

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 59 (1984)

Artikel: Quasiregular mappings and metrics on the n -sphere with punctures.
Autor: Rickman, Seppo
DOI: <https://doi.org/10.5169/seals-45387>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.12.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Quasiregular mappings and metrics on the n -sphere with punctures

SEPPO RICKMAN*

1. Introduction

Let D be a domain in the Euclidean n -space R^n and $f: D \rightarrow R^n$ continuous. We call f *quasiregular* if f belongs to the local Sobolev space $W_{n,\text{loc}}^1(D)$, i.e. f has generalized first order partial derivatives which are locally L^n -integrable and there exists K , $1 \leq K < \infty$, such that the distortion inequality

$$|f'(x)|^n \leq K J_f(x) \quad \text{a.e.} \quad (1.1)$$

holds. Here $f'(x)$ is the formal derivative of f at x defined by the partial derivatives, $|f'(x)|$ its operator norm, and $J_f(x)$ the Jacobian determinant. The definition extends immediately to maps $f: M \rightarrow N$ where M and N are oriented connected Riemannian n -manifolds, see for example [6]. If here N is $\bar{R}^n = R^n \cup \{\infty\}$, equipped with the spherical metric

$$d\sigma^2 = \frac{dx^2}{(1+|x|^2)^2},$$

where dx^2 is the Euclidean metric, and M is a domain in \bar{R}^n , we also call f *quasimeromorphic*. A quasiregular homeomorphism is called *quasiconformal*. The smallest K in (1.1) is the outer dilatation $K_0(f)$ of f and the smallest K in

$$J_f(x) \leq K \inf_{|h|=1} |f'(x)h|^n \quad \text{a.e.}$$

is the inner dilatation $K_I(f)$ of f . A quasiregular mapping f is called K -*quasiregular* if the dilatation $K(f) = \max(K_0(f), K_I(f))$ satisfies $K(f) \leq K$.

Quasiregular mappings form a natural generalization of analytic functions in plane to the real n -dimensional space. For the basic properties we refer to [2],

* This research was partially supported by Forschungsinstitut für Mathematik, ETH, Zürich.

[12]. For some years ago a Picard type theorem on omitted values was proved in the following form:

1.2. THEOREM [9]. *For $n \geq 3$ and $K \geq 1$ there exists a constant $q = q(n, K)$ such that every K -quasiregular mapping $f: R^n \rightarrow R^n \setminus \{a_1, \dots, a_q\}$, where a_1, \dots, a_q are distinct points in R^n , is constant.*

The proof of 1.2 in [9] is based on two basic tools in the theory of quasiregular mappings, namely, the method of moduli of path families and the theory of quasilinear partial differential equations. A proof which uses only the first of these methods is given in [11] by means of ideas from [10]. It was recently proved by the author that at least for $n = 3$ Theorem 1.2 is qualitatively best possible, in fact, any number of points can be omitted.

The purpose of this paper is to give some geometrical insight from a different point of view to Theorem 1.2. We shall study quasimeromorphic mappings of the unit ball $B = \{x \in R^n \mid |x| < 1\}$ into $Y = \bar{R}^n \setminus \{a_1, \dots, a_q\}$ where q is sufficiently large. We consider B as the Poincaré model of the hyperbolic n -space with the hyperbolic metric

$$d\rho^2 = \frac{4 dx^2}{(1 - |x|^2)^2}.$$

Our main result is that if Y is equipped with a metric with a certain natural singularity behavior near the points a_j , then f is a Lipschitz mapping if small distances are ruled out (Theorem 2.4).

Let us first take a look at the classical case $n = 2$. If $q \geq 3$, the analytic universal covering surface of Y is conformally equivalent to B . Let $\pi: B \rightarrow Y$ be an analytic covering projection. The map π induces a complete metric $d\tau^2$ on Y , called the Poincaré metric of Y . If $f: B \rightarrow Y$ is analytic, we can lift f to an analytic function $\tilde{f}: B \rightarrow B$ such that $\pi \circ \tilde{f} = f$. According to the Schwarz–Pick lemma \tilde{f} is distance decreasing, and with the metric $d\tau^2$ on Y , so is f . For the case $q = 3$ one gets from estimates on the metric $d\tau^2$ the Picard–Schottky theorem (see [1, Theorem 1–13]).

