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Subgroups with projective abelianization and trivial multiplier

MICHEAL DYER

1. Introduction

In this note we study the exact sequence

$$L \twoheadrightarrow G \twoheadrightarrow H$$

of groups and homomorphisms where the first homology H_1L of L is a projective H -module and the second homology H_2L is trivial. We call such subgroups *projective* subgroups. They arise as examples of projective G -crossed modules. See [R] for more details.

The motivating topological setting is as follows: let X be a connected subcomplex of a connected two-dimensional aspherical CW-complex Y and let $i: X \rightarrow Y$ be the inclusion. Let $L = \ker \{i_\#: \pi_1 X \rightarrow \pi_1 Y\}$ be the normal subgroup of $G = \pi_1 X$ and $H = \text{im } i_\#$. Then L is a projective subgroup ([D], [BD]).

Here are several examples of projective subgroups. Let X be a set and $F(X) = F$ be the free group on X . For any group H , consider $G = F * H$. Then setting elements of F equal to 1 gives an extension (which is split)

$$L \twoheadrightarrow G \twoheadrightarrow H$$

where L is the normal closure $\langle\langle F \rangle\rangle_G$ of F in G . It follows from the Kurosh theorem [Se, Theorem 14] that L is free on $\{h x h^{-1} \mid x \in X, h \in H\}$, that H_1L is a free H -module on X , and that $H_2L = 0$.

As a second example, consider a 1-relator presentation $\mathcal{P} = (X; r)$ of the group G . Let $F = F(X)$ and $R = \langle\langle r \rangle\rangle_F$. If the word r is *not* a proper power in F , then $H_1R \cong \mathbb{Z}G$ and $H_2R = 0$ (because R is free). This follows because the cellular 2-complex K modeled on \mathcal{P} is aspherical [C].

Of course, if $L = G$ is projective, we simply mean that H_1L is free abelian and $H_2L = 0$. A projective group $L = G$ which is not a free G -crossed module would be one whose $\text{weight} > \text{rank } H_1L$. Here the weight of G is the minimal number of normal generators of G .

Another example is $H_1L = H_2L = 0$ (L is *superperfect*). Any such L is a projective subgroup.

Projective subgroups are *hereditary* in the following sense. Suppose L is a projective subgroup of $G = K * F$. Then $\bar{L} = L \cap K$ is a projective subgroup of K . See section 3.

In this note we study the lower central series of projective subgroups. The main theorem states that if H_1L is a submodule of a free H -module and $H_2L = 0$, then each quotient L_n/L_{n+1} ($n \geq 1$) of the lower central series $\{L_n\}$ of L is the submodule of a free H -module. The proof is an extension of the proof devised by Ralph Strebel in [S]. Applications are given to the upper central series of G .

To fix notation, we let $[a, b] = aba^{-1}b^{-1}$ and $\mathbb{Z}G$ denote the integral group ring of the group G .

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2. The lower central series of L

For any group L , define $L_1 = L$ and $L_{n+1} = [L_n, L]$ ($n \geq 1$). This is called the *lower central series* of L . If L is a normal subgroup of G , then conjugation by elements of g ($l_n \rightarrow gl_ng^{-1}$) induces a left H ($H = G/L$)-module structure on each L_n/L_{n+1} ($n \geq 1$). Note that $H_1L = L_1/L_2$. The graded object $\text{gr } L = \{L_n/L_{n+1}\}_{n \geq 1}$ has the structure of a graded Lie-ring over $\mathbb{Z}H$ with the Lie bracket equal to $[\ , \]$.

If M is a left H -module, then the graded object

$$TM = \{\mathbb{Z}H, M, M \otimes M, M \otimes M \otimes M, \dots\}$$

has the structure of a graded $\mathbb{Z}H$ -algebra with multiplication given by tensoring: $m \cdot n = m \otimes n$. Here $M \otimes M$ means tensor product over \mathbb{Z} with the diagonal H -action.

We now state and prove the main result of this paper.

