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On the homology of Lie groups made discrete

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§1. Introduction

Let G be an arbitrary Lie group and let G^δ denote the same group with the discrete topology. Then the natural homomorphism $G^\delta \to G$ gives rise to a continuous mapping $\eta: BG^\delta \to BG$ between classifying spaces. This paper is organized around the following conjecture which was suggested to the author by E. Friedlander, at least in the complex case. (Compare Quillen, p. 176.)

ISOMORPHISM CONJECTURE. *This canonical mapping $BG^\delta \to BG$ induces isomorphisms of homology and cohomology with* mod p *coefficients, or more generally with any finite coefficient group.*

Here the homology of BG^δ is just the usual Eilenberg–MacLane homology of the uncountably infinite discrete group G^δ. These homology groups are of interest in algebraic K-theory (see for example Quillen), in the study of bundles with flat connection (Milnor, 1958), in the theory of foliations (Haefliger, 1973), and also in the study of scissors congruence of polyhedra (Dupont and Sah). They are difficult to compute, and tend to be rather wild. For example if G is non-trivial and connected, then Sah and Wagoner show that $H_2(BG^\delta; \mathbf{Z})$ maps *onto* an uncountable rational vector space. (See also Harris.) The homology and cohomology groups of BG, on the other hand, are much better behaved and better understood. (Borel, 1953.)

In §2 we will see that this Isomorphism Conjecture is true whenever the component of the identity in G is solvable. If it is true for simply-connected simple groups, then it is true for all Lie groups. It is always true for 1-dimensional homology, and is true in a number of interesting special cases for 2-dimensional homology. (See §4.) For higher dimensional computations which tend to support the conjecture, see Karoubi, p. 256, Parry and Sah, as well as Thomason.

Another partial result is the following (§3). *If G has only finitely many components, then for any finite coefficient group A the homomorphism $H_*(BG^\delta; A) \to H_*(BG; A)$ is split surjective.* Thus we obtain a direct sum decomposition

$$H_i(BG^\delta; A) \cong H_i(BG; A) \oplus (\text{unknown group}),$$

where the unknown summand is of course conjectured to be zero. The proof is based on Becker and Gottlieb, and generalizes a theorem of Bott and Heitsch. *As an immediate corollary, it follows that the integral cohomology* $H^*(BG; \mathbf{Z})$ *injects into* $H^*(BG^\delta; \mathbf{Z})$.

An appendix discusses the analogous homomorphisms with rational coefficients, which behave very differently. For example *the homomorphism* $H_i(BG^\delta; \mathbf{Q}) \to H_i(BG; \mathbf{Q})$ *is identically zero for* $i > 0$ *whenever* G *is compact, or complex and semi-simple with finitely many components.* More generally, even when these homomorphisms are not identically zero, it is often possible to describe the precise kernel of the associated ring homomorphism $H^*(BG; \mathbf{Q}) \to H^*(BG^\delta; \mathbf{Q})$.

The methods used in this note are all more or less well known. I am particularly grateful to J. F. Adams, E. Friedlander, A. Haefliger, and D. McDuff for pointing out some of the necessary tools to me, to A. Borel for pointing out an error in an earlier version, and to the Institut des Hautes Etudes Scientifiques for its hospitality.

§2. The solvable case

First some general definitions. We will always use singular homology theory with constant (i.e., untwisted) coefficients.

For any topological group G, *let* $\bar{G}$ *be the homotopy fiber of the map* $G^\delta \to G$. (Compare Thurston.) Thus $\bar{G}$ is the topological group consisting of all pairs (g, f) where g is a point of G^δ and f is a path from the identity element to the image of g in G. We will be particularly interested in the classifying space $B\bar{G}$. Mather calls the homology of $B\bar{G}$ the "local homology" of the topological group G, since it is completely determined by the germ of the group G at the identity element. (See also Haefliger 1978, which uses the notation $B\mathfrak{g}$ for our space $B\bar{G}$, and McDuff 1980, which uses the notation $\bar{B}G$.) If G is locally contractible, so that the identity component G_0 has a universal covering group U, note that the natural homomorphisms $U \to G_0 \to G$ induce isomorphisms $\bar{U} \to \bar{G}_0 \to \bar{G}$. *Hence the homology groups of* $B\bar{G}$ *depend only on the universal covering group of* G. In the case of a Lie group, it follows that they depend only on the Lie algebra of G.