Let then $n \geq 3$. To some extent the covering projection π in the 2-dimensional case can be replaced by a branched covering which is quasimeromorphic. In Section 3 we consider such maps $h: B \rightarrow Y = \bar{R}^n \setminus \{a_1, \dots, a_q\}$ which are automorphic with respect to some discrete group G of Möbius transformations acting on B and which are injective in each fundamental set. Such a map h induces a distance $\tau(y, z)$ for points y, z in Y from the hyperbolic metric in B . The singular behavior of the metric τ is similar to the behavior in the classical case as is shown

in Proposition 3.2. In dimension three we give explicitly an example of this type where the dilatation of h has an absolute bound and q is arbitrarily large. The possible sets $\{a_1, \dots, a_q\}$ in these constructions depend on G and the dilatation of h .

On the other hand, if we take an arbitrary sufficiently large set $\{a_1, \dots, a_q\}$ in \bar{R}^n and a metric τ on $Y = R^n \setminus \{a_1, \dots, a_q\}$ which has a singular behavior near each a_j like in Proposition 3.2, then we obtain a counterpart (Theorem 2.4) for the classical distance decreasing result mentioned above. As a corollary we get an analogue for the Picard–Schottky theorem and in this way also a new proof for Theorem 1.2.

1.3. Notation. The Euclidean (spherical) ball and the $(n-1)$ -dimensional sphere with center x and radius r are denoted by $B(x, r)$ ($D(x, r)$) and $S(x, r)$ ($C(x, r)$) respectively. We write $B(r) = B(0, r)$, $S(r) = S(0, r)$, $B = B(1)$, $S = S(1)$. The hyperbolic metric in B is denoted by ρ and the spherical metric in \bar{R}^n by σ .

2. The main result

Let a_1, \dots, a_q , $q \geq 3$, be distinct points in \bar{R}^n . We fix $\beta > 0$ such that

$$\beta \leq \frac{1}{4} \min_{j \neq k} \sigma(a_j, a_k)$$

and write $Y = \bar{R}^n \setminus \{a_1, \dots, a_q\}$, $U_j = D(a_j, \beta) \setminus \{a_j\}$, and

$$U = \bigcup_{j=1}^q U_j.$$

We shall consider metrics τ in Y which satisfy the conditions

$$\left| \tau(y_1, y_2) - \left| \log \frac{\log(1/\sigma(a_j, y_1))}{\log(1/\sigma(a_j, y_2))} \right| \right| \leq P \quad \text{if } y_1, y_2 \in U_j, \quad (2.1)$$

$$\tau(y_1, y_2) \leq Q\sigma(y_1, y_2) \quad \text{if } y_1, y_2 \in Y \setminus U, \quad (2.2)$$

for some positive constants P and Q .

Metrics τ satisfying (2.1) and (2.2) are for example obtained from conformal metrics

$$d\tau^2 = \gamma^2 d\sigma^2 \quad (2.3)$$

where γ is continuous in Y , constant in $Y \setminus U$, and

$$\gamma(y) = \frac{1}{\sigma(a_j, y) \log(1/\sigma(a_j, y))} \quad \text{if } y \in U_j.$$

We formulate our main result as follows.

2.4. THEOREM. *For each $K \geq 1$ and for each integer $n \geq 3$ there exists a number $\delta = \delta(n, K) > 0$ and a positive integer $q_0 = q_0(n, K)$ such that the following holds. If $f: B \rightarrow \bar{R}^n \setminus \{a_1, \dots, a_q\} = Y$ is a K -quasimeromorphic mapping where a_1, \dots, a_q are distinct and $q \geq q_0$, then*

$$\tau(f(x_1), f(x_2)) \leq C \max(\rho(x_1, x_2), \delta), \quad x_1, x_2 \in B, \quad (2.5)$$

where τ is a metric in Y satisfying (2.1) and (2.2) and C is a constant depending only on n, K, β, P , and Q .

The proof of 2.4 includes some value distribution results which we shall first list below.

