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## Subgroups with projective abelianization and trivial multiplicator

MICHEAL DYER

### 1. Introduction

In this note we study the exact sequence

$$L \rightarrowtail G \twoheadrightarrow H$$

of groups and homomorphisms where the first homology  $H_1 L$  of  $L$  is a projective  $H$ -module and the second homology  $H_2 L$  is trivial. We call such subgroups *projective* subgroups. They arise as examples of projective  $G$ -crossed modules. See [R] for more details.

The motivating topological setting is as follows: let  $X$  be a connected subcomplex of a connected two-dimensional aspherical CW-complex  $Y$  and let  $i: X \rightarrow Y$  be the inclusion. Let  $L = \ker \{i_{\#}: \pi_1 X \rightarrow \pi_1 Y\}$  be the normal subgroup of  $G = \pi_1 X$  and  $H = \text{im } i_{\#}$ . Then  $L$  is a projective subgroup ([D], [BD]).

Here are several examples of projective subgroups. Let  $X$  be a set and  $F(X) = F$  be the free group on  $X$ . For any group  $H$ , consider  $G = F * H$ . Then setting elements of  $F$  equal to 1 gives an extension (which is split)

$$L \rightarrowtail G \twoheadrightarrow H$$

where  $L$  is the normal closure  $\langle\langle F \rangle\rangle_G$  of  $F$  in  $G$ . It follows from the Kurosh theorem [Se, Theorem 14] that  $L$  is free on  $\{hxh^{-1} \mid x \in X, h \in H\}$ , that  $H_1 L$  is a free  $H$ -module on  $X$ , and that  $H_2 L = 0$ .

As a second example, consider a 1-relator presentation  $\mathcal{P} = (X; r)$  of the group  $G$ . Let  $F = F(X)$  and  $R = \langle\langle r \rangle\rangle_F$ . If the word  $r$  is *not* a proper power in  $F$ , then  $H_1 R \approx \mathbb{Z}G$  and  $H_2 R = 0$  (because  $R$  is free). This follows because the cellular 2-complex  $K$  modeled on  $\mathcal{P}$  is aspherical [C].

Of course, if  $L = G$  is projective, we simply mean that  $H_1 L$  is free abelian and  $H_2 L = 0$ . A projective group  $L = G$  which is not a free  $G$ -crossed module would be one whose weight  $>$  rank  $H_1 L$ . Here the weight of  $G$  is the minimal number of normal generators of  $G$ .

Another example is  $H_1L = H_2L = 0$  ( $L$  is *superperfect*). Any such  $L$  is a projective subgroup.

Projective subgroups are *hereditary* in the following sense. Suppose  $L$  is a projective subgroup of  $G = K * F$ . Then  $\bar{L} = L \cap K$  is a projective subgroup of  $K$ . See section 3.

In this note we study the lower central series of projective subgroups. The main theorem states that if  $H_1L$  is a submodule of a free  $H$ -module and  $H_2L = 0$ , then each quotient  $L_n/L_{n+1}$  ( $n \geq 1$ ) of the lower central series  $\{L_n\}$  of  $L$  is the submodule of a free  $H$ -module. The proof is an extension of the proof devised by Ralph Strehel in [S]. Applications are given to the upper central series of  $G$ .

To fix notation, we let  $[a, b] = aba^{-1}b^{-1}$  and  $\mathbb{Z}G$  denote the integral group ring of the group  $G$ .

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## 2. The lower central series of $L$

For any group  $L$ , define  $L_1 = L$  and  $L_{n+1} = [L_n, L]$  ( $n \geq 1$ ). This is called the *lower central series* of  $L$ . If  $L$  is a normal subgroup of  $G$ , then conjugation by elements of  $g$  ( $l_n \rightarrow gl_n g^{-1}$ ) induces a left  $H$  ( $H = G/L$ )-module structure on each  $L_n/L_{n+1}$  ( $n \geq 1$ ). Note that  $H_1L = L_1/L_2$ . The graded object  $\text{gr } L = \{L_n/L_{n+1}\}_{n \geq 1}$  has the structure of a graded Lie-ring over  $\mathbb{Z}H$  with the Lie bracket equal to  $[ , ]$ .

