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Relative cohomology of groups

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§0. Introduction

If G is a group and A is a G-module, the cohomology groups $H^n(G, A)$ can be interpreted either as the cohomology of the cochain complex $C^*(G, A)$ given by the bar resolution (see [M]) or as equivalence classes of n-fold extensions. This situation has reached perfection and is summarized in an historical note by MacLane in [Ho]. If N is a normal subgroup of G and B is a submodule of A on which N acts trivially (and hence B is a Q-module with $Q = G/N$), one may define the relative cohomology groups $H^n(Q, G; B, A)$ as the cohomology of the quotient complex $C^*(G, A)/C^*(Q, B)$. In this paper we present another cochain complex $L^*(G, N; B, A)$ whose cohomology is the relative cohomology but which is more explicit than the quotient complex and which grew out of certain "crossed" extensions which give an interpretation of the group $H^2(Q, G; B, A)$, analogous to the n-fold extensions mentioned above.

A crossed extension is an exact sequence $1 \to A \to C \xrightarrow{\partial} N \to 1$ together with an action $(g, c) \to gc$ of G on C such that $\partial(c)d = cdc^{-1}$ for $c, d \in C$ and $\partial(gc) = g\partial(c)g^{-1}$ for $g \in G$, $c \in C$. It follows that A is in the centre of C and that the action of G induces the structure of a G-module on A. This idea goes back to Whitehead ([W]) and was studied extensively in [H], [L], [O] and [R]. One may define a sum of two crossed extensions and equivalence classes of such form a group. It can be shown that this group is isomorphic to the second relative cohomology group. One way to establish this isomorphism is to give a cocycle description of crossed extensions. More precisely, if $s : N \to C$ is a section for ∂, define $\alpha_0 : N \times N \to A$ and $\alpha_1 : N \times G \to A$ by

$$\alpha_0(n, m) = s(n)s(m)s(nm)^{-1}$$
$$\alpha_1(n, g) = s(n)(gs(g^{-1}ng))^{-1}$$

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One can easily show (see [R]) the following cocycle relations

(1) $\alpha_0(m, s) - \alpha_0(nm, s) + \alpha_0(n, ms) - \alpha_0(n, m) = 0$

(2) $g\alpha_0(g^{-1}ng, g^{-1}mg) - \alpha_0(n, m) = \alpha_1(m, g) - \alpha_1(nm, g) + \alpha_1(n, g)$

(3) $g\alpha_1(g^{-1}ng, h) - \alpha_1(n, gh) + \alpha_1(n, g) = 0$

(4) $\alpha_1(n, m) = \alpha_0(n, m) - \alpha_0(m, m^{-1}nm)$

The dependence of α_0 and α_1 on the section s gives natural coboundary relations.

Crossed extensions arise naturally in algebraic topology and in the study of group actions on algebras. In [J] it was shown that the above relations can be interpreted as forming the beginning of a cochain complex derived from the double complex $C^*(G, C^*(N, A))$. In [O], this complex $L^*(G, N; A)$ is defined in all dimensions and even in the case where A is an arbitrary G-module. The pair (α_0, α_1) is then a 2-cocycle. Thus $H^2(L^*(G, N; A))$ classifies crossed extensions when A is a Q-module. It was conjectured in [O] that the cohomology of $L^*(G, N; A)$ is the same as the relative cohomology. The conjecture was proved for H^3. Also a "crossed module" formalism for arbitrary G-modules was worked out in [O] and [CW]. Another version appears in [H].

In §4 of this paper we will show that $L^*(G, N; B, A)$ has the cohomology of the quotient $C^*(G, A)/C^*(Q, B)$, thus establishing the conjecture of [O]. As an application, we immediately obtain the 8-term exact sequence of [H], [R], [L].

In §3 a spectral sequence converging to $H^*(Q, G; B, A)$ is given. For $B = \{a \mid ga = a \text{ for all } g \in N\}$ this has a small calculational advantage over the spectral sequence of Hochschild–Serre [H–S] in that the bottom line has disappeared.

In §5 we give an example where results are easily obtained using the cocycle relations of L^* which are not obvious using $C^*(G, A)/C^*(Q, B)$.

At this stage there is no "n-fold crossed extension" theory for the relative cohomology groups although some cases are treated in [O], and the third cohomology relations for the complex L^* do arise naturally in the study of Connes' invariant $\chi(M)$ for von Neumann algebras (see [C]).

§1. Shuffles and partial shuffles

Let σ be a permutation of $\{1, 2, \ldots, n\}$. For a group G we define a map $l_\sigma : G^n \to G^n$ by $l_\sigma(g_1, \ldots, g_n) = (b_1^{-1}a_1b_1, \ldots, b_n^{-1}a_nb_n)$ where $a_{\sigma(i)} = g_i$, $b_{\sigma(i)} = g_{j_1}g_{j_2} \cdots g_{j_l}$ where $i < j_1 < \cdots < j_l$ are all the j_k's satisfying $\sigma(i) > \sigma(j_k)$. In words, we move the variable g_i into the $\sigma(i)$th position, and then conjugate g_i by the product of those g_k, $k > i$, in their order, which are now in a position inferior to g_i. (Note that $b_1 = 1$).