2.6. Averages of the counting function over spheres. Let V be a ball $B(x_0, r_0)$ and $g: V \rightarrow \bar{R}^n$ a nonconstant K -quasimeromorphic mapping. For $y \in \bar{R}^n$ and for a Borel set E such that $\bar{E} \subset V$ we define

$$n(E, y) = \sum_{x \in g^{-1}(y) \cap E} i(x, g)$$

where $i(x, g)$ is the local topological index of g at x ; see [2, p. 6]. If E is as above and X is an $(n-1)$ -dimensional sphere in \bar{R}^n , we let $\nu(E, X)$ be the average of $n(E, y)$ over X with respect to the $(n-1)$ -dimensional spherical metric. Especially, if $E = \bar{B}(r)$ and $X = S(t)$, we call $n(r, y) = n(E, y)$ the counting function and write $\nu(r, t) = \nu(\bar{B}(r), S(t))$, in which case we also have that

$$\nu(r, t) = \frac{1}{\omega_{n-1}} \int_S n(r, ty) d\mathcal{H}^{n-1}y$$

where \mathcal{H}^{n-1} is the normalized $(n-1)$ -dimensional Hausdorff measure and $\omega_{n-1} = \mathcal{H}^{n-1}(S)$.

2.7. LEMMA. *For $r, s, t > 0$ and $\theta > 1$ such that $\bar{B}(\theta r) \subset V$ we have*

$$\nu(\theta r, t) \geq \nu(r, s) - \frac{K \left| \log \frac{t}{s} \right|^{n-1}}{(\log \theta)^{n-1}}.$$

This lemma is in a slightly weaker form in [8, 4.1]. The above form is due to M. Pesonen and A. Hinkkanen (independently) and the proof can be found in [7] and [11].

Let $A(r)$ be the average of $n(r, y)$ over \bar{R}^n with respect to the n -dimensional spherical measure. From 2.7 we obtain (see [8, p. 456]).

$$\nu(r/\theta, t) - \frac{K(a + a' |\log t|^{n-1})}{(\log \theta)^{n-1}} \leq A(r) \leq \nu(\theta r, t) + \frac{K(a + a' |\log t|^{n-1})}{(\log \theta)^{n-1}} \quad (2.8)$$

where $a, a' > 0$ depend only on n . Since $A(r)$ remains invariant if g is followed by a rotation in \bar{R}^n , we get from (2.8) the following lemma formulated with spherical radii.

2.9. LEMMA. *For $y, z \in \bar{R}^n$, for $0 < s, t \leq \pi/2$, and $r > 0$ and $\theta > 1$ such that $\bar{B}(\theta r) \subset V$ we have*

$$\nu(\theta r, C(y, s)) \geq \nu(r, C(z, t)) - \frac{K[b + b'(|\log s|^{n-1} + |\log t|^{n-1})]}{(\log \theta)^{n-1}}.$$

where $b, b' > 0$ depend only on n .

The next result is a variant of [9, 3.2] for spherical distances:

2.10. LEMMA. *There exists $\theta_0 = \theta_0(n, K) > 1$ such that the following holds. Let $r > 0$ and $\theta > \theta_0$ be such that $\bar{B}(\theta^2 r) \subset V$, let $u, v \in \bar{B}(r)$ and $y \in \bar{R}^n$ be points such that $s = \sigma(g(u), y) < t = \sigma(g(v), y)$. If y and some z in $\bar{R}^n \setminus D(y, t)$ are not in gV , then for some $d_n > 0$ depending only on n*

$$\nu(\theta^2 r, C(y, t)) \geq \frac{d_n \log \theta}{K} \left(\log \frac{t}{s} \right)^{n-1}.$$