MAIN THEOREM 2.1. *Suppose $L \twoheadrightarrow G \twoheadrightarrow H$ is an exact sequence of groups with H_1L a submodule of a free H -module and H_2L a torsion group. Then each successive quotient L_n/L_{n+1} of the lower central series L_n ($1 \leq n < \omega$) is a submodule of a free H -module.*

Proof. We follow the proof of Theorem 1 of [S], p. 149, and check that at each stage the maps defined are isomorphisms of H -modules. This yields a graded Lie

$\mathbb{Z}H$ -algebra isomorphism $\alpha: \text{gr } L \rightarrow TH_1L$ from the graded Lie $\mathbb{Z}H$ -algebra $\text{gr } L$ associated with L onto the Lie $\mathbb{Z}H$ -subalgebra of TH_1L generated by $H_1L = T^1H_1L$. It is clear that if H_1L is a submodule of a free H -module F , then TH_1L is a subalgebra of TF , with each $T^iF = F \otimes \cdots \otimes F$ (i times) a free H -module. Hence, each $\alpha(L_n/L_{n+1})$ is a submodule of T^nH_1L , which in turn is a submodule of T^nF .

Let $I = IL$ be the augmentation ideal of $\mathbb{Z}L$ and define a descending chain of normal subgroups of L by setting

$$D^j(L) = \{l \in L \mid l - 1 \in I^j\}$$

This series is a central series, and we can form the associated graded Lie $\mathbb{Z}H$ -algebra $\text{gr}\{D(L)\}$, because each $D^j(L)$ is *normal* in G .

In order to see that H acts on D^j/D^{j+1} via conjugation by elements of G , it is enough to show that if $l \in D^j(L)$, then, for any $\omega \in L$, $\omega l \omega^{-1} \equiv l \pmod{D^{j+1}(L)}$; i.e., $\omega l \omega^{-1} l^{-1}$ is a member of $D^{j+1}(L)$. But an easy calculation shows that $\omega l \omega^{-1} l^{-1} - 1 = (\omega - 1)(l - 1)l^{-1} - \omega(l - 1)(\omega^{-1} - 1)l^{-1}$. So if $l - 1 \in I^j$, then $\omega l \omega^{-1} l^{-1} - 1 \in I^{j+1}$ and we have verified H acts on $\{D^j/D^{j+1}\}$ via conjugation.

The inclusion $L_j \subseteq D^j(L)$ allows one to define a Lie $\mathbb{Z}H$ -algebra homomorphism $i: \text{gr } L \rightarrow \text{gr}(D(L))$.

Let $\text{gr } \mathbb{Z}L$ denote the graded $\mathbb{Z}H$ -algebra associated to the filtration $\{I^j\}_{0 \leq j < \omega}$ of $\mathbb{Z}L$. Here H acts on I^j/I^{j+1} via conjugation by elements of G . This is well-defined because conjugation by elements of L is trivial mod I^{j+1} . The function $g \mapsto g - 1$ then defines an *injective* Lie $\mathbb{Z}H$ -algebra homomorphism $\beta: \text{gr}(D(L)) \rightarrow \text{gr } \mathbb{Z}L$.

Finally, we use the isomorphism $\mu: H_1L \approx I/I^2$ ($l \cdot L' \mapsto (l - 1) + I^2$) to extend to a homomorphism $\mu: TH_1L \rightarrow \text{gr } \mathbb{Z}L$ of graded associative $\mathbb{Z}H$ -algebras, given in degree j by (H acts diagonally on TH_1L)

$$l_1L_2 \otimes l_2L_2 \otimes \cdots \otimes l_jL_2 \mapsto (l_1 - 1)(l_2 - 1) \cdots (l_j - 1) + I^{j+1}.$$

Clearly, μ is always surjective; it is also injective if H_1L is torsion free and H_2L is a torsion group (see [S], p. 150). The Lie $\mathbb{Z}H$ -algebra homomorphism $\alpha: \text{gr } L \rightarrow TH_1L$ is defined by $\alpha = \mu^{-1}\beta i$ in the following diagram:

$$\text{gr } L \xrightarrow{i} \text{gr}(D(L)) \xrightarrow{\beta} \text{gr } \mathbb{Z}L \xleftarrow{\mu} TH_1L.$$

