Some theorems on generation of ideals in affine algebras.

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Some theorems on generation of ideals in affine algebras

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§1. Introduction

Let A be an affine ring over a field k. We are interested in studying the following statements and their relation to Eisenbud–Evans type theorems.

C-1: Let A be reduced. If $I \subset A$ is any ideal which is a local complete intersection of height $=\dim A$, then I is a complete intersection.

C-2: Every maximal ideal of A is a complete intersection. (In particular, A is regular.)

C-3: $A^n(A) = \{\text{zero cycles modulo rational equivalence}\} = 0$.

If A is smooth, it is trivial to see that C-1 $\Rightarrow$ C-2 $\Rightarrow$ C-3. In this paper, we will prove that C-2 $\Rightarrow$ C-1 if k is algebraically closed. The equivalence of C-2 and C-3 is known only when $\dim A \leq 3$ and k is algebraically closed [MP].

Now we will state the three statements of Eisenbud–Evans conjectures, which were conjectured only for polynomial rings [EE]. We will not call these conjectures, since for general rings they are obviously false. The original conjectures are all proved [for e.g. see [BP]]. Let $\dim A = n$.

EE-1: *Let P, Q be two projective modules of rank n over A. If $P \oplus A \simeq Q \oplus A$, then $P \simeq Q$.*

EE-2: *If P is a rank n projective module over A, then $P \simeq Q \oplus A$, where Q is a rank $(n-1)$ projective module.*

EE-3: *Let M be any finitely generated module over A. Let $\mu_p(M)$ denote the minimal number of generators of M_p as an A_p-module where p is any prime ideal of A. Define $e(M) = \max\{\mu_p(M) + \dim A/p \mid \dim A/p < \dim A\}$. Then: minimal number of generators of $M = \mu(M) \leq e(M)$.*

Suslin's cancellation theorem asserts the validity of EE-1 for any A over an algebraically closed field. Amit Roy and M. P. Murthy have proved EE-1 when the base field is finite [AR]. EE-2 and EE-3 are easily seen to be false for such general rings. We will prove that EE-2 and EE-3 are equivalent to C-1 (at least when k is algebraically closed). We will show that EE-2 implies C-1 when k is a finite field. The major obstacle we face in proving C-1 assuming EE-2, is the analogue of Theorem 3.1 in [MP]. There we prove that when k is algebraically closed, every local complete intersection maximal ideal of A is projectively '$\dim A$' generated (i.e. there exists a projective module of the correct rank mapping onto the maximal ideal). In fact our proof actually gives that this projective module can be chosen to have determinant trivial if $\dim A \geqslant 2$. We will construct examples of smooth 3-folds over a field and a point which is not the zero of a section of a rank 3 projective module with trivial determinant This is a deviation from the algebraically closed field case. This will give a rank two stably free non-free module over smooth 3-folds over such fields. Also using the same techniques, we will construct a rank two projective module over a smooth 4-fold over $\mathbb{C}$, which is stably trivial and not trivial. To the best of our knowledge, all examples of stably free non-free modules were also non trivial holomorphically (and hence topologically). But this example by [MS] is trivial holomorphically: So this is strictly an algebraic example!

§2

LEMMA 1. *Let A be a reduced noetherian ring and let P, Q be two constant rank projective modules of the same rank. Let $\varphi : P \to A$ and $\psi : Q \to A$ be any two homomorphisms such that $\varphi(P) \subset \psi(Q)$ and $\varphi(P)$ is not contained in any minimal prime ideal. Then there exists a monomorphism $\theta : P \to Q$ such that $\varphi = \psi \circ \theta$.*

Proof. Existence of a θ with $\psi \circ \theta = \varphi$ is trivial. We will modify θ to make it injective. Let $M = \operatorname{Ker} \psi$ and denote by i the inclusion $M \hookrightarrow Q$. Let K be the total quotient ring of A. $P \otimes_A K$ and $Q \otimes_A K$ are K-free modules of the same rank. So $\operatorname{Ker} \theta \otimes_A K \simeq \operatorname{Coker} \theta \otimes_A K$. Thus we can write $P \otimes K \simeq P_1 \oplus P_2$, $Q \otimes K = Q_1 \oplus Q_2$ and $\theta|_{P_1} : P_1 \to Q_1$ is an isomorphism, $P_2 = \operatorname{Ker} \theta \otimes K$ and $P_2 \simeq Q_2 \simeq \operatorname{Coker} \theta \otimes_A K$.