2.11. Proof of Theorem 2.4. We may assume that f is nonconstant. We write

$$c_1 = \frac{(b + 2b')K}{(\log 2)^{n-1}}, \quad \theta_1 = \max(\theta_0, \exp(3c_1 K d_n^{-1})),$$

where b, b', θ_0 and d_n are the constants appearing in 2.9 and 2.10. Let q_0 be the smallest integer such that

$$q_0 \geq \omega_{n-1} \Omega_{n-1}^{-1} 2^{3n-3} \theta_1^{2n-2} \quad (2.12)$$

and let

$$\delta = 2^{-5} \theta_1^{-2}. \quad (2.13)$$

Here Ω_{n-1} is the $(n-1)$ -measure of the unit ball in R^{n-1} . Because $\beta < \frac{1}{3}$, it is possible to choose $p \geq 3$ such that

$$(\log p)^{n-1} = \frac{1}{2} \left(\log \frac{p}{\beta} \right)^{n-1}. \quad (2.14)$$

Let $x_1, x_2 \in B$ be such that $\rho(x_1, x_2) = \delta$ and write $y_i = f(x_i)$, $i = 1, 2$. Because f is open, it suffices to find a suitable estimate for $\tau(y_1, y_2)$. We consider different cases according to the location of y_1 and y_2 .

Case 1. $y_1, y_2 \in D(a_k, \beta/p)$ for some k .

Set $s_i = \sigma(a_k, y_i)$, $i = 1, 2$, and assume $s_2 \leq s_1$. By (2.1) we have

$$\tau(y_1, y_2) \leq \log \frac{\log s_2^{-1}}{\log s_1^{-1}} + P. \quad (2.15)$$

Write $r_1 = |x_1 - x_2| \theta_1^2$. By (2.13) and by simple estimation of the hyperbolic distance we get $r_1 \leq 2^{-4}(1 - |x_1|)$. Lemma 2.10 gives

$$\nu(\bar{B}(x_1, r_1), C(a_k, s_1)) \geq \frac{d_n \log \theta_1}{K} \left(\log \frac{s_1}{s_2} \right)^{n-1}. \quad (2.16)$$

By Lemma 2.9 we obtain

$$\nu(\bar{B}(x_1, 2r_1), C(a_j, \beta/p)) \geq \nu(\bar{B}(x_1, r_1), C(a_k, s_1)) - c_1 \left(\log \frac{1}{s_1} \right)^{n-1} \quad (2.17)$$

for all j . The left hand side of (2.17) is positive if

$$\nu(\bar{B}(x_1, r_1), C(a_k, s_1)) > c_1 \left(\log \frac{1}{s_1} \right)^{n-1}.$$

By (2.16) this in turn is true if

$$\left(\frac{\log s_2^{-1}}{\log s_1^{-1}} - 1 \right)^{n-1} = \left(\frac{\log (s_1/s_2)}{\log s_1^{-1}} \right)^{n-1} > \frac{c_1 K}{d_n \log \theta_1}.$$

Suppose now that $\tau(y_1, y_2) > c_2$ where

$$c_2 = P + \log \left[\left(\frac{c_1 K}{d_n \log \theta_1} \right)^{1/(n-1)} + 1 \right]. \quad (2.18)$$

Then the left hand side of (2.17) is positive by (2.15).

Since a_j is omitted and $\nu(\bar{B}(x_1, 2r_1), C(a_j, \beta/p)) > 0$, we have $E_j = S(x_1, 2r_1) \cap f^{-1}C(a_j, \beta/p) \neq \emptyset$ for all j . Let b be the smallest of the Euclidean distances $d(E_j, E_i)$, $j \neq i$, and let $b = d(E_l, E_m)$. Then $q\Omega_{n-1}(b/2)^{n-1} \leq \omega_{n-1}(2r_1)^{n-1}$. By (2.12) $b \leq |x_1 - x_2|/2$. Let $x_1^2 \in E_l$ and $x_2^2 \in E_m$ be such that $b = |x_1^2 - x_2^2|$ and write $r_2 = |x_1^2 - x_2^2| \theta_1^2$. Since $f(x_1^2)$ and $f(x_2^2)$ are separated by the ring $D(a_i, \beta) \setminus \bar{D}(a_i, \beta/p)$, Lemma 2.10 implies

$$\nu(\bar{B}(x_1^2, r_2), C(a_i, \beta)) \geq \frac{d_n \log \theta_1}{K} (\log p)^{n-1}. \quad (2.19)$$

Lemma 2.9 gives then for all j

$$\nu(\bar{B}(x_1^2, 2r_2), C(a_j, \beta/p)) \geq \nu(\bar{B}(x_1^2, r_2), C(a_i, \beta)) - c_1 \left(\log \frac{p}{\beta} \right)^{n-1}. \quad (2.20)$$

The left hand side of (2.20) is positive because

$$\frac{d_n \log \theta_1}{K} (\log p)^{n-1} > c_1 \left(\log \frac{p}{\beta} \right)^{n-1}$$

according to the choices of θ_1 and p .