EXAMPLE. (1) if σ is the permutation (123) then

$$l_\sigma(g_1, g_2, g_3) = (g_3, g_3^{-1}g_1g_3, g_3^{-1}g_2g_3).$$

(2) if σ is (132) then

$$(g_1, g_2, g_3) = (g_2, g_3, g_3^{-1}g_2^{-1}g_1g_2g_3)$$

(3) if σ is (12) then

$$l_\sigma \circ l_\sigma(g, h) = l_\sigma(h, h^{-1}gh) = (h^{-1}gh, h^{-1}g^{-1}hgh)$$

Example (3) above shows that in general $l_\tau \circ l_\sigma \neq l_{\tau\circ\sigma}$. However

PROPOSITION 0. *Let σ be a perumtation of $\{1, \ldots, p+q\}$ fixing the sets $\{1, \ldots, p\}\{p+1, \ldots, p+q\}$ and let τ be a permutation satisfying $\tau(i)<\tau(i+1)$ for $i \neq p$. Then $l_\tau \circ l_\sigma = l_{\tau\sigma}$.*

Proof. Let $l_\sigma(g_1, \ldots, g_p, g_{p+1}, \ldots, g_{p+q}) = (x_1, \ldots, x_p, x_{p+1}, \ldots, x_{p+q})$ $x_i = b_i^{-1}a_ib_i$. For $p+1 \leq i \leq p+q$ we find in both $l_\tau \circ l_\sigma(g_1, \ldots, g_{p+q})$ and $l_{\tau\circ\sigma}(g_1, \ldots, g_{p+q})$ the variable x_i in the $\tau(i)$th place. However for $1 \leq i \leq p$ in $l_\tau \circ l_\sigma(g_i)$ we find x_i in the $\tau(i)$th place conjugated by $x_{p+1} \cdots x_{p+r}$ where r is the maximum j satisfying $\tau(p+j)<\tau(i)$, whereas in $l_{\tau\circ\sigma}(g_i)$ we find x_i in the $\tau(i)$th place conjugated by the product of the $a_{p+i} = g_{\sigma^{-1}(p+i)}, 1 \leq i \leq r$, in their original order. From the definition of the b_i, we see, by induction, that these products are identical.

DEFINITION. A permutation σ is a partial shuffle of type (a, b, c, d), $a+b+c+d=n$ if

(1) $\sigma(i)=i$ for $1 \leq i \leq a$ or $a+b+c+1 \leq i \leq n$

(2) $\sigma(i)<\sigma(i+1)$ for $a+1 \leq i \leq a+b-1$ or $a+b+1 \leq i \leq a+b+c-1$

(In words, the sets $\{1, \ldots, a\}\{a+b+c+1, \ldots, n\}$ are left fixed, while the sets $\{a+1, \ldots, a+b\}\{a+b+1, \ldots, a+b+c\}$ are shuffled).

A permutation is a 3-shuffle of type $\langle a, b, c\rangle$, $a+b+c=n$, if $\sigma(i)<\sigma(i+1)$ for $i \neq a$, $i \neq a+b$.

PROPOSITION 1. (A) *If σ is of type $(0, a, b, c)$ and τ of type $(0, a+b, c, 0)$ then $\tau\circ\sigma$ is of type $\langle a, b, c\rangle$. Any η of type $\langle a, b, c\rangle$ has a unique decomposition as above. Furthermore $l_{\tau\circ\sigma} = l_\tau \circ l_\sigma$.*

(B) *If σ is of type $(a, b, c, 0)$ and τ of type $(0, a, b+c, 0)$ then $\tau\circ\sigma$ is of type $\langle a, b, c\rangle$. Any η of type $\langle a, b, c\rangle$ has a unique decomposition as above. Furthermore $l_{\tau\circ\sigma} = l_\tau \circ l_\sigma$.*

Proof. The formula $l_{\tau\circ\sigma} = l_\tau \circ l_\sigma$ follows from Proposition 0. The rest is obvious. Q.E.D.

Let G be a group, N a nomal subgroup. If we let X, Y, Z denote either G or N then $l_\sigma(X^a \times Y^{b+c} \times Z^d) \subset X^a \times Y^{b+c} \times Z^d$ for σ of type (a, b, c, d). Thus if $\alpha : X^a \times Y^{b+c} \times Z^d \to A$ is any map, A a $\mathbb{Z}$-module, we may define $S(a, b, c, d)\alpha : X^a \times Y^{b+c} \times Z^d \to A$ by

$$\boxed{S(a, b, c, d)\alpha = \sum_{\substack{\sigma \text{ type} \\ (a,b,c,d)}} (-1)^{|\sigma|}\alpha \circ l_\sigma} \quad (-1)^{|\sigma|} = \text{signature } \sigma.$$

PROPOSITION 2. *Let $\alpha : G^n \to A$. We have the formula*

$$\boxed{S(a, b, c, 0)S(0, a, b+c, 0)\alpha = S(0, a, b, c)S(0, a+b, c, 0)\alpha}$$

Proof. By Proposition 1 both sides are equal to

$$\sum_{\substack{\mu \text{ type} \\ \langle a,b,c\rangle}} (-1)^{|\mu|}\alpha \circ l_\mu.$$

§2. Relative cohomology of groups

Let G be a group, N a normal subgroup. We write Q for the quotient G/N. If A is any G-module we denote by A^N the submodule of elements fixed by N. A^N is a Q-module. Let B be a Q-submodule of A^N.

We denote by $C^*(G; A)$ the complex of normalized cochains $\alpha : G^n \to A$ with coboundary

$$\begin{aligned} d\alpha(g_0, \ldots, g_n) = {} & g_0\alpha(g_1, \ldots, g_n) \\ & + \sum_{i=1}^{n} (-1)^i \alpha(g_0, \ldots, g_{i-1}g_i, \ldots, g_n) \\ & + (-1)^{n+1}\alpha(g_0, \ldots, g_{n-1}) \end{aligned}$$

The projection $G \to Q$ induces an inclusion of complexes $C^*(Q; B) \subset C^*(G; A)$. Let $C^*(Q, G; B, A)$ be the quotient complex.

