

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 58 (1983)

**Artikel:** Acyclic groups of automorphisms.  
**Autor:** De La Harpe, Pierre / McDuff, Dusa  
**DOI:** <https://doi.org/10.5169/seals-44588>

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## Acyclic groups of automorphisms

PIERRE DE LA HARPE and DUSA McDUFF<sup>(1)</sup>

### 1. Introduction

A discrete group  $\Gamma$  is said to be acyclic if its Eilenberg–MacLane homology groups  $H_i(\Gamma)$  with coefficients in the trivial  $\Gamma$ -module  $\mathbf{Z}$  are zero for all  $i > 0$ . In this paper we show that certain groups, such as the group  $GL(V)$  of all continuous linear automorphisms of an infinite dimensional Hilbert space  $V$ , are acyclic. This is a folk theorem which was surely known long ago to experts in the field such as Quillen and Segal. However it seems worthwhile to publish a proof in view of the recent interest shown in such questions. For example, Herman pointed out in [He] that the group of diffeomorphisms of a compact manifold admits a canonical representation in  $GL(V)$ . Therefore, if  $GL(V)$  had carried non-trivial cohomology, one might have been able to define non-trivial characteristic classes for groups of diffeomorphisms. See also section 2.6 in [Ma] and the concluding remark of [H2].

We will consider the following groups.

1. The group  $\Sigma(X)$  of all permutations of an infinite set  $X$ .
2. The group  $\mathcal{A}(\Omega)$  of measure preserving automorphisms of a Lebesgue measure space  $(\Omega, \mathcal{B}, \mu)$  where  $\mu$  is infinite and non-atomic. (As usual one identifies two automorphisms which agree  $\mu$ -a.e.)
3. The group  $GL(W)$  of all linear automorphisms of an infinite dimensional vector space  $W$ .
4. The group  $GL(V)$  of all continuous linear automorphisms of an infinite dimensional Hilbert space  $V$  over the real, complexes or quaternions, as well as the group  $U(V)$  of invertible isometries of  $V$ .
5. The group  $GL(M)$  of invertible elements in a properly infinite von Neumann algebra  $M$ , and the subgroup  $U(M)$  of unitary elements.

**THEOREM.** *The groups defined above are acyclic.*

The above list is by no means complete. One could add many “classical

<sup>1</sup> Partially supported by NSF grant no MCS 79 05795 A02.

groups” in the sense of [H3], and also the group of continuous linear automorphisms of an infinite dimensional topological vector space  $E$  for suitable  $E$ . The Banach spaces  $c_0$  and  $l_p$ ,  $1 \leq p < \infty$ , are possible candidates: see proposition 2.a.2 in [LT]. However Douady [D] constructs a Banach space  $E$  for which the group of connected components of  $GL(E)$  is isomorphic to  $\mathbf{Z}$ . It follows that  $GL(E)$  is not perfect and hence not acyclic. Therefore the above theorem does not hold for  $GL(E)$  where  $E$  is an arbitrary Banach space. See also [St]. For acyclic groups of a quite different nature from those of our list, see [BDH] and [BDM].

Here is one consequence of the theorem.

**COROLLARY.** *Let  $G$  be one of the groups above and let  $A$  be a finitely generated abelian group. Then any extension*

$$0 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

*is trivial.*

*Proof.* Any non-trivial normal subgroup of  $G$  is of uncountable index. (See Appendix 1.) In particular any homomorphism from  $G$  to  $\text{Aut}(A)$  is trivial and so  $G$  acts trivially on  $A$  in the above extension. Our main theorem implies that  $H^2(G; A)$  is zero. Hence the extension is a semi-direct product. Again using the fact that the action of  $G$  on  $A$  is trivial, we see that the product is direct. ■