Continuing similarly we get a sequence $(x_1, x_2) = (x_1^1, x_2^1), (x_1^2, x_2^2), (x_1^3, x_2^3), \dots$ of pairs in B such that $x_1^{m+1}, x_2^{m+1} \in \bar{B}(x_1^m, 2r_m)$ and $r_m = |x_1^m - x_2^m| \theta_1^2 \leq r_{m-1}/2$. Then $|x_1^m - x_1| < 4r_1 \leq 2^{-2}(1 - |x_1|)$ which implies that $x_1^m, x_2^m \rightarrow x_0 \in B$. But $\sigma(f(x_1^m), f(x_2^m)) > \beta$ for all m which contradicts the continuity of f at x_0 .

We have thus proved that

$$\tau(y_1, y_2) \leq c_2 \quad (2.21)$$

where c_2 is defined in (2.18).

Case 2. $y_1 \in D(a_k, \beta/p)$, $y_2 \notin D(a_k, \beta/p)$ for some k .

Assume first that $y_1 \in D(a_k, \beta/p^2)$ or $y_2 \notin D(a_k, \beta)$. Then y_1 and y_2 are separated by the ring $D(a_k, \beta/p) \setminus \bar{D}(a_k, \beta/p^2)$ or $D(a_k, \beta) \setminus \bar{D}(a_k, \beta/p)$. Starting as in

Case 1 from the inequality (2.19) we get a contradiction with continuity if $\tau(y_1, y_2) > c_2$.

If $y_1 \notin D(a_k, \beta/p^2)$, $y_2 \in D(a_k, \beta)$, we get

$$\tau(y_1, y_2) \leq P + \log \frac{\log(p^2/\beta)}{-\log \beta} = c_3. \quad (2.22)$$

Case 3. $y_1, y_2 \notin \bigcup_j D(a_j, \beta/p) = U'$.

From (2.1) and (2.2) we obtain

$$\tau(y_1, y_2) = P + 2 \log \frac{\log(p/\beta)}{-\log \beta} + \frac{\pi}{2} Q = c_4. \quad (2.23)$$

Our conclusion from the inequalities (2.21), (2.22), and (2.23) is that in any case

$$\tau(y_1, y_2) \leq \max(c_2, c_3, c_4) = C_1.$$

For the constant C in the theorem we can by (2.13) take

$$C = 2^6 \theta_1^2 C_1.$$

The theorem is proved.

As a corollary of Theorem 2.4 we obtain a substitute for the Picard–Schottky theorem in the following form.

2.24. COROLLARY. *Let $f: B \rightarrow R^n \setminus \{a_1, \dots, a_{q-1}\}$, $n \geq 3$, be K -quasiregular and $q \geq q_0$ where q_0 is as in 2.4. Then*

$$\log |f(x)| \leq C_0(-\log s_0 + \log^+ |f(0)|)(1 - |x|)^{-C} \quad (2.25)$$

where

$$s_0 = \frac{1}{4} \min_{j \neq k} \sigma(a_j, a_k)$$

and C_0 and C are constants which depend only on n , K , and s_0 .

