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The fundamental group at infinity of affine surfaces

R. V. GURJAR and A. R. SHASTRI

§0. Introduction

The main motivation for the results of this paper is the following question, which arose in connection with the results in [G]:

(*) Suppose V is a contractible affine smooth surface/$\mathbb{C}$.

Can the fundamental group at infinity of V be a finite, nontrivial group? The analogous topological question is

(**) can a homology 3-sphere Σ with nontrivial finite fundamental group be the boundary of a smooth, contractible 4-manifold M?

An affirmative answer to (*) would have given an affirmative answer to (**), which in turn, would have given an example of a homology 3-sphere with nontrivial finite π_1 other than the Poincaré-sphere. The Poincaré homology 3-sphere is the only known example of a homology 3-sphere with nontrivial finite fundamental group. It is also known that it cannot be the boundary of a contractible smooth 4-manifold. This further motivated the study of (*).

However, the answer to (*) turned out to be negative. We do not know any answer to (**). (However, if M is not required to be smooth, the answer is yes; see [F]).

It turns out, that the only possible finite nontrivial group in (*) and (**) is the binary icosahedral group $P=\langle x, y \mid x^2=y^2=(xy)^5\rangle$, being the only nontrivial, finite perfect group that acts freely on a homotopy 3-sphere. See [M]. As in [CPR] we are led to the study of a finite connected system of nonsingular rational curves on an algebraic surface X whose dual graph is a tree. If N is a tubular neighbourhood of this system of curves, it turns out that the fundamental group at infinity of V is $\pi_1(\partial N)$. (See [CPR] or §2 for precise definition of the fundamental group at infinity). In §1 we classify all such trees with $\pi_1(\partial N)\simeq P$, under certain conditions which arise due to geometric considerations. The method of proof is purely combinatorial and closely follows that in [CPR]. As such it turns out that we need to classify trees with $\pi_1(\partial N)$ as a cyclic group of order ≤ 5 and in particular, the results of [CPR] about trees is also included. For all this we need a stronger group theoretic result than the proposition in III of [MU]. We find that the proof of this proposition as presented in [MU] is incomplete. So we have included the proof of this also in §1 (see Proposition 1).

Using the results in §1, (*) is answered negatively in §2. While this work was in progress, thanks to M. Miyanishi, we received a preprint from him in which he proves the following interesting result:

THEOREM [Miyanishi]. *Let* $\mathbb{C}^2 \xrightarrow{\varphi} V$ *be a proper morphism onto a normal, affine surface* V. *Then* $V \simeq \mathbb{C}^2/G$ *for a small, finite subgroup* G *of* $GL(2;\mathbb{C})$. *If* V *is smooth, then* $V \simeq \mathbb{C}^2$. *If the coordinate ring* $\Gamma(V)$ *is a UFD, then* V *is isomorphic to the affine surface* $X^2 + Y^3 + Z^5 = 0$.

Miyanishi has used the theory of logarithmic Kodaira dimension. As it turns out, our method for answering (*) is readily applicable for giving a topological proof of this result. This has been incorporated in §3. See also [G] for earlier partial results in this direction. Finally in §4 we give some examples of normal, affine surfaces whose fundamental group at infinity is P.

§1. Intersection trees

We shall use the terminologies of [CPR]. Consider the following geometric situation: Let X be any nonsingular, irreducible, surface/$\mathbb{C}$ and let $F \subset X$ be a Zariski closed subset of codimension one with irreducible components $C_1, \dots, C_n$ satisfying the following conditions:

(i) For each $i \neq j$ either $C_i \cap C_j = \emptyset$ or $C_i \cap C_j$ consists of a single point at which C_i and C_j intersect transversally.

(ii) For three distinct indices i, j, k, $C_i \cap C_j \cap C_k = \emptyset$.

We shall call such a pair (X, F) a normal pair.

Associated to a normal pair (X, F) is its weighted dual graph $T = T(X, F)$ defined as follows: The irreducible components $\{C_i\}$ are the vertices of T. Two vertices C_i and C_j are linked in T if and only if $C_i \cap C_j \neq \emptyset$. We express this by writing $[C_i, C_j]$ is a link in T. The weight at C_i, denoted by Ω_{C_i}, is the self intersection number of C_i i.e.

$$\Omega_{C_i} = C_i \cdot C_i$$

Here we shall recall some generalities about weighted graphs. We shall consider only finite, weighted graphs, and from now on simply refer to them as graphs, and denote them by T, T' etc. Vertices will be denoted by u, v, w etc. A vertex v of T is free if it is linked to at most one other vertex. It is linear if it is linked to at most two vertices and it is a branch point if it is linked to at least three other vertices. (Thus a free vertex is also a linear vertex).

A graph is connected if given any two vertices v and v' there exists a chain of links $[v_i; v_{i+1}]$, $i=0, \ldots, n$, such that $v=v_0$ and $v'=v_{n+1}$. A connected graph is a tree if there is no chain of links $[v_i; v_{i+1}]$, $i=1 \cdots n$, such that $v_1=v_{n+1}$. From now on we shall consider only trees, though most of the terminologies can be used for a general graph also with suitable modifications.

Let T be a tree and $v \in T$ be any vertex. By $T-\{v\}$ we mean the subgraph of T obtained by removing the vertex v and all the links at v from T, and keeping the weights unchanged. Obviously $T-\{v\}$ need not be connected. Its components are called branches of T at v. A branch $\mathfrak{S}$ of T at v is called simple if it does not have any branch points of T. An extremal branch point is a branch point at which at most one branch is not simple. Clearly a finite tree always has an extremal branch point. A tree is linear if it does not have any branch points. For instance a simple branch is necessarily a linear tree.

Associated to T is the bilinear form $B(T)$, on the real vector space spanned by the vertices $\{v_i\}$ of T as basis, defined as follows:

$$v_i \cdot v_i = \Omega_{v_i}$$

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } [v_i; v_j] \text{ is a link in } T \\ 0 & \text{otherwise for } i \neq j. \end{cases}$$

The discriminant of this form will be denoted by $d(T)$.

We say T is unimodular, or negative definite if $B(T)$ is unimodular or negative definite etc.

Clearly, if $T=T(X; F)$, is a tree of a normal pair (X, F) then $B(T)$ is the intersection form of the set of curves $\{C_i\}$ in F.

The fundamental group $\pi(T)$ of a tree T is defined as follows: Fix an indexing of the vertices arbitrarily. Let $\pi(T)$ be the quotient of the free group on $\{v_i\}$ by the relations:

(a) $[v_i, v_j]=e$ if $[v_i; v_j]$ is a link

(b) $v_{i_1} \cdots v_{i_k} \cdot v^{\Omega_v}=1$ for each vertex v, where $i_1 < \cdots < i_k$ and $\{v_{i_1}, \ldots, v_{i_k}\}$ is the set of vertices in T linked to v.

This presentation of $\pi(T)$ will be used heavily, in this section. It is easily seen that $\pi(T)$ does not depend, upto isomorphism, on the choice of indexing the vertices, and the abelianized group, $ab\pi(T)$ is of finite order if and only if

$d(T) \neq 0$ and then its order $=|d(T)|$. In particular T is unimodular if and only if $ab\pi(T)$ is trivial.

We say T is spherical or cyclic or of order $\leqslant n$ if $\pi(T)=e$ or cyclic of order $\leqslant n$ respectively.

For a normal pair (X, F) such that all the irreducible components of F are isomorphic to $\mathbb{P}^1$, if $T=T(X, F)$ is a tree then it is proved in [CPR] that $\pi(T) \simeq \pi_1(\partial N)$ where ∂N is the boundary of a small tubular neighbourhood N of F in X.

Definition of "blow-up" and "blow-down"

Let $[u; v]$ be a link in T. By "blow-up at $[u; v]$" we mean to obtain a new tree T' as follows: Introduce a new vertex w in T, delete the link $[u; v]$ and introduce links $[u; w]$ and $[w; v]$. Define the new weights Ω' by

$$\Omega'_x = \begin{cases} \Omega_x, & \text{if } x \neq u, v, w \\ \Omega_x - 1, & \text{if } x = u \text{ or } v \\ -1 & \text{if } x = w. \end{cases}$$

Let now v be a free vertex in T. By "blow-up at v" we mean to obtain a new tree T' as follows: Introduce a new vertex w and a new link $[v; w]$, and define the new weights Ω' by

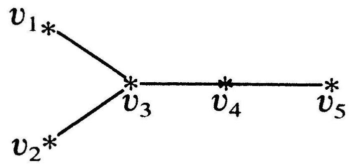
$$\Omega'_x = \begin{cases} \Omega_x, & \text{if } x \neq v, w \\ \Omega_v - 1, & \text{if } x = v \\ -1 & \text{if } x = w. \end{cases}$$

"Blow-down" is described precisely as the inverse process of blow-up and as such, we need to have a linear vertex w with $\Omega_w = -1$ to perform the blow-down, on a given tree T.