On page 151 of [S], Strebel shows that α is a monomorphism. ■

The following example shows that even if H_1L is a free H -module and

$L \twoheadrightarrow G \rightarrow H$ is split, the quotients L_n/L_{n+1} are not necessarily projective. Let $H = \mathbb{Z}_2 = \{e, h\}$ and $G = \mathbb{Z} * H$, where \mathbb{Z} generated by x . Then $L = \langle\langle \mathbb{Z} \rangle\rangle_G$ is the free group of rank 2 with basis x and $y = h x h^{-1}$. We order $x < y$, as weight one basic commutators. The only basic commutator of weight 2 is $c_1 = [y, x]$. The action of H on $L_2/L_3 \cong \mathbb{Z}$, generated by \bar{c}_1 , is non-trivial, because $h[y, x]h^{-1} = [hyh^{-1}, h x h^{-1}] = [x, y] = [y, x]^{-1}$. Hence L_2/L_3 is not projective, but is still a submodule of $\mathbb{Z}H$, as guaranteed by Theorem 2.1.

It is intriguing to ask just when the L_n/L_{n+1} might themselves be projective. The next theorem gives a partial result in this direction.

A group H is said to be *ordered* if there is a linear ordering \leq on H such that if $a \leq b$ in H , then $ah \leq bh$ and $ha \leq hb$ for all h in H . Note that $1 \leq a$ in H iff $a^{-1} \leq 1$. For example, any torsion free nilpotent group is ordered [P, p. 581].

THEOREM 2.2. *Suppose $L \twoheadrightarrow G \rightarrow H$ is a split exact sequence of groups, with $H_2L = 0$, H_1L a free H -module, and H an ordered group. Then $\text{gr } L$ is a free graded H -module.*

First, we prove the following.

LEMMA 2.3. *Suppose $F = F(X)$ is a free group with basis X and H is any ordered group. Form the group $G = F * H$ and the split exact sequence $L \xrightarrow{i} G \xrightarrow{\varphi} H$ where φ is obtained by setting elements of F equal to 1 and L is the normal closure $\langle\langle F \rangle\rangle_G$ of F in G . Then each free abelian group L_n/L_{n+1} ($n \geq 1$) is a free H -module.*

Proof. If X is a basis for F , then $\bar{X} = \{h x h^{-1} \mid x \in X, h \in H\}$ is a basis for the free group $L = \langle\langle F \rangle\rangle_G$. We order X arbitrarily and \bar{X} lexicographically according to the pairing (x, h) . We consider the basic commutators in \bar{X} (see [H], p. 166). Each basic commutator c_k of weight k is of the form (uniquely, as L is free)

$$c_k = [c_i, c_j]$$

where $\text{wt}(c_i) + \text{wt}(c_j) = k$, c_i, c_j are basic commutators and $c_i > c_j$. If $c_i = [c_r, c_s]$, then $c_j \geq c_s$. We *order* the basic commutators of weight k *lexicographically* by using the pairing (c_i, c_j) .

We now will prove inductively the following: if c_i, c_m are basic commutators of weight k and $h \in H$, then (1) $h c_i h^{-1}$ is a basic commutator of weight k and (2) $c_i > c_m$ implies that $h c_i h^{-1} > h c_m h^{-1}$, using the lexicographic ordering given above. It is clearly true for $k = 1$, using the ordering on \bar{X} and that H is an ordered group. If $c_i = [c_i, c_j]$ with $c_i > c_j$, then $h c_i h^{-1} = [h c_i h^{-1}, h c_j h^{-1}]$, with $h c_i h^{-1} > h c_j h^{-1}$. Also if $c_i = [c_r, c_s]$ and $c_j \geq c_s$, then $h c_j h^{-1} \geq h c_s h^{-1}$, by induction. Finally,

we must show that if $c_i > c_m$, then $hc_i h^{-1} > hc_m h^{-1}$. Let $c_m = [c_a, c_b]$ with $c_a > c_b$. If $c_i > c_a$, then $hc_i h^{-1} > hc_a h^{-1}$, while if $c_i = c_a$ and $c_j > c_b$, then $hc_i h^{-1} > hc_b h^{-1}$ is true by induction. Thus (1) and (2) are true for all basic commutators.