Proof. We choose a metric τ in $Y = R^n \setminus \{a_1, \dots, a_{q-1}\}$ given by (2.3) with $a_q = \infty$ and $\beta = s_0$. Since $|f(x)| \leq \pi/(2\sigma(f(x), \infty))$, we may assume that $f(x) \in$

$D(\infty, s_0)$. If $f(0) \in \bar{D}(\infty, s_0)$,

$$\begin{aligned} \frac{\log |f(x)|}{\log |f(0)|} &\leq \frac{4 \log (1/\sigma(\infty, f(x)))}{\log (1/\sigma(\infty, f(0)))} \leq 4 \exp \tau(f(0), f(x)) \\ &\leq 4 \exp (C(\rho(0, x) + \delta)) \leq C_0(1 - |x|)^{-C} \end{aligned}$$

and (2.25) holds. If $f(0) \notin \bar{D}(\infty, s_0)$, we choose a point $z \in C(\infty, s_0)$ with $\tau(f(0), f(x)) > \tau(z, f(x))$ and obtain

$$\frac{\log |f(x)|}{\log (1/s_0)} \leq 4 \exp \tau(z, f(x)) < 4 \exp \tau(f(0), f(x)) \leq C_0(1 - |x|)^{-C}$$

and (2.25) holds also in this case.

2.26. Remark. Similarly as in the classical case we use Corollary 2.24 to give a new proof of Theorem 1.2 as follows. Let q be as in 2.24 and let $f: R^n \rightarrow R^n \setminus \{a_1, \dots, a_{q-1}\}$ be K -quasiregular. Let $z \in R^n$ and h be the map $x \mapsto 2|z|x$ of the unit ball. Then 2.24 applied to $f \circ h$ gives

$$\log |f(z)| \leq C_0(-\log s_0 + \log^+ |f(0)|)2^C.$$

It follows that f is bounded and thus constant by [3, 3.7].

3. Branched coverings of sphere with punctures

Let M and N be oriented connected n -manifolds. A continuous map $f: M \rightarrow N$ is called a *branched covering* if

- (a) f is discrete, open, and surjective,
- (b) for each $y \in N$ there exists a neighborhood V of y such that each component of $f^{-1}V$ is relatively compact.

If $f: M \rightarrow N$ is a branched covering and V is as in (b) and connected, then every component D of $f^{-1}V$ is a normal domain, i.e. $f \partial D = \partial fD$, f maps D surjectively onto V , and the index (see 2.6)

$$\mu(y, f, D) = \sum_{x \in f^{-1}(y) \cap D} i(x, f)$$

is constant for all $y \in V$.

We shall consider special branched coverings from B onto some $Y = \bar{R}^n \setminus \{a_1, \dots, a_q\}$. These will be quasimeromorphic and automorphic with respect to certain discrete Möbius groups G acting on B .

Let P be a convex (open) hyperbolic polyhedron in B which satisfies the following conditions:

- (1) P has a finite number of faces and finite volume.
- (2) Each dihedral angle in P is π/k for some integer $k > 1$.
- (3) The set of vertices of P in ∂B is nonempty.

Let Γ be the group generated by reflections in the faces of P . Then Γ is a discrete group acting on B and P is a fundamental polyhedron for Γ [13]. Let G be the subgroup of Γ generated by an even number of reflections in the faces of P . Then G is a Möbius group. If T is the reflection in some (open) face A of P , $Q = \text{int}(\bar{P} \cup T\bar{P})$ is a fundamental polyhedron for G .

3.1. LEMMA. *There exists a homeomorphism $\varphi: \bar{P} \rightarrow \bar{B}$ such that $\varphi|_P$ is quasiconformal.*

The proof of this lemma can be carried out as in [5, 3.4]. Fix Q as above. We extend φ to a continuous map $\psi: \bar{Q} \rightarrow \bar{R}^n$ by reflection in A and ∂B . Then ψ maps $P \cup TP \cup A$ quasiconformally onto $B \cup (\bar{R}^n \setminus \bar{B}) \cup \varphi A$. Let $\{b_1, \dots, b_q\}$ be the set of vertices of P in ∂B and let $a_j = \varphi(b_j)$. We extend ψ to a quasimeromorphic mapping h of B by setting

$$h|_{g(\bar{Q} \cap B)} = \psi \circ g^{-1}|_{g(\bar{Q} \cap B)}, \quad g \in G.$$

Then h is a branched covering onto $Y = \bar{R}^n \setminus \{a_1, \dots, a_q\}$, it is automorphic with respect to the group G , and it is injective in each fundamental set.

