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## On the Nehari univalence criterion and quasicircles

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### 1. Jordan domains

We assume throughout the paper that the function  $f$  is meromorphic and locally univalent in the unit disk  $\mathbb{D}$ . The Schwarzian derivative

$$S_f(z) = \frac{d}{dz} \frac{f''(z)}{f'(z)} - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \tag{1.1}$$

is analytic in  $\mathbb{D}$ . It satisfies

$$S_{\varphi \circ f \circ \psi}(z) = S_f(\psi(z))\psi'(z)^2 + S_\psi(z) \tag{1.2}$$

for  $\varphi \in \text{Möb}$ , where Möb denotes the group of Möbius transformations.

Nehari [13] has shown that if

$$(1 - |z|^2)^2 |S_f(z)| \leq 2 \quad \text{for } z \in \mathbb{D}, \tag{1.3}$$

then  $f$  is univalent in  $\mathbb{D}$ .

The bound 2 cannot be improved because

$$f(z) = [(1+z)/(1-z)]^{i\varepsilon}, \quad \varepsilon > 0, \tag{1.4}$$

satisfies (1.3) with 2 replaced by  $2(1 + \varepsilon^2)$  but assumes some values infinitely often in  $\mathbb{D}$ .

The univalent function

$$f^*(z) = \log \frac{1+z}{1-z} \quad (z \in \mathbb{D}) \tag{1.5}$$

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satisfies  $(1 - z^2)^2 S_{f^*}(z) \equiv 2$  and maps  $\mathbb{D}$  onto the parallel strip

$$T = \left\{ w : -\frac{\pi}{2} < \text{Im } w < \frac{\pi}{2} \right\}. \tag{1.6}$$

Hence  $f(\mathbb{D})$  need not be a Jordan domain in  $\hat{\mathbb{C}}$  under the assumption (1.3).

Duren and Lehto [5] asked for conditions of the form

$$(1 - |z|^2)^2 |S_f(z)| \leq 2\lambda(|z|) \quad (r_0 < |z| < 1)$$

that imply that  $f(\mathbb{D})$  is a Jordan domain. They proved that  $\lambda(r) = 1 + \varepsilon/\log(1 - r)$  with  $\varepsilon > 0$  is a possible choice, and this was improved by Becker [3] to  $\lambda(r) = 1 + 2(1 + \varepsilon)(1 - r)/\log(1 - r)$ .

We shall show that the function  $f^*$  defined in (1.5) is essentially the only exception.

**THEOREM 1.** *Let  $f$  be meromorphic in  $\mathbb{D}$  and let*

$$(1 - |z|^2)^2 |S_f(z)| \leq 2 \quad \text{for } z \in \mathbb{D}. \tag{1.7}$$

*Then  $f$  has a spherically continuous extension to  $\bar{\mathbb{D}}$  and  $f(\mathbb{D})$  is a Jordan domain or the image of the parallel slit  $T$  under a Möbius transformation. Moreover if  $z_0 \in \partial\mathbb{D}$  and  $f(z_0) \neq \infty$ , then*

$$|f(rz_0) - f(z_0)| = O(\text{dist}(f(rz_0), \partial f(\mathbb{D}))^{1/2}) \quad \text{as } r \rightarrow 1 - 0. \tag{1.8}$$

The estimate (1.8) means geometrically that the Jordan curve  $\partial f(\mathbb{D})$  can at most have first order cusps (like two tangent circles).

In the second (exceptional) case, we can write

$$f = \varphi \circ f^* \circ \psi \quad \text{with } \varphi, \psi \in \text{Möb}, \psi(\mathbb{D}) = \mathbb{D}.$$

Thus  $(1 - |z|^2)^2 |S_f(z)| = 2$  on some hyperbolic geodesic, by (1.2) and (1.5). Hence we conclude from Theorem 1:

**COROLLARY 1.** *If*

$$(1 - |z|^2)^2 |S_f(z)| < 2 \quad \text{for } z \in \mathbb{D},$$

*then  $f(\mathbb{D})$  is a Jordan domain.*

The following more precise result will be stated under the normalization  $f''(0) = 0$ .

**THEOREM 2.** *Let the assumptions of Theorem 1 be satisfied and let  $f''(0) = 0$ . Then either*

$$f(z) = a \log \frac{e^{i\theta} + z}{e^{i\theta} - z} + b, \quad a, b \in \mathbb{C}, a \neq 0, \quad 0 \leq \theta < 2\pi, \quad (1.9)$$

or  $f$  has a homeomorphic extension to  $\bar{\mathbb{D}}$  with

$$|f(z) - f(z')| \leq M_1 \left( \log \frac{3}{|z - z'|} \right)^{-1} \quad (z, z' \in \bar{\mathbb{D}}), \quad (1.10)$$

$$|f(re^{i\theta}) - f(e^{i\theta})| \leq M_2 [\text{dist}(f(re^{i\theta}), \partial f(\mathbb{D}))]^{1/2} \quad (0 \leq r < 1, 0 \leq \theta < 2\pi) \quad (1.11)$$

for some constants  $M_1$  and  $M_2$ .

