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K_2-analogs of Hasse's norm theorems

Anthony Bak and Ulf Rehmann

§ 1. Introduction

The classical norm theorem of Hasse for fields and the classical norm theorem of Hasse and Schilling for simple algebras can be stated in the language of algebraic K-theory in terms of the functor K_1. The goal of this paper is to show that K_2-analogs of these results are valid. It turns out that whereas the classical norm theorem for fields is valid essentially only for cyclic extensions of global fields, its K_2-analog is valid for all finite extensions of global fields. Our results lead us to ask whether or not there are K_n-analogs of the classical norm theorems for all $n > 1$.

The K_2-analog of the Hasse-Schilling norm theorem would not have been possible without a reduced norm homomorphism for the functor K_2. The establishment of the reduced norm as well as several other spectacular, fundamental results for K_2 appeared recently in articles of Merkurjev and Suslin [12] and Suslin [18]. Moreover, the special case of our K_2-analog for fields (resp. simple algebras) such that the index of the field extension (resp. simple algebra) is square free can be deduced from the results in Merkurjev and Suslin [12, § 17]. A significant contrast between the methods of Merkurjev and Suslin and those in the current paper is that the former use in an essential way the connection between K_2 and Galois cohomology uncovered by Tate in [19] and [20] (cf. also [6]), whereas the latter do not use this connection. One consequence of the contrast is that Merkurjev and Suslin require taking special care in positive characteristic when the characteristic divides the index, whereas we can work uniformly. An excellent survey of the results of Merkurjev and Suslin is available in Soulé [17].

The reduced norm for K_2 of simple algebras of square free index is developed in [12] and for simple algebras of arbitrary finite index in [18]. It is reasonable to expect in view of [18] that one could remove the index restrictions in [12, § 17], using the methods employed already in [12]. Suslin has informed us, in fact, how to do this in computing the cokernel of the reduced norm homomorphism $N_{D/K}: K_2(D) \to K_2(K)$ when K is a global field and D is a finite, K-central, division algebra. The key to his proof is afforded by the following smooth transition from K-theory to Galois cohomology. To begin, one uses transfer methods, in particular [12, 6.1.3], to reduce to the case that the index of D is a prime number. Next, one reduces as in [12, § 17] the computation of coker $(N_{D/K})$ to that of the cokernel of the transfer homomorphism $N_{L/K}: K_2(L) \to K_2(K)$ for maximal cyclic subfields L of D. Then Suslin makes the key observation that the image of either homomorphism above contains the subgroup $[L:K]K_2(K)$, because composition of either homomorphism with the canonical homomorphism from $K_2(K)$ is multiplication by $[L:K]$. Thus, one can reduce the problems above mod $[L:K]$ and thereby pass in the usual way to Galois cohomology, whenever Char $(K) \nmid [L:K]$. The computations required in Galois cohomology are fairly routine. If Char $(K) \nmid [L:K]$, one can use the reciprocity law for $N_{D/K}$ to complete the proof. It is worth noting that the procedures above allow one also to compute coker $(N_{L/K})$ for any finite extension L of K such that Char $(K) \nmid [L:K]$. However, the reciprocity law for $N_{L/K}$ does not allow one to handle as above the case that Char $(K) \mid [L:K]$.

Colliot-Thélène informed us that the K_2-analog of Hasse's norm theorem for number fields is established in his paper [9, Lemma 2c)]. His proof is also via Galois cohomology.

The proofs in the current paper do not invoke Galois cohomology. Instead, they deal directly with the functor K_2, treating it as an object in number theory. Accordingly, much of the burden of proof is carried by classical number theory. The rest of the burden is carried by three number theoretic results of more recent vintage. They are the Moore–Weil reciprocity law [8] for K_2, the Bass–Garland–Tate finiteness theorem [10], [6] for the wild kernel, and Tate's theorem [19] that each element in the wild kernel has a pth root in $K_2(K)$ for each prime p not dividing the characteristic of K. Our approach was inspired by procedures developed in [3] to solve the congruence subgroup problem and metaplectic problem for SL_n, $n > 1$. In fact, it was our interest to eliminate an ambiguity of ± 1 in our solution to these problems that brought us to the current results. The ambiguity will be eliminated in [4]. The results permit also eliminating a similar ambiguity in the solution [2] of the congruence subgroup problem and metaplectic problem for the remaining classical groups of rank >1. This application will appear in a future paper of the first named author.