The map h induces from the hyperbolic metric ρ in B a metric τ on Y defined by

$$\tau(y, z) = \min \{ \rho(u, v) \mid u \in h^{-1}(y), v \in h^{-1}(z) \}.$$

3.2. PROPOSITION. *There exist a constant $a(n, K)$, depending only on n and $K = K(h)$, and a number $\beta > 0$ such that*

$$\left| \tau(y_1, y_2) - \left| \log \frac{\log(1/\sigma(a_j, y_1))}{\log(1/\sigma(a_j, y_2))} \right| \right| \leq a(n, K) \quad (3.3)$$

whenever $y_1, y_2 \in D(a_j, \beta) \setminus \{a_j\}$, $j = 1, \dots, q$.

Proof. Let $y_1, y_2 \in Y$ and let $x_i \in h^{-1}(y_i)$ be such that $\tau(y_1, y_2) = \rho(x_1, x_2)$. Suppose that for some j x_i belongs to the horosphere $S((1-r_i)b_j, r_i)$, $i = 1, 2$. We may assume $r_1 \geq r_2$. Since b_j is a parabolic fixed point for G , [4, 6.16] implies that for some $s_j > 0$

$$C_1 e^{-\gamma/r_1} \leq \sigma(a_j, y_i) \leq C_2 e^{-\delta/r_1} \quad \text{if } \sigma(a_j, y_i) \leq s_j, \quad (3.4)$$

where C_1, C_2, γ , and δ are positive constants with $1 \leq \gamma/\delta \leq b(n, K)$. A similar statement is included also in [4, 6.17(ii)] where, however, the r in the exponent should be replaced by $1/r$.

The inequalities (3.4) give for $\sigma(a_j, y_i) \leq s_j$, $i = 1, 2$,

$$\frac{-\log C_2 + \delta/r_2}{-\log C_1 + \gamma/r_1} \leq \frac{\log(1/\sigma(a_j, y_2))}{\log(1/\sigma(a_j, y_1))} \leq \frac{-\log C_1 + \gamma/r_2}{-\log C_2 + \delta/r_1}.$$

By choosing s_j smaller if necessary we get

$$\log \frac{r_1}{r_2} - \log \frac{2\gamma}{\delta} \leq \log \frac{\log(1/\sigma(a_j, y_2))}{\log(1/\sigma(a_j, y_1))} \leq \log \frac{r_1}{r_2} + \log \frac{2\gamma}{\delta}$$

and $\log(r_1/r_2) - br_1 \leq \rho(x_1, x_2) \leq \log(r_1/r_2) + br_1$ where b is some positive constant. The proposition follows with $\beta = \min(s_1, \dots, s_q)$.

Sources for examples of groups G of the type above are mentioned for instance in [13]. The possible configurations of the set $\{a_1, \dots, a_q\}$ depend on G and the dilatation of h . We shall in the following give an example in dimension three where the set $\{a_1, \dots, a_q\}$ is arbitrarily large and h has an absolute bound for its dilatation.

3.5. Example. We shall give the definition of a hyperbolic polyhedron in $H^3 = \{x \in \mathbb{R}^3 \mid x_3 > 0\}$. Let Σ be the set of spheres $S(x, 1)$ in \mathbb{R}^3 where x runs through the points of the lattice $\{x \in \partial H^3 \mid x = j\sqrt{3}e_1 + k(\sqrt{3}e_1/2 + 3e_2/2), j, k \in \mathbb{Z}\}$. Here e_i is the i th standard coordinate vector. We let m be a positive integer and define planes

$$\begin{aligned} A_1 &= \{x \in \mathbb{R}^3 \mid x_2 = 0\}, \\ A_2 &= \{x \in \mathbb{R}^3 \mid x_2 - \sqrt{3}x_1 = 0\}, \\ A_3 &= \{x \in \mathbb{R}^3 \mid x_2 + \sqrt{3}x_1 = 3m\}. \end{aligned}$$

Let Δ be the bounded open triangle in ∂H^3 bounded by the planes A_i . Let Σ_m be

the subset of Σ consisting of spheres which meet Δ , and let P' be the open hyperbolic convex polyhedron in H^3 bounded by the spheres in Σ_m and the planes A_i . Let T be a Möbius transformation which maps H^3 onto B^3 and $T(\sqrt{3} m/2, m/2, m) = 0$. Set $P = TP'$. The dihedral angle between any two adjacent faces of P is $\pi/3$ or $\pi/2$. Hence P defines a group G as described before. We shall next give a more detailed definition of the map $\varphi: \bar{P} \rightarrow \bar{B}^3$.

