Values of pseudoriemannian sectional cuvature.

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1. Introduction

In a Riemannian space (X, g) all two dimensional tangent planes are nondegenerate and the sectional curvature is a continuous function. If p is a fixed point of X, the planes of the tangent space T_pX form a compact set and it follows that the sectional curvature is bounded at p. If (X, β) is pseudoriemannian the situation is quite different. In this case the sectional curvature is only defined on nondegenerate planes and those of T_pX form a noncompact subset of the Grassmannian $G_2(T_pX)$ whenever $\dim X \geq 3$. Thorpe [9] proved that the sectional curvature can only be continuously extended to all null planes in the case of constant curvature. In general, the sectional curvature will not be bounded on the noncompact subset of $G_2(T_pX)$ consisting of the nondegenerate planes. Kulkarni [7] has shown that if $\dim X \geq 3$, then the sectional curvature function is either bounded from above or from below at p only when it is a constant at p. Harris [6] and Dajczer and Nomizu [4] noted that the sectional curvature function is bounded both above and below on all *timelike* planes at p only when the space has constant sectional curvature at p. Nomizu [8] has also investigated boundedness conditions on the sectional curvature restricted to nondegenerate planes which contain some fixed (spacelike) vector v of T_pX. He has shown that if *every* pencil of planes determined by a spacelike vector v has the property that the sectional curvature of all spacelike (resp. timelike) planes in the pencil is bounded, then (X, β) has constant sectional curvature at p. Sectional curvature of pseudoriemannian manifolds has also been investigated in [3] and [5]. Spaces of constant sectional curvature have been extensively studied by Wolf [10].

In this paper we study the sectional curvature of pseudoriemannian manifolds (X, β) of $\dim \geq 3$. Part of the original motivation for this paper came from our use of sectional curvature in [2]. Our approach differs from previous studies in that we begin by expressing the sectional curvature K_β at some point p of a *three* dimensional Lorentzian manifold (X, β) as a rational function from $\mathbb{R}P^2$ to $\mathbb{R}$ which is a ratio of quadrics. The denominator Q_2 of this ratio is normalized as $x_3^2 - x_1^2 - x_2^2$ on $\mathbb{R}P^2$ (resp. $1 - x^2 - y^2$ on $\mathbb{R}^2$). The numerator Q_1 is a quadric

$Ax_1^2+Bx_1x_2+Cx_2^2+Dx_1x_3+Ex_2x_3+Fx_3^2$ on $\mathbb{R}P^2$ (resp. $Ax^2+Bxy+Cy^2+Dx+Ey+F$ on $\mathbb{R}^2$) which may be degenerate. The *null locus* $Q_2=0$ corresponds to the set of *degenerate* or *null* planes in T_pX. A point of the null locus where Q_1 is nonvanishing corresponds to a null plane $\Pi_0 \subseteq T_pX$ where $|K_\beta(\Pi)| \to \infty$ as $\Pi \to \Pi_0$. If Q_2 is not a scalar multiple of Q_1, then a point where Q_1 and Q_2 both vanish corresponds to a plane Π_0 of T_pX where the sectional curvature and its absolute value are *indeterminate* in $\mathbb{R}^1 \cup \{\infty\} = \mathbb{R}P^1$ as $\Pi \to \Pi_0$. We find that for a fixed point p of a three-dimensional Lorentzian manifold there are at most 4 null planes of T_pX where the sectional curvature is indeterminate in $\mathbb{R}^1 \cup \{\infty\} = \mathbb{R}P^1$. Thus in dimension three the sectional curvature must become unbounded near all null planes at p with at most four exceptions whenever K is not constant at p. Corresponding to these (at most) 4 exceptions are (at most) 6 spacelike directions in T_pX such that the sectional curvature is constant on each pencil of planes determined by one of these 6 directions. For all other pencils of planes determined by a spacelike direction the sectional curvature is unbounded. In higher dimensions there may be infinitely many degenerate planes which are indeterminate (but these lie in a set of codimension at least 3) and infinitely many spacelike directions such that the sectional curvature is bounded on all planes containing one of these directions. On the other hand, our three-dimensional results imply that the set of spacelike directions which determine pencils of planes with unbounded sectional curvature form an open dense subset of the set of all spacelike directions. (It can be shown that the complement is of codimension at least 2.)

2. Preliminaries

Let (X, β) be a pseudoriemannian manifold of type $(s, n-s)$. This means β can be represented at any point $p \in X$ as a diagonal matrix with s negative eigenvalues and $n-s$ positive eigenvalues. There is always an associated pseudoriemannian manifold $(X, -\beta)$ of type $(n-s, s)$ and results for (X, β) always translate into corresponding results for $(X, -\beta)$ after appropriate sign changes. We shall always take $\dim X \geq 3$ and $2 \leq s \leq n-1$. A vector $v \in TM$ is *spacelike* (resp. *null, timelike*) if $\beta(v, v) < 0$ (resp. $=0, >0$). There are always 2-dimensional linear subspaces of each T_pM which are negative definite, but there are positive definite two dimensional linear subspaces of each T_pM only when $s \leq n-2$ (i.e., $n-s \geq 2$). Whenever $2 \leq s \leq n-2$ all of our results which hold for spacelike vectors also hold for timelike vectors. But in the Lorentzian case (i.e., $s=n-1$), this is not true.