LEMMA 1. *The Isomorphism Conjecture of* §1 *is true for a connected Lie group* G *if and only if the associated space* $B\bar{G}$ *has the* mod p *homology of a point, for every prime* p. *If it is true for a connected group* G, *then it is true for any Lie group* H, *connected or not, which is locally isomorphic to* G.

Proof. This follows easily from the mod p homology spectral sequences associated with the fibrations $B\bar{G} \to BG^\delta \to BG$ and $B\bar{G} \to BH^\delta \to BH$. (Note that

BG is simply-connected.) The passage from mod p coefficients to arbitrary finite coefficients can be carried out by induction on the order of the abelian coefficient group A, making use of the homology exact sequence associated with a coefficient sequence $A' \to A \to A/A'$, where A' is some non-trivial proper subgroup of A. Details will be omitted. ■

LEMMA 2. *If a discrete abelian group Γ is uniquely divisible, then its classifying space $B\Gamma$ has the* mod p *homology of a point.*

Proof. A "uniquely divisible" group is just one which is isomorphic to a vector space over the rational numbers $\mathbf{Q}$. First suppose that this vector space is 1-dimensional. Then Γ is a direct limit of free cyclic groups, hence its homology is trivial in all dimensions greater than one; and evidently the group

$$H_1(B\Gamma; \mathbf{Z}/p\mathbf{Z}) \cong H_1(B\Gamma; \mathbf{Z}) \otimes \mathbf{Z}/p\mathbf{Z} \cong \Gamma \otimes \mathbf{Z}/p\mathbf{Z}$$

is also zero. Next suppose that Γ is finite dimensional over $\mathbf{Q}$. Then the conclusion follows inductively, using the Künneth Theorem. Finally, the infinite dimensional case follows by a straightforward direct limit argument. ■

Combining these two results, we obtain the following.

LEMMA 3. *If the component of the identity of G is solvable, then the Isomorphism Conjecture is true for G.*

Proof by induction on the dimension. By Lemma 1 it suffices to consider the case of a simply-connected solvable group. In the 1-dimensional case, $G \cong \mathbf{R}$, the conclusion follows immediately, since $B\mathbf{R}$ is contractible, and $B\mathbf{R}^\delta$ has the mod p homology of a point by Lemma 2. In the case of a higher dimensional simply-connected solvable group, choose a homomorphism from G onto $\mathbf{R}$ with kernel N. Then the short exact sequence $N \to G \to \mathbf{R}$ gives rise to a Serre fibration $B\bar{N} \to B\bar{G} \to B\bar{\mathbf{R}}$. We may assume inductively that $B\bar{N}$ has the mod p homology of a point, and a spectral sequence computation shows that $B\bar{G}$ does also. ■

More generally, for any Lie group G, the associated Lie algebra $\mathfrak{g}$ has a maximal solvable ideal $\mathfrak{n}$, and the quotient $\mathfrak{g}/\mathfrak{n}$ splits as a direct product of simple Lie algebras $\mathfrak{s}_i$. Let S_i be corresponding simple Lie groups.