Let b be a vertex of P in ∂B , let $8r = 8r_b$ be the Euclidean distance from b to the set of other vertices of P . Let $U = U_b$ be the component of $P \cap (B \setminus \bar{B}(1-2r))$ such that $b \in \bar{U}$. In the following K_1 and K_2 are some absolute constants > 1 . By the technique in [5, p. 128] we first construct a K_1 -quasiconformal mapping $g = g_b$ of $V = V_b = P \cap B(b, 4r)$ onto $V \setminus \bar{U}$ such that

(1) g is the identity on $\partial V \cap B(1-2r)$,

(2) $U \cap B(b, r)$ is mapped onto $W_b = (V \setminus \bar{U}) \cap B(b', r/32)$ and $b' = g(b)$ is a point in $S(1-2r) \cap \bar{U}$ such that $d(b', \partial P) \geq r/8$,

(3) $|g(x) - b'| = c \exp(-1/|x - b|)$ if $x \in U \cap B(b, r)$ for some constant c .

Let φ_1 be the map of P such that $\varphi_1|_{V_b} = g_b$ if b is a vertex of P in ∂B and identity elsewhere. Furthermore, there exists a K_2 -quasimeromorphic mapping φ_2 of $E = \varphi_1 P$ onto B such that $\varphi_2|_{W_b}$ is the radial stretching $x \mapsto (1-2r_b)^{-1}x$ for each vertex b in ∂B . The required map $\varphi|_P$ is defined as $\varphi_2 \circ \varphi_1$.

REFERENCES

- [1] AHLFORS, L. V. *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, 1973.
- [2] MARTIO, O., RICKMAN, S. and VÄISÄLÄ, J. *Definitions for quasiregular mappings*. Ann. Acad. Sci. Fenn. A I 448 (1969), 1–40.
- [3] MARTIO, O., RICKMAN, S. and VÄISÄLÄ, J. *Distortion and singularities of quasiregular mappings*. Ann. Acad. Sci. Fenn. A I 465 (1970), 1–13.
- [4] MARTIO, O. and SREBRO, U. *Automorphic quasimeromorphic mappings in R^n* . Acta Math. 135 (1975), 221–247.
- [5] MARTIO, O. and SREBRO, U. *On the existence of automorphic quasimeromorphic mappings in R^n* . Ann. Acad. Sci. Fenn. A I 3 (1977), 123–130.
- [6] MATTILA, P. and RICKMAN, S. *Averages of the counting function of a quasiregular mapping*. Acta Math. 143 (1979), 273–305.
- [7] PESONEN, M. I. *A path family approach to Ahlfors's value distribution theory*. Ann. Acad. Sci. Fenn. A I Dissertationes 39 (1982), 1–32.
- [8] RICKMAN, S. *On the value distribution of quasimeromorphic maps*. Ann. Acad. Sci. Fenn. A I 2 (1976), 447–466.
- [9] RICKMAN, S. *On the number of omitted values of entire quasiregular mappings*. J. Analyse Math. 37 (1980), 100–117.
- [10] RICKMAN, S. *A defect relation for quasimeromorphic mappings*. Annals of Math. 114 (1981), 165–191.
- [11] RICKMAN, S. *Value distribution of quasiregular mappings*. Lecture Series in the Nordic Summer School in Mathematics, Joensuu 1981, Lecture Notes in Mathematics 981, Springer-Verlag, 1983, 220–245.

- [12] VÄISÄLÄ, J. *A survey of quasiregular maps in R^n* . Proceedings of the International Congress of Mathematicians, Helsinki 1978 (1980), 685–691.
- [13] VINBERG, E. B. *Discrete linear groups generated by reflections*. Math. USSR Izvestija 5 (1971), 1083–1119.

*Department of Mathematics
University of Helsinki
Helsinki, Finland*

Received February 1, 1983