A notable feature of the groups in 2, 4 and 5 is that they are contractible when given their natural topologies. (See [Ke] for  $\mathcal{A}(\Omega)$ , [DD] for  $U(V)$  and  $U(M)$  with the strong topology, [Ku] for  $GL(V)$  and  $U(V)$  with the uniform topology, and [BW] for  $GL(M)$  and  $U(M)$  with the uniform topology.) There are many other contractible groups of automorphisms which are acyclic when considered as discrete groups: for example, the group of compactly supported homeomorphisms of  $\mathbf{R}^n$  [M], and the group of diffeomorphisms of  $\mathbf{R}^n$  which are the identity near the origin [Se]. On the other hand, Sah pointed out that the universal cover  $\widetilde{SL(2, \mathbf{R})}$  of  $SL(2, \mathbf{R})$  is contractible as a topological group but is not acyclic as a discrete group [SW]. The main tool which we use in proving acyclicity is the infinite repetition argument of Mather [M] and Wagoner [W]. (See also [BDH] §4 and [Be] ch. 3.) There are several contractible groups which are more “flexible” than  $\widetilde{SL(2, \mathbf{R})}$ , but are still not large enough for this argument to be used. We have in mind groups such as  $\mathcal{A}(\Omega)$ , where  $\Omega$  has finite measure, or the group of compactly supported homeomorphisms of  $\mathbf{R}^n$  which preserve Lebesgue measure, for  $n > 2$ . These groups are known to be perfect [F1], [F2], and it would be interesting to know if they are acyclic. One could also consider much bigger groups such as the group of all homeomorphisms of a Hilbert cube or a Hilbert

space. These are shown to be contractible in [Re]. The groups  $GL(M)$  and  $U(M)$ , where  $M$  is a finite continuous von Neumann factor, are not contractible. They are discussed further in section 4.

The theorem is not hard to prove. We first show that the subgroup  $G_F$  of elements in  $G$  which are the identity on an appropriately defined “flag”  $F$  is acyclic. Then we show, using a technique due to Segal (§2 in [Se]), that this forces the whole group  $G$  to be acyclic. The first of these two steps uses the infinite repetition argument of [M] and [W] and, in the general case, an elegant algebraic trick due to Quillen [Q2]. The second step works essentially because the Tits building (or partially ordered set) formed by the flags is contractible. We give the proof for  $GL(V)$  in full detail, and in section 4 sketch the modifications needed for the other groups.

We discuss in Appendix 1 the results about normal subgroups of  $G$  needed for the corollary above. Though these are old results, we indicate for  $GL(W)$  and  $GL(V)$  a proof much shorter than the originally published ones. Doing this, we again show that  $G$  is perfect, namely that  $H_1(G)$  is trivial. This is what our main result and proof reduce to when cleared from homological machinery.

Finally Appendix 2 describes a result due to Quillen according to which the monoids (or semi-groups) related to our groups are contractible and hence acyclic.

The authors thank Charles Kratzer, Han Sah and Pierre Vogel for instructive conversations.

## 2. Subgroups of $GL(V)$

In this section and the next one,  $V$  denotes an infinite dimensional Hilbert space. Let  $F$  be a *flag* in  $V$ : we mean by this that  $F$  is a nested sequence  $S_1 \supset S_2 \supset S_3 \supset \dots$  of closed subspaces of  $V = S_0$  such that  $S_{i-1}/S_i$  is isomorphic to  $V$  for each  $i \geq 1$ . Define

$$G_i = \{g \in GL(V) \mid g = \text{id on } S_i\}$$

and

$$G'_i = \{g \in G_i \mid g(S_i^\perp) = S_i^\perp\}$$

for each  $i \geq 0$ . Define also  $G_\infty$  to be the union of the  $G_i$ 's and  $G'_\infty$  that of the  $G'_i$ 's. Then

$$\begin{array}{ccccccc} 1 = G_0 & \subset & G_1 & \subset & \dots & \subset & G_i & \subset & \dots & \subset & G_\infty \\ & & \parallel & & \cup & & \cup & & \cup & & \\ & & G'_0 & \subset & G'_1 & \subset & \dots & \subset & G'_i & \subset & \dots & \subset & G'_\infty. \end{array}$$