The remainder of the paper is organized as follows. In § 2, we recall the classical norm theorems and state our main results. The latter will include the K_2-analogs mentioned above and a result showing that the reduced norm on K_2 splits for finite, simple algebras over a global field. The proof of the K_2-analogs will be reduced to showing that a certain sequence is exact. The exactness of the sequence is shown in § 3. In an appendix, we give an elegant, alternative proof of the K_2-analog of the Hasse–Schilling theorem, communicated to us by O. Gabber.

§ 2. Statement of main results

We suggest as a general reference for number theory (resp. simple algebras) (resp. algebraic K-theory) Cassels–Fröhlich [7] and Serre [16] (resp. Reiner [15]) (resp. Milnor [13]).

We fix the following notation. Let K denote a global field and L a finite field extension of K. Let v denote a prime of K and w an extension of v to L. Let K_v and L_w denote respectively the completions of K at v and L at w. If v is noncomplex, let $\mu(K_v)$ denote the group of all roots of unity in K_v and if v is complex, let $\mu(K_v)=1$. Adopt the same convention for $\mu(L_w)$. If A is a ring, let $A^{\cdot}$ denote the multiplicative group of units in A. Let $(\ ,\ )_v : K_v^{\cdot}\times K_v^{\cdot}\to \mu(K_v)$ denote the norm residue symbol at v.

The Hasse norm theorem for fields is as follows: *If L is a cyclic extension of K then an element of $K^{\cdot}$ lies in the image of the norm homomorphism $N_{L/K}: L^{\cdot}\to K^{\cdot}$ if and only if its image in each completion $K_v^{\cdot}$ of $K^{\cdot}$ lies in the image of the local norm homomorphism $N_{L_w/K_v}: L_w^{\cdot}\to K_v^{\cdot}$.* In order to translate Hasse's theorem into algebraic K-theory, we make a few simple observations. The determinant map on matrices with coefficients in a field F induces an isomorphism $K_1(F)\cong F^{\cdot}$. Furthermore, if E is any finite field extension of F then the transfer homomorphism [13] $K_1(E)\to K_1(F)$ in algebraic K-theory will correspond under the isomorphism above to the usual norm homomorphism $E^{\cdot}\to F^{\cdot}$ in field theory. We shall denote the transfer homomorphism in K-theory also by the notation $N_{E/F}$. The translation of Hasse's theorem into an equivalent theorem concerning the functor K_1 is now obvious. However, the obvious translation of Hasse's is not the one we want, because its K_2-analog is valid for only finite Galois extensions L of K. To get a better translation, we group together in Hasse's theorem all primes w lying above a given prime v and then translate the theorem into an equivalent one concerning the functor K_1. The grouping together doesn't matter as far as Hasse's theorem is concerned, because image $(N_{L_w/K_v}) = \text{image}\,(\prod_{w|v} N_{L_w/K_v})$ whenever L is Galois over K. However, the K_2-analog we get has now the broadest possible validity, namely:

THEOREM 1. *If L is any finite extension of K then an element of $K_2(K)$ lies in the image of the transfer homomorphism $N_{L/K}: K_2(L)\to K_2(K)$ if and only if its image in each $K_2(K_v)$ lies in the image of the transfer homomorphism $\prod_{w|v} N_{L_w/K_v}: \prod_{w|v} K_2(L_w)\to K_2(K_v)$.*

We want to show next that the theorem above is equivalent to the one below. The theorem below will be proved in § 3. To state the theorem, we need to introduce a little more notation. By Matsumoto's theorem [13, § 11], K_2 of any field F is the universal Steinberg symbol $(\,,)$ on $F^{\bullet}\times F^{\bullet}$. Since the norm residue symbol $(\,,)_v$ is a Steinberg symbol on $K_v^{\bullet}\times K_v^{\bullet}$, there is by universality a homomorphism $K_2(K_v)\to\mu(K_v)$. Composing this homomorphism with the canonical homomorphism $K_2(K)\to K_2(K_v)$ given by the functoriality of K_2, we get a homomorphism

$$\lambda_v : K_2(K)\to\mu(K_v)$$
$$(a, b)\mapsto(a, b)_v.$$

THEOREM 2. *Let L be any finite extension of K.* Let $\Sigma_{L/K}=\{v\mid v$ *real, all extensions of v to L are complex*$\}$. *Then the sequence below is exact.*

$$K_2(L)\xrightarrow{N_{L/K}}K_2(K)\xrightarrow{\amalg\lambda_v}\coprod_{v\in\Sigma_{L/K}}\mu(K_v).$$