LEMMA 4. *If the Isomorphism Conjecture is true for each simple Lie group S_i, then it is true for G.*

The proof, based on the fibration $B\bar{N} \to B\bar{G} \to \prod B\bar{S}_i$, is easily supplied. ■

§3. The Gottlieb transfer

Let $\pi: E \to B$ be the projection map of a smooth fiber bundle, with a closed manifold as fiber. The *Gottlieb transfer* $\mathrm{tr}: H^i E \to H^i B$ can be defined intuitively as the cup product with the Euler characteristic along the fiber, followed by integration along the fiber. (For a precise definition see Gottlieb.) *Here, and throughout most of this section, some fixed coefficient group A is to be understood.* There is a completely analogous transfer homomorphism in homology. One basic property is that the composition

$$H_i B \xrightarrow{\mathrm{tr}} H_i E \xrightarrow{\pi_*} H_i B$$

is equal to multiplication by the Euler characteristic of the fiber.

Let G be any Lie group with finitely many components, and let K be a maximal compact subgroup. According to Mostow, the quotient space G/K is contractible, hence the natural map $BK \to BG$ is a homotopy equivalence. Let N be the normalizer of a maximal torus in K. According to Hopf and Samelson, the quotient manifold K/N has Euler characteristic $+1$. Note that there is a canonical fibration $\pi: BN \to BK$ with fiber K/N. Following Becker and Gottlieb, this implies the existence of a transfer homomorphism $\mathrm{tr}: H_i BK \to H_i BN$ such that the composition $H_i BK \to H_i BN \to H_i BK$ is just the identity map of $H_i BK$. *Therefore the natural homomorphism $\pi_*: H_i BN \to H_i BK$ is a split surjection.* A similar argument shows that the corresponding cohomology homomorphism $\pi^*: H^i BK \to H^i BN$ is a split injection.

Now let us assume that the coefficient group A is finite. Then $H_* BN^\delta \cong H_* BN$ by §2. We continue to assume that G has only finitely many components.

THEOREM 1. *The canonical homomorphism $\eta_*: H_i BG^\delta \to H_i BG$ is a split surjection. That is some direct summand of $H_i BG^\delta$ maps isomorphically onto $H_i BG$. Similarly, the cohomology homomorphism $\eta^*: H^i BG \to H^i BG^\delta$ is a split injection.*

Proof. This follows by inspection of the commutative diagram

$$\begin{array}{ccc} H_i BN^\delta & \longrightarrow & H_i BG^\delta \\ \downarrow \cong & & \downarrow \eta_* \\ H_i BN \xrightarrow{\pi_*} H_i BK & \xrightarrow{\cong} & H_i BG, \end{array}$$

or the analogous cohomology diagram. ■

COROLLARY 1. *The homomorphism* $\eta^*: H^i(BG; \mathbf{Z}) \to H^i(BG^\delta; \mathbf{Z})$ *of integral cohomology is injective.*

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} H^i(BG;\mathbf{Z}) \xrightarrow{n} H^i(BG;\mathbf{Z}) & \to & H^i(BG;\mathbf{Z}/n\mathbf{Z}) \\ \downarrow & & \downarrow \\ H^i(BG^\delta;\mathbf{Z}) & \to & H^i(BG^\delta;\mathbf{Z}/n\mathbf{Z}), \end{array}$$

using the fact that $H^i(BG; \mathbf{Z})$ is finitely generated, so that the intersection of the subgroups $nH^i(BG; \mathbf{Z})$ is zero; and using the fact that the right hand vertical arrow is injective. ■

The corresponding statement in homology would be false. For example if G is the unitary group $U(n)$ or the special linear group $SL(n, \mathbf{C})$, then we will see in the Appendix that $\eta_*: H_i(BG^\delta; \mathbf{Z}) \to H_i(BG; \mathbf{Z})$ is identically zero for $i > 0$. However we can prove the following weaker statement.

COROLLARY 2. *Every element of finite order* n *in* $H_i(BG; \mathbf{Z})$ *lifts to an element of order* n *in* $H_i(BG^\delta; \mathbf{Z})$.