Since $\varphi(P)$ is not contained in any minimal prime ideal, $\varphi(P) \otimes_A K = K = \psi(P) \otimes_A K$. Using this, it is trivial to verify that $M \otimes_A K$ surjects onto $\operatorname{Coker} \theta \otimes_A K$. So write, $M \otimes K = M_1 \oplus M_2$ with $i(M_1)$ maps isomorphically onto $\operatorname{Coker} \theta \otimes K$ and $i(M_2)$ maps to zero. Thus we get that $M_1 \simeq P_2$. Define $f : P \otimes K \to M \otimes K$ as follows. f maps P_1 to zero, P_2 isomorphically onto M_1. Multiplying f by a non-zero divisor of A we may further assume that $f \in \operatorname{Hom}_A(P, M)$. So f naturally defines a map $f : P \to Q$. Let $\theta' = \theta + f \in \operatorname{Hom}_A(P, Q)$. Since $f \in \operatorname{Hom}(P, M)$, it is trivial to see that $\psi \circ \theta' = \varphi$. We will check that θ' is injective. It is sufficient to check that $\theta' : P \otimes K \to Q \otimes K$ is injective. So let $p = (p_1, p_2)$ belong to $P \otimes K$, $p_i \in P_i$ and $\theta'(P) = 0$. So the image of $\theta'(p)$ in Coker θ is zero. i.e. image of $f(p)$ in Coker θ is zero. But $f(p) \in i(M_1)$ and $i(M_1)$ maps isomorphically onto $\operatorname{Coker} \theta \otimes_A K$. Therefore $f(p) = 0$. By definition of f we see that $p_2 = 0$. Since θ maps P_1 injectively into Q, we get that $p_1 = 0$. i.e. $p = 0$.

THEOREM 1. *Let A be a reduced affine ring of dimension n over k. Then* C-1 *implies* EE-2 *if k is finite or algebraically closed. If k is algebraically closed and* char $k \nmid (n-1)!$, *then* EE-2 $\Rightarrow$ C-1.

Proof. C-1 $\Rightarrow$ EE-12.

Let P be any rank n projective module. Taking a generic section of P^*, we can find a surjective map $\varphi : P \to I$, where I is a local complete intersection ideal of height n. By C-1, I is actually a complete intersection. So we have a surjective map $\psi : F \to I$, F a free A-module of rank n. By Lemma 1, we can find a $\theta : P \to F$ such that $\psi \circ \theta = \varphi$ and such that θ is injective. So θ is an isomorphism at every minimal prime of A. Also using the fact that I/I^2 is a free rank n module over A, we get that θ is an isomorphism at every maximal ideal containing I. Putting these together, we see that there exists a non-zero divisor $x \in A$ which is not contained in any maximal ideal containing I and such that x annihilates $K = \text{Coker}\ \theta$. So $K/xK = K$ and $I/xI = A/xA$. Denote by going modulo x. We get the following diagram:

$$\begin{tikzcd}
\bar{P} \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{\theta}"] & \bar{I} = \bar{A} \arrow[d, equal] \\
\bar{F} \arrow[r, "\bar{\psi}"] \arrow[d] & \bar{I} = \bar{A} \\
K \arrow[d] & \\
0 &
\end{tikzcd}$$

Let $p \in \bar{P}$ such that, $\bar{\varphi}(p) = \bar{I} \in \bar{A}$. Then clearly $\bar{\theta}(p) \in \bar{F}$ is unimodular. So $\bar{F}/\bar{\theta}(P)$. $\bar{A}$ is an $\bar{A}$-free module of rank $n-1$, by [Su, AR]. Thus $\mu(K) \leq n-1$. So we have an exact sequence, $0 \to Q \to A^{n-1} \to K \to 0$. Since $pd_A K = 1$, we get that Q is a projective module of rank $n-1$. Now by Schanuel's lemma, we get that P and Q are stably isomorphic and hence by [Su, AR], $P \simeq Q \oplus A$.
EE-2 $\Rightarrow$ C-1 if k is algebraically closed and Char $k \mid (n-1)!$