On $\mathbb{R}P^2$ we shall use homogeneous coordinates (x_1, x_2, x_3) and always take $\mathbb{R}^2$ to be the subset $x_3 \neq 0$ of $\mathbb{R}P^2$. On $\mathbb{R}^2$ we let $x = x_1/x_3$ and $y = x_2/x_3$. The Grassmannian of k-planes in $\mathbb{R}^n$ will be denoted by $G_k(\mathbb{R}^n)$. Then $\mathbb{R}P^2 \cong G_1(\mathbb{R}^3)$ and, using the usual Euclidean inner product on $\mathbb{R}^3$, we may identify $G_2(\mathbb{R}^3)$ and $G_1(\mathbb{R}^3)$ via the correspondence $\Pi \leftrightarrow \Pi^\perp$ where $\Pi \in G_2(\mathbb{R}^3)$.

3. Sectional curvature of 3-manifolds

In this section (X, β) will always denote a three dimensional Lorentzian manifold and thus have signature $(--+)$. We fix some point $p \in X$ and investigate the sectional curvature K_β at p. We assume that local coordinates have been chosen near p such that the metric tensor is represented by $\operatorname{diag}(-1, -1, +1)$ at p. Using the induced natural coordinates on the tangent space $T_pX \cong \mathbb{R}^3$, we obtain two inner products β_p and e_p on T_pX. If $u, v \in T_pX$ have coordinate representations $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ respectively, then $\beta_p(u, v) = -u_1v_1 - u_2v_2 + u_3v_3$ is the Lorentzian inner product on T_pX and $e_p(u, v) = u_1v_1 + u_2v_2 + u_3v_3$ is the Euclidean inner product on T_pX.

Let Π be a plane (i.e., a two dimensional linear subspace) in T_pX. If Π is nondegenerate, then there is a Lorentzian orthonormal basis u, v of Π and the sectional curvature [1, p. 409] of Π is given by

$$K_\beta(\Pi) = \pm \sum R_{ijkh} u_i v_j u_k v_h. \tag{3.1}$$

Here we have assumed that v is spacelike (i.e., $\beta_p(v, v) = -1$); the $-$ sign is to be taken if u is timelike, and the $+$ if u is spacelike.

The plane Π is determined by a Euclidean normal (a, b, c). Since Π is nondegenerate we have $-a^2 - b^2 + c^2 \neq 0$. If Π is spacelike, then (a, b, c) is timelike and we have $-a^2 - b^2 + c^2 > 0$; if Π is timelike, then (a, b, c) is spacelike and we have $-a^2 - b^2 + c^2 < 0$. Using $w = (a, b, c)$ we have the following equations:

$$e_p(w, u) = e_p(w, v) = \beta_p(u, v) = 0; \tag{3.2}$$

$$\pm \beta_p(u, u) = \beta_p(v, v) = -1. \tag{3.3}$$

If Π is timelike we may assume w.l.o.g. that $v_3 = 0$, $u_3 > 0$, $v_1 \geq 0$, and $v_1 = 0$ implies $v_2 > 0$. In this case w is spacelike and $a^2 + b^2 \neq 0$. Using equations (3.2) and (3.3) we obtain

$$\begin{aligned} au_1+bu_2+cu_3&=0,\\ av_1+bv_2&=0,\\ -u_1^2-u_2^2+u_3^2&=1,\\ -v_1^2-v_2^2&=-1,\\ u_1v_1+u_2v_2&=0. \end{aligned}$$

Using these five equations we may solve for u_1, u_2, u_3, v_1, and v_2 in terms of a, b, c:

$$\begin{aligned} &v_1=|b|/\sqrt{a^2+b^2},\\ &v_2^2=a^2/(a^2+b^2),\\ &u_1^2=a^2c^2/(a^2+b^2)(a^2+b^2-c^2),\\ &u_2^2=b^2c^2/(a^2+b^2)(a^2+b^2-c^2),\\ &u_3=\sqrt{a^2+b^2}/\sqrt{a^2+b^2-c^2}. \end{aligned}$$

If $ab\leq 0$, then v_2 is positive and the variables u_1 and u_2 have opposite signs. The sign of u_1 is positive if $bc\geq 0$ and negative if $bc<0$. If $ab>0$, then v_2 is negative and the variables u_1 and u_2 have the same sign. Here u_1 is positive for $ac\leq 0$ and negative for $ac>0$. Using this information and well known curvature identities, equation (3.1) yields the following formula for $K_\beta(\Pi)$:

$$K_\beta(\Pi)=\frac{c^2R_{2121}+a^2R_{3232}+b^2R_{3131}+2(acR_{2132}+bcR_{1231}-abR_{3132})}{c^2-a^2-b^2}. \tag{3.4}$$