We say two trees are equivalent if there is a finite chain of blow-ups and blow-downs to obtain one tree from the other. A tree T is minimal if it has no linear (or free) vertex v with $\Omega_v = -1$. Every (finite) tree is equivalent to a minimal one (which may be an empty one). It is easily seen that $\pi(T)$ is an invariant of this equivalence relation. If (X, F) is a normal pair with all the irreducible curves in F being nonsingular and rational, the blow-up and blow-down operations on $T=T(X, F)$ precisely correspond to the geometric "blow-up" and "blow-down" on (X, F). In particular, if T' is equivalent to $T=T(X, F)$, then there is another normal pair (X', F') with $F'=T(X', F')$ and $X-F \simeq X'-F'$ as varieties. Finally, if T' is obtained by blowing-up T once, then $B(T') \simeq B(T) \oplus (-1)$.

Remark. In [MU] it is proved that a nonempty, negative definite, spherical tree cannot be minimal.

DEFINITIONS. We say T satisfies the hypothesis (E) if every positive semidefinite subspace W of $B(T)$ is of real dimension ≤ 1. We say T satisfies the hypothesis (H) if no tree equivalent to T has a subtree of the form

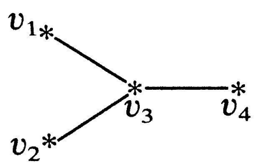


with $\Omega_{v_3} = -1$ and $\Omega_{v_5} \geq 0$.

Remarks. (a) It is clear that if T satisfies (E) then every subtree of T also satisfies (E), and every tree equivalent to T also satisfies (E). Further, there can be at most two vertices with nonnegative weights and if there are two of them then these two vertices should be linked, and one of the weights should be zero.

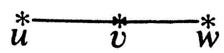
(b) If $T = T(X, F)$ where (X, F) is a normal pair obtained by resolving a normal singularity p of a surface V, then it is known that T is negative definite. On the other hand if V is a nonsingular affine surface and $V \subset X$ is a projective imbedding with X non-singular, so that (X, F) is a normal pair, where $F = X - V = \bigcup_{i=1}^{n} C_i$ and the irreducible curves C_i are linearly independent in the Neron-Severi vector space, then $T = T(X, F)$ has exactly one positive eigen-value. Thus in both the above geometric siutations $T = T(X, F)$ satisfies (E).

(c) If T satisfies (H), then it does not contain a subtree of the form



with $\Omega_{v_3} = 0$ and $\Omega_{v_4} > 0$. In particular if $T = T(X, F)$ and $H^1(X, \mathcal{O}_X) = 0$, then T satisfies (H). (See [CPR] Lemma 6).

LEMMA 1. *Suppose a tree T has a subtree of the form*



where v is a linear vertex in T, with $\Omega_v = 0$. Then T is equivalent to a tree with the same number of vertices and links and only the weights at u and w changed to $\Omega_u + 1$ and $\Omega_w - 1$ respectively.

Proof. "Blow-up" $[v; w]$ to obtain $\underset{u}{*}\text{---}\underset{v_1}{*}\text{---}\underset{v_2}{*}\text{---}\underset{w}{*}$ with weights $\Omega_u, -1, -1$, and $\Omega_w - 1$ respectively. Now blow-down the vertex v_1.

LEMMA 2. *Let T be a minimal tree with a linear subtree $\mathfrak{S} = \underset{u_1}{*}\text{---}*\text{----}\underset{u_r}{*}$ with a nonnegative weight, $r \geq 2$, and u_i being linear in T for $i \geq 2$. Assume that u_r is either free or is joined to a branch point w in T. Then T is equivalent to a minimal tree T' obtained by replacing $\mathfrak{S}$ by a linear tree $\mathfrak{S} = \underset{v_1}{*}\text{---}*\text{-----}*\text{---}\underset{v_s}{*}$ with the weight at $v_1 \geq 0$, and perhaps the weight at w being altered.*

Proof. If $\Omega_{u_1} \geq 0$ there is nothing to prove. By induction we can assume $\Omega_{u_i} \leq -2$, $i < k$, $\Omega_{u_k} \geq 0$. Blow up on the right of u_k successively, till the weight at u_k becomes 0. Using Lemma 1, make the weight at $u_{k-1} = 0$. In this process we may have introduced certain vertices on the right of u_k with weight -1. Blow down as many times as possible, to obtain a minimal tree. This of course does not change the weight at u_{k-1} and so we can use Lemma 1 repeatedly, to complete the proof.

LEMMA 3. *Let T be a minimal tree with a branch point v. Let $\mathfrak{S}$ be a simple branch at v, with some nonnegative weights. Then T is equivalent to a minimal tree with $\mathfrak{S}$ replaced by a simple branch $\mathfrak{S}'$ with the free vertex having weight 0 and the weight at v possibly being changed.*

Proof. By Lemma 2 we can assume that the free vertex u of T in $\mathfrak{S}$ has weight ≥ 0. If it is zero there is nothing more to prove. Suppose it is > 0.

$$\mathfrak{S} = \underset{u}{*}\text{------}*\text{-------}$$

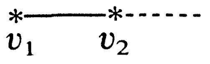
Blow up successively at the right of u till the weight at u has become 0. We now have

$$\underset{u}{*}\text{------}\underset{u_1}{*}\text{------}*\text{-------}$$

with $\Omega_u = 0$, $\Omega_{u_1} = -1$. Blow up at the free end at u to obtain

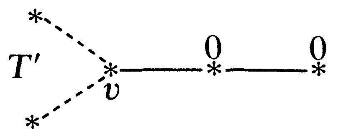
$$\underset{u'}{*}\text{------}\underset{u}{*}\text{------}\underset{u_1}{*}\text{------}$$

with weights -1, -1, and -1. Now blow down u to obtain



with weights at $v_i = 0$, $i = 1, 2$. By blowing down as many times as needed on the right of v_2 we can now obtain a minimal tree with the free vertex v_1 having weight 0. This completes the proof of the lemma.

LEMMA 4. *Let T' be any tree, $v \in T'$ be some vertex. Let T be obtained by joining the tree* $\overset{0}{*}$——$\overset{0}{*}$ *to T' at v:*



Then

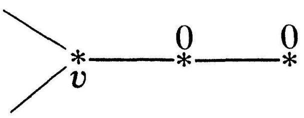
(i) $\pi(T) \simeq \pi(T')$

(ii) $B(T) \simeq B(T') \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *where* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *denotes the hyperbolic space*

(iii) *T satisfies (E) if and only if T' is negative definite.*

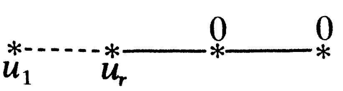
(iv) *T is minimal and satisfies (H) implies T' is minimal.*

Proof. (i) and (ii) are obvious and (iii) follows from (ii). To see (iv) we note that T' may fail to become minimal only if v is linear in T' and $\Omega_v = -1$. Since T is minimal v is not a free vertex in T'. Hence T will have a subtree of the form



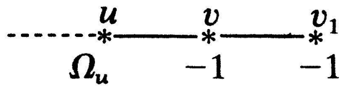
with $\Omega_v = -1$ contradicting (H).

LEMMA 5. *Let T be a minimal tree satisfying (E) and (H). Suppose T has a simple branch $\mathfrak{S}$ with nonnegative weights and $\pi(\mathfrak{S})$ is finite. Then T is equivalent to a tree T' obtained from T by replacing $\mathfrak{S}$ by a tree of the form*

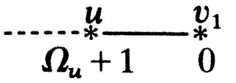


with $\Omega_{u_i} \leq -2$ and the vertex with weight zero at the right end being free in T'.

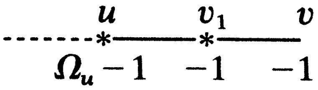
Proof. From Lemma 3 we can assume that $\mathfrak{S}$ has the free vertex v with weight zero. Since $\pi(\mathfrak{S})$ is finite it follows that $\mathfrak{S}$ is not $*_0$. Let the vertex adjacent to v in T be u. If $\Omega_u < 0$, blow up at the free vertex v to obtain



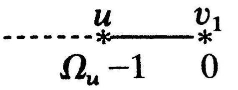
and then blow-down the vertex v, to obtain



Repeat this process till the weight at u becomes zero. On the other hand suppose $\Omega_u > 0$, then, first blow up the link $[u, v]$ to obtain

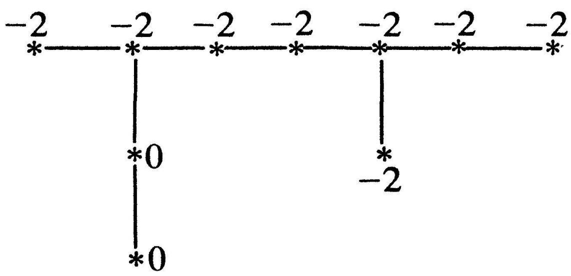


and then blow-down the free vertex v to obtain

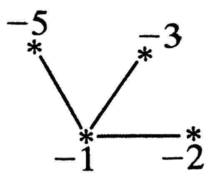


Repeat this process till the weight at u becomes zero.

