III.4 into a group valued invariant as follows. We fix an Euler characteristic $=k$ and let $C_n(k)$ be the full sub-category of $\mathscr{C}_n$ consisting of objects of Euler characteristic $k$. Similarly $\mathscr{M}_{2n}(k)$ will denote the image of $\mathscr{C}_n(k)$ under the functor $M$. Fix an object $\bar{\mathbb{C}}_*\in\mathscr{C}_n(k)$ and denote $\pi_n(\bar{\mathbb{C}}_*)$ by $\Pi$. From the above proposition if $M(\mathbb{C}_*)\in\mathscr{M}_{2n}(k)$ there exists a diagonal isometry $\varphi_C:((\Pi^*\oplus\Pi)^G,e^G)\to(\pi_n(\mathbb{C}_*)^*\oplus\pi_n(C_*),e^G_{\mathbb{C}})$ inducing a diagonal isometry $\hat{\varphi}_C$ from $\hat{e}$ to $\hat{e}_{\mathbb{C}_*}$. For each $M(\mathbb{C}_*)\in\mathscr{M}_{2n}(k)$ choose such a $\varphi_C$ and for $\mathbb{D}_*=M(B_*)$, $\mathbb{E}_*=M(\mathbb{C}_*)\in\mathscr{M}_{2n}(k)$ let $I(\mathbb{D}_*,\mathbb{E}_*)=\hat{\varphi}_C^{-1}\sigma(\mathbb{D},\mathbb{E})\hat{\varphi}_B\in\operatorname{Diag\,Isom}(\hat{e})$. To obtain a well defined invariant we proceed as follows.

PROPOSITION III.6. *Given $\hat{f}:\pi_1\to\pi_2$ a homomorphism, there exists a unique $g:\hat{\pi}_2^*\to\hat{\pi}_1^*$ such that $\hat{e}_1(gy,x)=\hat{e}_2(y,fx)$ for $y\in\hat{\pi}_2^*$, $x\in\hat{\pi}_1$.*

*Proof.* Lift $f$ to a map $\bar{f}:\pi_1^G\to\pi_2^G$. This restricts to a map $\bar{f}:N\pi_1\to N\pi_2$ and hence from (2′) to a map $|G|\,\pi_1^{*G*}\to|G|\,\pi_2^{*G*}$. This last obviously lifts to a map $\bar{g}^*:(\pi_1^{*^G})^*\to(\pi_2^{*^G})^*$ since these are free abelian. Note that if $\bar{f}$ is an isomorphism so is $\bar{g}^*$. It is clear that

$$\begin{array}{ccc}
(\pi_1^{G^*})^* & \xrightarrow{\text{rest}^*} & (\pi_1^{*^G})^* \\
\big\downarrow{\scriptstyle j(\bar{f})} & & \big\downarrow{\scriptstyle \bar{g}^*} \\
(\pi_2^{G^*})^* & \xrightarrow{\text{rest}} & (\pi_2^{*^G})^*
\end{array}$$

commutes.

Hence $\bar{g}=\bar{g}^{**}:\pi_2^{*G}\to\pi_1^{*G}$ maps $N\pi_2^*\to N\pi_1^*$ and so induces a map $g:\hat{\pi}_2^*\to\hat{\pi}_1^*$ so that $g,f$ have the desired property. The uniqueness of $g$ is a consequence of the fact $\hat{e}_1$ is non-singular (Theorem III.1).

COROLLARY. *There exists a bijection* (Diag Isom $(\hat{e}_1,\hat{e}_2)$, Diag Isom $(e_1^G,e_2^G)$) *with* (Iso $(\hat{H}^0(\pi_1),\hat{H}^0(\pi_2))$, Iso $(\pi_1^G,\pi_2^G)$). If $\mathbb{C}_{1*}=\mathbb{C}_{2*}$, *this is an isomorphism of group pairs.*

*Proof.* If $f:\hat{H}^0(\pi_1)\to\hat{H}^0(\pi_2)$ is an isomorphism, then $f^{-}:\hat{\pi}_2\to\hat{\pi}_1$ gives rise to an isomorphism $g:\hat{\pi}_1^*\to\hat{\pi}_2^*$ from the proposition and $g\oplus f$ is diagonal. The inverse map is given by $g\oplus f\to f$. These clearly map the subsets into each other and the last statement is obvious.