As the proof will show (see (3.4)), it is sufficient to assume instead of (1.7) that

$$\text{Re} [e^{2i\theta} S_f(re^{i\theta})] \leq \frac{2}{(1-r)^2} \quad (0 \leq \theta < 2\pi, 0 \leq r < 1) \quad (1.12)$$

in order to prove (1.10). This condition was considered by Steinmetz [16] who proved (1.10) with an extra factor  $1 - 2(1 - r^2)/\log [8/(1 - r^2)]$  in (1.12).

## 2. Quasidisks

The Jordan curve  $\Gamma$  is called a *quasicircle with constant  $M$*  if

$$\min [\text{diam } \Gamma_1, \text{diam } \Gamma_2] \leq M |w_1 - w_2| \quad \text{for } w_1, w_2 \in \Gamma \quad (2.1)$$

where  $\Gamma_1$  and  $\Gamma_2$  are the components of  $\Gamma \setminus \{w_1, w_2\}$ . A domain bounded by a quasicircle will be called a quasidisk. If  $f$  is univalent in  $\mathbb{D}$ , the  $f(\mathbb{D})$  is a quasidisk if and only if  $f$  has a quasiconformal extension to  $\hat{\mathbb{C}}$  as Ahlfors [1] has shown.

**THEOREM 3.** *If  $f$  is meromorphic in  $\mathbb{D}$  and if*

$$(1 - |z|^2)^2 |S_f(z)| \leq b < 2 \quad \text{for } z \in \mathbb{D}, \quad (2.2)$$

then  $f(\mathbb{D})$  is a quasidisk with constant

$$M \leq 8 \left(1 - \frac{b}{2}\right)^{-1/2}. \tag{2.3}$$

This result was proved by Ahlfors and Weill [2] except for the above estimate for the constant  $M$ . When  $b < 2$  the function

$$f(z) = \frac{[(1+z)/(1-z)]^a - 1}{[(1+z)/(1-z)]^a + 1} \quad (z \in \mathbb{D}), \quad a = \left(1 - \frac{b}{2}\right)^{1/2},$$

satisfies (2.2) while (2.1) holds for  $\Gamma = \partial f(\mathbb{D})$  only if

$$M \geq \left(2 \sin \frac{\pi a}{4}\right)^{-1} \geq \frac{2}{\pi} \left(1 - \frac{b}{2}\right)^{-1/2}.$$

Thus the order of the bound for  $M$  in (2.3) is best possible as  $b \rightarrow 2$ .

We give an extension of the Ahlfors–Weill theorem.

**THEOREM 4.** *Let  $f$  be meromorphic in  $\mathbb{D}$  and let*

$$\limsup_{|z| \rightarrow 1} (1 - |z|^2)^2 |S_f(z)| < 2. \tag{2.4}$$

*Then  $f$  has a spherically continuous extension to  $\bar{\mathbb{D}}$  and there exists  $p < \infty$  such that  $f$  assumes every value at most  $p$  times in  $\bar{\mathbb{D}}$ . If  $p = 1$  then  $f(\mathbb{D})$  is a quasidisk.*

The number  $p$  can be arbitrarily large because every function that is meromorphic and locally univalent in  $\bar{\mathbb{D}}$  satisfies (2.4).

The last assertion was conjectured by Becker [4]. He proved it under the additional hypothesis

$$\limsup_{|z| \rightarrow 1} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| < 2.$$

If  $f$  is not injective on  $\partial\mathbb{D}$ , then  $f(\mathbb{D})$  need not be a quasidisk as the example  $f(z) = e^{\pi z}$  shows.

**COROLLARY 2.** *If the meromorphic function  $f$  satisfies (1.7) and (2.4), then  $f(\mathbb{D})$  is a quasidisk.*

This follows at once from Theorems 1 and 4; the exceptional case in Theorem 1 cannot occur because of (2.4).

Our next result is a quantitative version of a theorem of Sullivan [17]. It is a consequence of a result of Mañé, Sad, and Sullivan [11] for which we give an invariant version in terms of the cross ratio

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}. \quad (2.5)$$

The Jordan curve  $\Gamma \subset \hat{\mathbb{C}}$  is a quasicircle if and only if [1, p. 295]

$$|(z_1, z_2, z_3, z_4)| \leq K_0 \quad (2.6)$$

for all ordered quadruples  $z_1, z_2, z_3, z_4$  on  $\Gamma$  and some constant  $K_0$ .