Let $I \subset A$ be any local complete intersection ideal of height n. Then by [MP], there exists a rank n projective module P with trivial determinant mapping onto I. (Notice that if dim $A = 1$, there is nothing to prove). By EE-2, we get a map

$\varphi : Q \oplus A \to I$, Q, a projective module of rank $n-1$. By general arguments we may modify this so that height of $\varphi(Q)$ is $n-1$. Let $\varphi(Q)=J$. Then $I=J+Ax$, $x=\varphi(0,1)$. Also Q/JQ surjects onto J/J^2. Since $\dim A/J=1$, and $\det(Q)$ is trivial, Q/JQ is A/J-free of rank $n-1$. So $\mu(J/J^2)=n-1$. Now by [[MK] Lemma 1] $\mu(I)\leqslant n$.

Remark. If A is smooth, then EE-2 $\Rightarrow$ C-2, when k is algebraically closed. Since in [Theorem 3.1 [MP]] no characteristic assumption was necessary.

THEOREM 2. *Let A be any reduced affine ring of dimension n over k, where k is algebraically closed or finite. Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$ be elements of A such that, $(x_1, \dots, x_n) = I \cap (y_1, \dots, y_n)$, (y_i) is a complete intersection ideal of height n, ht $I \geqslant 1$ and $I + (y_1, \dots, y_n) = A$. Then $\mu(I) \leqslant n$.*

Proof. If $n=1$, theorem is trivial. The case when $n=2$ is slightly different from the case $n\geqslant 3$.

We may assume by Lemma 1 that $x_i = \sum a_{ij}y_j$ and $\det(a_{ij})=d$ is a non-zero divisor in A. Write $J=(y_1, \dots, y_n)$. We have an obvious surjection from $A^n \to (y_1, \dots, y_n)=J$. Let K be the kernel. Similarly we have an obvious exact sequence

$$0 \to L \to A^n \to I \cap J = (x_1, \dots, x_n) \to 0 \tag{$*$}$$

It is easy to see that $(x_1, \dots, x_n, d) = I$. Thus dA is comaximal with J. So we get an exact sequence,

$$0 \to K/dK \to (A/dA)^n \to J/dJ = A/dA \to 0.$$

So K/dK is stably free of rank $n-1$ over A/dA and hence free, [Su, AR]. Since rank $K=n-1$, it is easy to see that there exists an element $e=1+fd$ in J, which is a non-zero divisor and K_e is free. In other words there exists a free module F of rank $n-1$, with $K \subset F$ and $K_e = F_e$.

Case when $n=2$. In this case, $K \simeq A$, since J is a complete intersection ideal of height 2. Let L be as in $(*)$. Then $\mathrm{Ext}^1(I \cap J, L) \simeq \mathrm{Ext}^1(I, L) \oplus \mathrm{Ext}^1(J, L)$. $(*)$

corresponds to an element (a, b) in this module. Since d annihilates $\mathrm{Ext}^1(I, L)$ and e annihilates $\mathrm{Ext}^1(J, L)$, we have, $ea=(1+fd)\,a=a$ and $eb=0$. Consider the following push-out diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d, "e"] & A^2 \arrow[r] \arrow[d] & I\cap J \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & I\cap J \arrow[r] & 0 \\
 & L/eL \arrow[r, equal] \arrow[d] & Q/A^2 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Then, $0\to L\to Q\to I\cap J\to 0$ corresponds to the element $(a, 0)$ in $\mathrm{Ext}^1(I\cap J, L)$. Also $(L/eL)\simeq(L/eL)_d\simeq L_d/eL_d\simeq A_d/eA_d\simeq A/eA$. Thus $Q/A^2\simeq A/eA$. So Q has finite homological dimension and $[Q]=[A^2]$ in $K_0(A)$. Since $(a, 0)$ is in the image of $\mathrm{Ext}^1(I, L)$ in $\mathrm{Ext}^1(I\cap J, L)$, we get an exact sequence, $0\to L\to P\to I\to 0$, which when pulled back by the inclusion $I\cap J\hookrightarrow I$, gives $0\to L\to Q\to I\cap J\to 0$. Since I_d and L_d are free P_d is free of rank 2. Also $P_e=Q_e\simeq A_e^2$. So P is projective of rank 2. Also $P/Q\cong I/I\cap J\cong A/J$ and hence $[P/Q]=[A/J]=0$ in $K_0(A)$. Thus $[P]=[Q]=[A^2]$. P is stably free and hence free of rank 2.