For  $g \in G_\infty$ , observe that  $g = \text{id}$  on  $S_\infty = \bigcap S_i$ . For notational convenience, we assume  $S_\infty = \{0\}$ . (But proposition 1 as well as its consequences in section 3 and the variations of section 4 would obviously hold without this assumption.) The result of this section is:

**PROPOSITION 1.** *The groups  $G'_\infty$  and  $G_\infty$  are acyclic.*

We shall recall the following facts from §2 in [W]. A *flabby* group is a group  $\Gamma$  such that there exist homomorphisms

$$\begin{aligned} \mu : \Gamma \times \Gamma &\rightarrow \Gamma \quad (\text{direct sum}) \\ \tau : \Gamma &\rightarrow \Gamma \quad (\text{infinite repetition}) \end{aligned}$$

with the following properties: For any finite subset  $\Phi \subset \Gamma$ , there are elements  $a, b, c$  in  $\Gamma$  satisfying

- (1)  $g\mu 1 = aga^{-1}$ ,  $1\mu g = bgb^{-1}$  where  $1$  is the identity element in  $\Gamma$ ,
- (2)  $g\mu\tau(g) = c\tau(g)c^{-1}$

for all  $g \in \Phi$ .

**LEMMA 2** (Wagoner). *A flabby group is acyclic.*

*Sketch of proof.* Any inner automorphism of  $\Gamma$  acts trivially on homology. By (1), this implies first that  $\mu$  induces a (non associative) ring structure  $\mu_* : H_*(\Gamma) \otimes H_*(\Gamma) \rightarrow H_*(\Gamma)$  on homology, with two-sided unit the number 1 in  $H_0(\Gamma) = \mathbb{Z}$ . By (2), this implies also that  $\mu(\text{id} \times \tau)\Delta$  and  $\tau$  act the same way on homology, where  $\Delta : \Gamma \rightarrow \Gamma \times \Gamma$  is the diagonal map.

Let  $i$  be an integer,  $i > 0$ , and assume inductively that  $H_n(\Gamma)$  is trivial for  $0 < n < i$  (this holds trivially if  $i = 1$ ). Choose  $z \in H_i(\Gamma)$ . By the Künneth formula

$$\Delta_*(z) = z \otimes 1 + 1 \otimes z \in H_i(\Gamma) \otimes H_0(\Gamma) + H_0(\Gamma) \otimes H_i(\Gamma) = H_i(\Gamma \times \Gamma)$$

so that

$$(\mu(\text{id} \times \tau)\Delta)_*(z) = \mu_*(z \otimes 1 + 1 \otimes \tau_*(z)) = z + \tau_*(z) \in H_i(\Gamma).$$

As this must coincide with  $\tau_*(z)$  one has  $z = 0$ . Hence  $H_i(\Gamma)$  is trivial. ■

**LEMMA 3.** *The group  $G'_\infty$  is flabby.*

*Proof.* Let  $T_0^0$  be a Hilbert space isomorphic to  $V$ . For any pair  $(j, k)$  of positive integers, let  $T_j^k$  be a copy of  $T_0^0$ . We identify  $V$  and  $T = \bigoplus_k \bigoplus_j T_j^k$  in such

a way that

$$S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$$

(where  $\bigoplus_k$  means  $\bigoplus_{k=0}^{\infty}$ ). For each  $j \geq 0$  define an isometry  $\rho_j$  from  $\bigoplus_k T_j^k$  onto  $T_j^0$  and an isometry (shift)  $\sigma_j$  from  $\bigoplus_k T_j^k$  onto  $\bigoplus_{k=1}^{\infty} T_j^k$  with  $\sigma_j(T_j^k) = T_j^{k+1}$  for all  $k \geq 0$ . Denote by  $\rho$  the isometry  $\bigoplus_j \rho_j$  from  $T$  onto  $\bigoplus_j T_j^0$  and by  $\sigma$  the shift  $\bigoplus_j \sigma_j$ . Define the maps