If  $M$  is a left  $H$ -module, then the graded object

$$TM = \{\mathbb{Z}H, M, M \otimes M, M \otimes M \otimes M, \dots\}$$

has the structure of a graded  $\mathbb{Z}H$ -algebra with multiplication given by tensoring:  $m \cdot n = m \otimes n$ . Here  $M \otimes M$  means tensor product over  $\mathbb{Z}$  with the diagonal  $H$ -action.

We now state and prove the main result of this paper.

**MAIN THEOREM 2.1.** *Suppose  $L \rightarrowtail G \twoheadrightarrow H$  is an exact sequence of groups with  $H_1L$  a submodule of a free  $H$ -module and  $H_2L$  a torsion group. Then each successive quotient  $L_n/L_{n+1}$  of the lower central series  $L_n$  ( $1 \leq n < \omega$ ) is a submodule of a free  $H$ -module.*

*Proof.* We follow the proof of Theorem 1 of [S], p. 149, and check that at each stage the maps defined are isomorphisms of  $H$ -modules. This yields a graded Lie

$\mathbb{Z}H$ -algebra isomorphism  $\alpha : \text{gr } L \rightarrow TH_1 L$  from the graded Lie  $\mathbb{Z}H$ -algebra  $\text{gr } L$  associated with  $L$  onto the Lie  $\mathbb{Z}H$ -subalgebra of  $TH_1 L$  generated by  $H_1 L = T^1 H_1 L$ . It is clear that if  $H_1 L$  is a submodule of a free  $H$ -module  $F$ , then  $TH_1 L$  is a subalgebra of  $TF$ , with each  $T^i F = F \otimes \cdots \otimes F$  ( $i$  times) a free  $H$ -module. Hence, each  $\alpha(L_n/L_{n+1})$  is a submodule of  $T^n H_1 L$ , which in turn is a submodule of  $T^n F$ .

Let  $I = IL$  be the augmentation ideal of  $\mathbb{Z}L$  and define a descending chain of normal subgroups of  $L$  by setting

$$D^i(L) = \{l \in L \mid l - 1 \in I^i\}$$

This series is a central series, and we can form the associated graded Lie  $\mathbb{Z}H$ -algebra  $\text{gr } \{D(L)\}$ , because each  $D^i(L)$  is *normal in G*.

In order to see that  $H$  acts on  $D^i/D^{i+1}$  via conjugation by elements of  $G$ , it is enough to show that if  $l \in D^i(L)$ , then, for any  $\omega \in L$ ,  $\omega l \omega^{-1} \equiv l \pmod{D^{i+1}(L)}$ ; i.e.,  $\omega l \omega^{-1} l^{-1}$  is a member of  $D^{i+1}(L)$ . But an easy calculation shows that  $\omega l \omega^{-1} l^{-1} - 1 = (\omega - 1)(l - 1)l^{-1} - \omega(l - 1)(\omega^{-1} - 1)l^{-1}$ . So if  $l - 1 \in I^i$ , then  $\omega l \omega^{-1} l^{-1} - 1 \in I^{i+1}$  and we have verified  $H$  acts on  $\{D^i/D^{i+1}\}$  via conjugation.

The inclusion  $L_j \subseteq D^j(L)$  allows one to define a Lie  $\mathbb{Z}H$ -algebra homomorphism  $i : \text{gr } L \rightarrow \text{gr } \{D(L)\}$ .