We show now the equivalence of Theorems 1 and 2. Theorem 1 is equivalent to the assertion that the canonical homomorphism $\operatorname{Coker}(N_{L/K}:K_2(L)\to K_2(K))\to\prod_{\text{all }v}\operatorname{Coker}(\prod_{w\mid v}N_{L_w/K_v}:\prod_{w\mid v}K_2(L_w)\to K_2(K_v))$ is injective. We shall show that if $v\notin\Sigma_{L/K}$ then $\operatorname{Coker}(\prod_{w\mid v}N_{L_w/K_v})=1$ and that if $v\in\Sigma_{L/K}$ then the norm residue symbol on $K_v^{\bullet}\times K_v^{\bullet}$ induces an isomorphism $\operatorname{Coker}(\prod_{w\mid v}N_{L_w/K_v})\xrightarrow{\cong}\mu(K_v)$. It will follow that Theorem 1 is equivalent to the exactness of the sequence $K_2(L)\xrightarrow{N_{L/K}}K_2(K)\xrightarrow{\amalg\lambda_v}\coprod_{v\in\Sigma_{L/K}}\mu(K_v)$. But, one knows that the latter homomorphism in the sequence is surjective, because the norm residue symbol $(\,,)_v$ on $K_v^{\bullet}\times K_v^{\bullet}$ is continuous and image $(K^{\bullet}\to\coprod_{v\in\Sigma_{D/K}}K_v^{\bullet})$ is dense by weak approximation [7, II § 6]. Thus, the exactness of the sequence above will imply Theorem 2, and conversely. By a theorem of Bass and Tate (cf. Milnor [13, A.15]), $\operatorname{Coker}(N_{L_w/K_v})=1$ whenever v is nonarchimedean. If v is complex or if both v and w are real then $K_v=L_w$; thus, N_{L_w/K_v} is the identity homomorphism and $\operatorname{Coker}(N_{L_w/K_v})=1$. Thus, $\operatorname{Coker}(\prod_{w\mid v}N_{L_w/K_v})=1$ whenever $v\notin\Sigma_{L/K}$. Suppose that v is real and w is complex. By [13, A.1], the norm residue symbol on $K_v^{\bullet}\times K_v^{\bullet}$ induces an isomorphism $K_2(K_v)/K_2(K_v)^2\xrightarrow{\cong}\mu(K_v)$. Since each element of $L_w^{\bullet}$ is a square, it follows that $\operatorname{image}(N_{L_w/K_v})\subseteq K_2(K_v)^2$. We shall show that $\operatorname{image}(N_{L_w/K_v})=K_2(K_v)^2$. The group $K_2(K_v)^2$ is generated by all elements (a, b) such that $b>0$. If $\beta\mapsto\bar\beta$ denotes complex conjugation on L_w and if β is chosen such that $\beta\bar\beta=b$ then by the Frobenius reciprocity law for transfer [13, 14.7], $(a, b)=N_{L_w/K_v}(a,\beta)$. Thus, $\operatorname{image}(N_{L_w/K_v})=K_2(K_v)^2$. Thus, if $v\in\Sigma_{L/K}$ then $\operatorname{image}(N_{L_w/K_v})=K_2(K_v)^2$ for all $w\mid v$. Thus, if $v\in\Sigma_{L/K}$ then the norm residue symbol on $K_v^{\bullet}\times K_v^{\bullet}$ induces an isomorphism $\operatorname{Coker}(\prod_{w\mid v}N_{L_w/K_v})\xrightarrow{\cong}\mu(K_v)$.

If A is any finite, central, simple algebra over the field F, let $N_{A/F}$ denote the reduced norm on either $A^{\bullet}$, $K_1(A)$ or $K_2(A)$.

Let D be a finite, K-central division algebra. The Hasse–Schilling norm theorem says the following: *An element of K lies in the image of the reduced norm homomorphism $N_{D/K}: D^{\cdot} \to K^{\cdot}$ if and only if it is positive at each real prime of K not splitting D.* Let

$$\Sigma_{D/K}$$

denote all real primes of K not splitting D. After identifying in the usual way $K_1(D) \cong D^{\cdot}/[D^{\cdot}, D^{\cdot}]$ (cf. Bass [5, V(9.5)]) and $K_1(K) \cong K^{\cdot}$ and after noting that an element $a \in K^{\cdot}$ is positive at the real prime v if and only if $(-1, a)_v = 1$, one deduces easily that the classical theorem above is equivalent to the exactness of *the sequence*

$$K_1(D) \xrightarrow{N_{D/K}} K_1(K) \to \coprod_{v \in \Sigma_{D/K}} \mu(K_v).$$
$$a \mapsto \coprod (-1, a)_v.$$

Moreover, one knows by weak approximation [7, II §6] that the latter homomorphism is surjective. The K_2-analog of the above is as follows.