It is not difficult to see that if $\hat{\pi}\simeq\mathbb{Z}/t_1\times\cdots\times\mathbb{Z}/t_n$ with $t_1\mid t_2\mid\cdots\mid t_n$ and $\pi^G\simeq\mathbb{Z}^n$, then Iso $(\pi^G)$ is normal in Iso $(\hat{\pi})$ and Iso $(\hat{\pi})$/Iso $(\pi^G)\simeq(\mathbb{Z}/t_1)^*/(\pm1)$ via reduction mod $t_1$, and the determinant map [17]. Note that we may have $t_1=1$ and hence Iso $(\hat{\pi})$/Iso $(\pi^G)=(0)$. If $\mathbb{C}_*\in\mathscr{C}_n$ then by taking fixed points we see $rk_{\mathbb{Z}}\pi^G=\chi(\mathbb{C}_*)+(-1)^{n+1}$. Also by degree shifting $\hat{H}^0(\pi)\simeq\hat{H}^{-(n+1)}(G,\mathbb{Z})=H_n(G,\mathbb{Z})$. Hence if $d(H_n(G,\mathbb{Z}))<\chi(\mathbb{C}_*)+(-1)^{n+1}$, where $d(B)=$ minimal number of generators of $B$, then Iso $(\hat{\pi})$/Iso $(\pi^G)=(0)$.

From the corollary we have Diag Isom $(\hat{e})$/Diag Isom $(e^G)\simeq$ Iso $(\hat{\pi})$/Iso $(\pi^G)$ which is abelian. Clearly Diag Isom $(e^G)\subseteq$ Isom $(e^G)\cap$ Diag Isom $(\hat{e})$ which is normal in Diag Isom $(\hat{e})$. Let $\Gamma_n^k(G)=$ Diag Isom $(\hat{e})$/Isom $(e^G)\cap$ Diag Isom $(\hat{e})$ where $n=$ length of the complex and $k=$ Euler characteristic. $\Gamma_n^k(G)$ is independent of $\bar{\mathbb{C}}_*\in\mathscr{C}_n(k)$ by Proposition III.5 and is a quotient of $(\mathbb{Z}/t_1)^*/(\pm1)$ by the preceding remarks. Let $\{I(\mathbb{D}_*,\mathbb{E}_*)\}$ denote the class of $I(\mathbb{D}_*,\mathbb{E}_*)$ in $\Gamma_n^k(G)$.

This is well defined for if $\hat{\varphi}_B$, $\hat{\varphi}_{\mathbb{C}}$ are different choices for the diagonal isometry $(\pi^*\oplus\pi)^\wedge\to(\pi(B)^*\oplus\pi(B))^\wedge$ then $\hat{\varphi}_C^{-1}\sigma(\mathbb{D}_*,\mathbb{E}_*)\hat{\varphi}_B\hat{\varphi}_B\sigma(\mathbb{D}_*,\mathbb{E}_*)^{-1}\hat{\varphi}_C\in$ Isom $(e^G)\cap$ Diag Isom $(\hat{e})$ since this is normal in Diag Isom $(\hat{e})$.

PROPOSITION III.7. (i) $\{I(\mathbb{D}_*,\mathbb{E}_*)\}\cdot\{I(\mathbb{E}_*,\mathbb{F}_*)\}=\{I(\mathbb{D}_*,\mathbb{F}_*)\}$

(ii) $\{I(\mathbb{D}_*,\mathbb{E}_*)\}^{-1}=\{I(\mathbb{E}_*,\mathbb{D}_*)\}$.

(iii) If there exists a chain homotopy equivalence $\mathbb{D}_*\to\mathbb{E}_*$ inducing an isometry of $(H_n(\mathbb{D}_*),e_{\mathbb{D}})$ with $(H_n(\mathbb{E}_*),e_{\mathbb{E}})$ then $\{I(\mathbb{D}_*,\mathbb{E}_*)\}=1$.