Now it is easy to find basic commutators of wt k which form an H -basis for L_k/L_{k+1} . These will consist of basic commutators of wt k whose first occurrence of an element of \bar{X} is actually an element of X . For example, basic commutators of weight 2 look like $[h x h^{-1}, g y g^{-1}]$ where $x, y \in X$, $g, h \in H$ and $(x, h) > (y, g)$. Then the set $\{[x, h y h^{-1}] \mid x, y \in X, h \in H \text{ and } (x, 1) > (y, h)\}$ is a $\mathbb{Z}H$ -basis because

$$\{h_1[x, h y h^{-1}]h_1^{-1} \mid h_1, h \in H, x, y \in X, (x, 1) > (y, h)\}$$

yields all weight 2 basic commutators with no repetitions. ■

Proof of Theorem 2.2. Let $s: H \rightarrow G$ denote a splitting of the sequence $L \twoheadrightarrow G \twoheadrightarrow H$. Because $H_1 L$ is a free H -module and $H_2 L = 0$, we may choose a subset X of L so that the image of X in $H_1 L$ is a $\mathbb{Z}H$ -basis for $H_1 L$ and so that the corresponding $\bar{X} = \{(sh)x(sh^{-1}) \mid L \in H, x \in X\}$ (see [HS], p. 204) is the basis for a free subgroup $\bar{L} < L$. Let $F = F(X)$ be the free subgroup of L generated by X . Let $\bar{G} = F * H$ and consider the split exact sequence and commutative diagram:

$$\begin{array}{ccccc} \bar{L} = \langle\langle F \rangle\rangle_{\bar{G}} & \twoheadrightarrow & \bar{G} & \hookrightarrow & H \\ \downarrow i & & \downarrow \omega & & \parallel \\ L & \twoheadrightarrow & G & \xrightarrow{s} & H \end{array}$$

The map ω is defined by the inclusion $i: F \rightarrow L$ and the splitting s . Because $H_1(i)$ is an isomorphism and $H_2(i)$ is zero we have that the map

$$\bar{L}_n / \bar{L}_{n+1} \rightarrow L_n / L_{n+1}$$

is an isomorphism of $\mathbb{Z}H$ -modules, which are free by the lemma. ■

3. Applications to groups

In this section we apply the main theorem about the structure of $\text{gr } L$ to show that often the center $\mathcal{Z}G$ of G must be “buried” inside L ; i.e., $\mathcal{Z}G \subset \bigcap_{n < \omega} L_n = L_\omega$.

First we state a simple lemma about certain elements in group rings which are not zero divisors.

LEMMA 3.1. *Let H be a group and h be an element in H . Then $(h - 1) \in \mathbb{Z}H$ is a zero divisor iff the order of h is finite: $(h + 1)$ is a zero divisor in $\mathbb{Z}H$ iff the order of h is even. If $|n| \neq 1$, then $(h - n)$ is never a zero divisor in $\mathbb{Z}H$. ■*

THEOREM 3.2. *Let $\succrightarrow G \xrightarrow{\varphi} H$ be an exact sequence of groups with H torsion free, H_1L isomorphic to a submodule of a free H -module and $H_2L=0$. Let $g \in G-L$. Then any $l \in L$ which commutes with g must live in $L_\omega = \bigcap L_n$.*