Notation. By joining $\underset{0}{*}$——$\underset{0}{*}$ to E_8 at eight different vertices v_i, $i = 1, 2, \ldots, 8$ we obtain eight different trees E_8^i. e.g. E_8^2 is shown below:



We shall denote by E_4 the following tree:

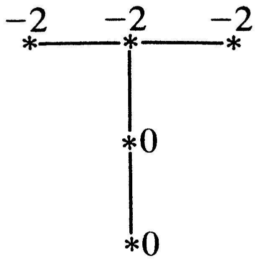


Note that $\pi(E_4) \simeq P$ and $B(E_4)$ has one positive eigen value.

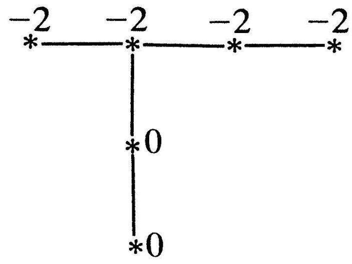
The main result of this section can be stated now.

THEOREM 1. *Let T be a minimal tree satisfying (E) and (H). Suppose $\pi(T)$ is a cyclic group of order ≤ 5 or is isomorphic to P, the binary icosahadral group. Then either T is linear or is equivalent to E_4 or E_8 or one of the E_8^i or*

to

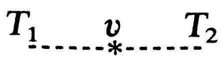


or to



We shall prove a sequence of lemmas, studying trees with increasing complexities, before proving the theorem.

LEMMA 6. *Let $\mathfrak{S}$ be a linear tree of the form*

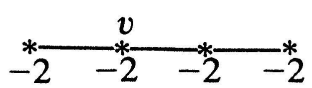


with T_i having weights ≤ -2, and $T_i \neq \varnothing$ $i=1,2$. If $|d(\mathfrak{S})| \leq 5$, then $-2 \leq \Omega_v \leq 0$.

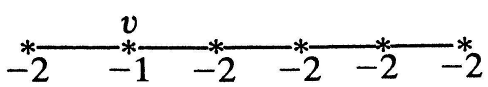
Proof. Using Lemma 2 of [CPR] it is easily seen that $d(\mathfrak{S}) = p_1p_2\Omega_v + p_1q_2 + p_2q_1$ for some positive integers p_1, p_2, q_1, q_2 such that $(p_i, q_i) = 1$, and $0 < q_i/p_i < 1$, $i = 1, 2$. Now one can easily see that $|d(\mathfrak{S})| \leq 5$ implies $-2 \leq \Omega_v \leq 0$.

LEMMA 7. *Let $\mathfrak{S}$ be a linear tree of the form*

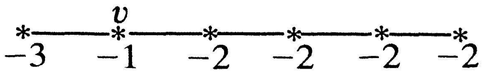
with T_i nonempty and having weights ≤ -2. Suppose $d(\mathfrak{S}) = 2, 3$ or 5. Then $\mathfrak{S}$ is one of the following trees, with $\pi(\mathfrak{S})$ isomorphic to the cyclic group of order shown in the bracket:

I. 

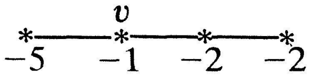
(5)

II. 

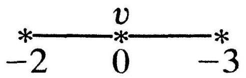
(3)

III. 

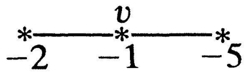
(2)

IV. 

(2)

V. 

(5)

VI. 

(3)

Proof. Use the Lemma 6 and compute directly.

For the study of trees with branch points we need a stronger version of a group theoretic result due to Mumford. Let $G_1, \ldots, G_n$ be any nontrivial groups, $a_i \in G_i$ $i = 1, \ldots, n$, be any elements. Let $\tau(G_1, \ldots, G_n)$ denote the quotient of the free $G_1 * \cdots * G_n$ by the single relation $a_1 * \cdots * a_n = e$. For $n = 3$, and $G_i \simeq \mathbb{Z}/(\lambda_i)$, and $a_i \in G_i$, the generators, $\tau(G_1, G_2, G_3)$ is denoted by $\tau(\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_i \geqslant 2$ are some integers. These are classically known as triangle groups. They are all nontrivial, noncyclic and those which are finite among them are all known. In particular, order $a_i = \lambda_i$ in $\tau(\lambda_1, \lambda_2, \lambda_3)$. These facts will be used heavily.

PROPOSITION 1. *Let $G_1, \ldots, G_n$ be any nontrivial groups, $a_i \in G_i$ be any elements. Then*

(i) *For $n \geqslant 4$, $\tau(G_1, \ldots, G_n)$ is infinite*
(ii) *$\tau(G_1, \ldots, G_n)$ is nontrivial for $n \geqslant 3$.*
(iii) *$\tau(G_1, G_2, G_3)$ is finite $\Rightarrow G_i$ are cyclic groups generated by a_i, $i = 1, 2, 3$.*

Proof. We shall repeatedly use the following basic fact which is a direct consequence of Schreier's construction of amalgamated products.

"Suppose K is a subgroup of the groups G and H. Then both G and H are subgroup of $G \overset{*}{_K} H$. If K is a proper subgroup of both G and H then $G \overset{*}{_K} H$ is infinite".

Now (i) follows from the fact that $\tau(G_1, \ldots, G_n)$ is isomorphic to the amalgamated product of $G_1 * G_2$ and $G_3 * \cdots * G_n$ over the infinite cyclic subgroups generated by $a_2^{-1} * a_1^{-1} \in G_1 * G_2$ and $a_3 * \cdots * a_n \in G_3 * \cdots * G_n$.

Assume $n = 3$. If one of the a_i is trivial then $\tau(G_1, G_2, G_3)$ is a free product and hence nontrivial. So, let $2 \leqslant$ order $a_i = \lambda_i \leqslant \infty$, $i = 1, 2, 3$.

Consider the three cyclic subgroups $(a_i) \subseteq G_i$, $i = 1, 2, 3$; and form the group

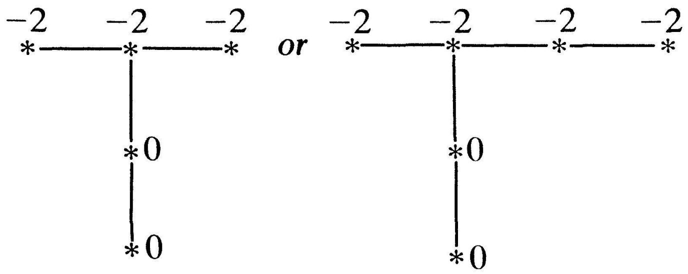
$\tau((a_1),(a_2),(a_3)) \simeq \tau(\lambda_1,\lambda_2,\lambda_3)$. Since order $a_1=\lambda_1$ in $\tau(\lambda_1,\lambda_2,\lambda_3)$, it follows that $\tau(G_1,(a_2),(a_3))$ is an amalgamated product of G_1 and $\tau((a_1),(a_2),(a_3))$ over the cyclic group (a_1). In particular $\tau((a_1),(a_2),(a_3))$ is a subgroup of $\tau(G_1,(a_2),(a_3))$ and hence order $a_2=\lambda_2$ in $\tau(G_1,(a_2),(a_3))$. As before it follows that $\tau(G_1,G_2,(a_3))$ is an amalgamated product of $\tau(G_1,(a_2),(a_3))$ and G_2, and similarly, $\tau(G_1,G_2,G_3)$ is an amalgamated product of $\tau(G_1,G_2,(a_3))$ and G_3. Thus we have

$$\tau(\lambda_1,\lambda_2,\lambda_3)=\tau((a_1),(a_2),(a_3))\subseteq\tau(G_1,(a_2),(a_3))\subseteq\tau(G_1,G_2,(a_3))\subseteq\tau(G_1,G_2,G_3)$$

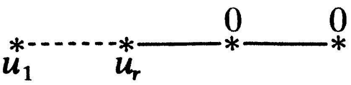
and hence $\tau(G_1,G_2,G_3)$ is nontrivial. Finally, if $\tau(G_1,G_2,G_3)$ is finite, then all the groups in above sequence are finite. Since $\tau((a_1),(a_2),(a_3))$ is not cyclic, (a_1) is a proper subgroup of $\tau((a_1),(a_2),(a_3))$. Hence $(a_1)=G_1$. Similarly $(a_2)=G_2$, and $(a_3)=G_3$.

Remark. The first and the second part of the above proposition are due to Mumford. However, we note that the proof of it as presented in III of [MU] is incomplete and needs modification.