*Proof.* (i) is obvious from the definitions. (ii) follows from (i) and the fact that $\{I(\mathbb{D}_*,\mathbb{D}_*)\}=1$. If $h:\mathbb{D}_*\to\mathbb{E}_*$ is the chain homotopy equivalence inducing the isometry then since $H_n(h)^G$ is an isometry of $e_{\mathbb{D}}^G$ to $e_{\mathbb{E}}^G$ and by Proposition III.4 and Lemma III.5 $\sigma(\mathbb{D}_*,\mathbb{E}_*)$ differs from $\hat{H}^0(H_n(h))$ by an isometry which lifts to an isometry of $e_{\mathbb{D}}^G$ with $e_{\mathbb{E}}^G$, $\{I(\mathbb{D}_*,\mathbb{E}_*)\}=1$.

From the remarks in Section III.1 and Proposition III.7 (iii) in order to show $MX$ is not homotopy equivalent to $MY$ we must show $\{I(MC_*(X,\Lambda),{}_\alpha MC_*(Y;\Lambda))\}\neq1$ for all $\alpha\in\operatorname{Aut}G$. Now if $h:MB_*\to MC_*$ is a chain map then $h:{}_\alpha MB_*\to{}_\alpha MC$ is also a chain map for any $\alpha\in\operatorname{Aut}G$. Since for any $\Lambda$-module $A$, ${}_\alpha A^G=A^G$ we have $\sigma(\mathbb{D}_*,\mathbb{E}_*)=\sigma({}_\alpha\mathbb{D}_*,{}_\alpha\mathbb{E}_*)$. Consider the map $g:\operatorname{Aut}G\to\Gamma_n^k(G)$ given by $g(\alpha)=\{I(\bar{\mathbb{C}}_*,{}_\alpha\bar{\mathbb{C}}_*)\}$. The above remark together with 3.5 (i) shows $g$ is a homomorphism and $\{I(\mathbb{D}_*,{}_\alpha\mathbb{E}_*)\}=g(\alpha)^{-1}\{I(\mathbb{D}_*,\mathbb{E}_*)\}$. Hence if we denote by $B_n^k(G)\subseteq\Gamma_n^k(G)$ the image of $g$ and by $[I(\mathbb{D}_*,\mathbb{E}_*)]$ the class of $\{I(\mathbb{D}_*,\mathbb{E}_*)\}$ in $\Gamma_n^k(G)/B_n^k(G)$, we have the following theorem.

THEOREM III.2. *If* $[I(\mathbb{D}_*, \mathbb{E}_*)] \neq 1 \in \Gamma_n^k(G)/B_n^k(G)$, *then for all* $\alpha \in \text{Aut}\, G$ *there does not exist an isometry inducing chain homotopy equivalence of* $\mathbb{D}_*$ *with* ${}_\alpha\mathbb{E}_*$.

## III.4. The examples

From the first section and Theorem III.2 we see that in order to produce examples where $MX \neq MY$ it is sufficient to show $\Gamma_n^k(G)/B_n^k(G)$ is non-zero and that we may realize these non-zero invariants. From the remark after the corollary to Proposition III.6, $\Gamma_n^k(G) = (0)$ unless $k = d(H_nG) + (-1)^n$. We will after [10] say that a group $G$ with the property that there exists $\mathbb{C}_* \in \mathcal{FC}_n$ with $\chi(\mathbb{C}_*) = d(H_nG) + (-1)^n \equiv \chi_{\min}(G)$ satisfies the minimality hypothesis in dimension $n$. It is shown in [10] that all finite abelian groups and all finite $p$-groups satisfy the minimality hypothesis for all dimensions $n$. We will from now on assume $G$ satisfies the minimality hypothesis in dimension $n$ and that all complexes considered have $\chi(C_*) = \chi_{\min}(G)$. We will denote $\Gamma_n^k(G)/(B_n^k(G))$ by $\Gamma_n(G)/(B_n(G))$.