DEFINITION. The group $H^*(Q, G; B, A) = H^*(C^*(Q, G; B, A))$ is called the relative cohomology group. The associated long exact sequence

$$\cdots \to H^*(Q, B) \to H^*(G, A) \to H^*(Q, G; B, A) \to \cdots$$

is called the fundamental long exact sequence.

Remark. We justify our use of the term "relative".

If A is a trivial G module, and $B = A$, then $H^n(Q, G; B, A) = H^{n+1}(M_p, BG; A)$ where M_p is the mapping cylinder of the fibration $p: BG \to BQ$.

Proof. If $C_*(X, A)(C^*(X, A))$ denotes singular (co)chains then $C^*(G, A) \simeq C^*(BG, A)$, $C^*(Q, A) \simeq C^*(BQ, A)$ so

$$\frac{C^*(G, A)}{C^*(Q, A)} \simeq \frac{\mathrm{Hom}_{\mathbf{Z}}(C_*(BG), A)}{\mathrm{Hom}_{\mathbf{Z}}(C_*(BQ), A)} = \mathrm{Hom}_{\mathbf{Z}}(\ker p_*, A)$$

where $1 \to \ker p_* \to C_*(BG) \xrightarrow{p_*} C_*(BQ) \to 1$ is exact. Finally, the suspension of $\ker p_*$ is isomorphic to coker i_*, where

$$1 \to C_*(BG) \xrightarrow{i_*} C_*(M_p) \to \operatorname{coker} i_* \to 1 \text{ is exact.}$$

§3. A spectral sequence for relative cohomology

We review briefly parts of the fundamental paper of Hochschild and Serre [1]: Regard $C^*(N, A)$ as a G-module by $(g\beta)(n_1, \ldots, n_s) = g\beta(g^{-1}n_1 g, \ldots, g^{-1}n_s g)$. Then $C^*(G, C^*(N, A))$ is a double complex with coboundaries

$$\begin{aligned} d_N\alpha(g_1, \ldots, g_r)(n_0, \ldots, n_s) = {} & n_0\alpha(g_1, \ldots, g_r)(n_1, \ldots, n_s) \\ & + \sum_{i=1}^{s} (-1)^i \alpha(g_1, \ldots, g_r)(n_0, \ldots, n_{i-1}n_i, \ldots, n_s) \\ & + (-1)^{s+1}\alpha(g_1, \ldots, g_r)(n_0, \ldots, n_{s-1}) \end{aligned}$$

and

$$\begin{aligned} d_G\alpha(g_0, \ldots, g_r)(n_1, \ldots, n_s) = {} & g_0\alpha(g_1, \ldots, g_r)(g_0^{-1}n_1 g_0, \ldots, g_0^{-1}n_s g_0) \\ & + \sum_{i=1}^{r} (-1)^i \alpha(g_0, \ldots, g_{i-1}g_i, \ldots, g_r)(n_1, \ldots, n_s) \\ & + (-1)^{r+1}\alpha(g_0, \ldots, g_{r-1})(n_1, \ldots, n_s) \end{aligned}$$

For reasons of convenience we will write $\alpha(n_1, \ldots, n_s, g_1, \ldots, g_r)$ in place of $\alpha(g_1, \ldots, g_r)(n_1, \ldots, n_s)$.

Let $T^*(G, N; A) = \oplus_{p=q=*} C^p(G, C^q(N, A))$ be the total complex associated to the double complex with total differential $d_G + (-1)^{i+1} d_N$ on $C^*(G, C^i(N, A))$.

We define a map $l: C^*(G, A) \to T^*(G, N; A)$ by

$$l(\alpha) = (\alpha_0, \ldots, \alpha_n), \quad \boxed{\alpha_p = S(0, q, p, 0)\alpha|_{N^q \times G^p}} \quad p+q=n.$$

Note in particular that $\alpha_n = \alpha$ and $\alpha_0 = \alpha|_{N^n}$.

PROPOSITION 3. *$l: C^*(G, A) \to T^*(G, N; A)$ is a map of complexes.*

Proof. This is the "general identity" of [H–S]. Q.E.D.

Define a filtration R_p of $C^*(G, A)$ where $\alpha \in R_p$ if $\alpha(h_1, \ldots, h_q, g_1, \ldots, g_p)$ depends only on the h_i and the classes mod N of the g_j. Also $R_p \cap C^n(G, A) = 0$ if $p > n$.

Define a filtration S_p of $T^*(G, N; A)$ by $S_p = \oplus_{i \leq p} C^i(G, C^*(N, A))$. If $\alpha \in R_p$ and $l(\alpha) = (\alpha_0, \ldots, \alpha_n)$, then $\alpha_j = 0$, $j < p$, so $l(\alpha) \in S_p$. Thus l is a map of filtered complexes.

THEOREM 4 (Hochschild–Serre).

$$l^*: E^1_{p,q}(R) \to E^1_{p,q}(S) = C^p(G, H^q(N, A))$$

is an injection and image $l^* = C^p(Q, H^q(N, A))$.

COROLLARY 5. $E^2_{p,q}(R) = H^p(Q, H^q(N, A))$

We now study the quotient complex $C^*(Q, G; B, A)$. Define a filtration of $C^*(Q, G; B, A)$ by $\bar{R}_j = \text{image of } R_j$.