Let  $\text{gr } \mathbb{Z}L$  denote the graded  $\mathbb{Z}H$ -algebra associated to the filtration  $\{I^i\}_{0 \leq i < \omega}$  of  $\mathbb{Z}L$ . Here  $H$  acts on  $I^i/I^{i+1}$  via conjugation by elements of  $G$ . This is well-defined because conjugation by elements of  $L$  is trivial mod  $I^{i+1}$ . The function  $g \mapsto g - 1$  then defines an *injective* Lie  $\mathbb{Z}H$ -algebra homomorphism  $\beta : \text{gr } \{D(L)\} \rightarrow \text{gr } \mathbb{Z}L$ .

Finally, we use the isomorphism  $\mu : H_1 L \approx I/I^2$  ( $l \cdot L' \mapsto (l - 1) + I^2$ ) to extend to a homomorphism  $\mu : TH_1 L \rightarrow \text{gr } \mathbb{Z}L$  of graded associative  $\mathbb{Z}H$ -algebras, given in degree  $j$  by ( $H$  acts diagonally on  $TH_1 L$ )

$$l_1 L_2 \otimes l_2 L_2 \otimes \cdots \otimes l_j L_2 \mapsto (l_1 - 1)(l_2 - 1) \cdots (l_j - 1) + I^{j+1}.$$

Clearly,  $\mu$  is always surjective; it is also injective if  $H_1 L$  is torsion free and  $H_2 L$  is a torsion group (see [S], p. 150). The Lie  $\mathbb{Z}H$ -algebra homomorphism  $\alpha : \text{gr } L \rightarrow TH_1 L$  is defined by  $\alpha = \mu^{-1} \beta i$  in the following diagram:

$$\text{gr } L \xrightarrow{i} \text{gr } \{D(L)\} \xrightarrow{\beta} \text{gr } \mathbb{Z}L \xleftarrow{\mu} TH_1 L.$$

On page 151 of [S], Strebel shows that  $\alpha$  is a monomorphism. ■

The following example shows that even if  $H_1 L$  is a free  $H$ -module and

$L \twoheadrightarrow G \twoheadrightarrow H$  is *split*, the quotients  $L_n/L_{n+1}$  are not necessarily *projective*. Let  $H = \mathbb{Z}_2 = \{e, h\}$  and  $G = \mathbb{Z} * H$ , where  $\mathbb{Z}$  generated by  $x$ . Then  $L = \langle\langle \mathbb{Z} \rangle\rangle_G$  is the free group of rank 2 with basis  $x$  and  $y = hxh^{-1}$ . We order  $x < y$ , as weight one basic commutators. The only basic commutator of weight 2 is  $c_1 = [y, x]$ . The action of  $H$  on  $L_2/L_3 \cong \mathbb{Z}$ , generated by  $\bar{c}_1$ , is non-trivial, because  $h[y, x]h^{-1} = [hyh^{-1}, hxh^{-1}] = [x, y] = [y, x]^{-1}$ . Hence  $L_2/L_3$  is not projective, but is still a submodule of  $\mathbb{Z}H$ , as guaranteed by Theorem 2.1.

It is intriguing to ask just when the  $L_n/L_{n+1}$  might themselves be projective. The next theorem gives a partial result in this direction.

A group  $H$  is said to be *ordered* if there is a linear ordering  $\leq$  on  $H$  such that if  $a \leq b$  in  $H$ , then  $ah \leq bh$  and  $ha \leq hb$  for all  $h$  in  $H$ . Note that  $1 \leq a$  in  $H$  iff  $a^{-1} \leq 1$ . For example, any torsion free nilpotent group is ordered [P, p. 581].

**THEOREM 2.2.** *Suppose  $L \twoheadrightarrow G \twoheadrightarrow H$  is a split exact sequence of groups, with  $H_2L = 0$ ,  $H_1L$  a free  $H$ -module, and  $H$  an ordered group. Then  $\text{gr } L$  is a free graded  $H$ -module.*

First, we prove the following.