**THEOREM 5.** *Let the domain  $G \subset \hat{\mathbb{C}}$  be bounded by a quasicircle  $\Gamma$  satisfying (2.6). Let the function*

$$g = g(z, \lambda) : G \times \mathbb{D} \rightarrow \hat{\mathbb{C}}$$

*be injective in  $z$  (for fixed  $\lambda$ ) and meromorphic in  $\lambda$  (for fixed  $z$ ). Let  $g(z, 0) \equiv z$ . If  $\lambda \in \mathbb{D}$ , then  $g(G, \lambda)$  is bounded by a quasicircle  $g(\Gamma, \lambda)$  with*

$$|(w_1, w_2, w_3, w_4)| \leq \frac{1}{16} \exp \left[ (\pi + \log K_0) \frac{1 + |\lambda|}{1 - |\lambda|} \right] \quad (2.7)$$

*for all ordered quadruples  $w_1, w_2, w_3, w_4$  on  $g(\Gamma, \lambda)$ .*

Let now  $G$  be a simply connected domain and let  $\rho_G$  denote the hyperbolic (Poincaré) metric of  $G$ . Let the functions  $f$  be meromorphic and locally univalent in  $G$ . Ahlfors [1] and Gehring [8] have proved that, if and only if  $G$  is a quasidisk, there is a constant  $a > 0$  such that

$$|S_f(z)| \leq a \rho_G(z)^2 (z \in G) \quad \text{implies } f \text{ univalent in } G.$$

It follows from the argument given in [8] that also the image  $f(G)$  is a quasidisk if  $a$  is replaced by a smaller number.

We show now that the last fact holds in a much more general context.

**THEOREM 6.** *Let  $G$  be bounded by a quasicircle  $\Gamma$  satisfying (2.6) and let  $\rho$*

be any positive function. Suppose that

$$|S_f(z)| \leq a\rho(z)^2 (z \in G) \quad \text{implies } f \text{ is univalent in } G. \tag{2.8}$$

If  $0 \leq b < a$  and

$$|S_f(z)| \leq b\rho(z)^2 \quad (z \in G), \tag{2.9}$$

then  $f(G)$  is bounded by a quasicircle  $f(\Gamma)$  with

$$|(w_1, w_2, w_3, w_4)| \leq \frac{1}{16} \exp \left[ (\pi + \log K_0) \frac{a+b}{a-b} \right] \tag{2.10}$$

for all ordered quadruples  $w_1, w_2, w_3, w_4$  on  $g(\Gamma)$ .

In we choose  $G = \mathbb{D}$ ,  $\rho(z) = (1 - |z|^2)^{-1}$  and  $a = 2$ , then (2.8) becomes the Nehari criterion. Hence we obtain a new proof of the Ahlfors–Weill theorem. It turns out however that, for  $b$  close to 2, the bound is substantially larger than the one obtained in Theorem 3.

*Remark.* A similar argument can be used to prove the following analogue of Theorem 6. Let the functions  $f$  be analytic and locally univalent in the simply connected domain  $G \subset \mathbb{C}$ . If there is a constant  $a > 0$  such that

$$\left| \frac{f''(z)}{f'(z)} \right| \leq a\rho(z) (z \in G) \quad \text{implies } f \text{ univalent in } G \tag{2.11}$$

and if  $0 \leq b < a$ , then

$$\left| \frac{f''(z)}{f'(z)} \right| \leq b\rho(z) (z \in G) \quad \text{implies } f(G) \text{ is a quasidisk.} \tag{2.12}$$

Martio and Sarvas [12, Theorem 4.9] have shown that (2.11) holds for some  $a > 0$  and  $\rho = \rho_G$  if  $G$  is a quasidisk. Astala and Gehring have just established the converse of this result, namely that (2.11) holds for some  $a > 0$  and  $\rho = \rho_G$  only if  $G$  is a quasidisk.

### 3. Proof of Theorem 2

(a) Let  $0 \leq \theta < 2\pi$ . The function

$$h(t) = e^{i\theta} \frac{e^t - 1}{e^t + 1} \quad (t \in T) \tag{3.1}$$

maps the strip  $T$  conformally onto  $\mathbb{D}$  and

$$g = f \circ h \tag{3.2}$$

is meromorphic and (at least) locally univalent in  $T$ . Computation shows that

$$|g'(t)| = \frac{1}{2}(1-r^2) |f'(re^{i\theta})| \quad \text{for } t \in \mathbb{R}, h(t) = re^{i\theta}. \tag{3.3}$$

Since  $S_h(t) = -\frac{1}{2}$ , it follows from (1.2) and (1.12) that

$$\operatorname{Re} S_g(t) = -\frac{1}{2} + \frac{1}{4}(1-r^2)^2 \operatorname{Re} [e^{2i\theta} S_f(re^{i\theta})] \leq 0 \tag{3.4}$$

for  $t \in \mathbb{R}$  and  $h(t) = re^{i\theta}$ .