THEOREM 6 (Spectral sequence for relative cohomology).

$$E_2^{p,q}(\bar{R}) = \begin{cases} H^p(Q, H^q(N, A)) & q > 0 \\ H^p\left(Q, \dfrac{A^N}{B}\right) & q = 0 \end{cases}$$

$$\Rightarrow H^*(Q, G; B, A)$$

Remark. If $B = A^N$, the E_2 term of this spectral sequence is identical to the Hochschild–Serre E_2 term except that the bottom line has disappeared.

Proof. Considering $B \subset A \subset C^0(N, A)$ we have an inclusion $C^*(G, B) \to T^*(G, N; A)$ of complexes since $d_N(C^*(G, B)) = 0$. Since $l(C^n(Q, B)) \subset C^n(G, B)$, we get

$$\bar{l} : C^n(Q, G; B, A) \to \frac{T^N(G, N; A)}{C^n(G, B)}.$$

If we let $\bar{S}_j$ = image of S_j in $T^n(G, N; A)/C^n(G, B)$ then $\bar{l}$ is a map of filtered complexes. Theorem 6 is then a corollary of the following analog of Theorem 4.

PROPOSITION 7. *$\bar{l}_* : E^1_{p,q}(\bar{R}) \to E^1_{p,q}(\bar{S})$ is an inclusion with image*

$$\begin{cases} C^p(Q, H^q(N, A)) & q > 0 \\ C^n\left(Q, \dfrac{A^N}{B}\right) & q = 0 \end{cases}$$

Proof. $C^n(Q, B) \subset R_n \subset R_p$, if $p \le n$, so $E^0_{p,q}(\bar{R}) \simeq E^0_{p,q}(R)$ if $q > 0$ and

$$E^0_{n,0}(\bar{R}) \simeq \frac{E^0_{n,0}(R)}{C^n(Q, B)}.$$

Since $d(C^n(Q, B)) \subset C^{n+1}(Q, B) \subset R_{n+1}$, it follows that

$$\begin{array}{llll} 0 \to C^n(Q, B) \to 0 & & \to \cdots & \text{is a subcomplex of} \\ 0 \to E^0_{n,0}(R) & \to E^0_{n,1}(R) & \to \cdots & \text{with quotient} \\ 0 \to E^0_{n,0}(\bar{R}) & \to E^0_{n,1}(\bar{R}) & \to \cdots & \end{array}.$$

From the associated long exact sequence, we get

$$0 \to C^n(Q, B) \to E^1_{n,0}(R) \to E^1_{n,0}(\bar{R}) \to 0$$

is exact and $E^1_{p,q}(R) = E^1_{p,q}(\bar{R})$, $q > 0$.

Similarly $0 \to C^n(G, B) \to E^1_{n,0}(S) \to E^1_{n,0}(\bar{S}) \to 0$ is exact and $E^1_{p,q}(S) \simeq$

$E^1_{p,q}(\bar{S})$, $q>0$. A look at the commutative diagrams

$$\begin{array}{ccc} E^1_{p,q}(R) \simeq C^p(Q, H^q(N, A)) \subset E^1_{p,q}(S) & & q>0 \\ \downarrow \wr \qquad\qquad\qquad \downarrow \wr & & \\ E^1_{p,q}(\bar{R}) \longrightarrow E^1_{p,q}(\bar{S}) & & \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & C^n(Q, B) & \to & E^1_{n,0}(R) & \to & E^1_{n,0}(\bar{R}) & \to & 0 \\ & & \Big\downarrow & & \wr\!\!\mid\!\!\mid & & \Big\downarrow & & \\ & & & & C^n(Q, A^N) & & & & \\ & & & & \cap & & & & \\ & & & & C^n(G, A^N) & & & & \\ & & & & \wr\!\!\mid\!\!\mid & & & & \\ 0 & \to & C^n(G, B) & \to & E^1_{n,0}(S) & \to & E_{n,0}(\bar{S}) & \to & 0 \end{array}$$

establishes the proposition.

§4. The complex $L^*(G, N; A, B)$

We write an element of $T^n(G, N; A)/C^n(G, B)$ as $(\alpha_0, \ldots, \alpha_{n-1}, \bar{\alpha}_n)$ where

$$\bar{\alpha}_n \in \frac{C^n(G, A)}{C^n(G, B)} \simeq C^n(G, A/B).$$

DEFINITION

$$L^n(G, N; A, B) = \Big\{ (\alpha_0, \ldots, \alpha_{n-1}, \bar{\alpha}_n) \in \frac{T^n(G, N; A)}{C^n(G, B)}$$

$$\text{such that } \alpha_k \equiv S(0, n-k, k, 0)\bar{\alpha}_n|_{N^{n-k}\times G^k} \bmod B$$

$$\text{and } S(0, n-m, m-k, k)\alpha_k = S(n-m, m-k, k, 0)\alpha_m|_{N^{n-k}\times G^k} \Big\}$$

Remark. If $A = A^N = B$ we make abbreviations such as $L^n(G, N; A, A)$ to $L^n(G, N; A)$. Note that in this case

$$\bar{\alpha}_n \in \frac{C^n(G, A)}{C^n(G, A)} = 0$$

so we abbreviate $(\alpha_0, \ldots, \alpha_{n-1}, \bar{\alpha}_n)$ to $(\alpha_0, \ldots, \alpha_{n-1})$. The reader may check that if $A = A^N = B$, then (α_0, α_1) is in $L^2(G, N; A)$ if and only if the relation (4) of the introduction is satisfied and a cocycle iff (1) (2) and (3) are satisfied.