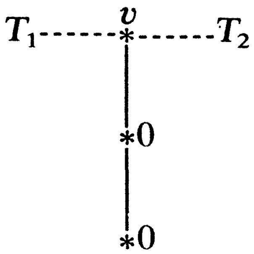
LEMMA 8. *Let T be a minimal tree with at most one branch point. Suppose T satisfied (E) and (H) and $\pi(T)$ is cyclic of order $\leqslant 5$. Then T is either linear or is equivalent to*



Proof. Let v be the branch point of T. Since T satisfies (E) it follows that at most one of the branches at v has nonnegative weights. Since a minimal linear tree with negative weights cannot be spherical at most one branch at v can be spherical. On the other hand putting $v=e$ in the presentation of $\pi(T)$, we obtain a quotient of $\pi(T)$ of the form $\tau(G_1,\ldots,G_n)$, with $G_i\simeq\pi(T_i)$ where T_i are the branches of T at v. Since $\pi(T)$ is finite cyclic, using the Proposition 1, we conclude that except possibly for two, say G_1 and G_2, all the G_i are trivial, $i\geqslant 3$. From the above observation it now follows that $n=3$. In particular, T_3 is the spherical branch at v, and carries some nonnegative weights. By Lemma 5, we can assume that T_3 is of the form



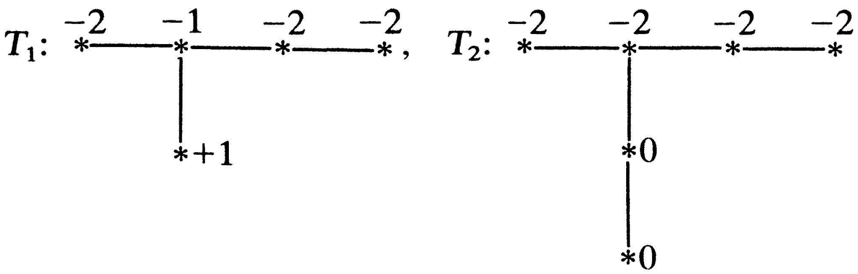
with $[v; u_1]$ being a link in T. But T_3 is spherical implies $r=0$, and hence T has the form



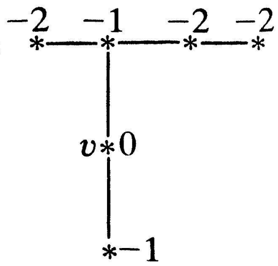
Then $\pi(T)\simeq\pi(\mathfrak{S})$ where $\mathfrak{S}$ is the horizontal linear subtree above. Lemma 7 now shows that $\mathfrak{S} = \overset{-2}{*}\!-\!\overset{-2}{*}\!-\!\overset{-2}{*}$ or $\overset{-2}{*}\!-\!\overset{-2}{*}\!-\!\overset{-2}{*}\!-\!\overset{-2}{*}$. Hence the result.

Remarks. (a) The argument used in the above lemma is very typical and occurs repeatedly in what follows; viz., constructing the quotient of $\pi(T)$ by putting a branch point $v=e$. The basic fact we use about P is that the only nontrivial quotient of P is $\tau(2,3,5)$ which is isomprohic to A_5. We shall be much brief, in using the above argument, in what follows.

(b) The following two trees are equivalent

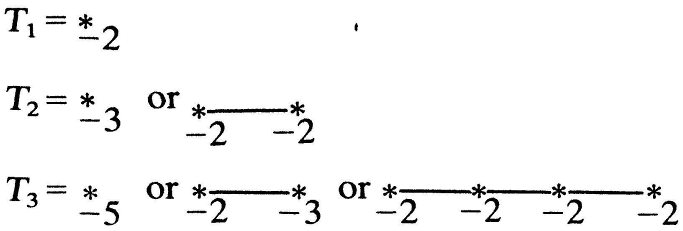


For, blow up T_1 at the free vertex with weight $+1$ to obtain and use Lemma 1 with weight 0 at v.

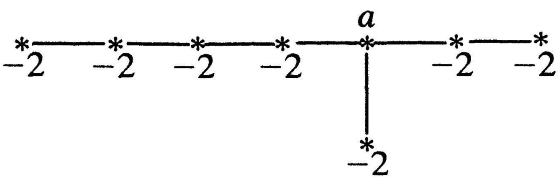


LEMMA 9. *Let T be a tree with a single branch point v and weights on each branch at $v \leqslant -2$. Suppose $\pi(T)\simeq P$. Then T is either E_4 or E_8.*

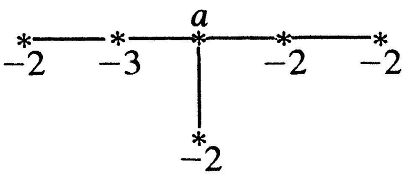
Proof. Here we use the fact that the only nontrivial quotient of P is $\tau(2,3,5) \simeq A_5$. Thus putting $v = e$ in $\pi(T)$ it follows that there are exactly three branches at v, say T_1, T_2 and T_3, with $\pi(T_i)$ of order 2, 3 and 5 respectively $i = 1, 2, 3$. (Since T_i have weights ≤ -2, $\pi(T_i)$ are nontrivial finite cyclic groups). Thus the possible choices for T_i can be listed as follows:



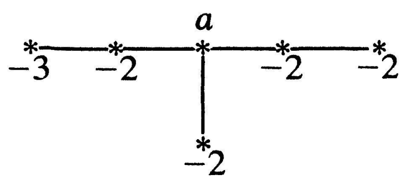
Taking different choices for T_i and joining them at v, we obtain different choices for T. Since $\underset{-2}{*}\!-\!\!-\underset{-3}{*}$ can be joined essentially in two different ways, we obtain the following eight possibilities for T. Out of these only the first and the last have discriminant ± 1, for $\Omega_v - 2$ and -1 respectively. One can directly check that these two graphs T do have $\pi(T) \simeq P$. $(a = \Omega_v)$:



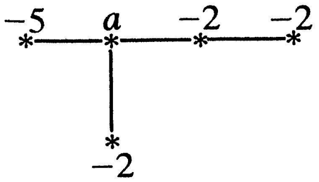
$d(T) = -(30a + 59) = 1$ if $a = -2$.



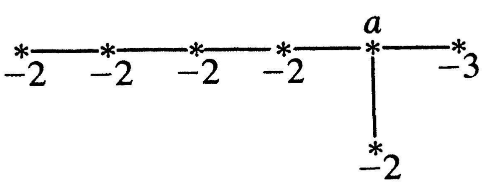
$d(T) = -(30a + 47) \neq \pm 1$



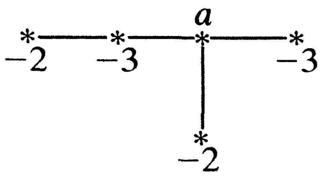
$d(T) = -(30a + 53) \neq \pm 1$



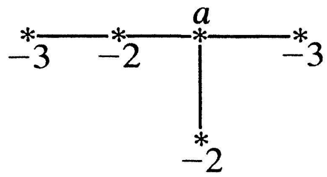
$d(T) = 30a + 41 \neq \pm 1.$



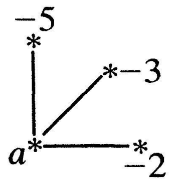
$d(T) = 30a + 49 \neq \pm 1$



$d(T)=30a+37\neq\pm1.$



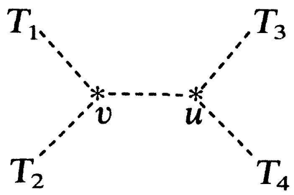
$d(T)=30a+43\neq\pm1.$



$d(T)=-(30a+31)=-1$ if $a=-1$.

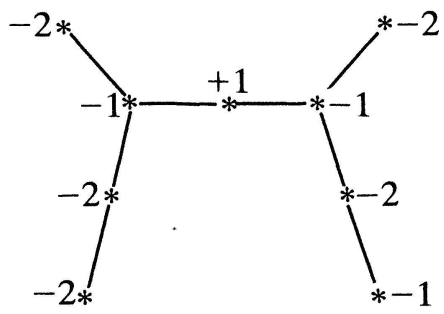
Remark. In particular, when $B(T)$ is negative definite and has exactly one branch point and $\pi(T)\simeq P$, then $T=E_8$.

LEMMA 10. *There is no minimal tree T satisfying* (E) *and* (H) *with* $\pi(T)$ *as a cyclic group of order* $\leqslant 5$ *or* $\pi(T)\simeq P$ *and* T *having the form*



with T_i *nonempty simple branches with negative weights.*

Proof. We first claim that v and u are linked. If not let $\mathfrak{S}$ be the linear subtree between v and u, $\mathfrak{S}\neq\varnothing$. Putting $v=e$ and using the Proposition 1, we conclude that the nonsimple branch T' at v is cyclic of order $\leqslant 5$. By Lemma 8, it follows that $\mathfrak{S}$ is spherical. So we can as well assume $\mathfrak{S}=\overset{+1}{*}$ by Lemma 5 of [CPR]. Arguing as above at v as well as at u, and using Lemma 8 and the Remark (b) below it we see that T is equivalent to



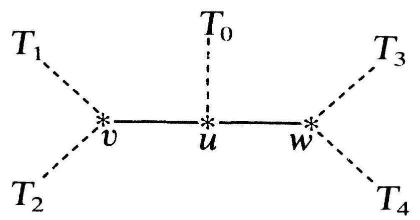
with discriminant $=-11$. This is absurd.

So u and v are linked. Let δ_i be the discriminant of T_i and Δ_1, Δ_2 denote the discriminants of $T_1 \text{-----} \overset{v}{*} \text{----} T_2 = \gamma_1$ and $T_3 \text{-----} \overset{u}{*} \text{------} T_4 = \gamma_2$ respectively. As in [CPR] one can easily see that $|d(T)| = |\Delta_1 \Delta_2 - \delta_1 \delta_2 \delta_3 \delta_4|$. Since weights on T_i are $\leqslant -2$, we have $|\delta_i| \geqslant 2$.