A similar calculation for the case of a spacelike plane Π yields the same final equation (3.4). Thus this formula is valid for all nondegenerate planes at p. We define two quadratic forms Q_1 and Q_2 on $\mathbb{R}^3$ by

$$Q_1=Ax_1^2+Bx_1x_2+Cx_2^2+Dx_1x_3+Ex_2x_3+Fx_3^2$$

and $Q_2=x_3^2-x_1^2-x_2^2$. Here $A=R_{3232}$, $B=-2R_{3132}$, $C=R_{3131}$, $D=2R_{2132}$, $E=2R_{1231}$ and $F=R_{2121}$. Restricting these forms to the unit sphere in $\mathbb{R}^3$ and identifying antipodal points we may regard Q_1 and Q_2 as being defined on $\mathbb{R}P^2$.

Then

$$K_\beta(\Pi) = \frac{Q_1(x_1, x_2, x_3)}{Q_2(x_1, x_2, x_3)}$$

where $\Pi^\perp$ is the one-dimensional vector space perpendicular in the Euclidean sense to Π with homogeneous coordinates (x_1, x_2, x_3). Taking $x_3 = 0$ as the line at infinity in $\mathbb{R}P^2$, we obtain

$$K_\beta(\Pi) = \frac{Ax^2 + Bxy + Cy^2 + Dx + Ey + F}{1 - x^2 - y^2}$$

where $x = x_1/x_3$ and $y = x_2/x_3$.

Two conics N and H may be defined on $\mathbb{R}P^2$ by $N = \{(x_1, x_2, x_3);\ Q_2(x_1, x_2, x_3) = 0\}$ and $H = \{(x_1, x_2, x_3);\ Q_1(x_1, x_2, x_3) = 0\}$. The conic N is an ellipse which is represented in $\mathbb{R}^2$ by $x^2 + y^2 = 1$. We call N the *null locus* since each point of N represents a null plane in T_pX. The second conic H will be called the *homaloidal locus*, since the nondegenerate planes in it are flat. Unlike N, H may be degenerate: it may be all of $\mathbb{R}P^2$, an ellipse, two lines in $\mathbb{R}P^2$, one line (counted twice), a single point, or the empty set. From classical projective geometry the (real) intersection $N \cap H$ may be N, four points (counted with multiplicity), two points (counted with multiplicity) or the empty set. If $N \cap H = N$, then Q_1 is a scalar multiple of Q_2 and $K_\beta(\Pi)$ is constant at p.

DEFINITION 3.1. (a) If $\Pi_0 \in N \backslash H$, then $|K_\beta(\Pi)| \to \infty$ as the plane Π approaches the plane Π_0. The point Π_0 is called a *pole* of K_β.

(b) If $\Pi_0 \in N \cap H$ and $N \cap H \neq N$, then Π_0 is called an *indeterminate* or *ambiguous point*.

An indeterminate point Π_0 may have a positive even multiplicity (two or four) or else an odd multiplicity (one or three). At an indeterminate point of odd multiplicity, N and H have a nonempty transverse intersection.

The spacelike planes of T_pX correspond to the set $S = \{(x_1, x_2, x_3);\ x_3^2 > x_1^2 + x_2^2\}$ of $\mathbb{R}P^2$ which lies "inside" the conic N and the spacetime planes (of signature $(-+)$) correspond to the set $T = \{(x_1, x_2, x_3);\ x_3^2 < x_1^2 + x_2^2\}$ of $\mathbb{R}P^2$ which lies "outside" N. The *spacelike image set* $I_s := K_\beta(S)$ will be the values of the sectional curvature on the spacelike planes and the *timelike image set* $I_t := K_\beta(T)$ will be the values on the spacetime planes. The *gap set* will be $G := \mathbb{R} \backslash (I_s \cup I_t)$.

Remark 3.2. *Each of the sets I_s and I_t is the continuous image of a connected set and is thus connected.*

A *simple transverse intersection* of N with H willl be a transverse intersection of multiplicity one. If $N \cap H$ has a point of multiplicity three, then they must have an intersection of total multiplicity four, and thus $N \cap H$ has a point where N and H have a simple transverse intersection. We will prove that $I_t = I_s = \mathbb{R}$ whenever N and H have a simple transverse intersection. The set H may be an ellipse which meets N two to four times, or a pair of distinct lines at least one of which meets N at two points.