$$\mu: \begin{cases} GL(T) \times GL(T) \rightarrow GL(T) \\ (g, h) \rightarrow \rho g \rho^* + \sigma h \sigma^* \end{cases}$$

and

$$\tau: \begin{cases} GL(T) \rightarrow GL(T) \\ g \rightarrow \sum_k \sigma^k \rho g \rho^* \sigma^{*k} \end{cases}$$

(The series converges strongly, and  $\rho^*$  is the adjoint of  $\rho$ ; in view of section 4, it is appropriate to define  $\rho^*$  by  $\rho^*(\xi) = \eta$  if  $\eta = \rho(\xi) \in \text{Im}(\rho)$  and  $\rho^*(\xi) = 0$  if  $\xi \perp \text{Im}(\rho)$ .)

It is easy to check that  $\mu$  and  $\tau$  are homomorphism because  $\rho$  and  $\sigma$  are isometries with orthogonal complementary ranges. Similarly  $\mu(\text{id} \times \tau)\Delta = \tau$ . For each  $i \geq 0$  one has  $\mu(G'_i \times G'_i) \subset G'_i$  and  $\tau(G'_i) \subset G'_i$  because  $\rho_i \rho_i^* + \sigma_i \sigma_i^*$  coincides with the identity on  $\bigoplus_k T_j^k$  for  $j \geq i$ . It follows that  $\mu$  and  $\tau$  induce homomorphisms  $G'_\infty \times G'_\infty \rightarrow G'_\infty$  and  $G'_\infty \rightarrow G'_\infty$ , denoted below by  $\mu$  and  $\rho$  again. Requirement (2) in the definition of a flabby group obviously holds (with  $c = 1$ ).

Consider some integer  $i \geq 0$ . Let  $a_i$  be an invertible isometry of  $T$  which acts as  $\bigoplus_{j=0}^{i-1} \rho_j$  on  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ , as the identity on  $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$ , and (thus) maps in some way  $\bigoplus_k T_i^k$  onto

$$\left( \bigoplus_{k=1}^{\infty} \bigoplus_{j=0}^{i-1} T_j^k \right) \oplus \left( \bigoplus_k T_i^k \right).$$

One has  $a_i \in G'_{i+1} \subset G'_\infty$  and  $a_i g a_i^* = g \mu 1$  for all  $g \in G'_i$ . Similarly, let  $b_i$  be an invertible isometry of  $T$  which acts as  $\bigoplus_{j=0}^{i-1} \sigma_j$  on  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$  and as the identity on  $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$ . Then  $b_i \in G'_{i+1}$  and  $b_i g b_i^* = 1 \mu g$  for all  $g \in G'_i$ . It follows that requirement (1) above holds. ■

We know thus that  $G'_\infty$  is acyclic. The reader who is interested in  $U(V)$  and not in  $GL(V)$  may skip the end of this section since  $G_\infty \cap U(V) = G'_\infty \cap U(V)$ .

Let us now recall what we need from a result due to Quillen (theorem 1' of [Q2]). Let  $A$  be a  $\mathbf{Q}$ -algebra with unit, let  $\Gamma$  be the group of invertible  $(2 \times 2)$ -matrices over  $A$  which have the form  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ , let  $\Gamma'$  be the subgroup of  $\Gamma$  consisting of diagonal matrices and let  $\pi: \Gamma \rightarrow \Gamma'$  be the homomorphism defined by

$$\pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If  $R$  is a  $\mathbf{Z}[\Gamma]$ -module, we denote by  $H_i(\Gamma, R)$  the  $i^{\text{th}}$  Eilenberg–MacLane homology group of  $\Gamma$  with coefficients in  $R$ ; moreover  $R$  is assumed to have the trivial  $\mathbf{Z}[\Gamma]$ -structure if there is no strong reason for any other one (such as  $R = H_t(N; \mathbf{K})$  below).