Proof. Let the abelianization $L \rightarrow H_1L$ be denoted by $l \mapsto \bar{l}$ and $\varphi(g) = \hat{g}$ ($l \in L, g \in G$). Then $glg^{-1}l^{-1} = 1 \Rightarrow \hat{g}\bar{l} - \bar{l} = 0$ in $H_1L \Rightarrow (\hat{g} - 1)\bar{l} = 0$ in $H_1L \subset \bigoplus_{i \in I} \mathbb{Z}H_i$. Write $\bar{l} = (\bar{l}_i)_{i \in I}$, where each $\bar{l}_i \in \mathbb{Z}H$. Thus $((\hat{g} - 1)\bar{l}_i) = 0$ and it follows from the lemma that each $\bar{l}_i = 0$; hence $\bar{l} = 0$. So $l \in L_2$. We use Theorem 2.1 and a similar argument to show that $l \in L_n$ for all $n \geq 2$. ■

Note 3.3. (a) A similar argument will show that if $g \in G-L$ and $l \in L$ satisfies $glg^{-1} = l^n$ ($n \in \mathbb{Z}$), then $l \in L_\omega$.

(b) Other identities may be used. For example, if $g_1, g_2 \in G-L$ and $l \in L$ with $[[g_1, l], g_2] = 1$, then $l \in L_\omega$. This follows because $[[g_1, l], g_2] = (1 - \hat{g}_2)[g_1, l] = (1 - \hat{g}_2)(\hat{g}_1 - 1)\bar{l} = 0$. Then apply the argument twice.

Recall that, if G is a group, then the n th order center of G , $\mathcal{Z}^n G$ ($n \geq 1$), is inductively defined as $\mathcal{Z}^1 G = \text{center of } G$, $\mathcal{Z}^{n+1} G = \{g \in G \mid \varphi_n(g) \in \mathcal{Z}^1(G/\mathcal{Z}^n G)\}$, where $\varphi : G \rightarrow G/\mathcal{Z}^n G$ is the natural map.

COROLLARY 3.4. *Suppose that the exact sequence of groups is as in Theorem 3.2 with $G-L \neq \emptyset$ and $H_1L \neq 0$. Then all the centers $\mathcal{Z}^n G$ ($n \geq 1$) are contained in L_ω .*

Proof. (a) We show that $\mathcal{Z}^1 G \subset L_\omega$. Suppose there is an element $g \in (G-L) \cap \mathcal{Z}^1 G$. Choose any $l \in L$ and observe that $[g, l] = 1$. By Theorem 3.2, we see that $l \in L_\omega$. Hence $L = L_\omega$, which contradicts the hypothesis that $H_1L \neq 0$. Hence $\mathcal{Z}^1 G \subset L$.

Now choose any $g \in G-L$ and $l \in \mathcal{Z}^1 G \subset L$. Again, the proposition shows that $l \in L_\omega$. Hence $\mathcal{Z}^1 G \subset L_\omega$.

(b) We observe that $g \in \mathcal{Z}^2 G$ iff for all $g_1, g_2 \in G$ the commutator $[[g_1, g], g_2] = 1$. Now suppose $g \in G-L \cap \mathcal{Z}^2 G$. Choose any $l \in L$ and observe that $[[g, l], g_2] = 1$ for all $g_2 \in G$. Choosing $g_2 \in G-L$, we see that $l \in L_\omega$ for all $l \in L$ by note (b). The rest of the argument is similar to (a).

(c) One can either prove $\mathcal{Z}^n G \subset L$ for $n \geq 2$ by induction or by studying the higher commutators $[\dots, [[g_1, l], g_2] \dots, g_n]$. ■

Note 3.5. Corollary 3.4 also follows without using 3.2 because L/L_ω is a non-abelian free group (under the assumption imposed above) and in that case $\mathcal{Z}^n G \subseteq \mathcal{Z}^n L \subseteq L_\omega$.

Note 3.6. The above corollary is false if $H_1L = 0$ or if $G = L$. Let L be a

superperfect group, let $G = \mathbb{Z} \times L$ and $G \rightarrow \mathbb{Z}$ be the projection with kernel L . Then $\mathbb{Z} \subset \mathcal{Z}G$ is not contained in L . Also, if $G = L = \mathbb{Z}$, then $\mathcal{Z}G = \mathbb{Z}$ is not contained in $L_\omega = 0$.