Consider the case when T is cyclic of order $\leqslant 5$. Putting $v = e$ (and respectively $u = e$) in $\pi(T)$, we obtain that $\mathfrak{S}_2$ (respectively $\mathfrak{S}_1$) is spherical. I.e. $|\Delta_1| = |\Delta_2| = 1$. Hence $|d(T)| \geqslant 7$ which is a contradiction.

On the other hand, when $\pi(T) \simeq P$, $|d(T)| = 1$. Hence it follows that $|\Delta_1 \Delta_2| \geqslant 7$. But, as before, $|\Delta_i| \leqslant 5$. Hence $|\Delta_i| \neq 1$. This means each of $\mathfrak{S}_i$ is cyclic of order 2, 3 or 5. Thus $\mathfrak{S}_i$ is one of the six linear trees listed in Lemma 7. This implies that at least two of the δ_i are greater than or equal to 3 in absolute value. Hence $|\delta_1 \delta_2 \delta_3 \delta_4| \geqslant 36$. This mean $|\Delta_1 \Delta_2| \geqslant 35$ which is absurd since $|\Delta_i| \leqslant 5$, $i = 1, 2$.

LEMMA 11. *There is no minimal tree T, with $\pi(T)$ of order $\leqslant 5$ or $\pi(T) \simeq P$ and T having the form*



where T_i are simple branches with negative weights.

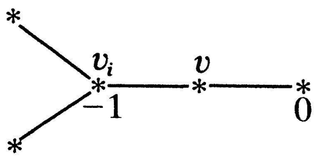
Proof. Let $\mathfrak{S}$ and $\mathfrak{S}'$ denote the nonsimple branches of T at v and w respectively. Putting $v = e$ (or $w = e$) in $\pi(T)$ we conclude that $\mathfrak{S}$ (or $\mathfrak{S}'$ respectively) is cyclic of order $\leqslant 5$. Now putting $w = e$ (or $v = e$) in $\pi(\mathfrak{S})$ (in $\pi(\mathfrak{S}')$ resp.) one concludes that $T_0 \text{----} \underset{u}{*}$ is spherical. Since T_0 has weights $\leqslant -2$, $\Omega_u = -1$. By Lemma 8, it follows that $\mathfrak{S}$ (respectively $\mathfrak{S}'$) is equivalent to a linear tree. Clearly, this is possible, only if all the weights on T_0 are $= -2$ and then $\mathfrak{S}$ can be blown down to $T_3 \text{---} \underset{w}{*} \text{---} T_4$ with weight at w changed to $\Omega_w + r + 1$. By Lemma 6, we have $-2 \leqslant \Omega_w + r + 1 \leqslant 0$. Similarly, we conclude that $-2 \leqslant \Omega_v + r + 1 \leqslant 0$.

By putting $u = e$ in $\pi(T)$, it is seen that both $T_1 \text{----} \underset{v}{*} \text{------} T_2$ and $T_3 \text{----} \underset{w}{*} \text{-----} T_4$ cannot have fundamental groups of order >5. So we may assume that $T_1 \text{----} \underset{v}{*} \text{----} T_2$ is of order $\leqslant 5$. Again by Lemma 6, it follows that $-2 \leqslant \Omega_v \leqslant 0$. Together with $-2 \leqslant \Omega_v + r + 1 \leqslant 0$, this implies $r = 1$ and $\Omega_v = -2$. In particular, $T_1 \text{----} \underset{v}{*} \text{----} T_2$, having at least three vertices with weights $\leqslant -2$, it is of order $\geqslant 4$. Now again purring $u = e$ in $\pi(T)$ we conclude that $T_3 \text{-----} \underset{w}{*} \text{-------} T_4$ is of order $\leqslant 3$. Hence, by Lemma 7 $\Omega_w = -1$, contradicting the earlier observation that $\Omega_w + r + 1 \leqslant 0$. This completes the proof of the lemma.

Proof of the theorem

Let k denote the number of branch points in T. We shall induct on k. Clearly if $k=0$, there is nothing to prove. So assume $k \geqslant 1$.

We first observe that at a branch point v, $*_0$ cannot occur as a branch. For if so let $T_1, \ldots, T_n$ be the other branches at v, $n \geqslant 2$, with vertices $v_1, \ldots, v_n$ linked to v. Then $\pi(T)$ is isomorphic to $\pi(T_1) * \pi(T_2) * \cdots * \pi(T_n)$ and hence $\pi(T_i) = (e)$ for $i \geqslant 2$, say. Also each T_i is negative definite. Hence, as in [MU], it follows that $\Omega_{v_i} = -1$, $i \geqslant 2$. In particular, by the minimality of T, v_i are not linear in T. Hence T has a subtree of the form



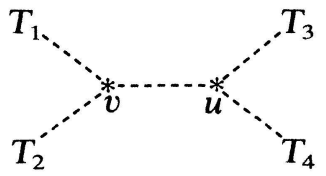
which contradicts hypothesis (H).

It is enough to show that T is E_4 or E_8 under the additional hypothesis that all simple branches of T at any branch point of T carry negative weights. For if there is a (unique!) simple branch with nonnegative weights, using Lemma 5, we see that T is equivalent to a tree $\bar{T}$ obtained by joining $\underset{0}{*}$——$\underset{0}{*}$ to a tree T' where all simple branches of T' carry negative weights. Moreover, number of branch points of $\bar{T} = k$ and hence number of branch points of $T' \leqslant k$. All the hypothesis of the theorem are satisfied by T' also. So T' is either linear, or E_4 or E_8 according to the above claim. But, clearly T' is negative definite and so it is not E_4. If it is E_8 then $\bar{T}$ is one of the E_8^i and so we are through. If T' is linear, since $\bar{T}$ should have a branch point, T' has at least three vertices. The only minimal negative definite linear trees with at least three vertices and of discriminant less than or equal to 5 in absolute value are $\overset{-2}{*}$——$\overset{-2}{*}$——$\overset{-2}{*}$ and $\overset{-2}{*}$——$\overset{-2}{*}$——$\overset{-2}{*}$——$\overset{-2}{*}$. Joining $\underset{0}{*}$——$\underset{0}{*}$ to them we get the other two possibilities for $\bar{T}$.

Thus we shall assume that all simple branches of T at any branch point have negative weights and show that T is E_4 or E_8.

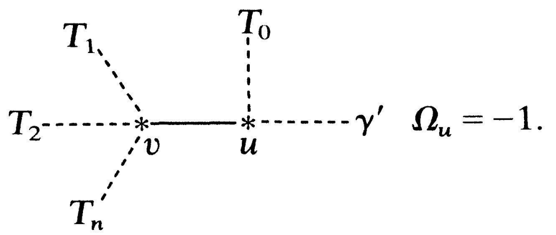
First consider $k=1$. Let v be the branch point and put $v=e$ in $\pi(T)$. Using Proposition 1, we conclude that $\pi(T)$ cannot be cyclic and so $\pi(T) \simeq P$. Lemma 9 now says that T is either E_4 or E_8.

We shall claim that there is no tree T satisfying all the conditions of the theorem with $k \geqslant 2$, by induction on k. So consider first the case $k=2$. Let u and v be the branch points of T. If possible let there be more than two simple branches, say at v. Putting $u=e$ in $\pi(T)$ we obtain the nonsimple branch $\mathfrak{S}$ at u is of order $\leqslant 5$. $\mathfrak{S}$ has a branch point v of which there are at least three simple branches carrying negative weights. Clearly $\mathfrak{S}$ is minimal and hence cannot be of order ≤ 5, by Lemma 8, a contradiction. Thus T is of the forms

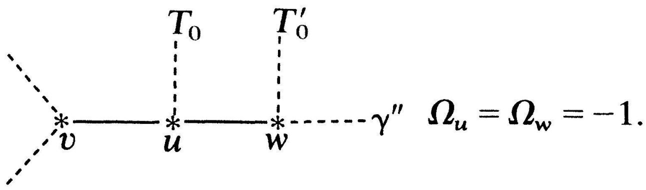


as in Lemma 10, and case $k=2$ is done.

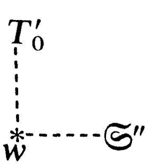
Now assume $k \geq 3$. We first claim that at an extremal branch point there are exactly two simple branches. If not let v be an extremal branch point and $T_1, \ldots, T_n$, $n \geq 3$ be the simple branches at v, $\mathfrak{S}$ be the nonsimple branch. Since all T_i have negative weights, $\pi(T_i) \neq e$ and hence by the Proposition 1, putting $v=e$ in $\pi(T)$ we conclude that $\pi(\mathfrak{S})=e$. By induction hypothesis, it follows that there is a vertex $u \in \mathfrak{S}$, linked to v in T, linear in $\mathfrak{S}$, with $\Omega_u=-1$. Further, there is exactly one simple branch T_0 and one nonsimple branch $\mathfrak{S}'$ of T at u.