The corollary to III.6 shows there exists an epimorphism $\phi : \text{Iso}\,(\hat{\pi})/\text{Iso}\,(\pi^G) \to \Gamma_n(G)$. If $\mathbb{D}_* = MB_*$ and $\mathbb{E}_* = M\mathbb{C}_*$ and one defines $\{b(\mathbb{B}_*, \mathbb{C}_*)\} \in \text{Iso}\,(\hat{\pi})/\text{Iso}\,(\pi^G)$ where $b$ is defined analogously to $I$ using $\mu$ instead of $\sigma$, then the uniqueness part of III.6 shows $\{I(\mathbb{D}_*, \mathbb{E}_*)\} = \phi\{b(\mathbb{B}_*, \mathbb{C}_*)\}$ i.e., we have, somewhat loosely, a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_n \times \mathcal{C}_n & \xrightarrow{M\times M} & \mathcal{M}_{2n} \times \mathcal{M}_{2n} \\ \Big\downarrow{\scriptstyle \{b\}} & & \Big\downarrow{\scriptstyle \{I\}} \\ \text{Iso}\,\hat{\pi}/\text{Iso}\,\pi^G & \xrightarrow{\Phi} & \Gamma_n(G) \end{array}$$

Clearly the map $g : \text{Aut}\, G \to \Gamma_n(G)$ factors as $\phi \circ \bar{g}$ where $\bar{g} : \text{Aut}\, G \to$ Isp $\hat{\pi}/\text{Iso}\,\pi^G$ is the map $\bar{g}(\alpha) = \{b(\bar{B}_*, {}_\alpha\bar{B}_*)\}$.

The following two propositions may be found in [10]. Recall that if $\hat{\pi} \simeq \mathbb{Z}/t_1 \times \cdots \times \mathbb{Z}/t_r$, $t_1 \mid \cdots \mid t_r$ then there exists an isomorphism $\det : \text{Iso}\,(\hat{\pi})/\text{Iso}\,(\pi^G) \xrightarrow{\simeq} (\mathbb{Z}/t_1)^*/(\pm 1)$.

PROPOSITION III.8. *Let* $G$ *be a finite abelian group with* $d(G) = s$ *and* $H_n(G) \simeq \mathbb{Z}/t_1 \times \cdots \times \mathbb{Z}/t_r$, $t_1 \mid \cdots \mid t_r$. *Then* $\text{Image}\,(\det \circ \bar{g}) \subseteq (\mathbb{Z}/t_1)^*/\pm 1$ *equals* $(\mathbb{Z}/t_1)^{*e(n,s)}$ *for some integer* $e(n, s)$.

The integer $e(n, s)$ is defined as follows. Let $M(n, s) =$ set of zero or odd partitions of $n$ of length $s$ i.e., $n=n_1+n_2+\cdots+n_s$, $0\leqq n_1\leqq n_2\leqq\cdots\leqq n_s$ and all $n_i$ are zero or odd. Let $N(n,s)\subseteq M(n,s)$ be those partitions where all $n_i\neq 0$. If $\alpha\in M(n,s)$, let $l_j=$ number of $n_i=j$.

Define $f_n: M(n,s)\to\mathbb{Z}$ by $f_n(\alpha)=\dfrac{n+s-l_0}{2s}\dfrac{s!}{l_0!\,l_1!\cdots}$ where of course $0!=1$. Then

$$e(n,s)=\sum_{\alpha\in M(n,s)} f_n(\alpha)+\sum_{\beta\in N(n-1,s)} f_{n-1}(\beta).$$

It is not hard to see

$$e(2,s)=s-1,$$

$$e(3,s)=\binom{s-1}{2}+2 \text{ if } s>2,$$
$$=3 \qquad \text{if } s=2,$$

$$e(4,s)=\binom{s-1}{3}+3(s-1) \text{ for } s\geqq 4,\ e(4,2)=3,\ e(4,3)=7.$$

PROPOSITION III.9. *Let $G$ be finite abelian, say $G\simeq\mathbb{Z}/m_1\times\cdots\times\mathbb{Z}/m_s$; $m_1\mid\cdots\mid m_s$. Given $y\in\operatorname{Iso}(\hat{\pi})/\operatorname{Iso}\pi^G$, there exists $B_*$, $\mathbb{C}_*\in\mathcal{FC}_n$ such that $\{b(B_*,\mathbb{C}_*)\}=y$. More explicitly, for each $q$ such that $(q,m_1)=1$ there exists a chain complex $\mathbb{C}(q)_*\in\mathcal{FC}_n$ and $\det\{b(\mathbb{C}(1)_*,\mathbb{C}(q)_*)\}=[q]\in(\mathbb{Z}/m_1)^*/\pm 1$.*