We define

$$v(t) = |g'(t)|^{-1/2} \quad \text{for } t \in \mathbb{R}; \tag{3.5}$$

this function is zero at a possible pole of  $g$ . We see that

$$\frac{v'}{v} = -\frac{1}{2} \operatorname{Re} \frac{g''}{g'}, \quad \frac{v''}{v} - \left(\frac{v'}{v}\right)^2 = -\frac{1}{2} \operatorname{Re} \left[ \frac{d}{dt} \frac{g''}{g'} \right] \tag{3.6}$$

and therefore

$$v''(t) = p(t)v(t) \quad \text{for } t \in \mathbb{R} \tag{3.7}$$

(except where  $g$  has a pole) where

$$p(t) = -\frac{1}{2} \operatorname{Re} S_g(t) + \left(\frac{1}{2} \operatorname{Im} \frac{g''(t)}{g'(t)}\right)^2 \geq 0 \tag{3.8}$$

by (3.4). Hence  $v$  is non-negative and convex in  $\mathbb{R}$ ; this is also true if  $g$  has a pole at  $t_0 \in \mathbb{R}$  in which case  $v(t_0) = 0$ .

(b) We use now the hypothesis that  $f''(0) = 0$ . It follows from (3.2) that  $g''(0) = 0$ . Hence (3.6) shows that  $v'(0) = 0$ . Therefore  $v$  has its minimum at 0 where  $v(0) > 0$ , and we conclude that  $g(t) \neq \infty$  for  $t \in \mathbb{R}$ .

Let first  $v'(t_0) = 0$  for some  $t_0 \neq 0$ , say  $t_0 > 0$ . Since  $v$  is convex, we conclude that  $v'(t) = 0$  for  $0 \leq t \leq t_0$  and thus  $v''(t) = 0$ . Hence  $\operatorname{Re} [g''/g'] = 0$  by (3.6) and  $\operatorname{Im} [g''/g'] = 0$  by (3.4) and (3.8). We conclude that  $g''(t) = 0$  for  $0 \leq t \leq t_0$  and thus

for  $t \in T$  by the identity theorem. It therefore follows from (3.1) and (3.2) that  $f$  has the form (1.9).

Suppose next that  $f$  is not of the form (1.9). Then the above argument shows that  $v'(1) > 0$  for each choice of the constant  $\theta$  in (3.1). It follows by continuity that

$$v'(t) \geq \alpha > 0 \quad \text{for } 1 \leq t < \infty$$

for some constant  $\alpha$  and therefore

$$v(t) \geq v(t_0) + \alpha(t - t_0) \quad \text{for } 1 \leq t_0 \leq t < \infty. \tag{3.9}$$

In view of (3.5) this means that

$$|g'(t)| \leq \frac{1}{[v(t_0) + \alpha(t - t_0)]^2} \quad \text{for } 1 \leq t_0 \leq t < \infty. \tag{3.10}$$

(c) We obtain from (3.1), (3.3), and (3.10) that

$$|f'(z)| \leq 2\alpha^{-2}(1 - |z|^2)^{-1} \left( \log \frac{1 + |z|}{1 - |z|} - 1 \right)^{-2} \quad \text{for } |z| \geq \frac{e - 1}{e + 1}.$$

Hence there are constants  $a$  and  $b$  such that

$$|f'(z)| < \frac{a}{1 - |z|} \left( \log \frac{8}{1 - |z|} \right)^{-2} + b \quad \text{for } z \in \mathbb{D}. \tag{3.11}$$

We apply now a standard method (see for instance [15]) to derive (1.10) from (3.11). It is sufficient to consider  $z, z' \in \mathbb{D}$  because then (1.10) shows that  $f$  is uniformly continuous in  $\mathbb{D}$  and hence has a continuous extension to  $\bar{\mathbb{D}}$ . Let  $\Gamma$  be the hyperbolic segment joining  $z$  and  $z'$  in  $\mathbb{D}$ . Then  $\Gamma$  has length  $l \leq \pi |z - z'|/2$  and

$$\min(s, l - s) \leq \frac{\pi}{2} (1 - |\zeta|) \tag{3.12}$$

for each  $\zeta \in \Gamma$ , where  $s$  is the length of the part of  $\Gamma$  between  $z$  and  $\zeta$ . We see

from (3.11) and (3.12) that

$$\begin{aligned}
 |f(z) - f(z')| &\leq \int_{\Gamma} |f'(\zeta)| |d\zeta| \\
 &\leq \int_{\Gamma} \frac{a}{1-|\zeta|} \left( \log \frac{8}{1-|\zeta|} \right)^{-2} |d\zeta| + bl \\
 &\leq 2a \int_0^{1/2} \frac{\pi}{2s} \left( \log \frac{4\pi}{s} \right)^{-2} ds + bl \\
 &\leq \pi a \left( \log \frac{16}{|z-z'|} \right)^{-1} + \frac{\pi b}{2} |z-z'| \leq M_1 \left( \log \frac{3}{|z-z'|} \right)^{-1}
 \end{aligned}$$

because  $\frac{1}{x} \left( \log \frac{8}{x} \right)^{-2}$  is decreasing in  $(0, 1)$ .