PROPOSITION 3.3. *If the null locus N and homaloidal locus H have a simple transverse intersection at Π_0, then $I_s = I_t = \mathbb{R}$. Furthermore, for each neighborhood $U(\Pi_0)$ of Π_0 and each real number $\alpha \in \mathbb{R}$ there exist points $\Pi_1, \Pi_2 \in U(\Pi_0) \setminus N$ such that $\Pi_1 \in S$, $\Pi_2 \in T$, and $K_\beta(\Pi_1) = K_\beta(\Pi_2) = \alpha$.*

Proof. Given $U(\Pi_0)$ and $\alpha \in \mathbb{R}$ we shall prove the existence of Π_1. The existence of Π_2 may be demonstrated by the same method. Assume (as we may) that $U(\Pi_0) \subseteq \mathbb{R}^2$ and choose a curve $\gamma : [0, 1] \to \mathbb{R}^2$ given by $\gamma(t) = (x(t), y(t))$ with $1 - x^2 - y^2 \geq 0$ and equality iff $t = 0, 1$. Since N and H have a nonempty transverse intersection at Π_0 we may choose γ such that $\gamma(t) \in U(\Pi_0) \setminus \Pi_0$ for $0 \leq t \leq 1$; $\gamma(t) \in \{(x, y) \mid Q_1(x, y, 1) < 0\}$ for $t < \frac{1}{2}$, and $\gamma(t) \in \{(x, y) \mid Q_1(x, y, 1) > 0\}$ for $t > \frac{1}{2}$. Then $K_\beta(\gamma(t))$ is a continuous function of t for $0 < t < 1$ with $K_\beta(\gamma(t)) \to -\infty$ as $t \to 0^+$ and $K_\beta(\gamma(t)) \to +\infty$ as $t \to 1^-$. Thus there is some $\Pi_1 = \gamma(t_1)$ with $0 < t_1 < 1$ and $K_\beta(\Pi_1) = \alpha$. $\square$

The intersection $N \cap H$ is empty when H is empty, a point of either T or S, a line in T, two lines in T, an ellipse in T or an ellipse in S. In all these cases I_s and I_t are closed half-lines which are disjoint and point in opposite directions.

PROPOSITION 3.4. *If $H \cap N = \phi$, then I_s and I_t are closed oppositely oriented half lines and $\mathbb{R} \setminus (I_s \cup I_t)$ is a nonempty open interval.*

Proof. Since $N \cap H = \phi$, the value of Q_1 must be always positive or always negative on N. We assume that Q_1 is always positive on N, the other case being similar. Then K_β uniformly approaches $+\infty$ as Π approaches N through points of S (inside N) and K_β uniformly approaches $-\infty$ as Π approaches N through points of T (outside N). Since K_β is continuous on S and T, it follows from a collaring argument that there must be some minimum b of K_β on S and maximum a of K_β on T. Using Remark 3.2 we find $I_t = (-\infty, a]$ and $I_s = [b, +\infty)$.

In order to prove that $a < b$ let Π_1 be a point of S where $K_\beta(\Pi_1) = b$ and let Π_2 be a point of T where $K_\beta(\Pi_2) = a$. By making a projective change of coordinates which leaves N fixed as a set, we may map Π_1 to $(0, 0, 1)$ corresponding to the origin of $\mathbb{R}^2$, and Π_2 to a point of the projective line containing the

x-axis of $\mathbb{R}^2$. The value of K_β along the x-axis is then

$$f(x)=\frac{A'x^2+D'x+F'}{1-x^2}$$

where the coefficients A', D', F' may be different from the original, A, D, F. The derivative of f is

$$f'(x)=\frac{D'x^2+x(2A'+2F')+D'}{(1-x^2)^2}.$$

Since K_β has a minimum at the origin, $f'(0)=0$ and consequently $D'=0$. Thus $f(x)=(A'x^2+F')(1-x^2)^{-1}$. Using the fact that $A'x^2+F'$ is positive at $x=\pm 1$, we obtain $A'+F'>0$ and hence $F'>-A'$. Since the minimum of f on $-1<x<1$ occurs at $x=0$, we find $b=f(0)=F'$. Elementary calculus shows that the supremum of f on $|x|>0$ corresponds to the limiting value of f as $x\to\infty$. Thus $a=f(\infty)=-A'$. Consequently, $F'>-A'$ implies $b>a$ and $\mathbb{R}\backslash(I_s\cup I_t)$ must be a nonempty open interval. $\square$

There are three ways in which $H\cap N$ may be a single point of multiplicity four. If H is nondegenerate, then H can be an ellipse "inside" the ellipse N or else H can be an ellipse which is "outside" N. If H is degenerate, then H must be a single line (counted twice) which is tangent to N.