**LEMMA 2.3.** *Suppose  $F = F(X)$  is a free group with basis  $X$  and  $H$  is any ordered group. Form the group  $G = F * H$  and the split exact sequence  $L \xrightarrow{i} G \xrightarrow{\varphi} H$  where  $\varphi$  is obtained by setting elements of  $F$  equal to 1 and  $L$  is the normal closure  $\langle\langle F \rangle\rangle_G$  of  $F$  in  $G$ . Then each free abelian group  $L_n/L_{n+1}$  ( $n \geq 1$ ) is a free  $H$ -module.*

*Proof.* If  $X$  is a basis for  $F$ , then  $\bar{X} = \{hxh^{-1} \mid x \in X, h \in H\}$  is a basis for the free group  $L = \langle\langle F \rangle\rangle_G$ . We order  $X$  arbitrarily and  $\bar{X}$  lexicographically according to the pairing  $(x, h)$ . We consider the basic commutators in  $\bar{X}$  (see [H], p. 166). Each basic commutator  $c_k$  of weight  $k$  is of the form (uniquely, as  $L$  is free)

$$c_k = [c_i, c_j]$$

where  $\text{wt}(c_i) + \text{wt}(c_j) = k$ ,  $c_i, c_j$  are basic commutators and  $c_i > c_j$ . If  $c_i = [c_r, c_s]$ , then  $c_j \geq c_s$ . We *order* the basic commutators of weight  $k$  lexicographically by using the pairing  $(c_i, c_j)$ .

We now will prove inductively the following: if  $c_l, c_m$  are basic commutators of weight  $k$  and  $h \in H$ , then (1)  $hc_lh^{-1}$  is a basic commutator of weight  $k$  and (2)  $c_l > c_m$  implies that  $hc_lh^{-1} > hc_mh^{-1}$ , using the lexicographic ordering given above. It is clearly true for  $k = 1$ , using the ordering on  $\bar{X}$  and that  $H$  is an ordered group. If  $c_l = [c_i, c_j]$  with  $c_i > c_j$ , then  $hc_lh^{-1} = [hc_ih^{-1}, hc_jh^{-1}]$ , with  $hc_ih^{-1} > hc_jh^{-1}$ . Also if  $c_i = [c_r, c_s]$  and  $c_i \geq c_s$ , then  $hc_ih^{-1} \geq hc_sh^{-1}$ , by induction. Finally,

we must show that if  $c_i > c_m$ , then  $hc_ih^{-1} > hc_mh^{-1}$ . Let  $c_m = [c_a, c_b]$  with  $c_a > c_b$ . If  $c_i > c_a$ , then  $hc_ih^{-1} > hc_ah^{-1}$ , while if  $c_i = c_a$  and  $c_j > c_b$ , then  $hc_jh^{-1} > hc_bh^{-1}$  is true by induction. Thus (1) and (2) are true for all basic commutators.

Now it is easy to find basic commutators of  $\text{wt } k$  which form an  $H$ -basis for  $L_k/L_{k+1}$ . These will consist of basic commutators of  $\text{wt } k$  whose first occurrence of an element of  $\bar{X}$  is actually an element of  $X$ . For example, basic commutators of weight 2 look like  $[hxh^{-1}, gyg^{-1}]$  where  $x, y \in X, g, h \in H$  and  $(x, h) > (y, g)$ . Then the set  $\{[x, hyh^{-1}] \mid x, y \in X, h \in H \text{ and } (x, 1) > (y, h)\}$  is a  $\mathbb{Z}H$ -basis because

$$\{h_1[x, hyh^{-1}]h_1^{-1} \mid h_1, h \in H, x, y \in X, (x, 1) > (y, h)\}$$

yields all weight 2 basic commutators with no repetitions. ■

*Proof of Theorem 2.2.* Let  $s: H \rightarrow G$  denote a splitting of the sequence  $L \rightarrow G \rightarrow H$ . Because  $H_1L$  is a free  $H$ -module and  $H_2L = 0$ , we may choose a subset  $X$  of  $L$  so that the image of  $X$  in  $H_1L$  is a  $\mathbb{Z}H$ -basis for  $H_1L$  and so that the corresponding  $\bar{X} = \{(sh)x(sh^{-1}) \mid L \in H, x \in X\}$  (see [HS], p. 204) is the basis for a free subgroup  $\bar{L} < L$ . Let  $F = F(X)$  be the free subgroup of  $L$  generated by  $X$ . Let  $\bar{G} = F * H$  and consider the split exact sequence and commutative diagram:

$$\begin{array}{ccccc} \bar{L} & = & \langle\langle F \rangle\rangle_{\bar{G}} & \rightarrow & \bar{G} \xrightarrow{\omega} H \\ & \downarrow i & & \downarrow \omega & \parallel \\ L & \longrightarrow & G & \xrightarrow{s} & H \end{array}$$

The map  $\omega$  is defined by the inclusion  $i: F \rightarrow L$  and the splitting  $s$ . Because  $H_1(i)$  is an isomorphism and  $H_2(i)$  is zero we have that the map

$$\bar{L}_n/\bar{L}_{n+1} \rightarrow L_n/L_{n+1}$$

is an isomorphism of  $\mathbb{Z}H$ -modules, which are free by the lemma. ■

### 3. Applications to groups

In this section we apply the main theorem about the structure of  $\text{gr } L$  to show that often the center  $\mathcal{Z}G$  of  $G$  must be “buried” inside  $L$ ; i.e.,  $\mathcal{Z}G \subset \bigcap_{n < \omega} L_n = L_\omega$ .

First we state a simple lemma about certain elements in group rings which are not zero divisors.

**LEMMA 3.1.** *Let  $H$  be a group and  $h$  be an element in  $H$ . Then  $(h - 1) \in \mathbb{Z}H$  is a zero divisor iff the order of  $h$  is finite:  $(h + 1)$  is a zero divisor in  $\mathbb{Z}H$  iff the order of  $h$  is even. If  $|n| \neq 1$ , then  $(h - n)$  is never a zero divisor in  $\mathbb{Z}H$ . ■*

**THEOREM 3.2.** *Let  $\rightarrowtail G \twoheadrightarrow H$  be an exact sequence of groups with  $H$  torsion free,  $H_1L$  isomorphic to a submodule of a free  $H$ -module and  $H_2L=0$ . Let  $g \in G-L$ . Then any  $l \in L$  which commutes with  $g$  must live in  $L_\omega = \bigcap L_n$ .*

*Proof.* Let the abelianization  $L \rightarrow H_1L$  be denoted by  $l \mapsto \bar{l}$  and  $\varphi(g) = \hat{g}$  ( $l \in L, g \in G$ ). Then  $glg^{-1}l^{-1} = 1 \Rightarrow \hat{g}\bar{l} - \bar{l} = 0$  in  $H_1L \Rightarrow (\hat{g}-1)\bar{l} = 0$  in  $H_1L \subset \bigoplus_{i \in I} \mathbb{Z}H_i$ . Write  $\bar{l} = (\bar{l}_i)_{i \in I}$ , where each  $\bar{l}_i \in \mathbb{Z}H$ . Thus  $((\hat{g}-1)\bar{l}_i) = 0$  and it follows from the lemma that each  $\bar{l}_i = 0$ ; hence  $\bar{l} = 0$ . So  $l \in L_2$ . We use Theorem 2.1 and a similar argument to show that  $l \in L_n$  for all  $n \geq 2$ . ■

**Note 3.3.** (a) A similar argument will show that if  $g \in G-L$  and  $l \in L$  satisfies  $glg^{-1} = l^n$  ( $n \in \mathbb{Z}$ ), then  $l \in L_\omega$ .