Proof. This follows from the commutative diagram

$$\begin{array}{ccccc} H_{i+1}(BG^\delta;\mathbf{Z}/n\mathbf{Z}) & \to & H_i(BG^\delta;\mathbf{Z}) & \xrightarrow{n} & \\ \downarrow \text{onto} & & \downarrow & & \\ H_{i+1}(BG;\mathbf{Z}/n\mathbf{Z}) & \to & H_i(BG;\mathbf{Z}) & \xrightarrow{n} & . \end{array}$$ ■

§4. Examples for H_2

Homology with *integer* coefficients is to be understood throughout this section. We will need the following observation to relate integer homology to mod p homology.

LEMMA 5. *A path-connected space* X *has the* mod p *homology of a point for every prime* p *if and only if the integer homology group* H_iX *is uniquely divisible for* $i > 0$.

In particular, a connected group G satisfies the Isomorphism Conjecture if and only if the integer homology $H_iB\bar{G}$ is uniquely divisible for $i > 0$.

Proof. This follows from the homology exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$. ■

Recall from §2 that it would suffice to prove the Isomorphism Conjecture for connected semi-simple groups.

LEMMA 6. *If G is connected and semi-simple, then $H_1B\bar{G}$ is zero, and there is a split exact sequence $0 \to H_2B\bar{G} \to H_2BG^\delta \to H_2BG \to 0$.*

Here H_2BG can be identified with the fundamental group $\pi_1 G$, since G is connected. So the last statement means that H_2BG^δ splits as the direct sum of the finitely generated group $\pi_1 G$, and a group $H_2B\bar{G}$ which is conjectured to be a rational vector space.

Proof. For the computation of $H_1B\bar{G}$, we may assume that G is simply-connected (compare §2), and hence that $H_2BG = 0$. Since G is perfect, the group $H_1BG^\delta \cong G/[G, G]$ is zero. The statement that $H_1B\bar{G} = 0$ then follows from the spectral sequence of the fibration $B\bar{G} \to BG^\delta \to BG$.

For any connected Lie group G, note that H_3BG is finite, since the rational cohomology of BG is a polynomial algebra on even dimensional generators (Borel, 1953). Therefore H_3BG^δ maps onto H_3BG by Corollary 2 of §3. If G is semi-simple, so that $H_1B\bar{G} = 0$, an elementary spectral sequence argument now yields the required short exact sequence; and it follows from Corollary 2 that this exact sequence splits. ■

LEMMA 7. *If G is a Chevalley group over the real or complex numbers, then $H_2B\bar{G}$ is uniquely divisible and uncountably infinite.*

For the proof, which is based on deep results of Steinberg, Moore and Matsumoto, the reader is referred to Sah and Wagoner, p. 623. ■

Note that any *complex* simply-connected simple Lie group is automatically a Chevalley group. In the complex case, the proof shows that $H_2B\bar{G}$ is naturally isomorphic to the group $K_2\mathbf{C}$ of algebraic K-theory, which is uniquely divisible by a theorem of Bass and Tate.

Typical examples of real Chevalley groups are special linear group $SL(n, \mathbf{R})$, the rotation groups $SO(n, n)$ and $SO(n, n+1)$, and the symplectic group consisting of automorphisms of a skew form on $\mathbf{R}^{2n}$. In the real case, $H_2B\bar{G}$ is isomorphic to the "real part" of $K_2\mathbf{C}$, that is the subspace fixed under the involution arising from complex conjugation.

For non-Chevalley groups, the known information is rather sparse. Alperin and Dennis have proved an analogous result for the stable special linear group over the quaternions. Their paper also contains an ingenious argument due to Mather, which proves the following. *If $T \cong S^1$ is a maximal torus in the 3-sphere group $SU(2)$, then H_2BT^δ maps onto $H_2BSU(2)^\delta$.* Since H_2BT^δ is known to be uniquely divisible, it follows that $H_2BSU(2)^\delta$ is at least divisible. I do not know how to prove the corresponding statement even for $SU(3)$. Alperin has shown that the successive homomorphisms

$$H_2BSU(3)^\delta \to H_2BSU(4)^\delta \to \cdots$$

are surjective (and bijective from $SU(6)$ on); but no more precise information about these groups seems to be available.