PROPOSITION 3.5. *If the null locus N and homaloidal locus H intersect in a single point of multiplicity four, then I_t is a closed half line and I_s is the complement set $\mathbb{R}\backslash I_t$.*

Proof. We shall give only the proof for the degenerate case in which H is a single line which is tangent to N at Π_0. By a projective change of coordinates, we may move H to the line which intersects $\mathbb{R}^2$ in the line $x=1$. Then Q_1 must be a nonzero multiple of the quadric $x_1^2-2x_1x_3+x_3^2$. Thus in the xy-plane the sectional curvature is given by

$$f(x, y)=\frac{c(x^2-2x+1)}{1-x^2-y^2} \tag{3.5}$$

where $c\neq 0$. Since the argument is the same for positive or negative c, we consider only $c>0$. If $y=0$ and $-1<x<1$, then equation (3.5) shows that the interval $(0,\infty)$ is contained in I_s. Letting $y=0$ and $x\in\mathbb{R}\backslash[-1, 1]$ shows that the interval

$(-\infty, 0)$ is contained in I_t. Since $f(x, y) \equiv 0$ on $x=1$, it follows that I_t at least contains the closed interval $(-\infty, 0]$. It only remains to show that $I_t \cap I_s = \phi$. If I_t and I_s were not disjoint, then the above arguments together with Remark 3.2 would yield some point (x_1, y_1) inside the disk $\{(x, y); x^2+y^2<1\}$ with $f(x, y)=0$. But equation (3.5) shows that this cannot happen. □

There are several ways in which $H \cap N$ can consist of a single point of multiplicity two. If H is nondegenerate, then all but one point of H may be contained in the set S or all but one point of H may be contained in the set T. If H is degenerate it may be a single point of N or it may be two lines (one tangent to N and one disjoint from N).

PROPOSITION 3.6. *Assume $H \cap N$ consists of a single point of multiplicity two. Then I_s and I_t are oppositely directed open half lines and $\mathbb{R} \setminus (I_s \cup I_t)$ is a single point.*

Proof. We shall give only the proof for the case where H is a degenerate conic consisting of two lines, one of which is tangent to N and the other disjoint from N. We first make a projective change of coordinates which leaves N fixed and maps H to a pair of lines, one of which intersects $\mathbb{R}^2$ in the euclidean line $x=1$. The quadric Q_1 is then a scalar multiple of $(x-1)(ax+by+c)$. The fact that the line of H given by $ax+by+c=0$ does not meet N means $ax+by+c$ is either always positive or always negative on $S \cup N$. Assuming (as we may) that $ax+by+c>0$ on $S \cup N$, then $|a|<c$. The sectional curvature (in $\mathbb{R}^2$) is given by

$$K_\beta(x, y) = R\frac{(x-1)(ax+by+c)}{1-x^2-y^2}$$

where $R \neq 0$. Let us assume $R>0$, the case $R<0$ being similar. Along the euclidean line $y=m(x-1)$ the value of the sectional curvature is

$$K_\beta(x, m(x-1)) = R\frac{(x-1)(ax+bm(x-1)+c)}{1-x^2-m^2(x-1)^2} = \frac{R(ax+bmx+c-bm)}{(-1-m^2)x+m^2-1}. \tag{3.6}$$

The intersection of S with $y=m(x-1)$ corresponds to $\{x; (m^2-1)/(1+m^2)<x<1\}$. Using equation (3.6), we find that the sectional curvature has image $(-\infty, -R(a+c)/2)$ on this interval. Notice that the image is independent of m (and b). Hence $I_s=(-\infty, -R(a+c)/2)$. The intersection of T with the line $y=m(x-1)$ corresponds to $\{x; x<(m^2-1)/(1+m^2)$ or $x>1\}$. It follows, using equation (3.6), that the image of K_β on the intersection of T with the *projective* line corresponding to $y = m(x-1)$ is the open interval $(-R(a+c)/2, +\infty)$. Again the image is independent of m (and b). The image of K_β along the projective line corresponding to $x = 1$ is $\{0\}$ and $|a| < c$ yields $0 \in (-R(a+c)/2, +\infty)$. It follows that I_t is the open half line $(-R(a+c)/2, +\infty)$. $\square$

In the above proof of Proposition 3.6 we made essential use of the fact that the line $y = m(x-1)$ intersected N in two points and exactly one of these (namely $(1,0)$) was a point of H. Using the same types of techniques as above we can establish the following result.

LEMMA 3.7. *Let L be a projective line which intersects $H \cap N$ in exactly two points. Then the sectional curvature is a constant on the set $L \setminus (H \cap N)$.*

In the case of $H \cap N$ consisting of exactly two points, each of which has multiplicity two, the proof used in Proposition 3.6 must be modified slightly to take into account the fact that one of the lines of the form $y = m(x-1)$ intersects $H \cap N$ in two points.

PROPOSITION 3.8. *If $H \cap N$ consists of exactly two points and each has multiplicity two, then I_s and I_t are oppositely directed closed half lines with a common endpoint. The gap set $G = \mathbb{R} \setminus (I_s \cup I_t)$ is empty.*

In dimension three the six components R_{2121}, R_{3232}, R_{3131}, R_{2132}, R_{1231} and R_{3132} are all independent. Given any six numbers one may always construct a Lorentzian manifold (X, β) with a point p such that the given six numbers are equal to the respective components R_{2121}, R_{3232}, R_{3131}, R_{2132}, R_{1231} and R_{3132} at p. It follows that all of the forms of I_t and I_s given in Propositions 3.3, 3.4, 3.5, 3.6, and 3.8 actually occur in examples.