(b) Other identities may be used. For example, if  $g_1, g_2 \in G-L$  and  $l \in L$  with  $[[g_1, l], g_2] = 1$ , then  $l \in L_\omega$ . This follows because  $\overline{[[g_1, l], g_2]} = (1 - \hat{g}_2)[\bar{g}_1, \bar{l}] = (1 - \hat{g}_2)(\hat{g}_1 - 1)\bar{l} = 0$ . Then apply the argument twice.

Recall that, if  $G$  is a group, then the  $n$ th order center of  $G$ ,  $\mathcal{Z}^nG$  ( $n \geq 1$ ), is inductively defined as  $\mathcal{Z}^1G = \text{center of } G$ ,  $\mathcal{Z}^{n+1}G = \{g \in G \mid \varphi_n(g) \in \mathcal{Z}^1(G/\mathcal{Z}^nG)$ , where  $\varphi : G \rightarrow G/\mathcal{Z}^nG$  is the natural map}.

**COROLLARY 3.4.** *Suppose that the exact sequence of groups is as in Theorem 3.2 with  $G-L \neq \emptyset$  and  $H_1L \neq 0$ . Then all the centers  $\mathcal{Z}^nG$  ( $n \geq 1$ ) are contained in  $L_\omega$ .*

*Proof.* (a) We show that  $\mathcal{Z}^1G \subset L_\omega$ . Suppose there is an element  $g \in (G-L) \cap \mathcal{Z}^1G$ . Choose any  $l \in L$  and observe that  $[g, l] = 1$ . By Theorem 3.2, we see that  $l \in L_\omega$ . Hence  $L = L_\omega$ , which contradicts the hypothesis that  $H_1L \neq 0$ . Hence  $\mathcal{Z}G \subset L$ .

Now choose any  $g \in G-L$  and  $l \in \mathcal{Z}G \subset L$ . Again, the proposition shows that  $l \in L_\omega$ . Hence  $\mathcal{Z}^1G \subset L_\omega$ .

(b) We observe that  $g \in \mathcal{Z}^2G$  iff for all  $g_1, g_2 \in G$  the commutator  $[[g_1, g], g_2] = 1$ . Now suppose  $g \in G-L \cap \mathcal{Z}^2G$ . Choose any  $l \in L$  and observe that  $[[g, l], g_2] = 1$  for all  $g_2 \in G$ . Choosing  $g_2 \in G-L$ , we see that  $l \in L_\omega$  for all  $l \in L$  by note (b). The rest of the argument is similar to (a).

(c) One can either prove  $\mathcal{Z}^nG \subset L$  for  $n \geq 2$  by induction or by studying the higher commutators  $[\dots, [[g_1, l], g_2], \dots, g_n]$ . ■

**Note 3.5.** Corollary 3.4 also follows without using 3.2 because  $L/L_\omega$  is a non-abelian free group (under the assumption imposed above) and in that case  $\mathcal{Z}^nG \subseteq \mathcal{Z}^nL \subseteq L_\omega$ .

**Note 3.6.** The above corollary is false if  $H_1L = 0$  or if  $G = L$ . Let  $L$  be a

superperfect group, let  $G = \mathbb{Z} \times L$  and  $G \rightarrow \mathbb{Z}$  be the projection with kernel  $L$ . Then  $\mathbb{Z} \subset \mathcal{Z}G$  is not contained in  $L$ . Also, if  $G = L = \mathbb{Z}$ , then  $\mathcal{Z}G = \mathbb{Z}$  is not contained in  $L_\omega = 0$ .

In order to give the next application, we need the notion of a  $C$ -subgroup. We say that a subgroup  $N < G$  is a  $C$ -subgroup if there is a  $G$ -projective resolution  $P_* \rightarrow \mathbb{Z}$  of the trivial module  $\mathbb{Z}$  for which the homomorphism  $N \otimes \partial_3 : \mathbb{Z} \otimes_N P_3 \rightarrow \mathbb{Z} \otimes_N P_2$  is trivial. See [BDS] for properties and applications of this concept.

The next result shows that, in some sense,  $H$  is close to being a two-dimensional group, the “closeness” being measured by  $H_2 L$ .