LEMMA 4 (Quillen). *Let  $\mathbf{K}$  be a field which is either finite or the rationals. Then  $\pi$  induces an isomorphism on  $H_*(-; \mathbf{K})$ .*

*Proof.* Let  $N$  be the subgroup of  $\Gamma$  consisting of matrices of the form  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ , which is isomorphic to the additive group of the algebra  $A$ . As  $N$  is torsion-free and abelian,  $H_*(N; \mathbf{Z})$  is isomorphic to the additive group  $\bigwedge_{\mathbf{Z}} N$ . (This holds for finitely generated free abelian groups, as one checks knowing homology of compact tori; this holds in general because  $N$  and the inductive limit of finitely generated subgroups of  $N$  have the same homology.) It follows that  $H_*(N; \mathbf{K}) \approx (\bigwedge_{\mathbf{Z}} N) \otimes_{\mathbf{Z}} \mathbf{K}$  for any field  $\mathbf{K}$ . In particular  $H_*(N; \mathbf{K}) = H_0(N; \mathbf{K}) = \mathbf{K}$  if  $\mathbf{K}$  is finite (because  $N$  is divisible) and  $H_*(N; \mathbf{Q}) = \bigwedge_{\mathbf{Q}} A$ . (This is a highly degenerate form of the results described in §8 of [Q2].)

Consider the Hochschild–Serre spectral sequence

$$E_{s,t}^2 = H_s(\Gamma'; H_t(N; \mathbf{K})) \Rightarrow H_{s+t}(\Gamma; \mathbf{K})$$

corresponding to the extension

$$0 \rightarrow N \rightarrow \Gamma \rightarrow \Gamma' \rightarrow 1.$$

If  $\mathbf{K}$  is a finite field, one has  $H_t(N; \mathbf{K}) = 0$  for  $t > 0$  and  $H_0(N; \mathbf{K}) = \mathbf{K}$ . The spectral sequence therefore degenerates, giving the desired result.

Suppose  $\mathbf{K} = \mathbf{Q}$ . Make  $\mathbf{Q}^*$  act on  $\Gamma$  by

$$\lambda \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & \lambda b \\ 0 & 1 \end{pmatrix}.$$

Thus  $\lambda \in \mathbf{Q}^*$  acts on the exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & N & \rightarrow & \Gamma & \rightarrow & \Gamma' \rightarrow 1 \\ & & \downarrow \lambda & & \downarrow \lambda & & \downarrow \text{id} \\ 0 & \rightarrow & N & \rightarrow & \Gamma & \rightarrow & \Gamma' \rightarrow 1 \end{array}$$

and consequently also on the spectral sequence. As  $\lambda \in \mathbf{Q}^*$  acts on  $H_t(N; \mathbf{Q}) = \Lambda_{\mathbf{Q}}^t(N \otimes_{\mathbf{Z}} \mathbf{Q})$  by multiplying by  $\lambda^t$ , and acts trivially on  $\Gamma'$ , it follows that  $\lambda$  acts on  $E_{s,t}^2$  by multiplying by  $\lambda^t$ . Assume  $\lambda \neq \pm 1$ ; as the differentials commute with the  $\mathbf{Q}^*$ -action and as  $\lambda^t \neq \lambda^{t'}$  for  $t \neq t'$ , all differentials are zero. It follows that