4. Higher dimensional results

In this section we consider pseudoriemannian manifolds (X, β) of arbitrary dimension ≥ 3. At each $p \in X$ there will exist nondegenerate planes of signature $(+\,-)$ and $(-\,-)$. If (X, β) does not have Lorentzian signature, there will also be nondegenerate planes of signature $(+\,+)$. We let I_{+-} be the image under K_β of all planes of signature $(+\,-)$ and in similar fashion define the image sets I_{--} and I_{++}. In the Lorentzian case, $I_t = I_{+-}$, $I_s = I_{--}$, and $I_{++} = \phi$.

One consequence of the classification given by Propositions 3.3, 3.4, 3.5, 3.6 and 3.8 is the following theorem for n-dimensional Lorentzian manifolds.

THEOREM 4.1. *Let p be a point of a Lorentzian manifold of dimension* ≥ 3. *If the sectional curvature is not constant at p, then both* I_s *and* I_t *are intervals of infinite length.*

Proof. Propositions 3.3, 3.4, 3.5, 3.6 and 3.8 clearly imply this result for $\dim X = 3$.

If $\dim X > 3$, we first note that I_t and I_s must be connected subsets of $\mathbb{R}^1$ because the sectional curvature is continuous on the set of timelike planes and on the set of spacelike planes and these are both connected sets of $G_2(T_pX)$. If either I_s or I_t were not of infinite length, then the sectional curvature would be constant on every nondegenerate three-dimensional subspace of T_pX with Lorentzian signature. We claim that the value of this sectional curvature constant is the same for all three-dimensional subspaces of this type. If V and W are such subspaces, let V and W have orthonormal bases $\{e_1, e_2, e_3\}$ and $\{u_1, u_2, u_3\}$, respectively, where e_1 and u_1 are timelike. Define $V_1 = V$, $V_2 = \operatorname{span}\{e_1, e_2, u_2\}$, $V_3 = \operatorname{span}\{e_1, u_2, u_3\}$, and $V_4 = W$. Each V_i is a linear subspace of dimension at least two and is nondegenerate since it contains a time-like vector and the metric on T_pX is Minkowskian. Each $V_i \cap V_{i+1}$ contains at least one nondegenerate plane, so the sectional curvature constant is the same for V_i and V_{i+1}. Consequently, V and W have the same sectional curvature constant. The result now follows from the fact that every nondegenerate plane at p lies in some nondegenerate three-dimensional subspace of T_pX with Lorentzian signature. $\square$

Consideration of the sectional curvature on three-dimensional subspaces of Lorentzian signature can also be used to establish the following pseudoriemannian result.

THEOREM 4.2. *Let p be a fixed point of the pseudoriemannian manifold* (X, β) *of dimension* ≥ 4. *Assume that* (X, β) *is* not *Lorentzian. Then the sectional curvature is constant at p iff any of the following three conditions holds*:

(1) *the sectional curvature is bounded above and below on planes of signature* $(+\,-)$ (*i.e.*, I_{+-} *has finite length*);

(2) *the sectional curvature is bounded above and below on planes of signature* $(+\,+)$ (*i.e.*, I_{++} *has finite length*);

(3) *the sectional curvature is bounded above and below on planes of signature* $(-\,-)$ (*i.e.*, I_{--} *has finite length*).

We now consider the sectional curvature on the collection of all nondegenerate planes containing some spacelike vector $v \in T_pX$. The results of Section 3 show that if (X, β) is a Lorentzian manifold of dimension three, then for each spacelike vector $v \in T_pX$ the sectional curvature restricted to the pencil of planes containing v must be constant or must be unbounded. The sectional curvature is constant on the pencil exactly when the line in $\mathbb{R}P^2$ corresponding to this pencil intersects the null locus N in two points which also lie on the homaloidal locus H. If N is not a subset of H, then $H \cap N$ can have at most four points and thus there are at most six pencils of planes (each determined by a spacelike vector) such that the sectional curvature is constant on each pencil. If the sectional curvature is not constant on a pencil of planes determined by a spacelike vector $v \in T_pX$, then the results of Section 3 show that the sectional curvature is unbounded either above or below on all spacetime planes containing v and also is unbounded either above or below on all spacelike planes which contain v. We now show that the sectional curvature restricted to the pencil of planes containing a spacelike vector v is either constant or unbounded in any pseudoriemannian manifold.