Case when $n\geqslant 3$. Using the inclusion $I\cap J\hookrightarrow J$ we get a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d, "(a_{ij})"] & I\cap J \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & K \arrow[r] & A^n \arrow[r] & J \arrow[r] & 0
\end{tikzcd}$$

Thus $L\subset K$ and $L_d=K_d$. We can think of all these modules as submodules of F_{de}. Let $M=L_e\cap F_d$. Since $L_e\subset K_e=F_e$, we get, $M\subset F_e\cap F_d=F$ and hence M is a finitely generated A-module. Also $M_e=L_e$ and $M_d=F_d$. Notice that $L\subset M$ and $M/L\simeq F/K$. So M/L has finite homological dimension since K has.

Using $L \subset M$ and $(*)$, we get a pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & I \cap J \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & I \cap J \arrow[r] & 0 \\
F/K \simeq & M/L \arrow[d] \simeq & Q/A^n \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \qquad (**)$$

Also the natural map $\mathrm{Ext}^1(I, M) \to \mathrm{Ext}^1(I \cap J, M)$ is an isomorphism, since $\mathrm{Ext}^1(I \cap J, M) \simeq \mathrm{Ext}^1(I, M) \oplus \mathrm{Ext}^1(J, M)$ and $\mathrm{Ext}^1(J, M) \simeq \mathrm{Ext}^1(J_d, M_d) \simeq \mathrm{Ext}^1(J_d, F_d) \simeq \mathrm{Ext}^1(J, F) = 0$ since $n \geq 3$.

So there exists an exact sequence, $0 \to M \to P \to I \to 0$, which when pulled back by the inclusion $I \cap J \hookrightarrow I$ gives $(**)$. So we get $P/Q \simeq I/I \cap J \simeq A/J$. Also since $P_e \simeq Q_e \simeq A^n$ and $P_d \simeq A_d \oplus M_d \cong A_d \oplus F_d$, P is a projective module of rank n. Since Q has finite projective dimension, in $K_0(A)$, we have $[P]-[Q]=[A/J]=0$. Also $[Q]-[A^n]=[F]-[K]=[F]+[A]-[A^n]=0$. So $[P]\cong[A^n]$. P is stably free and hence free by [Su, AR]. This proves the theorem.

Remark. If height $I \geq 2$ in Theorem 2, one can give a more elementary proof of the Theorem.

COROLLARY 1. *Let A be as in Theorem 2. Let I and J be two local complete intersection height n co-maximal ideals. If any two of I, J and $I \cap J$ are complete intersections, then so is the third.*

Proof. If I and $I \cap J$ are complete intersections, then J is also a complete intersection by Theorem 2. Assume that I and J are complete intersections. Then we can find a sequence $x_1, \ldots, x_n$ in $I \cap J$ which generate $I \cap J$ modulo its square. So, $(x_1, \ldots, x_n) = I \cap J \cap K$, K co-maximal with $I \cap J$, height n and local complete intersection. Since I is a complete intersection, by Theorem 2, $\mu(J \cap K) = n$. Since J is a complete intersection, again by Theorem 2, $\mu(K) = n$. So K is a complete intersection and again by Theorem 2, $I \cap J$ is a complete intersection.

COROLLARY 2. *If A is a regular affine domain, over $k = \bar{k}$, then* C-2 $\Rightarrow$ C-1.