It is a direct consequence of Proposition 2 that the map

$$\bar{l}: \frac{C^n(G, A)}{C^n(Q, B)} \to \frac{T^n(G, N; A)}{C^n(G, B)}$$

of §3 (given by $\bar{l}(\bar{\alpha}) = (\alpha_0, \ldots, \alpha_{n-1}, \bar{\alpha}_n)$, $\alpha_k = S(0, n-k, k, 0)\alpha|_{N^{n-k} \times G^k}$) has image contained in $L^n(G, N; A, B)$. In fact

PROPOSITION 8.

Image $\bar{l} = L^n(G, N; A, B)$

COROLLARY 9. *$L^*(G, N; A, B)$ is a subcomplex of $T^*(G, N; A)/C^*(G, B)$.*

DEFINITION. $\Lambda^*(G, N; A, B) = H^*(L^*(G, N; A, B))$

THEOREM 10. *$\bar{l}^*: H^*(Q, G; B, A) \to \Lambda^*(G, N; A, B)$ is an isomorphism.*

Proof of Proposition 8 *and Theorem* 10. Consider the following diagram of complexes

$$\begin{array}{ccccccc} 0 \to & \dfrac{C^*(G, B)}{C^*(Q, B)} & \longrightarrow & \dfrac{C^*(G, A)}{C^*(Q, B)} & \longrightarrow & \dfrac{C^*(G, A)}{C^*(G, B)} & \to 0 \\ & \downarrow{\scriptstyle \bar{l}_1} & & \downarrow{\scriptstyle \bar{l}_2} & & \| & \\ 0 \to & L^*(G, N; B, B) & \to & L^*(G, N; A, B) & \xrightarrow{\pi} & \dfrac{C^*(G, A)}{C^*(G, B)} & \to 0 \end{array}$$

where $\pi(\alpha_0, \ldots, \bar{\alpha}_n) = \bar{\alpha}_n$. The top row obviously is exact and the first square commutative. Commutativity of the second square follows from the identity $S(0, 0, n, 0)\alpha = \alpha$, so that if $l(\alpha) = (\alpha_0, \ldots, \alpha_n)$, then $\alpha_n = \alpha$. Hence π is surjective since $\pi \circ \bar{l}_2$ is.

Furthermore, since $\alpha_k \equiv S(0, n-k, k, 0)\bar{\alpha}_n|_{N^{n-k} \times G^k} \bmod B$, it follows that if $\bar{\alpha}_n = 0$ then α_k takes values in B. This proves exactness of the bottom line.

From the diagram it follows that $\bar{l}_1$ is surjective if and only if $\bar{l}_2$ is. The five lemma applied to the long exact sequences in homology shows that $\bar{l}_1$ induces an isomorphism on homology if and only if $\bar{l}_2$ does.

So it suffices to prove 8 and 10 in case $A = A^N = B$. Then

$$\frac{T^n(G, N; A)}{C^n(G, A)} = \bigoplus_{\substack{p+q=n \\ q>0}} C^p(G, C^q(N, A))$$

and

$$L^n(G, N; A) = \Big\{(\alpha_0, \ldots, \alpha_{n-1}) \in \frac{T^n(G, N; A)}{C^n(G, A)} \text{ such that for }$$
$$k < m \leq n-1, S(n-m, m-k, k, 0)\alpha_m|_{N^{n-k} \times G^k} = S(0, n-m, m-k, k)\alpha_k\Big\}$$

We begin with the following

LEMMA 11. *Let* $a = (\alpha_0, \ldots, \alpha_{n-1})$, $b = (\beta_0, \ldots, \beta_{n-1})$ *be elements of* $L^n(G, N; A)$. *Then* $a = b$ *if and only if* $\alpha_0 = \beta_0$ *and* $\alpha_k(m_1, \ldots, m_{n-k}, g_1, \ldots, g_k) = \beta_k(m_1, \ldots, m_{n-k}, g_1, \ldots, g_k)$ *whenever* $g_1 \notin N$.

Proof of Lemma 11. Since $\alpha_k|_{N^n} = S(0, n-k, k, 0)\alpha_0 = S(0, n-k, k, 0)\beta_0 = \beta_k|_{N^n}$ we may assume some $g_i \notin N$. Suppose we have shown that $\alpha_i = \beta_i$ for $i < k$, and that $\alpha_k = \beta_k$ whenever the first l for which $g_l \notin N$ satisfies $l \leq j < k$. We show $\alpha_k(m_1, \ldots, m_{n-k}, h_1, \ldots, h_j, g_{j+1}, \ldots, g_k) = \beta_k(m_1, \ldots, m_{n-k}, h_1, \ldots, h_j, g_{j+1}, \ldots, g_k)$ whenever $h_1, \ldots, h_j \in N$ but $g_{j+1} \notin N$.