LEMMA 4.3. *Let p be a point of the pseudoriemannian manifold (X, β) and assume v is a spacelike vector in T_pX. The sectional curvature is either constant on all nondegenerate planes containing v or else is unbounded on this pencil. If the sectional curvature is unbounded on this pencil, then it is unbounded either above or below on the set of spacetime planes which contain v and unbounded either above or below on the set of spacelike planes which contain v.*

Proof. Assume that the sectional curvature is not constant on the set of nondegenerate planes which contain v. The results of Section 3 imply we need only show that there is some three-dimensional linear subspace L of T_pX such that $v \in L$, the metric tensor on L has Lorentzian signature, and K_β is not constant on the planes of L containing v. Choose two nondegenerate planes Π_1 and Π_2 containing v with $K_\beta(\Pi_1) \neq K_\beta(\Pi_2)$. Both may be spacetime planes (signature $(+\,-)$), both spacelike $(-\,-)$, or one spactime and the other spacelike. We consider the case where both are spacetime planes, the others being similar. Choose a spacelike plane Π_3 containing v. If $K_\beta(\Pi_3) \neq K_\beta(\Pi_1)$, let $L = \Pi_1 + \Pi_3$; if $K_\beta(\Pi_3) = K_\beta(\Pi_1)$, let $L = \Pi_2 + \Pi_3$. $\square$

The conclusion of Lemma 4.3 is valid when v is a timelike vector in a pseudoriemannian manifold of signature $(s, n-s)$ with $2 \leq s \leq n-2$. On the other hand, the situation when v is a timelike vector in a Lorentzian manifold (of arbitrary dimension ≥ 3) is quite different. In this case all planes containing v are nondegenerate and thus the sectional curvature restricted to planes containing v is a continuous function defined on a compact set. Consequently, the sectional curvature is bounded both above and below on the pencil of planes determined by a timelike vector in a Lorentzian manifold.

We now obtain a generalization of some of the results of Nomizu [8]. We show that generically the sectional curvature is unbounded on the set of spacelike vectors.

PROPOSITION 4.4. *Let* (X, β) *be a pseudoriemannian manifold of dimension at least three which has signature* $(s, n-s)$ *with* $2 \leq s \leq n-1$. *If the sectional curvature is not constant at* $p \in X$, *then the set of spacelike vectors* $v \in T_pX$ *such that the sectional curvature is unbounded on the pencil of planes containing* v *forms an open dense subset of the collection of all spacelike vectors in* T_pX.

Proof. Let W be the set of spacelike vectors v such that K_β is unbounded on the pencil of planes containing v.

We first prove W is dense in the set of all spacelike vectors at p. If v_0 is a spacelike vector not in W, then Lemma 4.3 shows K_β is a contant k_0 on the set of planes containing v_0. Since K_β is not constant on T_pX, Theorems 4.1 and 4.2 imply there is a spacetime plane Π_1 with $K_\beta(\Pi_1) = k_1 \neq k_0$. Then $v \notin \Pi_1$ and $K_\beta(\Pi) \neq k_0$ for all planes Π sufficiently close to Π_1 in the usual topology on $G_2(T_pX)$. By redefining Π_1 if necessary, we may assume w.l.o.g. that the three dimensional linear subspace L containg v and Π_1 is nondegenerate. Since v is spacelike and Π_1 is a spacetime plane, the subspace L is either Lorentzian of signature $(+--)$ or Lorentzian of signature $(++-)$. In either case, $k_1 \neq k_0$ implies that K_β is not constant on L and this yields the existence of at most six spacelike directions in L such that K_β is contant on the pencils in L determined by these directions. It follows that there are spacelike vectors in L arbitrarily close to v_0 such that K_β is not constant on the pencils determined by these vectors. It follows that v_0 is in the closure of the set W.

That W is open follows easily from the fact that if K_β is unbounded on the nondegenerate planes containing $v_1 \in W$, then K_β cannot be constant on any pencil determined by a spacelike vector v sufficiently close to v_1. □

Remark 4.5. *The conclusion of Proposition* 4.4 *remains valid for timelike vectors* v *as well as spacelike vectors* v *provided the signature* $(s, n-s)$ *satisfies* $2 \leq s \leq n-2$.

If $\dim X \geq 4$, then there may be uncountably many spacelike vectors v such that the sectional curvature is bounded both above and below on the set of nondegenerate planes containing v. Let (X_0, β_0) be any two-dimensional Lorentzian manifold with $R_{1212} \neq 0$ at some point $p_0 \in X_0$, and let $X = X_0 \times \mathbb{R}^{n-2}$ have the Lorentzian product structure $\beta_0 \oplus (-dx_3^2 - \cdots - dx_n^2)$. At $p \in X$ of the form $(p_0, x_3, \ldots, x_n)$, all components of the curvature tensor will vanish except for $R_{1212} = R_{2121} = -R_{1221} = -R_{2112} \neq 0$. It is easy to check that if $v \in T_p(X)$ is tangent to $\mathbb{R}^{n-2}$, then any nondegenerate plane containing v must have sectional curvature zero. The set of all such $v \in T_pX$ clearly forms a codimension-2 linear subspace.