(d) We also obtain from (3.5) and (3.10) that

$$\int_{t_0}^{\infty} |g'(t)| dt \leq \int_{t_0}^{\infty} \frac{dt}{[v(t_0) + \alpha(t-t_0)]^2} = \frac{1}{\alpha v(t_0)} = \frac{1}{\alpha} |g'(t_0)|^{1/2}$$

for  $1 \leq t_0 < \infty$ . Hence we see from (3.1), (3.2), and (3.3) that

$$|f(e^{i\theta}) - f(re^{i\theta})| \leq \frac{1}{\alpha} \left[ \frac{1}{2} (1-r^2) |f'(re^{i\theta})| \right]^{1/2}, \quad (3.13)$$

and (1.11) follows from a consequence of the Koebe distortion theorem [14, p. 22]. This completes the proof of Theorem 2 except for the statement that  $f$  is injective on  $\partial\mathbb{D}$ .

#### 4. Proof of Theorem 1

There exists  $\varphi \in \text{Möb}$  such that  $(\varphi \circ f)''(0) = 0$ . Hence it follows from Theorem 2 that  $\varphi \circ f$  and therefore  $f$  has a spherically continuous extension to  $\bar{\mathbb{D}}$ .

Suppose now that  $f$  is not injective on  $\partial\mathbb{D}$ . Since  $S_f$  is invariant under Möbius transformations, we may assume that

$$f(z_1) = f(z_2) = \infty, \quad z_1, z_2 \in \partial\mathbb{D}, \quad z_1 \neq z_2. \quad (4.1)$$

Let  $\Gamma$  be the hyperbolic geodesic joining  $z_1$  and  $z_2$  in  $\mathbb{D}$  and let  $h$  map the strip  $T$  conformally onto  $\mathbb{D}$  such that  $h(\mathbb{R}) = \Gamma$ .

We set  $g = f \circ h$ . Then  $g$  is analytic in  $T$  and we see as in part (a) of the proof of Theorem 2 that

$$v(t) = |g'(t)|^{-1/2} \quad (t \in \mathbb{R})$$

is convex and positive. Suppose that  $v'(t_0) \neq 0$  for some  $t_0 \in \mathbb{R}$ . If  $v'(t_0) = \alpha > 0$  then we obtain (3.10) as in part (b) of the proof of Theorem 2. This implies  $g(+\infty) \neq \infty$  in contradiction to (4.1). Similarly  $v'(t_0) < 0$  leads to  $g(-\infty) \neq \infty$  contradicting (4.1). Thus  $v'(t) \equiv 0$ ,  $g''(t) \equiv 0$  and  $g \in \text{Möb}$ . Hence  $f(\mathbb{D})$  is the image of  $T$  under the Möbius transformation  $g$ .

### 5. Proofs of Theorems 3 and 4

We need the following characterization of quasidisks. We say that the domain  $G \subset \mathbb{C}$  has a *c-accessible boundary* if each  $z_1, z_2 \in \partial G$  can be joined by an open arc  $A \subset G$  such that

$$\min_{j=1,2} |z - z_j| \leq c \text{ dist}(z, \partial G) \quad \text{for } z \in A. \tag{5.1}$$

It follows from (5.1) that  $c \geq 1$ .

**LEMMA 1.** *Let  $G$  be a Jordan domain in  $\mathbb{C}$ . Suppose that there is a constant  $c$  such that, for all  $\varphi \in \text{Möb}$  with  $\varphi(G) \subset \mathbb{C}$ , the domains  $\varphi(G)$  have  $c$ -accessible boundaries. Then  $\partial G$  is a quasi-circle with constant  $M \leq 2c$ .*

It easily follows from [9, Theorem III.2.3] that the converse holds except for the constants.

*Proof.* We show first that each  $w_1, w_2 \in \partial G$  can be joined by an open arc  $B \subset G$  such that

$$|w - w_1| \leq c |w_1 - w_2| \quad \text{for } w \in B. \tag{5.2}$$

We may assume that  $w_1, w_2$  are finite and set

$$\varphi(w) = (w - w_1)/(w - w_2).$$

Then  $\varphi(G) \subset \mathbb{C}$  with  $0, \infty \in \partial\varphi(G)$ . By hypothesis there is an open arc  $A$  joining 0

and  $\infty$  in  $\varphi(G)$  such that

$$|w| \leq c \operatorname{dist}(w, \partial\varphi(G)) \leq c |w - 1| \quad \text{for } w \in A$$

because  $1 \notin \varphi(G)$ . If  $w \in B = \varphi^{-1}(A)$  we deduce that

$$|w - w_1| = \frac{|\varphi(w)|}{|\varphi(w) - 1|} |w_1 - w_2| \leq c |w_1 - w_2|.$$

Now fix  $w_1, w_2 \in \partial G$  and suppose that

$$\min(\operatorname{diam} \Gamma_1, \operatorname{diam} \Gamma_2) > 2c |w_1 - w_2|$$

where  $\Gamma_1$  and  $\Gamma_2$  are the components of  $\partial G \setminus \{w_1, w_2\}$ . Then we can choose  $z_j \in \Gamma_j$  with

$$\min_{j,k=1,2} |z_j - w_k| > c |w_1 - w_2|. \quad (5.3)$$

Let  $C$  be the open segment  $(w_1, w_2)$  and suppose first that  $C \cap \partial G = \emptyset$ .