*Remark.* These chain complexes correspond to the chain complexes of finite $n$-dimensional $CW$-complexes. For $n=2$, $\mathbb{C}(q)$ is the chain complex of the standard 2-complex associated to the presentation $\{a_1,\ldots,a_s \mid a_i^{m_i}=1,\ (a_1^q,a_2)=1,\ (a_i,a_j)=1,\ 1\leqq i<j\leqq s\ (i,j)\neq(1,2)\}$ of $\mathbb{Z}/m_1\times\cdots\times\mathbb{Z}/m_s$. For $n>2$, $\mathbb{C}(q)_*$ corresponds to the complex whose $(n-1)$-skeleton is the $(n-1)$ skeleton of $K(G,1)=\prod_1^s K(\mathbb{Z}/m_i,1)$ but some of whose $n$ cells are attached differently than those of the $n$-skeleton of $K(G,1)$ (except of course for $\mathbb{C}(1)_*$).

We see from the above discussion that the non-zero elements of $\Gamma_n(G)/B_n(G)$ (if there are any) may be realized as obstructions $[I(\mathbb{D}_*,\mathbb{E}_*)]$ provided $G$ is finite abelian. Therefore we are left with showing $\Gamma_n(G)/B_n(G)\neq(0)$.

We shall restrict ourselves to the case of elementary abelian $p$-groups, $G=(\mathbb{Z}/p)^s$, where $p$ is an odd prime. It is obvious that $\hat{\pi}$ is also an elementary $p$-group and hence we see from the corollary to Lemma III.3 that we may assume the form $e^G$ to be the $\mathbb{Z}$-valued form $\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ and $\hat{e}$ to be the $\mathbb{Z}/p$-valued form

$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. The Wall group $L_1^0(R)$ detects equivalence classes of automorphisms of determinant 1 of the hyperbolic form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ over $R$ under stabilization [16]. Hence under the assumption $G$ is $p$-elementary we obtain the following diagram.

$$\begin{array}{ccccc}
\mathrm{Isom}\,(e^G)\cap \mathrm{Diag\,Isom}\,(e) & \hookrightarrow & \mathrm{Diag\,Isom}\,(e) & \longrightarrow & \Gamma_n(G) \\
\downarrow & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} \\
L_1^0(\mathbb{Z}) & \xrightarrow{\text{reduction}} & L_1^0(\mathbb{Z}/p) & \longrightarrow & \mathrm{coker\,(red)}
\end{array}$$

The following results may be extracted from [15].

PROPOSITION III.10. (I) *$L_1^0(\mathbb{Z})\simeq\mathbb{Z}/2$ generated by the class of* $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii) *$L_1^0(\mathbb{Z}/p)\simeq\mathbb{Z}/2$ generated by the class of* $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ *where $r$ is a non-square* mod $p$.

THEOREM III.3. *Suppose $G=(\mathbb{Z}/p)^s$ is elementary abelian where $p$ is a prime congruent to* 1 mod 4 *and both $n$ and $e(n,s)$ are even. Then there exist finite $n$-dimensional CW complexes $X$, $Y$ such that* (i) *$MX^{2n}$ and $MY^{2n}$ have isomorphic intersection forms* (ii) *$MX \# t(S^n\times S^n)$ is diffeomorphic to $MY \# t(S^n\times S^n)$ for some $t(t=1$ if $n\geqq 4)$.* (iii) *$MX$ is not homotopy equivalent to $MY$.*