Proof. By Corollary 1, if $I=$ finite intersection of maximal ideals, then I is a complete intersection. Let I be any local complete intersection height n ideal of A. By Bertini's Theorem [Sw], we can choose $x_1, \ldots, x_n \in I$, which generate $I \bmod^2 I^2$ and the residual intersection is a finite set of reduced points i.e. $(x_1, \ldots, x_n) = I \cap M$, where M is intersection of finitely many maximal ideals co-maximal to I. Now by Corollary 1 we are done.

COROLLARY 3. *Let A be an affine domain over k, where k is a finite field or algebraically closed. Then* C-1 $\Leftrightarrow$ EE-3.

Proof. EE-3 $\Rightarrow$ C-1 is trivial. So assume C-1. By the theorem of Sathaye [for e.g. see [MK1] (with obvious modifications)], we need to prove only that if I is any ideal of A with $\mu(I/I^2) = \dim A = n$, then $\mu(I) = n$. Again by general arguments we can find $x_1, \ldots, x_n \in I$, generating $I \bmod I^2$ and the residual intersection is a height n local complete intersection ideal J. By C-1, J is a complete intersection. Then Theorem 2 implies the result.

§3. Some examples

Let K be any field, p any prime number and n any positive integer. Let $a \in K$ such that, $X^{p^n} + a$ (and hence $X^{p^n} + a^{p-1}$) in $K[X]$ are irreducible polynomials. Then, consider the homogeneous polynomial, $F_n = ((((X_0^p + aX_1^p)^p + aX_2^{p^2})^p + aX_3^{p^3})^p + \cdots) + aX_n^{p^n}$ in $n+1$ variables. We claim that F is an irreducible polynomial. If not, when we specialise F at, $X_0 = X_1 = \cdots = X_{n-2} = 0$, we must get a reducible polynomial. But then we get $a^p X_{n-1}^{p^n} + aX_n^{p^n}$. This polynomial is irreducible since $X^{p^n} + a^{p-1}$ is irreducible. Let $X = \mathbb{P}_K^n \backslash \{F_n = 0\}$. Then X is a smooth affine n-fold. We claim that zero cycles modulo rational equivalence $= A^n(X) \simeq \mathbb{Z}/p\mathbb{Z}$. $A^n(X)$ is a quotient of $A^n(\mathbb{P}^n) \simeq \mathbb{Z}$ given by degree of the zero cycle. So to prove our claim we need to check that there exists a zero cycle of degree p on $F=0$ and any zero-cycle on $F=0$ has degree a multiplier of p.

If we intersect by $X_2 = X_3 = \cdots = X_n = 0$, we get a $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ and $F \cap \mathbb{P}^1$ is given by the equation, $X_0^p + aX_1^p = 0$. By our hypothesis on a, this point has degree p. To prove the other part of the claim, we need to check that for every point x on $F=0$, $\deg x$ is a multiple of p. Choose X_k with k maximum such that $X_k(x) \neq 0$. (k may be equal to n). Then x satisfies the equation F_k. Since $X_k(x) \neq 0$, 0, $x \in A_K^n$, given by $X_k \neq 0$. Let $f_k = F_k(X_0, \ldots, X_{k-1})$. Then f_k can be considered as a regular function on $\mathbb{A}^n \cap F$ and its image is zero in $K(x)$. But $f_k = g^p + a$, g a regular function on $\mathbb{A}^n \cap F$. Thus $k(x)$ contains $k[X]/(X^p + a)$. So by hypothesis $[k(x):k]$ is divisible by p.

EXAMPLE 1. The above example immediately gives projective modules over smooth rational surfaces which are not free plus an ideal if the base field is not algebraically closed.

EXAMPLE 2. Take $p=2$ and $n=3$ and any field K with an irreducible polynomial t^8+a over K. Then we get an affine open subset X of $\mathbb{P}^3_K$ such that $A^3(X) \simeq \mathbb{Z}/2$. Let $x \in X$ be any K-rational point. This exists since $\mathbb{P}^3$ has K-rational points and none of them can lie on $\mathbb{P}^3 - X$. Then we claim that x is not the zero of a section of a rank three projective module over x with trivial determinant. It is the zero of a section of rank 3-bundle namely $L \oplus L \oplus L$ where L is the restriction of hyperplane section on X.