THEOREM 3. *For any finite, K-central, division algebra D, the sequence below is exact*

$$K_2(D) \xrightarrow{N_{D/K}} K_2(K) \xrightarrow{\coprod \lambda_v} \coprod_{v \in \Sigma_{D/K}} \mu(K_v).$$

We shall reduce below the proof of Theorem 3 to that of Theorem 2. However, before doing this we want to record the following exercises. Let $D_v = D \otimes_K K_v$.

EXERCISE 1. Show that the Hasse–Schiling norm theorem is equivalent to the following result: *The canonical homomorphism* Coker $(N_{D/K}: D^{\cdot} \to K^{\cdot}) \to \prod_{\text{all } v}$ Coker $(N_{D_v/K_v}: D_v^{\cdot} \to K_v^{\cdot})$ *is injective.* Hint: Use the local norm theorem of Nakayama and Matsushima.

EXERCISE 2. Show that Theorem 3 is equivalent to the following result: *The canonical homomorphism* Coker $(N_{D/K}: K_2(D) \to K_2(K)) \to \prod_{\text{all } v}$ Coker $(N_{D_v}/K_v: K_2(D_v) \to K_2(K_v))$ *is injective.*

We reduce now the proof of Theorem 3 to that of Theorem 2. The surjectivity of the homomorphism $\coprod \lambda_v$ $(v \in \Sigma_{D/K})$ has been shown already above. Thus, it suffices to show that the sequence $(*): K_2(D) \xrightarrow{N_{D/K}} K_2(K) \xrightarrow{\coprod \lambda_v} \mu(K_v)$ is exact. The homomorphism $(\coprod \lambda_v)N_{D/K}$ is the composite of the homomorphisms $K_2(D) \to K_2(D_v) \xrightarrow{N_{D_v/K_v}} K_2(K_v) \xrightarrow{(\,,\,)_v} \mu(K_v)$ and from a result of Alperin and Dennis [1], it follows that $(\,,\,)_v N_{D_v/K_v}$ is trivial. Thus, $(*)$ is a zero-sequence. We shall show that Theorem 2 implies: $(*)$ is exact. Let L be a maximal subfield of D. From the commutativity (cf. [12, § 6]) of the diagram

$$\begin{array}{ccc} K_2(L) & \longrightarrow & K_2(D) \\ & {\scriptstyle N_{L/K}}\searrow \quad \swarrow{\scriptstyle N_{D/K}} & \\ & K_2(K) & \end{array}$$

and the fact that $(*)$ is a zero-sequence, it follows that the sequence $(**): K_2(L) \xrightarrow{N_{L/K}} K_2(K) \xrightarrow{\coprod \lambda_v} \coprod_{v \in \Sigma_{D/K}} \mu(K_v)$ is a zero-sequence. On the other hand, if $(**)$ is exact then $(*)$ is also exact. By Theorem 2, $(**)$ will be exact, providing $\Sigma_{L/K} = \Sigma_{D/K}$. We show next that there is an L such that $\Sigma_{L/K} = \Sigma_{D/K}$. If $i(D)$ denotes the index of D (cf. [15, p. 253]) and $i_v(D)$ denotes the index of D_v then $i_v(D) \mid i(D)$. Thus, by the Grundwald–Wang theorem [21], there is a (cyclic) extension L of K such that $[L:K] = i(D)$ and such that $[L_w:K_v] = i_v(D)$ for each $w \mid v$ such that v is real or $i_v(D) \neq 1$. Thus, by construction, $\Sigma_{L/K} = \Sigma_{D/K}$. On the other hand, by results of Hasse (cf. [15, 32.15]), L splits D. But, since $[L:K] = i(D)$, it follows (cf. [15, 28.10]) that L can be embedded as a maximal subfield of D.

We want to discuss next K_2-analogs of Wang's theorem [22]. We recall that Wang's theorem says the following: *For any finite, K-central division algebra D, the reduced norm homomorphism $N_{D/K}: K_1(D) \to K_1(K)$ is injective.* Merkurjev and Suslin have shown in their paper [12, 17.4] that the K_2-analog of Wang's theorem holds, proving the index of D is square free. Moreover, they conjectured that the full K_2-analog holds. We shall contribute additional evidence for their conjecture by showing that the reduced norm homomorphism splits.