REFERENCES

[1] J. K. BEEM and P. E. EHRLICH, *Global Lorentzian Geometry*, New York: Marcel Dekker, 1981.

[2] J. K. BEEM and P. E. PARKER, *Whitney stability of solvability*, Pac. J. Math., to appear.

[3] M. DAJCZER and K. NOMIZU, *On sectional curvature of indefinite metrics II*, Math. Ann. *247* (1980), 279–282.

[4] M. DAJCZER and K. NOMIZU, *On the boundedness of Ricci curvature of an indefinite metric*, Bol. Soc. Bras. Mat. *11* (1980), 25–30.

[5] L. GRAVES and K. NOMIZU, *On sectional curvature of indefinite metrics*, Math. Ann. *232* (1978), 267–272.

[6] S. G. HARRIS, *A triangle comparision theorem for Lorentz manifolds*, Indiana Math. J. *31* (1982), 289–308.

[7] R. S. KULKARNI, *The value of sectional curvature in indefinite metrics*, Comm. Math. Helv. *54* (1979), 173–176.

[8] K. NOMIZU, *Remarks on sectional curvature of an indefinite metric*, Proc. Amer. Math. Soc. *89* (1983), 473–476.

[9] J. A. THORPE, *Curvature and the Petrov canonical forms*, J. Math. Phys. *10* (1969), 1–7.

[10] J. A. WOLF, *Spaces of Constant Curvature*, 3rd edition, Boston: Publish or Perish, 1974.

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C. Apostol, C. M. Pearcy, B. Sz.-Nagy, D. Voiculescu, Manag. Edit. Gr. Arsene, **Dilation Theory, Toeplitz Operators, and Other Topics** (7th International Conference on Operator Theory, Timişoara and Herculane (Romania), June 7–17, 1982), Birkhäuser Verlag Basel-Boston-Stuttgart, 1983, 408 pp., SFr. 74.–

List of participants – Programme of the conference – Contributions by Albrecht E./Vasilescu F.-H., Alpay S., Ceauşescu Z./Suciu I., Clancey K. F., Coburn L. A., Constantinescu T., Gurarie D./Kon M. A., Haslinger F., Helson H., Herrero D. A., Holbrook J. A., Janas J., Kummerer B., Nikol'skii N. K., Pasnicu C., Putinar M., Riedel N., Salinas N., Schmüdgen K., Stochel J., Stochel J./Szafraniec F. H., Sz.-Nagy B./Foiaş C., Timotin D., Upmeier H.

D. Xia, **Spectral Theory of Hyponormal Operators,** Birkhäuser Verlag Basel-Boston-Stuttgart, 1983, 241 pp, SFr. 54.–

1. Elementary properties of hyponormal operators and semi-hyponormal operators – 2. Symbols – 3. Singular integral models – 4. Relations between the spectra of semi-hyponormal operators and those of the general polar symbols – 5. Mosaics and characteristic functions – 6. Spectral mapping – 7. Pincus principal functions, traces and determinants – Appendix: Spectral analysis of contractions

H. Triebel, **Theory of Function Spaces,** Birkhäuser Verlag Basel-Boston-Stuttgart, 1983, 284 pp., SFr. 78.–

I. Function Spaces and Elliptic Differential Equations – 1. Spaces of Entire Analytic Functions – 2. Function Spaces on R_n – 3. Function Spaces on Domains – 4. Regular Elliptic Differential Equations – II. Further Types of Function spaces – 5. Homogeneous Function Spaces – 6. Ultra-Distributions and Weighted Spaces of Entire Analytic Functions – 7. Weighted Function Spaces on R_n – 8. Weighted Function Spaces on Domains and Degenerate Elliptic Differential Equations – 9. Periodic Function Spaces – 10. Further Types of Function Spaces

L. H. Kauffmann, **Formal Knot Theory,** Princeton University Press, 1983, 167 pp., US$16.–

1. Introduction – 2. States, Trails, and the Clock Theorem – 3. State Polynomials and the Duality Conjecture – 4. Knots and Links – 5. Axiomatic Link Calculations – 6. Curliness and the Alexander Polynomial – 7. The Coat of Many Colors – 8. Spanning Surfaces – 9. The Genus of Alternative Links – 10. Ribbon Knots and the Arf Invariant – Appendix. The Classical Alexander Polynomial

M. Kashiwara, **Systems of Microdifferential Equations,** Birkhäuser Verlag Boston-Basel-Stuttgart, 1983, 159 pp., SFr. 42.–

Introduction (by J. L. Brylinski) – 1. Microfunctions – 2. Microdifferential Systems – 3. Structure of Coherent $\mathscr{E}_X$-modules – 4. Holomorphic Solutions of Systems of Partial Differential Equations – 5. Solutions of Holonomic Systems – 6. Index Theorem – Appendix 1: Derived Categories and Functors – Bibliography – Appendix 2: Whitney Stratification and Constructible Sheaves – Bibliography