Putting $u=e$ in $\pi(T)$, it now follows that $\mathfrak{S}'$ is spherical. Hence T looks like



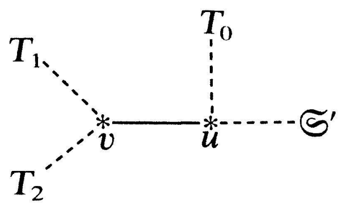
Suppose T_0 has r vertices, $r \geq 1$. Then it follows that after successive blow-downs beginning at the vertex u, the entire branch $T_0 \cup \{u\}$ of $\mathfrak{S}$ should disappear to give the tree $\mathfrak{S}'''$:



with the weight at $w = \Omega_w + r + 1 = r \geq 1$. In particular, $\mathfrak{S}'''$ is minimial. Being equivalent to $\mathfrak{S}$, it is spherical. By induction hypothesis, (and Lemma 8) $\mathfrak{S}'''$ is linear. But T_0' and $\mathfrak{S}''$ are nonempty and hence $|d(\mathfrak{S}''')/ \neq 1$. This contradiction shows that at an extremal branch point there are exactly two simple branches.

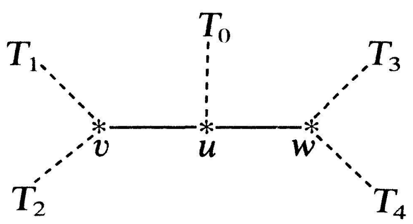
Further, let v be any extremal branch point, T_1, T_2 be the simple branches, and $\mathfrak{S}$ be the nonsimple branch at v. Then putting $v = e$ in $\pi(T)$, it follows that, since $\mathfrak{S}$ is of order $\leqslant 5$, there is a vertex u in $\mathfrak{S}$ with $\Omega_u = -1$, u is linked to v in T and u is linear in $\mathfrak{S}$.

In other words, we have: (*) At each extremal point v of T we have the following configuration for T:



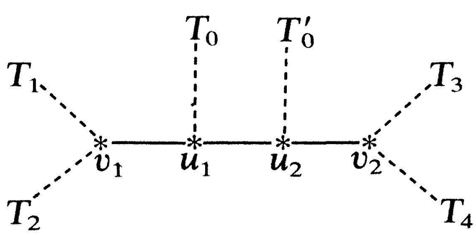
with T_i being simple, and $\Omega_u = -1$.

We shall dispose of the case $k = 3$ now. From the above observation (*) it follows that if v and w are the two extremal branch points of T, then T has the following configuration:

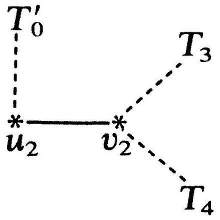


Hence we are in the situation of Lemma 11 completing the case $k = 3$.

Now assume $k \geqslant 4$. Consider the case wherein for all extremal branch points v, $\Omega_v = -1$. Let v_1 and v_2 be two distinct extremal branch points ($k \geqslant 4$). By (*) there are vertices u_1 and u_2 with $\Omega_{u_1} = -1$, and links $[v_1; u_1]$ and $[v_2; u_2]$. Since $k \geqslant 4$ it also follows from (*) that v_1 is not linked to v_2 or u_2 and v_2 is not linked to u_1. In particular $u_1 \neq u_2$. If u_1 is not linked to u_2 then it follows that $v_1 + u_1$ and $v_2 + u_2$ will span a two dimensional positive semidefinite subspace of $B(T)$ contradicting (E). Hence $[u_1; u_2]$ is a link. Thus T has the following configuration

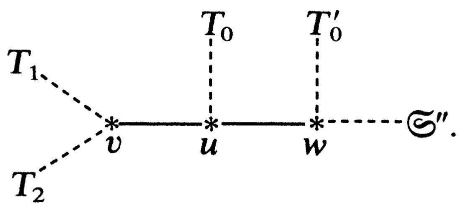


with T_i nonempty simple branches with weights ≤ -2. Let $\mathfrak{S}$ denote the nonsimple branch at v_1. Then putting $v_1 = e$ in $\pi(T)$ it follows from case $k=3$, that all the weights on T_0 are -2 and if r = number of vertices on T_0, then $\mathfrak{S}$ is equivalent to $\tilde{\mathfrak{S}}$:



with the weight at u_2 changed to $\Omega'_{u_2} = \Omega_{u_2} + r + 1 = r \geq 1$. But $\pi(\tilde{\mathfrak{S}}) \simeq \pi(\mathfrak{S})$ is of order ≤ 5 and hence putting $v_2 = e$ in $\pi(\tilde{\mathfrak{S}})$, it follows that T_0'- - - - -$\underset{u_2}{*}$ with $\Omega'_{u_2} = r \geq 1$ has to be spherical, which is absurd.

Hence there exists an extremal branch point v in T with $\Omega_v \neq -1$. In particular T_1- - - - - - -$\underset{v}{*}$- - - - - - -T_2 is not spherical. Hence putting $u = e$ in $\pi(T)$ yields that $\mathfrak{S}'$ is of order ≤ 5. By induction, it follows that there is a vertex $w \in \mathfrak{S}'$, linked to u, in T, linear in $\mathfrak{S}'$ with $\Omega_w = -1$. T looks like



with $\Omega_u = -1$, $\Omega_w = -1$, $\mathfrak{S}''$ having at least one branch point of T. Putting $v = e$, the nonsimple branch $\mathfrak{S}$ at v has to be of order ≤ 5. Since u is the only vertex which is linear and with $\Omega_u = -1$, it follows that $\mathfrak{S}$ is equivalent to a minimal tree $\mathfrak{S}_0$ obtained by successively blowing down at u. But then the weight at w will become ≥ 0 and hence $\mathfrak{S}_0$ will have at least two branch points (but fewer than k), contradicting the induction hypothesis. This completes the proof of the theorem.

§2. A generalization of C. P. Ramanujam's theorem

We will begin with the following

PROPOSITION 2. *Let V be a normal, quasi-projective, irreducible surface/$\mathbb{C}$ and $V \subset X$ with X a normal, projective surface containing V as a Zariski-dense open subset. Assume that X is smooth in a neighbourhood of $X - V$ and X is a minimal, normal compactification of V. Further assume that for a smooth, projective* *surface Y birational with X, $q(Y)=0$. Then the weighted dual graph of $X-V$ cannot be E_8^i for $i=1,\ldots,8$.*

Remark. If the dual graph of $X-V$ is E_8^i and if V is actually affine, then using a slight generalization of the Lefschetz hyperplane section theorem, we can see that actually $\pi_1(Y)=(1)$ where Y, is as above. Thus the condition $q(Y)=0$ is automatic in this case.

Proof of the Proposition. Assume that the weighted dual graph of $X-V$ is E_8^i for some i and C_1, C_2 are the non-singular rational curves with $C_1^2=0=C_2^2$ and C_1 joined to the E_8-configuration at the ith vertex.

Let $Y \xrightarrow{\sigma} X$ be a resolution of singularities such that $Y-\sigma^{-1}\{p_1,\ldots,p_r\}\to X-\{p_1,\ldots,p_r\}$ is an isomorphism, where $\{p_1,\ldots,p_r\}$ is the singular locus of X. Then we can think of the E_8^i configuration lying on Y. Thus it suffices to assume that V and hence X is smooth.

From $C_2^2+C_2\cdot K=-2$, we get $C_2\cdot K=-2$ and hence $|nK|=\emptyset$ for all $n\geq 1$. We have now $P_g(X)=0=q(X)$. By the Riemann-Roch Theorem, $\dim H^0(X,\mathcal{O}(C_2))\geq 2$ and from the exact sequence $0\to H^0(X,\mathcal{O})\to H^0(X,\mathcal{O}(C_2))\to H^0(C_2,\mathcal{O}(C_2)|_{C_2})\to 0$, it follows that $|C_2|$ has no base points. By taking a 2-dimensional subsystem of $|C_2|$ containing C_2, we get a morphism $X\xrightarrow{\varphi}\mathbb{P}^1$ which is a $\mathbb{P}^1$-fibration. C_2 is one fiber of φ and C_1 is a section of φ. Since the E_8 configuration occurring in E_8^i is connected and disjoint from C_2, the E_8 configuration is contained in a single fiber F of φ. φ is obtained from a minimal $\mathbb{P}^1$ fibration over $\mathbb{P}^1$ by successively blowing-up points. It follows that F contains at least one exceptional curve of the 1st kind. Blowing-down such a curve still gives a $\mathbb{P}^1$ fibration. The new fibration will also have a singular fiber containing an exceptional curve of the 1st kind. Blowing down this new curve also gives a $\mathbb{P}^1$-fibration, and so on until we get a minimal $\mathbb{P}^1$-fibration. Since each curve in the E_8 configuration has self-intersection-2 it can be easily seen that starting from φ the above process of blowing down exceptional curves will not yield a minimal $\mathbb{P}^1$-fibration. This contradiction shows that the dual graph of $X-V$ cannot be E_8^i.

Our next result is the following:

THEOREM 2. *Let V be an affine, irreducible, non-singular surface/$\mathbb{C}$. Assume the following conditions:*

(i) *The co-ordinate ring $\Gamma(V)$ of V is a U.F.D. and all the unit in $\Gamma(V)$ are constants.*

(ii) *for some non-singular, projective compactification $V\subset X$, $P_g(X)=0$ and*

(iii) *the fundamental group at infinity of V is finite.*

Then $V\approx\mathbb{C}^2$ as an affine variety.