If  $C \subset G$  then we join  $z_1, z_2$  by an open arc  $A \subset G$  satisfying (5.1). Since  $C$  separates  $z_1$  and  $z_2$  in  $G$  we can choose  $z \in A \cap C$  in which case

$$\operatorname{dist}(z, \partial G) \leq \frac{1}{2} |w_1 - w_2|.$$

Thus, by (5.1),

$$\min_{j=1,2} |z_j - w_k| \leq \frac{c}{2} |w_1 - w_2| + |z - w_k| \leq c |w_1 - w_2| \quad (5.4)$$

where  $w_k$  is the endpoint of  $C$  nearest to  $z$ .

If  $C \subset \mathbb{C} \setminus \bar{G}$  let  $B$  be an open arc joining  $w_1, w_2$  in  $G$  for which (5.2) holds. Then  $B \cup \bar{C}$  is a Jordan curve which separates  $z_1$  and  $z_2$ , and hence

$$\min_{i=1,2} |z_i - w_1| \leq \max_{w \in B \cup \bar{C}} |w - w_1| \leq c |w_1 - w_2|$$

by (5.2). Together with (5.4) this shows that

$$\min_{j,k=1,2} |z_j - w_k| \leq c |w_1 - w_2| \quad (5.5)$$

whenever  $C \cap \partial G = \emptyset$ .

Thus we see from (5.3) that  $C \cap \partial G \neq \emptyset$ . Let  $C_1$  and  $C_2$  denote the components of  $\partial G \setminus \{z_1, z_2\}$ . For  $j = 1, 2$  we choose  $w'_j \in \bar{C} \cap C_j$  such that

$$|w'_1 - w'_2| = \text{dist}(\bar{C} \cap C_1, \bar{C} \cap C_2)$$

and let  $C' = (w'_1, w'_2)$ . Then  $z_1$  and  $z_2$  lie in different components of  $\partial G \setminus \{w'_1, w'_2\}$ . Since  $C' \cap \partial G = \emptyset$  it follows from (5.5) that

$$\min_{j,k=1,2} |z_j - w'_k| \leq c |w'_1 - w'_2|.$$

It is easy to see that this is a contradiction to (5.3). Thus  $\partial G$  is a quasircle with constant  $M \leq 2c$ .

*Proof of Theorem 3.* We show first that  $G$  is  $c$ -accessible. We verify (5.1) where it is sufficient to consider  $z_1 = f(-1)$ ,  $z_2 = f(1)$  because of (1.2).

We employ the notation of Section 3 with  $\theta = 0$ . It follows from (2.2) and from (3.4) through (3.8) that

$$v''(t) \geq a^2 v(t) \quad \text{for } -\infty < t < \infty \tag{5.6}$$

where  $a^2 = (2 - b)/8$ . For given  $t_0$  we may assume that  $v'(t_0) \geq 0$ ; otherwise we replace  $g(t)$  by  $g(-t)$ .

We compare the differential inequality (5.6) with the initial value problem

$$u''(t) = a^2 u(t) (t \geq t_0), \quad u(t_0) = v(t_0), \quad u'(t_0) = 0$$

which is solved by

$$u(t) = v(t_0) \cosh a(t - t_0).$$

From a well-known comparison theorem, or directly from

$$\begin{aligned} \frac{d}{dt} \frac{v(t)}{u(t)} &= \frac{v'(t)u(t) - v(t)u'(t)}{u(t)^2} \\ &= u(t)^{-2} \int_{t_0}^t (v''u - vu'') ds + v'(t_0)v(t_0) \geq 0 \end{aligned}$$

for  $t \geq t_0$ , we deduce that  $v(t) \geq u(t)$  for  $t \geq t_0$ . Thus, by (3.5),

$$\int_{t_0}^{\infty} |g'(t)| dt \leq |g'(t_0)| \int_{t_0}^{\infty} [\cosh a(t - t_0)]^{-2} dt = \frac{1}{a} |g'(t_0)|.$$