Assume such a projective module P exists. Then $C_1(P)=0$, $C_3(P) \neq 0$, where C_1 denotes the Chern classes. $A^2(X)$ is generated by the class of $\mathbb{P}^1 \cap X$, where $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is any line.

Claim. There exists a projective module Q of rank 2 over X such that $C_1(Q)=0$ and $C_2(Q)=C_2(P)$.

We have seen in the construction of X, that there exists a $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $\mathbb{P}^1 \cap F_3 =$ a degree 2 point. Let $\mathbb{P}^1 \cap X = C$ and I denote its ideal in $\Gamma(X)=A$. Since $K_{\mathbb{P}^1} = \mathcal{O}(-2)$, and $\mathbb{P}^1 \cap F_3$ has degree 2, K_C is trivial.

$$K_C = \mathrm{Ext}^1_A(I, K_X) \simeq \mathrm{Ext}^1_A(I, A) \otimes K_X^{-1}.$$

But $K_X = \mathcal{O}_{\mathbb{P}^3}(-4)|_X$ and hence $K_X|_C$ is trivial by the same reasoning. So $\mathrm{Ext}^1(I, A)$ is trivial. i.e. $\mathrm{Ext}^1(I, A) \simeq A/I$. Thus we get a projective module Q of rank 2 with an exact sequence,

$$0 \to A \to Q \to I \to 0$$

Thus $C_1(Q)=0$ and $C_2(Q)$ is a generator of $A^2(X)$. Now by [[MP] Theorem 1], we can get projective modules of rank 2 with trivial first chern class and whose second chern class is any positive multiple of the generator of $A^2(X)$. Since $A^2(X)$ is torsion, we can arrange it to have any second chern class.

Now let Q be a rank two projective module with $C_1(Q)=0$ and $C_2(Q)=C_2(P)$. Using the fact that C_1 and C_2 are always isomorphisms from the appropriate filtration of $K_0(X)$, we get that $e=[P]-[Q \oplus A]$ is an element of $F^3K_0(X)$ and $C_3(e) \neq 0$. We have a surjection $\psi_3 : A^3(X) \to F^3K_0(X)$ and $c_3 \circ \psi_3$ is multiplication by 2. But $A^3(X) \simeq \mathbb{Z}/2\mathbb{Z}$ implies that $C_3=0$. This is a contradiction.

EXAMPLE 3. In the above example we saw that there exists a maximal ideal M of $A = \Gamma(X)$, such that no rank 3 projective module with trivial determinant maps onto it. Let f be some element in $A \backslash M$ such that M_f is 3-generated. We have an exact sequence,

$$0 \rightarrow K_f \rightarrow A_f^3 \rightarrow M_f \rightarrow 0.$$

Since $f \notin M$, there exists $g \in M$ such that, $fA + gA = A$. Localising the above exact sequence at g, we get K_g over A_{fg} to be a rank two stably free module. We claim that it is not free. If it were free, it is clear that the split exact sequence $0 \rightarrow A_g^2 \rightarrow A_g^3 \rightarrow A_g \rightarrow 0$ can be restricted to A_{fg} and patched up by a matrix whose determinant is trivial and thus a projective module of rank 3 with trivial determinant mapping onto I. This contradicts Example 2.

EXAMPLE 4. There exists a smooth affine rational 4-fold over $\mathbb{C}$ and a rank two projective module which is stably free and not free.

In Example 3, we can take any base field K such that there exists an element $a \in K$ and $X^8 + a \in K[X]$ is irreducible. In particular we can take $K = \mathbb{C}(t)$. Thus there exists a smooth rational affine 3-fold over K and a rank two stably free non-free module. By usual 'spreading' technique we can get a 3-fold over a smooth rational curve/$\mathbb{C}$ and a projective module as before. This 4-fold clearly does the trick.

Remark. It is not known whether stably free rank two projective modules over an affine 3-fold/$\mathbb{C}$ are free. The above method will not work to construct an example, since, Suslin has proved that such examples cannot exist for surfaces over a C^1-field. (I do not know a proof of this).

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