In order to give the next application, we need the notion of a C -subgroup. We say that a subgroup $N < G$ is a C -subgroup if there is a G -projective resolution $P_* \rightarrow \mathbb{Z}$ of the trivial module \mathbb{Z} for which the homomorphism $N \otimes_{\partial_3} \mathbb{Z} \otimes_N P_3 \rightarrow \mathbb{Z} \otimes_N P_2$ is trivial. See [BDS] for properties and applications of this concept.

The next result shows that, in some sense, H is close to being a two-dimensional group, the “closeness” being measured by H_2L .

THEOREM 3.7. *Let $L \twoheadrightarrow G \twoheadrightarrow H$ be an exact sequence of groups with H_1L a projective H -module, $H_2L = 0$ and L a C -subgroup. Then the cohomological dimension of H is ≤ 2 .*

Proof. Let $P_* \rightarrow \mathbb{Z}$ be the resolution assured by L being a C -subgroup of G . By tensoring P_* with $\mathbb{Z} \otimes_L -$, using that H_1L is projective as an H -module and that L is a C -subgroup, we obtain an exact sequence of H -modules

$$H_2L \twoheadrightarrow \mathbb{Z} \otimes_L P_2 \oplus H_1L \rightarrow \mathbb{Z} \otimes_L P_1 \rightarrow \mathbb{Z} \otimes_L P_0 \rightarrow \mathbb{Z}$$

with the inner three terms projective. The hypothesis that $H_2L = 0$ does the rest. ■

The next result yields new information about the aspherical question of J. H. C. Whitehead ([BD] and [BDS]): Are subcomplexes of aspherical 2-complexes aspherical?

PROPOSITION 3.8. *Suppose $G = K * N$ is a free product of groups K and N and that L is a projective subgroup of G . Then $\bar{L} = K \cap L$ is a projective subgroup of K .*

Proof. We observe that \bar{L} is a normal subgroup of K . It follows from the Kurosh structure theorem about subgroups of a free product [Se, Theorem 14] that \bar{L} is a free summand of $L = \bar{L} * M$. It is easy to check that, if $\bar{H} = K/\bar{L}$ ($\bar{H} < G/L$), then H_1L is isomorphic to $H_1\bar{L} \oplus H_1M$ as \bar{H} -modules. Because H_1L is a projective G/L -module, it is a projective \bar{H} -module. We have $H_2\bar{L} = 0$ because $H_2L \cong H_2\bar{L} \oplus H_2M = 0$. Thus \bar{L} is projective. ■

A $[G, 2]$ -complex X is a connected, 2-dimensional CW-complex with fundamental group isomorphic to G and a single zero cell. If X is a $[G, 2]$ -complex which is a subcomplex of an aspherical $[K, 2]$ -complex Y , we let $\bar{X} = X \cup Y^1$ be

the union of X together with the 1-skeleton of Y and $i: X \hookrightarrow Y$, and $i_1: \bar{X} \hookrightarrow Y$ denote the inclusion maps. Furthermore, let $L = \ker \pi_1(i_1)$ and $\bar{L} = \ker \pi_1(i)$. Then the fundamental group $\pi_1(\bar{X}) \cong G * F$, where F is a free group whose rank corresponds to the number of 1-cells of Y outside of X . It is well known (see [BD]) that $L \twoheadrightarrow G * F \twoheadrightarrow \text{im } \pi_1(i_1) = H$ is a free $G * F$ -crossed module. Thus $H_1 L$ is a free H -module and $H_2 L = 0$. Now $\bar{L} = \ker \pi_1(i) = L \cap G$ is a projective $\bar{H} = G/\bar{L}$ -module by the previous proposition (see [D]). Furthermore (and this seems to be new), the sequence

$$\hat{L} = F \cap L \twoheadrightarrow F \twoheadrightarrow F/F \cap L = \hat{H} < H$$

is also a projective subgroup of the free group F , with \hat{H} a 2-dimensional group.

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