For simplicity, write (m, h, g) for $(m_1, \ldots, m_{n-k}, h_1, \ldots, h_j, g_{j+1}, \ldots, g_k)$. Since $\alpha_{k-j} = \beta_{k-j}$ we have

$$\begin{aligned} S(n-k, j, k-j, 0)\alpha_k(m, h, g) &= S(0, n-k, j, k-j)\alpha_{k-j}(m, h, g) \\ &= S(0, n-k, j, k-j)\beta_{k-j}(m, h, g) = S(n-k, j, k-j, 0)\beta_k(m, h, g). \end{aligned}$$

Furthermore, for σ of type $(n-k, j, k-j, 0)$, $\sigma \neq id$, we have by our induction hypothesis that $\alpha_k \circ l_\sigma(m, h, g) = \beta_k \circ l_\sigma(m, h, g)$ since in $l_\sigma(m, h, g)$ the value $g_{l+1} \notin N$ will occur earlier. Thus

$$\begin{aligned} \alpha_k(m, h, g) &= S(n-k, j, k-j, 0)\alpha_k(m, h, g) - \sum_{\substack{\sigma \text{ type}(n-k,j,k-j,0) \\ \sigma \neq id}} (-1)^{|\sigma|}\alpha_k \circ l_\sigma(m, h, g) \\ &= S(n-k, j, k-j, 0)\beta_k(m, h, g) - \sum_{\substack{\sigma \text{ type}(n-k,j,k-j,0) \\ \sigma \neq id}} (-1)^{|\sigma|}\beta_k \circ l_\sigma(m, h, g) \\ &= \beta_k(m, h, g). \end{aligned}$$

End of proof of Proposition 8. Let $(\alpha_0, \ldots, \alpha_{n-1}) \in L(G, N; A)$. We will define $\beta \in C^n(G, A)$ so that if $\bar{l}(\bar{\beta}) = (\beta_0, \ldots, \beta_{n-1})$, then $\beta_0 = \alpha_0$ and $\beta_i(m_1, \ldots, m_{n-i}, g_1, \ldots, g_i) = \alpha_i(m_1, \ldots, m_{n-i}, g_1, \ldots, g_i)$ whenever $m_1, \ldots, m_{n-i} \in N$ and

$g_1 \notin N$. By Lemma 11, it will follow that $\beta_i = \alpha_i$, all i, i.e. $\bar{l}(\bar{\beta}) = (\alpha_0, \ldots, \alpha_{n-1})$.

Set $\beta \mid N^n = \alpha_0$. Then $\beta_0 = \alpha_0$. Set $\beta(g_1, \ldots, g_n) = 0$ if $g_1 \notin N$. Suppose β is defined on $N^{n-i} \times (G-N) \times G^{i-1}$, for $i > j > 0$. Define

$$\beta(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j) = -\sum_{\substack{\sigma \text{ type}(0,n-j,j,0) \\ \sigma \neq id}} (-1)^{|\sigma|} \beta \circ l_\sigma(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j) + \alpha_j(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j),$$

for $m_i \in N$ and $g_1 \notin N$. (Note that $\beta \circ l_\sigma(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j)$ has already been defined). It follows that

$$\beta_j(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j) = \sum_{\sigma \text{ type}(0,n-j,j,0)} (-1)^{|\sigma|} \beta \circ l_\sigma(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j) = \alpha_j(m_1, \ldots, m_{n-j}, g_1, \ldots, g_j).$$

Proof of Theorem 10 ($A = A^N = B$). It suffices to show the following: Let K^* be the kernel of the composite

$$C^*(G, A) \to \frac{C^*(G, A)}{C^*(Q, A)} \xrightarrow{\bar{l}} L^*(G, N; A).$$

Then the inclusion $C^*(Q, A) \subset K^*$ induces an isomorphism on homology.

Let B_i be the filtration of $C^*(G, A)$ where $\alpha \in B_i \cap C^n(G, A)$ if α is unchanged whenever any of the last i values is multiplied by an element of N, $B_i \cap C^n(G, A) = C^n(Q, A)$ if $i \geq n$. For $i < p \leq n$, let $B_{i,p} = \{\alpha \in B_i \mid \alpha(g_1, \ldots, g_n) = 0$ if $g_{p-i}, \ldots, g_{n-i}$ are all in $N\}$, $B_{i,i} = B_i$, and $B_{i,p} \cap C^n(G, A) = B_{i+1} \cap C^n(G, A)$ if $p > n$. Set $K_i = K \cap B_i$, $K_{i,p} = K \cap B_{i,p}$.

We will show the following

(a) $K_i = K_{i,i+1}$

(b) $H^n\left(\dfrac{K_{i,n}}{K_{i,n+1}}\right) = 0 \qquad i < n$

(c) $H^n\left(\dfrac{K_{i,p}}{K_{i,p+1}}\right) = 0 \qquad i < p < n.$

It follows that

$$H^n\left(\frac{K_i}{K_{i+1}}\right) = H^n\left(\frac{K_{i,i+1}}{K_{i,n+2}}\right) = 0$$

and hence that

$$H^n\left(\frac{K}{C^*(Q, A)}\right) = H^n\left(\frac{K_0}{K_{n+1}}\right) = 0.$$

Proof of (*a*). Let σ be of type $(0, n-i, i, 0)$. If $\sigma \neq id$ and $\alpha \in B_i$, then $\alpha \circ l_\sigma|_{N^{n-i}\times G^i} = 0$. Hence the formula, valid for $\alpha \in B_i : \alpha_i = \alpha|_{N^{n-i}\times G^i}$. In particular if $\alpha \in K_i$, $0 = \alpha_i = \alpha|_{N^{n-i}\times G^i}$, i.e. $\alpha \in K_{i,i+1}$.

Proof of (*b*). For $\alpha \in B_{i,n} \cap C^n(G, A)$, $d\alpha \in B_{i,n+1}$ we have for $m \in N$

$$\begin{aligned} 0 &= d\alpha(g_1, \ldots, g_{n-i-1}, g, m, h_1, \ldots, h_i) \\ &= (-1)^{n-i}(\alpha(g_1, \ldots, g_{n-i-1}, gm, h_1, \ldots, h_i) - \alpha(g_1, \ldots, g_{n-i-1}, g, h_1, \ldots, h_i)) \end{aligned}$$

i.e. $\alpha \in B_{i+1} \cap C^n(G, A) = B_{i,n+1} \cap C^n(G, A)$. It follows that if $\alpha \in K_{i,n} \cap C^n(G, A)$, $d\alpha \in K_{i,n+1}$, then $\alpha \in K_{i,n+1}$.