$$E_{s,t}^2 = E_{s,t}^\infty \quad \text{for all } s, t \geq 0.$$

Now  $\bigoplus_{s+t=n} E_{s,t}^\infty$  is the graded object associated to the natural filtration of  $H_n(\Gamma; \mathbf{Q})$  for each integer  $n \geq 1$ . Since  $\mathbf{Q}^*$  acts on  $\Gamma$  by inner automorphisms, the induced action on  $H_n(\Gamma; \mathbf{Q})$  is trivial; thus  $\mathbf{Q}^*$  acts trivially on each  $E_{s,t}^\infty$ . Hence  $E_{s,t}^\infty = 0$  for any  $(s, t)$  with  $s \geq 0$  and  $t > 0$ . This shows that  $H_s(\Gamma'; \mathbf{Q}) = H_s(\Gamma; \mathbf{Q})$  for any  $s \geq 0$ . ■

**COROLLARY 5** (a universal coefficient argument). *The homomorphism  $\pi: \Gamma \rightarrow \Gamma'$  induces an isomorphism on  $H_*(-) = H_*(-; \mathbf{Z})$ .*

*Proof.* We know that  $\pi$  induces an isomorphism for  $H_*(-; R)$  if  $R$  is the additive group of a finite field. Using direct products and extensions of the coefficients, one checks the same holds for  $R$  a finite abelian group. As homology commutes with inductive limits of coefficients, this holds also when  $R = \mathbf{Q}/\mathbf{Z}$ . Using the sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z} \rightarrow 0$$

and the fact that  $\pi_*$  is an isomorphism for  $R = \mathbf{Q}$  and  $R = \mathbf{Q}/\mathbf{Z}$ , one proves the claim. ■

*The proof of Proposition 1.* We use again the notations defined earlier in this section, and we denote by  $L(V)$  the algebra of all bounded operators on  $V$ . For each  $i > 0$  the spaces  $S_i^\perp$  and  $S_i$  are both isomorphic to  $V$ . It follows that  $G_i$  is isomorphic to

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in L(V) \text{ with } a \text{ invertible} \right\}$$

and that  $G'_i$  consists of matrices in  $G_i$  with  $b = 0$ . Quillen's argument shows that the inclusion of  $G'_i$  in  $G_i$  induces an isomorphism  $H_*(G'_i) \approx H_*(G_i)$ . It follows that the inclusion of  $G'_\infty$  in  $G_\infty$  induces also an isomorphism  $H_*(G') \approx H_*(G)$ , so that the proof of proposition 1 is complete. ■

Let us end this section by two observations. First the groups of our main theorem are not flabby. Consider for example  $G = U(V)$  with  $V$  an infinite dimensional separable complex Hilbert space, and suppose there exists a "direct sum" homomorphism  $\mu : G \times G \rightarrow G$  with property (1) preceding lemma 2; we shall reach a contradiction.

Choose an orthonormal basis  $(e_j)_{j \in \mathbb{N}}$  of  $V$  and a sequence  $(\lambda_j)_{j \in \mathbb{N}}$  of pairwise distinct numbers in the interval  $]-\pi, \pi[$ . Define  $r \in G$  by  $r(e_j) = \exp(i\lambda_j)e_j$  for  $j \in \mathbb{N}$ . The centralizer of  $r$  in  $G$  is the abelian group  $T$  of unitary operators which are diagonal with respect to the chosen basis.

Consider the homomorphism  $\mu_1 : G \rightarrow G$  given by  $g \mapsto \mu(g, 1)$ . By hypothesis  $\mu_1(g)$  is conjugate to  $g$ . Therefore,  $\mu_1$  is injective and, because its image commutes with  $\mu(1, r)$ , the centralizer of  $\mu(1, r)$  is not abelian. But there exists  $b \in G$  with  $\mu(1, r) = brb^{-1}$ . Therefore the centralizer of  $\mu(1, r)$  is the abelian group  $bTb^{-1}$ . This contradiction shows that  $G$  is not flabby.