Appendix: Real or rational coefficients

The cohomology of BG^δ with real or rational coefficients behaves quite differently from cohomology with finite coefficients, and is somewhat better understood. In fact, there are two basic tools which help to make the real case tractable, namely the Chern–Weil theory of characteristic classes expressed in terms of curvature forms, and the van Est theory of continuous cohomology. One consequence of these theories is the following.

LEMMA 8. *If G is compact, then the canonical homomorphism $H_iBG^\delta \to H_iBG$, with real or rational coefficients, is zero for $i > 0$.*

If the integer homology $H_i(BG; \mathbf{Z})$ happens to be free abelian, then it follows easily that the corresponding homomorphism with integer coefficients is also zero. This is the case, for example, when G is the unitary group $U(n)$.

More generally, let G be any Lie group with finitely many components, and let K be a maximal compact subgroup.

LEMMA 9. *In this case, the homomorphism $H_iBG^\delta \to H_iBG$ is zero for i greater than the dimension of G/K.*

Here and elsewhere, real or rational coefficients are to be understood. Evidently this reduces to the previous statement if G itself is compact.

Here is a different generalization. Let G be any Lie group which contains a discrete cocompact subgroup Γ. Such a subgroup exists, for example, whenever G

is connected and semi-simple (see Borel and Harish-Chandra), or whenever G is simply-connected and nilpotent with rational structure constants (Mal'cev).

LEMMA 10. *Then the image of* $\eta_*: H_iBG^\delta \to H_iBG$ *is precisely equal to the image of the composition*

$$H_i\Gamma \to H_iBG^\delta \to H_iBG.$$

Similarly, the kernel of the ring homomorphism $\eta^*: H^*BG \to H^*BG^\delta$ is equal to the kernel of $H^*BG \to H^*B\Gamma$. Here are some examples. If G is compact, then we can take Γ to be trivial, and recover Lemma 8. If G is the group $PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm I\}$, then a maximal compact subgroup K is a circle, and G can be identified with the group of all orientation preserving isometries of the hyperbolic plane G/K. In this case we can take Γ to be the fundamental group of a closed surface $\Gamma \backslash G/K \simeq B\Gamma$. The cohomology $H^*BG \cong H^*BK$ is a polynomial ring on one 2-dimensional generator, and it follows from either Lemma 9 or 10 that the square of this generator maps to zero in H^4BG^δ. However, the image of the generator itself in H^2BG^δ is non-zero. (Compare Milnor 1958, as well as Wood.)

Another closely related result is the following.

LEMMA 11. *If* G *is complex and semi-simple, with finitely many components, then again the homomorphism* $H_iBG^\delta \to H_iBG$ *is zero for* $i > 0$.

For a real semi-simple connected Lie group, the kernel of the cohomology homomorphism η^* can be computed as follows. Let $h: G \to G_{\mathbf{C}}$ be a *complexification* of G. That is, let $G_{\mathbf{C}}$ be a connected complex Lie group whose Lie algebra is the complexification $\mathfrak{g} \otimes \mathbf{C}$ of the Lie algebra of G, and let h be a homomorphism which induces the embedding of $\mathfrak{g}$ into its complexification. Note that the kernel of h is necessarily discrete and central.

THEOREM 2. *With these hypotheses, the sequence of ring homomorphisms* $H^*BG_{\mathbf{C}} \to H^*BG \to H^*BG^\delta$ *is "exact", in the sense that the kernel of the second homomorphism is equal to the ideal generated by the positive dimensional elements in the image of the first.*

Remark. If we use real coefficients, then the image of $h^*: H^*BG_{\mathbf{C}} \to H^*BG$ can be identified with the image of the Chern–Weil homomorphism associated with G.