PROPOSITION 1. *For any finite, K-central division algebra D over a global field K, the reduced norm homomorphism $N_{D/K}: K_2(D) \to K_2(K)$ decomposes into a split surjective map $K_2(D) \to \text{image}\,(N_{D/K})$ and a split injective map* $\text{image}\,(N_{D/K}) \to K_2(K)$.

Proof. The proof of the first assertion will be divided into two cases, according to whether or not $N_{D/K}: D^\bullet \to K^\bullet$ is surjective.

Suppose $N_{D/K}: D^\bullet \to K^\bullet$ is surjective. By a result of Rehmann and Stuhler [14, Theorem 2.2] there is a homomorphism $\psi: K_2(K) \to K_2(D)$ induced by $(a, N_{D/K}\beta)_K \mapsto (a, \beta)_D$ where $a \in K^\bullet$, $\beta \in D^\bullet$. By the diagram above in the case of a maximal subfield $L \subset D$ with $\beta \in L$ and by Frobenius reciprocity for transfer [13, 14.7], we get

$$\begin{aligned} N_{D/K}\psi(a, N_{D/K}(\beta))_K &= N_{D/K}(a, \beta)_D = N_{L/K}(a, \beta)_L \\ &= (a, N_{L/K}(\beta))_K = (a, N_{D/K}(\beta))_K. \end{aligned}$$

Hence, ψ splits $N_{D/K}$.

Suppose now $N_{D/K}: D^{\cdot} \to K^{\cdot}$ is not surjective. Then Char $K = 0$. As shown above, D contains a maximal subfield L such that L/K is cyclic and $\Sigma_{L/K} = \Sigma_{D/K}$. Consider the commutative diagram

$$\begin{array}{ccc} K_2(L) & \xrightarrow{\theta} & K_2(D) \\ & {\scriptstyle N_{L/K}}\searrow \quad \swarrow{\scriptstyle N_{D/K}} & \\ & K_2(K) & \end{array}$$

where θ denotes the canonical homomorphism induced by the embedding $L \to D$. By Theorems 2 and 3, image $(N_{L/K}) =$ image $(N_{D/K})$. To show that $N_{D/K}: K_2(D) \to$ image $(N_{D/K})$ has a splitting, it suffices to show that $\theta(\text{Ker}\,(N_{L/K})) = 1$. Let σ denote a generator of the Galois group of L/K. By the "Hilbert 90 theorem" of Merkurjev and Suslin [12, 14.1] for K_2, Ker $(N_{N/L}) = \{c^{-1}\sigma(c) \mid c \in K_2(L)\}$. But an easy application of the Skolem–Noether theorem shows that $\theta(c) = \theta(\sigma(c))$. Thus, $\theta(\text{Ker}\,(N_{L/K})) = 1$.

The second assertion of the proposition is very easy to dispose of. By Theorem 3, it suffices to show that the surjective homomorphism $\coprod_{v \in \Sigma_{D/K}} \lambda_v : K_2(K) \to \coprod_{v \in \Sigma_{D/K}} \mu(K_v)$ splits. To accomplish this, it suffices to show that for each $v \in \Sigma_{D/K}$ there is an element $c \in K_2(K)$ such that $c^2 = 1$, $\lambda_v(c) = -1$, and $\lambda_{v'}(c) = 1$ for each $v' \neq v$, $v' \in \Sigma_{D/K}$. By the weak approximation theorem [7, II § 15], there is an element $a \in K^{\cdot}$ such that a is negative at v and positive at all $v' \neq v$, $v' \in \Sigma_{D/K}$. Clearly, the element $(a, -1)$ has the desired properties.

§ 3. Proof of Theorem 2

We adopt the notation and conventions of the previous section. Thus, $\mu(K_v)$ denotes the group of all roots of unity in K_v, except when v is complex, in which case $\mu(K_v)$ denotes the trivial group. Let

$$m_v = |\mu(K_v)|$$
$$n_w = |\mu(L_w)|.$$

The quotient n_w/m_v is whole, except when v is real and w is complex, in which case it is $\frac{1}{2}$. When n_w/m_v is whole then raising elements of $\mu(L_w)$ to the (n_w/m_v)th power defines a surjective homomorphism n_w/m_v: $\mu(L_w) \to \mu(K_v)$. If $n_w/m_v = \frac{1}{2}$ then the notation $n_w/m_v: \mu(L_w) \to \mu(K_v)$ will denote the trivial homomorphism from $\mu(L_w)$ to $\mu(K_v)$.