**THEOREM 3.7.** *Let  $L \rightarrow G \rightarrow H$  be an exact sequence of groups with  $H_1 L$  a projective  $H$ -module,  $H_2 L = 0$  and  $L$  a  $C$ -subgroup. Then the cohomological dimension of  $H$  is  $\leq 2$ .*

*Proof.* Let  $P_* \rightarrow \mathbb{Z}$  be the resolution assured by  $L$  being a  $C$ -subgroup of  $G$ . By tensoring  $P_*$  with  $\mathbb{Z} \otimes_L -$ , using that  $H_1 L$  is projective as an  $H$ -module and that  $L$  is a  $C$ -subgroup, we obtain an exact sequence of  $H$ -modules

$$H_2 L \rightarrow \mathbb{Z} \otimes_L P_2 \oplus H_1 L \rightarrow \mathbb{Z} \otimes_L P_1 \rightarrow \mathbb{Z} \otimes_L P_0 \rightarrow \mathbb{Z}$$

with the inner three terms projective. The hypothesis that  $H_2 L = 0$  does the rest. ■

The next result yields new information about the aspherical question of J. H. C. Whitehead ([BD] and [BDS]): Are subcomplexes of aspherical 2-complexes aspherical?

**PROPOSITION 3.8.** *Suppose  $G = K * N$  is a free product of groups  $K$  and  $N$  and that  $L$  is a projective subgroup of  $G$ . Then  $\bar{L} = K \cap L$  is a projective subgroup of  $K$ .*

*Proof.* We observe that  $\bar{L}$  is a normal subgroup of  $K$ . It follows from the Kurosh structure theorem about subgroups of a free product [Se, Theorem 14] that  $\bar{L}$  is a free summand of  $L = \bar{L} * M$ . It is easy to check that, if  $\bar{H} = K/\bar{L}$  ( $\bar{H} < G/L$ ), then  $H_1 L$  is isomorphic to  $H_1 \bar{L} \oplus H_1 M$  as  $\bar{H}$ -modules. Because  $H_1 L$  is a projective  $G/L$ -module, it is a projective  $\bar{H}$ -module. We have  $H_2 \bar{L} = 0$  because  $H_2 L \cong H_2 \bar{L} \oplus H_2 M = 0$ . Thus  $\bar{L}$  is projective. ■

A  $[G, 2]$ -complex  $X$  is a connected, 2-dimensional CW-complex with fundamental group isomorphic to  $G$  and a single zero cell. If  $X$  is a  $[G, 2]$ -complex which is a subcomplex of an aspherical  $[K, 2]$ -complex  $Y$ , we let  $\bar{X} = X \cup Y^1$  be

the union of  $X$  together with the 1-skeleton of  $Y$  and  $i:X \hookrightarrow Y$ , and  $i_1:\bar{X} \hookrightarrow Y$  denote the inclusion maps. Furthermore, let  $L = \ker \pi_1(i_1)$  and  $\bar{L} = \ker \pi_1(i)$ . Then the fundamental group  $\pi_1(\bar{X}) \cong G * F$ , where  $F$  is a free group whose rank corresponds to the number of 1-cells of  $Y$  outside of  $X$ . It is well known (see [BD]) that  $L \rightarrow G * F \rightarrow \text{im } \pi_1(i_1) = H$  is a free  $G * F$ -crossed module. Thus  $H_1 L$  is a free  $H$ -module and  $H_2 L = 0$ . Now  $\bar{L} = \ker \pi_1(i) = L \cap G$  is a projective  $\bar{H} = G/\bar{L}$ -module by the previous proposition (see [D]). Furthermore (and this seems to be new), the sequence

$$\hat{L} = F \cap L \rightarrow F \rightarrow F/F \cap L = \hat{H} < H$$

is also a projective subgroup of the free group  $F$ , with  $\hat{H}$  a 2-dimensional group.

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