If  $z_0 \in (-1, +1)$  is given, we choose  $t_0$  such that  $z_0 = h(t_0)$  and obtain

$$\min_{j=1,2} |z_j - f(z_0)| \leq \frac{1}{a} |g'(t_0)| \leq \frac{2}{a} \text{dist}(f(z_0), \partial G)$$

by (3.3) and the Koebe distortion theorem. Thus (5.1) holds with

$$c = 4 \left(1 - \frac{b}{2}\right)^{-1/2}. \quad (5.7)$$

Since the Schwarzian derivative is Möbius invariant, we therefore conclude that the assumption of Lemma 1 is satisfied with (5.7) and  $G = f(\mathbb{D})$ . Thus  $f(\mathbb{D})$  is a quasidisk with constant

$$M \leq 2c = 8 \left(1 - \frac{b}{2}\right)^{-1/2}.$$

*Proof of Theorem 4.* By (2.4) there exist  $\delta > 0$  and  $r_1 < 1$  such that

$$(1 - |z|^2)^2 |S_f(z)| < 2 - 5\delta \quad \text{for } r_1 \leq |z| < 1. \quad (5.8)$$

Let  $\alpha > 0$ . The function

$$\varphi(\zeta) = e^{-i\pi\delta/2} \left(\frac{1+\zeta}{1-\zeta}\right)^{1-\delta} - i\alpha \quad (\zeta \in \mathbb{D}) \quad (5.9)$$

maps  $\mathbb{D}$  conformally onto a wedge of vertex  $-i\alpha$  and angle  $\pi(1-\delta)$  that lies in the right-hand halfplane and has  $[-i\alpha, -i\infty]$  as one boundary line. Hence

$$\psi(\zeta) = e^{i\theta} \frac{\varphi(\zeta) - 1}{\varphi(\zeta) + 1}, \quad 0 \leq \theta \leq 2\pi, \quad (5.10)$$

maps  $\mathbb{D}$  conformally onto a domain  $H$  in  $\mathbb{D}$  bounded by an arc of  $\partial\mathbb{D}$  together with a circle through  $e^{i\theta}$  and  $e^{i\theta}(\alpha - i)/(\alpha + i)$  that forms the angle  $\pi(1-\delta)$  with  $\partial\mathbb{D}$ . Hence we can choose  $\alpha$  so large that  $H \subset \{r_1 < |z| < 1\}$ . We see that, for some fixed  $\beta > 0$  independent of  $\theta$ ,

$$\{e^{it} : \theta - \beta \leq t \leq \theta\} \subset \partial H. \quad (5.11)$$

We obtain from (1.2), (5.10), and (5.9) that

$$S_\psi(\zeta) = S_\varphi(\zeta) = \frac{2\delta(2-\delta)}{(1-\zeta^2)^2} \quad (\zeta \in \mathbb{D}). \tag{5.10}$$

Since  $\psi(\mathbb{D}) = H \subset \{r_1 < |z| < 1\}$ , it follows from (1.2), (5.8), and (5.12) that the function  $h = f \circ \psi$  satisfies

$$\begin{aligned} |S_h(z)| &\leq |S_f(\psi(z))| \left( \frac{1-|\psi(z)|^2}{1-|z|^2} \right)^2 + |S_\psi(z)| \\ &\leq \frac{(2-5\delta)+4\delta}{(1-|z|^2)^2} = \frac{2-\delta}{(1-|z|^2)^2} \end{aligned}$$

for  $z \in \mathbb{D}$ . Hence we see from Theorem 3 that  $h$  maps  $\bar{\mathbb{D}}$  topologically onto a closed quasidisk with constant  $M = 8(2/\delta)^{1/2}$ .

Since the domains  $H$  are congruent for all  $\theta$ , it follows from (5.11) that some annulus  $\{r_2 < |z| < 1\}$  can be covered by finitely many domains  $H$ . Hence we obtain from the last paragraph that  $f$  has a continuous extension to  $\bar{\mathbb{D}}$  and assumes every value at most  $p$  times in  $\bar{\mathbb{D}}$  for some  $p < \infty$ .

Assume now that  $p = 1$ . Then  $\Gamma = f(\partial\mathbb{D})$  is a Jordan curve. We may assume that  $\text{diam } \Gamma \leq 1$  because the Schwarzian is Möbius invariant. Then there exists  $d > 0$  such that

$$|f^{-1}(w) - f^{-1}(w')| \leq \frac{\beta}{\pi} \quad \text{if } w, w' \in \Gamma, |w - w'| \geq d.$$

Choose  $w_1, w_2 \in \Gamma$  and let  $\Gamma_1, \Gamma_2$  denote the components of  $\Gamma \setminus \{w_1, w_2\}$ .

Let first  $|w_1 - w_2| \leq d/(2M)$ . We show that

$$\min(\text{diam } \Gamma_1, \text{diam } \Gamma_2) \leq 4M |w_1 - w_2|. \tag{5.13}$$

Otherwise we could find points  $z_1 \in \Gamma_1, z_2 \in \Gamma_2$  with

$$|z_j - w_1| = 2M |w_1 - w_2| \leq d \tag{5.14}$$

and a domain  $H$  such that  $z_1, z_2, w_1, w_2 \in \partial f(H)$ . Then  $z_1, z_2$  would lie in different components of  $\partial f(H) \setminus \{w_1, w_2\}$  and (5.14) would contradict the fact that  $\partial f(H)$  is a quasicircle with constant  $M$ .