As an example, if $G = SL(2n, \mathbf{R})$, then we can take $G_{\mathbf{C}} = SL(2n, \mathbf{C})$. The

cohomology ring H^*BG is a polynomial ring generated by the Pontrjagin classes $p_1, \ldots, p_n$, together with the Euler classs e, subject to the relation $e^2 = p_n$; and the image of h^* is equal to the subalgebra generated by the Pontrjagin classes. (See for example Milnor and Stasheff.) *Thus it follows that only the Euler class survives to* H^*BG^δ (*or to* $H^*B\Gamma$ *if* Γ *is a discrete cocompact subgroup*).

To begin the proofs, let us consider the *Chern–Weil homomorphism*

$$\theta: \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}'] \to H^*(BG; \mathbf{R})$$

associated with a Lie group G and its Lie algebra $\mathfrak{g}$. Here $\mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$ stands for the graded algebra consisting of all real valued polynomial functions on the vector space $\mathfrak{g}$ which are invariant under the adjoint action of G. Given such an invariant polynomial $f: \mathfrak{g} \to \mathbf{R}$, homogeneous of degree n, and given a smooth principal G-bundle over some manifold M, with a smooth G-invariant connection, the curvature 2-forms Ω of the connection give rise to a closed $2n$-form $f(\Omega)$, and hence to a characteristic cohomology class

$$(f(\Omega)) \in H^{2n}(M; \mathbf{R}).$$

This corresponds to the required class $\theta(f) \in H^{2n}(BG; \mathbf{R})$ under the canonical homomorphism $H^{2n}(BG; \mathbf{R}) \to H^{2n}(M; \mathbf{R})$. See Kobayashi and Nomizu or Spivak for details.

Chern–Weil Theorem. *If* G *is compact, then this homomorphism* $\theta: \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}'] \to H^*(BG; \mathbf{R})$ *is bijective.*

In particular, BG has only even dimensional cohomology with real coefficients. This theorem is proved in Cartan or Chern or Bott 1973.

Proof of Lemma 8. Any homology class in $H_{2n}(BG^\delta; \mathbf{Q})$ can be realized as the image of a homology class from some smooth open manifold which is mapped into BG^δ. To prove that its image in $H_{2n}(BG; \mathbf{Q})$ is zero, it evidently suffices to evaluate on an arbitrary real cohomology class in $H^*(BG; \mathbf{R}) \cong \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$. If $n > 0$, then choosing any homogeneous polynomial $f \in \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$ of degree n, the characteristic class $(f(\Omega))$ of the induced bundle over M is zero since this induced bundle has curvature $\Omega = 0$. The conclusion follows. ■

In the case of a *complex* Lie group, there is an analogous homomorphism

$$\mathrm{Inv}_G \mathbf{C}[\mathfrak{g}'] \to H^*(BG; \mathbf{C}),$$

where now $\mathbf{C}[\mathfrak{g}']$ must be interpreted as the graded algebra consisting of all complex polynomial functions on the complex vector space $\mathfrak{g}$,

LEMMA 12. *If G is complex and semi-simple, with only finitely many connected components, then this complex Chern–Weil homomorphism* $\mathrm{Inv}_G \mathbf{C}[\mathfrak{g}'] \to H^*(BG; \mathbf{C})$ *is also bijective.*

Proof of Lemmas 12 *and* 11. Let $K \subset G$ be a maximal compact subgroup. (Compare Mostow.) Since K is essentially unique, it coincides with the compact real form of G, as constructed by Weyl. Hence the Lie algebra $\mathfrak{g}$ can be identified with the complexification $\mathfrak{k} \otimes \mathbf{C}$ of the Lie algebra of K. It is then not difficult to check that $\mathrm{Inv}_G \mathbf{C}[\mathfrak{g}']$ can be identified with $\mathrm{Inv}_K \mathbf{R}[\mathfrak{k}'] \otimes \mathbf{C}$, so that Lemma 12 follows from the Chern–Weil Theorem applied to K. Evidently Lemma 11 follows easily. ■

Next consider the following construction. Let G be any Lie group (with a finite or countably infinite number of components). Fixing some large integer N, let $E \to X$ be a smooth *N-universal* principal G-bundle. That is, we assume that the total space E is $(N-1)$-connected. Then the base space $X = E/G$ is a finite dimensional manifold such that the natural map $X \to BG$ induces isomorphisms of homology and cohomology in dimensions less than N. Let $A(E)$ be the de Rham complex of smooth differential forms on E, and let $\mathrm{Inv}_G A(E)$ be the subcomplex of G-invariant forms. We will be interested in the cohomology groups $H^n(\mathrm{Inv}_G A(E))$ in dimensions $n < N$.