COROLLARY. *Let V be a nonsingular, contractible affine surface/$\mathbb{C}$. If the fundamental group at infinity of V is finite then $V \approx \mathbb{C}^2$ as an affine variety.*

Remark. The authors do not know whether a contractible affine nonsingular surface is necessarily rational.

Proof of Theorem 2

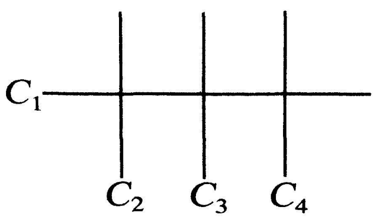
Embed $V \subset X$ where X is a nonsingular, projective surface such that the dual graph of $X - V$ is minimal and normal. $\Gamma(V)$ is a U.F.D. implies that Pic X is generated by the line bundles $[C_1], \ldots, [C_r]$, where C_i are the irreducible components of $X - V$. Also Pic X is finitely generated implies that $H^1(X, \mathcal{O}) = (0)$ (actually, it will follow soon that $\pi_1(X) = (1)$). Since $\Gamma(V)$ has no nontrivial units, Pic X is freely generated by the line bundles $[C_i]_{1 \leq i \leq r} \cdot P_g(X) = 0$ implies that $H^2(X, \mathbb{Z})$ is freely generated by the cohomology classes of the 2-cycles $C_1, \ldots, C_r$.

Let $F = X - V = \bigcup_{i=1}^r C_i$. The fundamental group at infinity of V can be found as follows. Let N be a sufficiently small tubular neighbourhood of F in X, such that F is a strong deformation retract of $\bar{N}$ and $\bar{N}$ is a strong deformation retract of $\bar{N} - F$, where $\bar{N}$ is the closure of N. Then $\pi_1(\partial \bar{N})$ is the fundamental group at infinity of V (see [CPR]). Since $\pi_1(\bar{N} - F)$ surjects onto $\pi_1(\bar{N})$, by the hypothesis it follows that $\pi_1(F)$ is finite. Hence each $C_i \simeq \mathbb{P}^1$ and (X, F) is a normal pair and $T = T(X, F)$ is a minimal tree. Note that the connectivity of F follows from the affineness of V.

By Poincaré duality, it follows that the intersection form $B(T)$ has determinant ± 1. Hence $ab\pi_1(T) \simeq H_1(\partial \bar{N})$ is trivial. Thus $\partial \bar{N}$ is a homology sphere of dimension 3. It follows that $\pi(T) = \pi_1(\partial \bar{N})$ is either trivial or P, the binary icosahedral group.

If $\pi_1(T) \simeq (e)$ then by [CPR] $V \simeq C^2$. We shall show that $\pi_1(T) \neq P$. So if possible, let $\pi_1(T) \simeq P$.

By Hodge index theorem it follows that $B(T)$ has exactly one positive eigen value. As seen above $H^1(X, \mathcal{O}) = 0$ and hence T satisfies (H). Hence from Theorem 1, it follows that T is equivalent to E_4 or E_8^i for some $i = 1, \ldots, 8$. The latter cases are not possible by the above Proposition 2. Hence T is equivalent to E_4 i.e. $\bigcup_{i=1}^r C_i$ has the following configuration:



with $C_1^2 = -1$, $C_2^2 = -2$, $C_3^2 = -3$ and $C_4^2 = -5$.

C_1 can be blown-down to a smooth point on a projective surface X_1. The image of C_2 in X_1 is an exceptional curve of the 1st kind, which can be blown-down to a smooth point on a smooth projective surface X_2. Here the image of C_3 is an exceptional curve of the 1st kind. Blowing-down this curve, we get a smooth projective surface X_3 in which the image of C_4 is a rational curve C with exactly one singular point p. C is defined locally at p by $Z_1^2 - Z_2^3 = 0$. Also $C^2 = 1$. Now $X_3 - C \simeq V$, so Pic X_3 is generated by $[C]$. $P_g(X_3) = 0$ and the topological Euler-characteristic of X_3 is 3. From these observations, we deduce easily that $X_3 \approx \mathbb{P}^2$ and C is a line in $\mathbb{P}^2$, a contradiction. This completes the proof of the theorem.

Proof of the Corollary

Assume that V is contractible, nonsingular and affine. It was proved in [G] that under these hypothesis $\Gamma(V)$ is a *UFD* and for any smooth compactification $V \subset X$, $P_g(X) = 0$. Clearly $\pi_1(V) = (1)$, hence $\Gamma(V)$ cannot have nontrivial units. Now the corollary follows from Theorem 2.

§3. A result of Miyanishi

THEOREM 3. (See [G] and [MI]). *Let V be a normal, affine surface/$\mathbb{C}$ and $\mathbb{C}^2 \xrightarrow{\pi} V$ be a proper morphism onto V. Then*

(i) *$V \simeq \mathbb{C}^2$ as an affine variety if V is nonsingular.*

(ii) *If $\{p_1, \ldots, p_r\}$ is the set of singular points of $V (r \geq 1)$ then $\pi_1(V - \{p_1 \cdots p_r\})$ is nontrivial.*

(iii) *$V \simeq \mathbb{C}^2/G$, where G is a small finite subgroup of $GL(2, \mathbb{C})$ (acting in the obvious manner on $\mathbb{C}^2$).*

(iv) *V is isomorphic to the affine surface $X^2 + Y^2 + Z^5 = 0$ in $\mathbb{C}^3$, if $\Gamma(V)$ is a UFD (and V is singular).*

Proof of (i). Assume V is nonsingular. Under these hypothesis it is proved in [G] that V is contractible, Since $\pi : \mathbb{C}^2 \to V$ is a proper morphism, the fundamental group at infinity of V is finite. Now appeal to the above corollary to conclude that $V \simeq \mathbb{C}^2$.

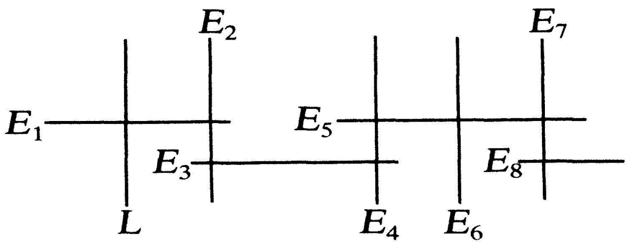
Proof of (ii). So, if possible let $V' = V - \{p_1 \cdots p_r\}$, $r \geq 1$, be simply connected. Since Pic $(\mathbb{C}^2 - \pi^{-1}\{p_1 \cdots p_r\})$ is trivial, it follows that Pic V' is finite. Any nontrivial torsion line bundle on V' defines a nontrivial unramified cover of V'. Since V' is simply connected it follows that Pic V' is trivial. This implies $\Gamma(V)$ is a *UFD.*

It is proved in [G] that V is contractible. Let U_i be a small neighbourhood of p_i in V, $i=1\cdots r$. Let $U=\bigcup_{i=1}^r U_i$. Then $V=V'\cup U$, and using the Meyer–Vietoris sequence for the couple $\{V', U\}$ it is easily seen that $H_1(V')=0=H_2(V')$, and $H_1(U-\{p_1\cdots p_r\})=0$. Hence $H_1(U_i-\{p_i\})=0$. As before $\pi_1(U_i-\{p_i\})$ are finite and nontrivial (since π is a finite proper map). Thus it follows that $\pi_1(U_i-\{p_i\})\simeq P$. It is also known that under these circumstances, the singularities p_i are locally defined by $x^2+y^3+z^5=0$ and the weighted dual graph of the minimal resolution of singularity at p_i is E_8.

Let $V\subset X$ be a normal projective, compactification such that X is smooth outside $p_1,\ldots,p_r$, and $X-V$ has minimal, normal dual graph. Let $\Psi: Y\to X$ be a minimal resolution of singularities of X, $F_i=\Psi^{-1}(p_i) i\leq r$, $F_{r+1}=\Psi^{-1}(X-V)$ and $F=\bigcup_{i=1}^{r+1}F_i$. Since $\Psi: Y-F\xrightarrow{\approx} V'$ is an isomorphism, $H_1(Y-F)=H_2(Y-F)=0$. By Lefschetz duality $H^3(Y,F)=0=H^2(Y,F)$. Hence by the cohomology exact sequence of (Y,F) it follows that $H^2(Y)\to H^2(F)$ is an isomorphism. In particular, the intersection matrix of the curves in F is unimodular. Since each F_i is a connected component of F, it follows that the intersection matrix of the curves in F_i is unimodular, for each i. Also by Hodge index theorem it follows that the intersection of F_{r+1} has exactly one positive eigen value. Further, it follows that the fundamental group at infinity of V is $\pi_1(\partial N)$, for a sufficiently nice neighbourhood N of F_{r+1}, and $\pi_1(\partial N)\simeq(e)$ or P. If $\pi_1(\partial N)\simeq(e)$ then using the result of [CPR] (viz. the proposition and Lemma 5), we can assume that $F_{r+1}\simeq\mathbb{P}^1$ with self intersection $F_{r+1}^2=1$. Using the fact that $P_g(Y)=0=q(Y)$ and using Riemann–Roch theorem, we see easily that the rational map given by the linear system $|F_{r+1}|$ on Y gives an imbedding of $Y\subset\mathbb{P}^2$ such that F_{r+1} is a line. Then $Y-F$ is $\mathbb{C}^2$ which means V is nonsingular.