*Proof.* We have seen that it is sufficient to show $\Gamma_n(G)/B_n(G)\neq(0)$. Now the above proposition shows $\varphi_1$ and hence $\varphi$ is onto. Moreover the discussion relating $B_n(G)$ and the image of $\bar{g}:\mathrm{Aut}\,G\to \mathrm{Iso}\,\hat{\pi}/\mathrm{Iso}\,\pi^G$ shows

$$\varphi(B_nG)=\left\{\text{class of}\begin{pmatrix} r^{e(n,s)} & 0 \\ 0 & r^{-e(n,s)} \end{pmatrix};\ (r,p)=1\right\}$$

in coker (reduction) which is zero by III.10(ii) since $e(n,s)$ is even. Since $p\equiv 1$ mod 4 the reduction map is zero and hence $\Gamma_n(G)/B_n(G)$ maps onto coker (reduction) $\simeq\mathbb{Z}/2$.

COROLLARY. *There exist examples satisfying* (i), (ii) *and* (iii) *in every dimension* $\equiv 0$ (mod 4).

*Proof.* From the above theorem it is sufficient to show that for any $n$, there exists an $s$ such that $e(n, s)$ is even. Referring to the definition of $e(n, s)$ given after Proposition III.8 claim: If $\alpha$ is a zero or odd partition of $n$ of length $s$ and has $s-k$ zeros, then $f_n(\alpha)$ is an integral multiple of $\binom{s-1}{k-1}$. Accepting the claim for the moment we see that if $s>n$, $e(n, s)=\sum_{\alpha\in M(n,s)} f_n(\alpha)=\sum_\alpha r_k\binom{s-1}{s-k}$ and if $s-1=2^N$ each binomial coefficient is even. To demonstrate the claim we have

$$f_n(\alpha)=\frac{n+s-(s-k)}{2s}\frac{s!}{(s-k)!\,l_1!\cdot\cdots\cdot l_r!}=\frac{n+k}{2}\frac{(s-1)!}{(s-k)!\,l_1!\cdot\cdots\cdot l_r!}$$

where $l_1+\cdots+l_r=k$, $n=l_1+3l_2+5l_3+\cdots$. Hence

$$f_n(\alpha)=\frac{n+k}{2}\frac{(k-1)!}{l_1!\cdot\cdots\cdot l_r!}\binom{s-1}{k-1}$$

and we must show $y=\dfrac{n+k}{2}\dfrac{k-1}{l_1!\cdot\cdots\cdot l_r!}$ is an integer. Since $n=k+2l_2+4l_3+\cdots$ implies $n+k=2k+2l_2+4l_3+\cdots$ and hence

$$y=\left(1+\frac{l_2}{k}+\frac{2l_3}{k}+\cdots\right)\frac{k!}{l_1!\cdot\cdots\cdot l_r!}.$$

The term

$$\frac{(j-1)l_j}{k}\frac{k!}{l_1!\cdot\cdots\cdot l_r!}=\frac{(j-1)l_j}{k}\frac{k!}{l_j!\cdot m_j!\,l_1!}\frac{m_j!}{\cdots\cdot\hat{l}_j!\cdot\cdots\cdot l_r!}$$

where $m_j=l_1+\cdots\hat{l}_j+\cdots+l_r$. So

$$y=\frac{k!}{l_1!\cdot\cdots\cdot l_r!}+\sum_{j\geq 2}(j-1)\binom{k-1}{l_j-1}\frac{m_j!}{l_1!\cdot\cdots\cdot\hat{l}_j!\cdot\cdots\cdot l_r!}.$$

As expressions $\dfrac{(x_1+\cdots+x_n)!}{x_1!\cdot\cdots\cdot x_n!}$ are integers $y$ is an integer.

Since $e(2, s)=s-1$ the simplest examples of such manifolds are as follows. Let $p\equiv 1 \bmod 4$, $s$ odd, and let $X(q)$ be the finite dimensional complex based on the presentation

$$\{a_1, \ldots, a_s \mid a_i^p = 1 (a_1^q, a_2) = 1, (a_i, a_j) = 1, 1 \leqq i < j \leqq s, (i, j) \neq (1, 2)\}$$

Then if $qq'^{-1}$ is a non square mod $p$, $M(X(q))^4$ and $M(X(q'))^4$ are a pair of examples.

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