If G has only finitely many components, then these groups $H^n(\mathrm{Inv}_G A(E))$ are isomorphic to the continuous (or the differentiable) Eilenberg–MacLane cohomology groups of G, as studied by van Est. (See for example Borel and Wallach, p. 279.) Furthermore $H^n(\mathrm{Inv}_G A(E))$ can also be identified with the group $H^n(\mathrm{Inv}_G A(G/K))$, where K is a maximal compact subgroup of G, or equivalently with the Lie algebra cohomology $H^n(\mathfrak{g}, K)$. *Thus this cohomology is zero in dimensions greater than the dimension of G/K.* (Compare van Est, Borel–Wallach, Dupont, or Haefliger 1973.) The following two lemmas are essentially due to van Est.

LEMMA 13. *The natural homomorphism* $\eta^*: H^n(BG; \mathbf{R}) \to H^n(BG^\delta; \mathbf{R})$ *factors through the group* $H^n(\mathrm{Inv}_G A(E))$, *providing that* $n < N$.

Clearly Lemma 9, with real coefficients, will follow as an immediate corollary once we have proved this statement; and the corresponding statement with rational coefficients will then also follow.

LEMMA 14. *If Γ is a discrete cocompact subgroup of G, then the composition $H^n(\mathrm{Inv}_G A(E)) \to H^n(BG^\delta; \mathbf{R}) \to H^n(B\Gamma; \mathbf{R})$ is injective for $n < N$.*

Proof of Lemmas 13 *and* 9. Evidently we can identify $H^n(BG; \mathbf{R})$ with the de Rham cohomology $H^n(A(E/G))$, which maps naturally to $H^n(\mathrm{Inv}_G A(E))$. On the other hand, if SE denotes the smooth singular complex of E, then G^δ operates freely and properly on SE, so the quotient complex SE/G^δ has the same cohomology groups as BG^δ in dimensions less than N. A canonical cochain homomorphism

$$\mathrm{Inv}_G A^n(E) \to C^n(SE/G^\delta; \mathbf{R})$$

is constructed by integrating G-invariant n-forms over smooth singular simplexes which are well defined up to right translation by G^δ. This cochain homomorphism induces the required homomorphism from $H^n(\mathrm{Inv}_G (A(E))$ to $H^n(BG^\delta; \mathbf{R})$. Further details will be left to the reader. ■

Proof of Lemmas 14 *and* 10. We can identify $H^n(B\Gamma; \mathbf{R})$ with the nth cohomology of the complex $\mathrm{Inv}_\Gamma A(E) \cong A(E/\Gamma)$ of Γ-invariant forms on E. Let α be a closed G-invariant n-form on E, and suppose that $\alpha = d\beta$ for some Γ-invariant $(n-1)$-form β. If we translate β by any element of the compact coset space $\Gamma\backslash G$, which acts on the right, then we obtain another $(n-1)$-form with coboundary α. Averaging these translates with respect to the Haar measure on this compact coset space, we obtain a G-invariant $(n-1)$-form with the same coboundary α. This proves Lemma 14; and Lemma 10 follows easily. ■

Proof of Theorem 2. Part of this Theorem, namely the statement that the composition $H^iBG_{\mathbf{C}} \to H^iBG \to H^iBG^\delta$ with real or rational coefficients is zero for $i > 0$, follows immediately from Lemma 11 together with the commutative diagram

$$\begin{array}{ccc} BG^\delta & \to & BG \\ \downarrow & & \downarrow \\ BG^\delta_{\mathbf{C}} & \to & BG_{\mathbf{C}}. \end{array}$$