Proof of (*c*). Choose a section $* : Q \to G$ of the quotient map $\pi : G \to Q$ such that $1^* = 1$.

We first show (c) in the special case $i = 0$.

Suppose $\alpha \in K_{0,p} \cap C^n(G, A)$, $d\alpha \in K_{0,p+1}$. Define for

$$\begin{aligned} \sigma, m_{p+2}, \ldots, m_n \in N, \beta(g_1, \ldots, g_{p-1}, q^*\sigma, m_{p+2}, \ldots, m_n) \\ = \alpha(g_1, \ldots, g_{p-1}, q^*, \sigma, m_{p+2}, \ldots, m_n). \end{aligned}$$

If $\tilde{\beta}$ is any extension of β to all G^{n-1}, then $\tilde{\beta} \in B_{0,p}$. Moreover

$$\begin{aligned} d\tilde{\beta}&(g_1, \ldots, g_{p-1}, q^*\sigma, m_{p+1}, \ldots, m_n) \\ &= (-1)^p\beta(g_1, \ldots, g_{p-1}, q^*\sigma m_{p+1}, m_{p+2}, \ldots, m_n) + \cdots \\ &\qquad + (-1)^p\beta(g_1, \ldots, g_{p-1}, q^*\sigma, m_{p+1}, \ldots, m_n) \\ &= (-1)^p\alpha(g_1, \ldots, g_{p-1}, q^*, \sigma m_{p+1}, m_{p+2}, \ldots, m_n) + \cdots \\ &\qquad + (-1)^n\alpha(g_1, \ldots, g_{p-1}, q^*, \sigma, m_{p+1}, \ldots, m_{m-1}) \\ &= -d\alpha(g_1, \ldots, g_{p-1}, q^*, \sigma, m_{p+1}, \ldots, m_n) \\ &\qquad + (-1)^p\alpha(g_1, \ldots, g_{p-1}, q^*\sigma, m_{p+1}, \ldots, m_n) \\ &= (-1)^p\alpha(g_1, \ldots, g_{p-1}, q^*\sigma, m_{p+1}, \ldots, m_n) \end{aligned}$$

since $d\alpha \in K_{0,p+1}$, i.e. $\alpha - (-1)^p\, d\tilde{\beta} \in B_{0,p+1}$.

Thus it suffices to show that β can be extended to a $\tilde{\beta} \in K$. Now β is already defined on $G^p \times N^{n-p-1}$. Extend β as follows:

Let $\beta(g_1, \ldots, g_{n-1}) = 0$ if $g_1 \notin N$ and some $g_i \notin N$, $p < i \leq n-1$. Assume now that β is defined if some $g_i \notin N$, $i \leq k < n-1$. Then if $g_{k+1} \notin N$ but $m_1, \ldots, m_k \in N$ (and if $k < p$ some $g_i \notin N, p < i \leq n-1$) define

$$\beta(m_1, \ldots, m_k, g_{k+1}, \ldots, g_{n-1}) = - \sum_{\substack{\sigma \text{ type}(0,k,n-k-1,0) \\ \sigma \neq id}} (-1)^{|\sigma|} \beta \circ l_\sigma(m_1, \ldots, m_k, g_{k+1}, \ldots, g_{n-1})$$

the right side already having been defined. It follows that $\beta_{n-k-1}(m_1, \ldots, m_k, g_{k+1}, \ldots, g_{n-1}) = 0$ for $g_{k+1} \notin N$ (and provided some $g_i \notin N$, $p < i \leq n-1$, if $k < p$). Thus, by Lemma 11, it will follow that $\beta \in K$ if we can show

$$\beta_{n-k-1}(m_1, \ldots, m_k, g_{k+1}, \ldots, g_p, m_{p+1}, \ldots, m_{n-1}) = 0 \quad \text{for} \quad m_{p+1}, \ldots, m_{n-1} \in N.$$

But since $\beta \in B_{0,p}$ and $\alpha \in K_{0,p}$, a calculation shows

$$\beta_{n-k-1}(m_1, \ldots, m_k, g_{k+1}, \ldots, g_{p-1}, q^*\sigma, m_{p+1}, \ldots, m_{n-1}) = \alpha_{n-k}(m_1, \ldots, m_k, g_{k+1}, \ldots, g_{p-1}, q^*, \sigma, m_{p+1}, \ldots, m_{n-1}) = 0.$$

Now suppose $\alpha \in K_{i,p}$, $d\alpha \in K_{i,p+1}$, $i > 0$. Define $\alpha'(q_1, \ldots, q_i)(g_1, \ldots, g_{n-i}) = \alpha(g_1, \ldots, g_{n-i}, q_1^*, \ldots, q_i^*)$. Then $\alpha'(q_1, \ldots, q_i) \in B_{0,p-i}$ and a calculation shows

$$\alpha'(q_1, \ldots, q_i)_k(m_1, \ldots, m_{n-i-k}, g_1, \ldots, g_k) = \alpha_{k+i}(m_1, \ldots, m_{n-i-k}, g_1, \ldots, g_k, q_1^*, \ldots, q_i^*) = 0$$

so $\alpha'(q_1, \ldots, q_i) \in K_{0,p-i}$. Moreover $d\alpha'(q_1, \ldots, q_i) \in K_{0,p-i+1}$ since

$$d\alpha'(q_1, \ldots, q_i)(g_1, \ldots, g_{n-i}, m) = d\alpha(g_1, \ldots, g_{n-i}, m, q_1^*, \ldots, q_i^*) \quad \text{for} \quad m \in N.$$

Thus there is a $\beta'(q_1, \ldots, q_i) \in K_{0,p-i}$ such that $\alpha'(q_1, \ldots, q_i) - d\beta'(q_1, \ldots, q_i) \in K_{0,p-i+1}$.