Now let $\pi_1(\partial N)\simeq P$. By Theorem 1, the weighted dual graph of F_{r+1} can be assumed to be E_4 or $E_8^{(i)}$ for some $i=1\cdots 8$. By the Proposition 2, $E_8^{(i)}$ are ruled out. Thus we can assume that the weighted dual graph of F_{r+1} is E_4 and as in the proof of Theorem 2, by successive "blowing-down" at F_{r+1} we obtain a smooth surface Y' containing a rational curve C with $C^2=1$, with a unique singular point $q\in C$, such that C has local equation $z_1^2-z_2^3=0$ at q. Also $Y'-C\simeq Y-F_{r+1}$. As before, we see that the linear system $|C|$ has dimension at least 2 (i.e. $\dim H^0(Y',\mathcal{O}(C))\geq 2$). Take a 2-dimensional linear subsystem $\mathcal{L}\subset|C|$ containing C. Since C is irreducible, $C^2=1$, $\mathcal{L}$ has a unique base point which is a simple point of every member of $\mathcal{L}$. Blow-up this base point to get a projective surface $\tilde{Y}$. Let E be the new exceptional curve and $\tilde{C}$ be the proper transform of C so that $\tilde{C}^2=0$. Using $\mathcal{L}$, we get a morphism $\tilde{Y}\xrightarrow{\varphi}\mathbb{P}^1$. φ is an elliptic fibration and $\tilde{C}$ is a (scheme theoretic) singular fiber. Since $E\cdot\tilde{C}=1$, E is a section of φ. Since $\tilde{Y}-(\tilde{C}\cup E)\simeq Y-F_{r+1}$, we can treat F_i as systems of curves on $\tilde{Y}(1\leq i\leq r)$. Let $S_1,\ldots,S_l$ be the singular fibers of φ other than $\tilde{C}$. Then it follows that each F_i (for $1\leq i\leq r$) is contained insome S_j (& hence $l\geq 1$).

Now $\chi_{\text{top}}(\tilde{Y})=4+8r$, because $H^2(\tilde{Y})$ is freely generated by $\tilde{C}$, E and the $8r$ irreducible curves in $\bigcup_{i=1}^r F_i$. Also one has the formula, $\chi_{\text{top}}(\tilde{Y})=\sum_{j=1}^l \chi_{\text{top}}(S_j)+\chi_{\text{top}}(\tilde{C})$. If one of the singular fiber S_j contains some of the $F_1,\dots,F_r$, say s of them, then from the list of singular fibres of φ given by Kodaira in [K] it follows that S_j should have at least one more curve so that $\chi_{\text{top}}(S_j)\geqslant 8\cdot s+2$. Since $\chi_{\text{top}}(\tilde{C})=2$, the equality $4+8r=\sum_{j=1}^l \chi_{\text{top}}(S_j)+\chi_{\text{top}}(\tilde{C})$ shows that there is exactly one singular fiber S_1 (other than $\tilde{C}$) and all F_i, $1\leqslant i\leqslant r$, are contained in S_1; and there is exactly one more curve L in S_1 other than $\bigcup_{i=1}^r F_i$, i.e. $S_1=\bigcup_{i=1}^r F_i\cup L$. Since S_1 is connected, L should meet each F_i transversally. Again looking at Kodaira's list of possible fibers of φ, it is easily inferred that $r=1$ and S_1 has the following configuration:



with each curve having self-intersection -2. Let $\varphi(\tilde{C})=p\in\mathbb{P}^1$, $\varphi(S_1)=q\in\mathbb{P}^1$, then clearly for any small neighbourhood U_ε of q in $\mathbb{P}^1$, $\varphi^{-1}(U_\varepsilon)$ is a strong deformation retract of $\tilde{Y}-\tilde{C}$. Since E is a section, $\varphi^{-1}(U_\varepsilon)-E$ is also a strong deformation retract of $\tilde{Y}-(\tilde{C}\cup E)$. One can choose U_ε such that $\varphi^{-1}(U_\varepsilon)=U_1\cup U_2$ where U_1 is a tubular neighbourhood of L and U_2 is a tubular neighbourhood of F_1. Also it is easily arranged that $U_2\cap E=\emptyset$, and $U_1\cap U_2$ is a strong deformation retract of U_1-E. Hence it follows that U_2 is a strong deformation retract of $\varphi^{-1}(U_\varepsilon)-E$. Hence $\pi_1(\tilde{Y}-(\tilde{C}\cup E\cup F_1))\simeq\pi_1(\varphi^{-1}(U_\varepsilon)-(E\cup F_1))\simeq\pi_1(U_2-F_1)\simeq P$. But $\tilde{Y}-(\tilde{C}\cup E\cup F_1)\simeq V'$ and hence is simply connected by assumption. This contradiction completes the proof of (ii).

(iii) Suppose $p_1,\dots,p_r$ are the singular points of V. Then $\mathbb{C}^2-\pi^{-1}\{p_1\cdots p_r\}\xrightarrow{\pi} V-\{p_1\cdots p_r\}$ is a proper morphism. Since $\mathbb{C}^2-\pi^{-1}\{p_1\cdots p_r\}$ is simply connected, it follows from Hopf's theorem that the fundamental group of $V-\{p_1\cdots p_r\}$ is finite. Let W' be the universal covering space of $V-\{p_1\cdots p_r\}$. The map π factors as $\mathbb{C}^2-\pi^{-1}\{p_1\cdots p_r\}\xrightarrow{\pi'} W'\to V-\{p_1\cdots p_r\}$. W' can be imbedded in a normal affine surface W such that π' extends to a proper morphism $\mathbb{C}^2\to W$ (since $W\to V$ is a finite, proper morphism and $\mathbb{C}^2$ is normal). From (ii), it follows that W is nonsingular. From (i) it follows that $W\simeq\mathbb{C}^2$. Hence the group of covering transformations G of W' extends to a group of algebraic automorphisms of W and V is the quotient.

But any finite group of automorphisms of $\mathbb{C}^2$ can be conjugated to a subgroup of $GL(2, \mathbb{C})$. It is also well-known that G can be assumed to contain no pseudoreflections.

This completes the proof of Part (iii) of Theorem 3.

(iv) Now assume that V has a singular point and $\Gamma(V)$ is a *UFD*. By Part (iii) above, $V \approx \mathbb{C}^2/G$, $G \subset GL(2, \mathbb{C})$ and G contains no pseudoreflections. Clearly the point p in V which is the image of $0 \in \mathbb{C}^2$ is the unique singular point of V. For a small neighbourhood $U \ni p$, $\pi_1(U-p)$ is finite.

Let $V \subseteq X$ be an embedding such that X is smooth in a neighbourhood of $X-V$ and X is a minimal, normal compactification. Let $Y \xrightarrow{\Psi} X$ be a minimal resolution of singularity at p. Then all topological 2-cycles on Y are algebraic and using the fact that $\Gamma(V)$ is a *UFD*, we see easily that Pic Y is freely generated by the line bundles given by the irreducible curves occurring in $\Psi^{-1}(p)$ and $\Psi^{-1}(X-V)$.

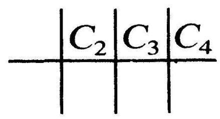
As before, we see that the dual graph of $\Psi^{-1}(p)$ is E_8 and $\pi_1(U-p)$ is isomorphic to P. From the known list of finite subgroups of $GL(2, \mathbb{C})$, we know that $G \approx P$ and $\mathbb{C}^2/G$ is the affine surface given by $X^2+Y^3+Z^5=0$ in $\mathbb{C}^3$.

This completes the proof of Theorem 3.

§4. Some examples

(1) Consider the affine normal surface V given by $X^2+Y^3+Z^5=0$. If X is a minimal, normal compactification of V, then the weighted dual graph of $X-V$ is equivalent to E_4.

For, by using the arguments before, we see that the fundamental group at infinity of V is either trivial or isomorphic to P. If it is trivial, we can get a contradiction as in the proof of part (ii) of Theorem 3. But the dual graph cannot be equivalent to $E_8^{(i)}$ for $i=1, \ldots, 8$ by the Proposition 2. Thus it is equivalent to E_4.

(2) Consider the curve $C: X^2 \cdot Z - Y^3 = 0$ in $\mathbb{P}^2$. Choose simple points $p_1, \ldots, p_8$ on C such that no three of the p_i lie on a line and no six of the p_i lie on a conic. Blowing-up $\mathbb{P}^2$ at $p_1, \ldots, p_8$ we get a nonsingular rational surface X containing the proper transform C' of C, $C'^2=1$. The map $C' \to C$ is an isomorphism. Also $X-C'$ is an affine surface V. By blowing up X at the singular point of C' and then at suitable infinitely near points on the blow-up, we get a configuration of curves C_1 

with weights as in the E_4 tree. Thus we get a smooth projective surface Y and an E_4 configuration on Y such that $Y - \bigcup_{i=1}^4 C_i$ is a nonsingular, affine surface.

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