If  $|w_1 - w_2| \geq d/(2M)$  then

$$\text{diam } \Gamma_1 \leq 1 \leq \frac{2M}{d} |w_1 - w_2|.$$

Hence we see from (5.13) that  $\Gamma$  is a quasicircle with constant  $M_1 \leq \max(2M/d, 4M)$ .

## 6. Proofs of Theorems 5 and 6

Theorem 5 is an immediate consequence (with  $A = G$ ) of the following lemma which is a quantitative and Möbius-invariant version of the surprising “ $\lambda$ -lemma” of Mañé, Sad and Sullivan [11].

**LEMMA 2.** *Let  $A$  be any set in  $\hat{\mathbb{C}}$  and let the function  $g = g(z, \lambda) : A \times \mathbb{D} \rightarrow \hat{\mathbb{C}}$  be injective in  $z$  (for fixed  $\lambda$ ) and meromorphic in  $\lambda$  (for fixed  $z$ ). Let  $g(z, 0) \equiv z$ . Then  $g(z, \lambda)$  has a spherically continuous extension to  $\bar{A} \times \mathbb{D}$  that is meromorphic in  $\lambda \in \mathbb{D}$  and satisfies*

$$|(w_1, w_2, w_3, w_4)| \leq \frac{1}{16} \exp \left[ (\pi + \log^+ |(z_1, z_2, z_3, z_4)|) \frac{1 + |\lambda|}{1 - |\lambda|} \right] \quad (6.1)$$

for every quadruple  $z_1, z_2, z_3, z_4$  in  $\bar{A}$  where  $w_j = g(z_j, \lambda)$ .

*Proof.* Fix distinct points  $z_j \in A$  ( $j = 1, 2, 3, 4$ ). The function

$$h(\lambda) = (g(z_1, \lambda), g(z_2, \lambda), g(z_3, \lambda), g(z_4, \lambda)) \quad (\lambda \in \mathbb{D}) \quad (6.2)$$

is meromorphic and omits the values 0, 1 and  $\infty$  because the points  $g(z_j, \lambda)$  are distinct. Hence we obtain

$$|h(\lambda)| \leq \frac{1}{16} \exp \left[ (\pi + \log^+ |h(0)|) \frac{1 + |\lambda|}{1 - |\lambda|} \right] \quad (6.3)$$

from the precise form of Schottky’s Theorem proved by Hempel [7] (see also [6]). Since  $h(0) = (z_1, z_2, z_3, z_4)$  this is our assertion (6.1) for the case  $z_j \in A$ . The general case will follow from the next paragraph by continuity.

Let now  $z_0 \in \bar{A}$  and let  $\zeta_n, \zeta'_n$  be distinct points in  $A \setminus \{z_2, z_4\}$  with  $\zeta_n \rightarrow z_0, \zeta'_n \rightarrow z_0$  as  $n \rightarrow \infty$ . The meromorphic functions

$$h_n(\lambda) = (g(\zeta_n, \lambda), g(z_2, \lambda), g(\zeta'_n, \lambda), g(z_4, \lambda)) \quad (\lambda \in \mathbb{D})$$

omit 0, 1,  $\infty$  and therefore form a normal sequence. Since  $h_n(0) = (\zeta_n, z_2, \zeta'_n, z_4) \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $h_n(\lambda) \rightarrow 0$  locally uniformly in  $\lambda \in \mathbb{D}$ . Hence  $g(\zeta, \lambda)$  has

a limit as  $\zeta \rightarrow z_0$ ,  $\zeta \in A$ , and it follows that  $g$  has a continuous extension to  $\bar{A} \times \mathbb{D}$  which is meromorphic in  $\lambda$ .

*Proof of Theorem 6.* Choose a point  $z_0 \in G$  with  $z_0 \neq \infty$ . Since the Schwarzian is Möbius invariant we may assume that  $f(z_0) = z_0$ ,  $f'(z_0) = 1$ ,  $f''(z_0) = 0$ . Let  $\lambda \in \mathbb{D}$ . Since  $G$  is simply connected, it follows from the theory of linear differential equations [10] that the initial value problem

$$S_g(z) = \lambda \frac{a}{b} S_f(z), \quad g(z_0) = z_0, g'(z_0) = 1, g''(z_0) = 0$$

has a unique solution  $g = g(z, \lambda)$  which is meromorphic in  $\lambda$ . Note that

$$g(z, 0) = z, g\left(z, \frac{b}{a}\right) = f(z). \quad (6.4)$$

We see from (2.9) that  $|S_g(z)| \leq a\rho(z)^2$  for  $z \in G$  so that  $g(z, \lambda)$  is univalent in  $G$  by condition (2.8). Hence our assertion follows from (6.4) and Theorem 5.

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