Define β by $\beta(g_1, \ldots, g_n) = \beta'(\pi(g_{n-i}), \ldots, \pi(g_n))(g_1, \ldots, g_{n-i-1})$. We have $\beta \in B_{i,p}$ (so $\beta_j = 0$ for $j < i$). Moreover

$$\beta_{i+k}(m_1, \ldots, m_{n-i-k-1}, g_1, \ldots, g_k, h_1, \ldots, h_i) = \beta'(\pi(h_1), \ldots, \pi(h_i))_k(m_1, \ldots, m_{n-i-k-1}, g_1, \ldots, g_k) = 0.$$

So $\beta \in K$. Since

$$(\alpha - d\beta)(g_1, \ldots, g_{n-i-1}, m, h_1, \ldots, h_i) = (\alpha' - d\beta')(\pi(h_1), \ldots, \pi(h_i))(g_1, \ldots, g_{n-i-1}, m) \quad \text{for} \quad m \in N,$$

it follows that $\alpha - d\beta \in K_{i,p+1}$.

§5. A calculation using $L^*(G, N; A)$

PROPOSITION 12. *Let A be a divisible abelian group. Let N be a central subgroup of a perfect group G. Then $H^3(G/N; A) \to H^3(G, A)$ is injective and $H^2(G/N; A) \to H^2(G, A)$ is surjective (trivial coefficients).*

Proof. It is equivalent to show $H^2(G/N, G; A) = 0$. By Theorem 10, it is equivalent to show $\Lambda^2(G, N; A) = 0$. So let $a = (\alpha_0, \alpha_1)$ be a cocycle of $L^*(G, N; A)$. We must show $a = db$, where $b = (\beta)$ and $db = (d_N\beta, d_G\beta)$. Now

$$0 = d_G\alpha_1(n, g, h) = g\alpha_1(g^{-1}ng, h) - \alpha_1(n, gh) + \alpha_1(n, g)$$
$$= \alpha_1(n, h) - \alpha_1(n, gh) + \alpha_1(n, g) \text{ (since } A \text{ is a trivial } G \text{ module and } N \text{ is central}$$

so $\alpha_1(n, \cdot): G \to A$ is a homomorphism. Since G is perfect this shows $\alpha_1 = 0$.

Now $0 = \alpha_1(n, m) = \alpha_0(n, m) - \alpha_0(n, m^{-1}nm) = \alpha_0(n, m) - \alpha_0(m, n)$ so $\alpha_0: N^2 \to A$ is a symmetric cocycle. The corresponding extension $1 \to A \to C \to N \to 1$ is thus abelian and represents an element in $\mathrm{Ext}^1_{\mathbb{Z}}(N, A) = 0$, since A is divisible (hence injective, c.f. [M2] p. 93). So this extension is split, i.e. $0 = [\alpha_0] \in H^2(N, A)$, i.e., $\alpha_0 = d_N\beta$ for $\beta: N \to A$. Now

$$d_G\beta(n, g) = g\beta(g^{-1}ng) - \beta(n) = \beta(n) - \beta(n) = 0$$

so

$$db = (d_N\beta, d_G\beta) = (d_N\beta, 0) = (\alpha_0, 0) = (\alpha_0, \alpha_1) = a.$$

The case N central, A trivial

Let $1 \to N \to G \to Q \to 1$ be a central extension and A a trivial G module. Then equations 2–4 become

(2) $0 = \alpha_1(m, g) - \alpha_1(mn, g) + \alpha_1(n, g)$ i.e. α_1 is bilinear

(3) $0 = \alpha_1(n, h) - \alpha_1(n, gh) + \alpha_1(n, g)$

(4) $\alpha_1(n, m) = \alpha_0(n, m) - \alpha_0(m, n)$

It follows easily that

PROPOSITION 13. *For a central extension with trivial coefficients we have an exact sequence*

$$0 \to \Lambda^2(G, N; A) \xrightarrow{i} H^2(N, A) \times \mathrm{Bilin}(N \times G, A) \xrightarrow{t} \mathrm{Bilin}(N \times N, A)$$

(*where* $i(\alpha_0, \alpha_1) = ([\alpha_0], \alpha_1)$ *and*

$$t([\alpha_0], \alpha_1) = \alpha_1|_{N\times N} - (\alpha_0 - T\alpha_0),\ T\alpha_0(n, m) = \alpha_0(m, n).)$$

COROLLARY 14. (1) $\mathrm{Ext}^1(N, A) \times \mathrm{Bilin}(N \times Q; A) \subset \Lambda^2(G, N; A)$
(2) *If* $\mathrm{Bilin}(N \times G; A) = 0$, $\Lambda^2(G, N; A) = \mathrm{Ext}^1(N, A)$
(3) *If* $H^2(N, A) = 0$, $\Lambda^2(G, N; A) = \mathrm{Bilin}(N \times Q; A)$

COROLLARY 15. *If G is perfect,* $\Lambda^2(G, N; A) \simeq \mathrm{Ext}^1(N, A)$

Remark. The results in this section may also be derived from known results from the homology theory of groups. [K] [EHS] [GS].

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