Note, by Lemmas 13 and 14, that an element of $H^i(BG; \mathbf{R})$ maps to zero in $H^i(BG^\delta; \mathbf{R})$ if and only if it maps to zero in the group $H^i \mathrm{Inv}_G A(E) \cong H^i \mathrm{Inv}_G A(G/K)$. *Thus, to prove the Theorem, we must check that the sequence*

$$H^*BG_{\mathbf{C}} \to H^*BG \to H^* \mathrm{Inv}_G A(G/K),$$

with real coefficients, is "exact" in the sense of Theorem 2.

A standard elementary argument shows that the chain complex $\mathrm{Inv}_G A(G/K)$ can be identified with the complex $C^*(\mathfrak{g}, K) \cong \mathrm{Inv}_K \Lambda^*(\mathfrak{g}/\mathfrak{k})'$ consisting of all multi-linear skew forms on the vector space $\mathfrak{g}/\mathfrak{k}$ which are invariant under the adjoint action of K, provided with a suitable coboundary operator. If we pass to complex coefficients, then the cohomology of this complex can be computed in terms of the complexification $h: G \to G_{\mathbf{C}}$ as follows. Choose a maximal compact subgroup L of $G_{\mathbf{C}}$ with $h(K) \subset L$. Then G and L are both real forms of the complex Lie group $G_{\mathbf{C}}$. Hence the corresponding real Lie algebras $\mathfrak{g}$ and $\mathfrak{l}$ have isomorphic complexifications. It follows easily that $H^*(\mathfrak{g}, K) \otimes \mathbf{C}$ is isomorphic to $H^*(\mathfrak{l}, h(K)) \otimes \mathbf{C}$. This can be identified with the cohomology of the complex $\mathrm{Inv}_L A(L/h(K)) \otimes \mathbf{C}$, in fact, since L is compact and connected, it can simply be identified with $H^*(L/h(K); \mathbf{C})$.

Note also that $h(K)$ is the quotient of K by a finite central subgroup, so that the cohomology of $Bh(K)$, with real or rational coefficients is isomorphic to the cohomology of BK or of BG. *To simplify the notation, let us assume that* $K \cong h(K)$, *so that we may think of* K *as a subgroup of* L. The statement to be proved then reduces to the following.

LEMMA 15 (Cartan, p. 69). *Given compact connected Lie groups* $K \subset L$, *the sequence* $H^*BL \to H^*BK \to H^*(L/K)$ *of ring homomorphisms (with real or rational or complex coefficients) is "exact" in the sense of Theorem* 2.

Proof. The fibration sequence $L \to L/K \to BK$ gives rise to a cohomology spectral sequence; or alternatively to the statement that $H^*(L/K)$ is isomorphic to the cohomology of the complex $H^*BK \otimes H^*L$ under a coboundary operator d which has the following properties. The image $d(H^*BK \otimes 1)$ is zero; and furthermore, if $v \in H^*L$ is universally transgressive so that its transgression $\bar{v}$ is defined and lies in the image of $H^*BL \to H^*BK$, then $d(1 \otimes v) = \bar{v} \otimes 1$. (See Borel, 1953 p. 187.) Since H^*L is an exterior algebra generated by universally transgressive elements, it follows easily that the image of d intersected with $H^*BK \otimes 1$ is the ideal spanned by the $\bar{v}$. This proves the Lemma. ■

To prove the Theorem, we must identify the sequence $H^*BL \to H^*BK \to H^*(L/K)$, of Lemma 15, with the required sequence $H^*BG_{\mathbf{C}} \to H^*BG \to H^* \mathrm{Inv}_G (A(G/K))$, using complex coefficients. This can be done, making use of a purely algebraic construction of the last homomorphism. (See Haefliger 1973, p. 6.) Details will be omitted. ■

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