PROPOSITION 2 (Habdank). *For any finite extension L of K, the diagram below commutes*

$$\begin{array}{ccc} K_2(L) & \xrightarrow{\prod_{w|v} \lambda_v} & \prod_{w|v} \mu(L_w) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \prod_{w|v} n_w/m_v} \\ K_2(K) & \xrightarrow[\lambda_v]{} & \mu(K_v). \end{array}$$

Proof. Let $D(L_w)$ and $D(K_v)$ denote respectively the kernels of the homomorphisms $K_2(L_w) \twoheadrightarrow \mu(L_w)$, $(a, b) \mapsto (a, b)_w$, and $K_2(K_v) \twoheadrightarrow \mu(K_v)$, $(a, b) \mapsto (a, b)_v$. By a theorem of C. Moore (cf. [13, A.14]), $D(L_w)$ and $D(K_v)$ are infinitely divisible groups. Thus, any homomorphism from $K_2(L_w)$ to $K_2(K_v)$ must take $D(L_w)$ to $D(K_v)$, because $\mu(K_v)$ is not infinitely divisible. It follows that any homomorphism $f: K_2(L_w) \to K_2(K_v)$ induces a homomorphism $f: \mu(L_w) \to \mu(K_v)$. Thus, there is a commutative diagram

$$\begin{array}{ccc} K_2(L) & \xrightarrow{\prod_{w|v} \lambda_w} & \prod_{w|v} \mu(L_w) \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle \prod_{w|v} N_{L_w/K_v}} \\ K_2(K) & \xrightarrow[\lambda_v]{} & \mu(K_v) \end{array}$$

where the N_{L_w/K_v}'s are induced from the corresponding homomorphisms on $K_2(L_w)$'s. It remains to show that $N_{L_w/K_v} = (n_w/m_v)$.

If w is complex then the equality above is clear, because $\mu(L_w) = 1$. On the other hand, if both v and w are real then $K_v = L_w$; thus, the transfer homomorphism N_{L_w/K_v} and the homomorphism n_w/m_v are the identity homomorphisms. Thus, the proposition is true whenever v is archimedean.

Suppose now that v is nonarchimedean. By a theorem of Bass and Tate (cf. [13, A.15]), the transfer homomorphism $N_{L_w/K_v}: K_2(L_w) \to K_2(K_v)$ is surjective. It follows that $N_{L_w/K_v}: \mu(L_w) \to \mu(K_v)$ is also surjective. The proof of Bass and Tate breaks the extension L_w/K_v into a tower of extensions L_i/L_{i-1} such that each extension has specific properties. Habdank [11] has shown that for such extensions L_i/L_{i-1}, the transfer homomorphism $N_{L_i/L_{i-1}}: \mu(L_i) \to \mu(L_{i-1})$ is raising to the power $|\mu(L_i)|/|\mu(L_{i-1})|$. The proposition follows.

The following technical lemma will be required in the proof of Theorem 2. Its proof will be postponed till after that of Theorem 2.

LEMMA 1. *Let k denote a natural number. Let $x \in K_2(K)$ such that $\lambda_v(x) = 1$ for each $v \in \Sigma_{L/K}$. If x has a k'th root in $K_2(K)$ then there is an element $y \in K_2(L)$ such that $xN_{L/K}(y) \in (\operatorname{Ker}(\coprod_{\text{all } v} \lambda_v))^k$.*

Proof of Theorem 2. The surjectivity of the homomorphism $\coprod \lambda_v$ $(v \in \Sigma_{L/K})$ has been shown already in § 2. Thus, it suffices to prove exactness at $K_2(K)$. If $v \in \Sigma_{L/K}$ then by Proposition 2 $\lambda_v N_{L/K}$ is trivial, because $\mu(L_w) = 1$ for each $w \mid v$. Thus, $(\coprod_{v \in \Sigma_{L/K}} \lambda_v)$ $N_{L/K}$ is trivial. It remains to prove that $\operatorname{Ker}(\coprod_{v \in \Sigma_{L/K}} \lambda_v) \subset \operatorname{image}(N_{L/K})$. Let $x \in \operatorname{Ker}(\coprod_{v \in \Sigma_{L/K}} \lambda_v)$. Let $N = \operatorname{Ker}(\coprod_{\text{all } v} \lambda_v)$. By Lemma 1, there is an element $y_0 \in K_2(L)$ such that $xN_{L/K}(y_0) \in N$. By results of Garland [10] and Bass–Tate [6], N is a finite group containing no $\operatorname{Char}(K)$-torsion whenever $\operatorname{Char}(K) \neq 0$ and by Tate [19] (cf. also [6]), each element of N has a p'th root in $K_2(K)$, for any natural prime $p \nmid \operatorname{Char}(K)$. We want now to filter N so that we can make good use of Lemma 1. Let $p_1, \ldots, p_r$ be natural primes such that the product $p_1 \cdots p_r$ annihilates N. Let $N_0 = N$, $N_1 = N^{p_1}, \ldots, N_r = N^{p_1 \cdots p_r} = 1$. By Tate's result above, each element of N_i has a $(p_1 \cdots p_i p_{i+1})$'th root in $K_2(K)$. Thus, by applying Lemma 1 r times, we can find elements $y_1, \ldots, y_r \in K_2(L)$ such that $xN_{L/K}(y_0 y_1 \cdots y_r) \in N_r = 1$.

Proof of Lemma 1. We want to reduce to the special case $k = 1$. So, let us assume that the case $k = 1$ has been proved and let $k > 1$. Let $x \in K_2(K)$ such that $\lambda_v(x) = 1$ for each $v \in \Sigma_{L/K}$. We show first that there is an element $\alpha \in K_2(K)$ such that $\alpha^k = x$ and $\lambda_v(\alpha) = 1$ for each $v \in \Sigma_{L/K}$. By hypothesis, there is an element $\alpha \in K_2(K)$ such that $\alpha^k = x$. But in the proof of Proposition 1, it was shown that the homomorphism $\coprod_{v \in \Sigma_{L/K}} \lambda_v : K_2(K) \to \coprod_{v \in \Sigma_{L/K}} \mu(K_v)$ splits. It follows that there is an α such that $\alpha^k = x$ and $\lambda_v(\alpha) = 1$ for each $v \in \Sigma_{L/K}$. By our assumption for the case $k = 1$, there is an element $\beta \in K_2(L)$ such that $\alpha N_{L/K}(\beta) \in \operatorname{Ker}(\coprod_{\text{all } v} \lambda_v)$. Thus, if $y = \beta^k$ then $xN_{L/K}(Y) = (\alpha N_{L/K}(\beta))^k \in (\operatorname{Ker}(\coprod_{\text{all } v} \lambda_v))^k$.

We treat now the case $k = 1$. Let $\mu(K)$ and $\mu(L)$ denote respectively the groups of roots of unity in K and L. Let $m = |\mu(K)|$ and $n = |\mu(L)|$. Consider the commutative diagram

$$\begin{array}{ccccc}
K_2(L) & \xrightarrow{\lambda_L} & \coprod_v \prod_{w \mid v} \mu(L_w) & \xrightarrow{\gamma_L} & \mu(L) \\
\downarrow{\scriptstyle N_{L/K}} & & \downarrow{\scriptstyle \coprod_v \prod_{w/v} (n_w/m_v)} & & \downarrow{\scriptstyle n/m} \\
K_2(K) & \xrightarrow[\lambda_K]{} & \coprod_v \mu(K_v) & \xrightarrow[\gamma_K]{} & \mu(K)
\end{array}$$

where $\lambda_K = \coprod_{\text{all } v} \lambda_v$, $\gamma_K = \prod_{\text{all } v} (m_v/m)$, $\lambda_L = \coprod_{\text{all } v} \prod_{w|v} \lambda_w$, and $\gamma_L = \prod_{\text{all } v} (\prod_{w|v} (n_w/n))$. The rows are exact by Moore reciprocity (cf. [8] or [3, 3.2]). Let $x \in K_2(K)$ such that $\lambda_v(x) = 1$ for all $v \in \Sigma_{L/K}$. Thus, $\lambda_K(x) \in \text{image}(\coprod_{\text{all } v} \prod_{w|v} (n_w/m_v))$. We want to show that $\lambda_K(x)$ has a lifting $z \in \coprod_v \prod_{w|v} \mu(L_w)$ such that $\gamma_L(z) = 1$. It will follow then from the exactness of the top row that there is a $y \in K_2(L)$ such that $\lambda_L(y) = z$. Clearly, y^{-1} has the properties required in the lemma.

Let z_0 be a lifting of $\lambda_K(x)$. Since the bottom row of the diagram is exact, we know that $\gamma_K \lambda_K(x) = 1$. Thus, $(\gamma_L(z_0))^{n/m} = 1$, i.e. $\gamma_L(z_0) \in \mu(L)^m$. Let $\theta = \coprod_{\text{all } v} \prod_{w|v} (n_w/m_v)$. To complete the proof, it suffices to show that $\mu(L)^m \subseteq \gamma_L(\text{Ker}\,\theta)$. Let δ be a generator of $\mu(L)$. It suffices to show that $\delta^m \subset \gamma_L(\text{Ker}\,\theta)$. By [3, 3.4], there is a finite set S of nonarchimedean primes of K such that $\{(m_v/m) \mid v \in S\}$ is relatively prime to $\frac{n}{m}$. Choose integers s_v such that $\sum s_v(m_v/m) \equiv 1 \bmod (\mathbb{Z}\frac{n}{m})$. Thus, $\sum_{v \in S} s_v m_v \equiv m \bmod (\mathbb{Z}n)$. For each $v \in S$, pick exactly one w in L above it and let T denote the resulting set of w's. Let ζ_w be a generator of $\mu(L_w)$ such that $\gamma_L(\zeta_w) = \delta$. Then, clearly the element $\zeta = \coprod_{w \in T} \zeta_w^{s_v m_v} \in \coprod_v \prod_{w|v} \mu(L_w)$ has the properties that $\theta(\zeta) = 1$ and $\gamma_K(\zeta) = \delta$.

§ A. Appendix

O. Gabber has told us an elegant, simple way to conclude the proof of Theorem 3 without making reference to Theorem 2. We want to record next his proof. Any oversights or other undesirable aspects of the proof are our own.

Our proof of Theorem 3 starts with two elementary steps: It shows that the homomorphism $\coprod \lambda_v : K_2(K) \to \coprod_{v \in \Sigma_{D/K}} \mu(K_v)$ is surjective and that the sequence $K_2(D) \xrightarrow{N_{D/K}} K_2(K) \xrightarrow{\coprod \lambda_v} \coprod_{v \in \Sigma_{D/K}} \mu(K_v)$ is a zero-sequence. Thus, it suffices to show that the group $K_2(K)/N_{D/K}(K_2(D))$ has order $\leq 2^{|\Sigma_{D/K}|}$. Gabber demonstrates this fact as follows.

A few elementary observations, made already in the proof of Proposition 1, show that $N_{D/K}(K_2(D))$ contains all elements (a, b) such that $a \in K^\bullet$ and $b \in N_{D/K}(D^\bullet)$. Since the symbol $(\,,)$ is skew symmetric, it follows that $N_{D/K}(K_2(D))$ contains all elements (b, a) where a and b are as above. Let $a_1, \ldots, a_r \in K^\bullet$ generating $K^\bullet/N_{D/K}(D^\bullet)$. From the bimultiplicativity of $(\,,)$, it follows that the elements (a_i, a_j) $(1 \leq i \leq r, 1 \leq j \leq r)$ generate $K_2(K)/N_{D/K}(K_2(D))$. Furthermore, since $(a_i, a_j)^2 = (a_i^2, a_j) = (a_i, a_j^2)$ and $a_i^2, a_j^2 \in N_{D/K}(D^\bullet)$, it is clear that $K_2(K)/N_{D/K}(K_2(D))$ has exponent 2. Thus, to complete the proof, it suffices to show one can choose $r = |\Sigma_{D/K}|$ and elements $a_1, \ldots, a_r$ such that $(a_i, a_j) \equiv 1 \bmod N_{D/K}(K_2(D))$ whenever $i \neq j$.

If v is real, let $|\ |_v$ denote the real archimedean valuation associated to v. The canonical homomorphism $K^\cdot/N_{D/K}(D^\cdot) \to \prod_{v\in\Sigma_{D/K}} K_v^\cdot/|K_v^\cdot|_v$ is clearly injective and by weak approximation, it is surjective. Let $r = |\Sigma_{D/K}|$ and let $v_1, \ldots, v_r$ denote the elements of $\Sigma_{D/K}$. Choose now $a_1, \ldots, a_r$ such that for each i, a_i is negative at v_i and positive at each v_j such that $j \neq i$. Clearly, $a_1, \ldots, a_r$ generate $K^\cdot/N_{D/K}(D^\cdot)$. Fix indices i and j, $i \neq j$. Let $<_k$ denote the ordering on K associated to v_k. Choose $a \in K^\cdot$ such that $a <_i 0$, $1 <_j a$, and $0 <_k a <_k 1$ for each $k \neq i$ and j. Clearly, $a \equiv a_i \bmod N_{D/K}(D^\cdot)$ and $1 - a \equiv a_j \bmod N_{D/K}(D^\cdot)$. Thus $(a_i, a_j) \equiv (a, 1-a) = 1$.

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