The second observation is that there are plenty of (non trivial)  $G$ -modules  $R$  with non trivial  $H_*(G, R)$  or  $H^*(G, R)$ . Consider for example a subgroup  $G_1$  of  $G$  and a  $G_1$ -module  $R_1$ . Let  $R = \text{Hom}_{\mathbb{Z}G_1}(\mathbb{Z}G, R_1)$ , where  $\mathbb{Z}G$  is considered as a left  $\mathbb{Z}G_1$ -module and as a right  $\mathbb{Z}G$ -module; then  $R$  is naturally a  $G$ -module (namely a left  $\mathbb{Z}G$ -module). A standard result known as Shapiro's lemma states that  $H^n(G_1, R_1)$  is naturally isomorphic with  $H^n(G, R)$  for all  $n \geq 0$ ; see for example §34.2 in [Bab]. Choose in particular a finite cyclic subgroup  $G_1$  of  $G$  and let  $R_1$  be a trivial  $G_1$ -module isomorphic to  $G_1$  as abelian group. Then  $H^n(G, R) \neq 0$  for all  $n > 0$ .

This is quite a general construction. Indeed, let  $\Gamma$  be any group with more than one element. One shows by induction from a (possibly infinite) cyclic subgroup of  $\Gamma$  that there exists a  $\Gamma$ -module  $M$  and an integer  $n > 0$  with  $H^n(\Gamma; M) \neq 0$ .

### 3. The set of flags

Let  $Gr$  be the set of those closed subspaces  $S$  of  $V$  which are isomorphic to  $V/S$ . (Thus  $Gr$  is the set of points in a Grassmannian space.)

LEMMA 6. *Let  $\{S_1, \dots, S_p\}$  be a finite subset of  $Gr$ . There exist  $S'_1, \dots, S'_p \in Gr$  with  $S'_m \subset S_m$  ( $1 \leq m \leq p$ ) and  $S'_m \perp S'_n$  ( $1 \leq m < n \leq p$ ).*

*Proof.* Any subspace of  $V$  whose codimension is strictly smaller than the dimension of  $V$  intersects non trivially any element of  $Gr$ . One may thus choose unit vectors as follows

$$v_{1,1} \in S_1, v_{2,1} \in S_2 \cap \{v_{1,1}\}^\perp, \dots, v_{p,1} \in S_p \cap \{v_{1,1}, \dots, v_{p-1,1}\}^\perp$$

and in general

$$\begin{aligned} v_{1,i} &\in S_1 \cap \{v_{1,1}, \dots, v_{p,1}, \dots, v_{1,i-1}, \dots, v_{p,i-1}\}^\perp, \\ &\dots, \\ v_{p,i} &\in S_p \cap \{v_{1,1}, \dots, v_{p,1}, \dots, v_{1,i}, \dots, v_{p-1,i}\}^\perp. \end{aligned}$$

(The index  $i$  runs over  $\mathbf{N}^*$  if  $V$  is separable and over some suitable infinite set if  $V$  is “larger”.) Define  $S'_m$  to be the closed linear span of the  $v_{m,i}$ ’s. Then  $S'_1, \dots, S'_p$  have the desired properties. ■

LEMMA 7. *Let  $S_1, \dots, S_p \in Gr$  and let  $h_1, \dots, h_p \in GL(V)$ . There exist  $S'_1, \dots, S'_p \in Gr$  with  $S'_m \subset S_m$  ( $1 \leq m \leq p$ ),  $S'_m \perp S'_n$  and  $h_m(S'_m) \perp h_n(S'_n)$  ( $1 \leq m < n \leq p$ ).*

*Proof.* By Lemma 6 there exist  $S''_1, \dots, S''_p \in Gr$  with  $S''_m \subset S_m$  ( $1 \leq m \leq p$ ) and  $S''_m \perp S''_n$  ( $1 \leq m < n \leq p$ ). Define  $T_m = h_m(S''_m)$  ( $1 \leq m \leq p$ ). There exist also  $T'_1, \dots, T'_p \in Gr$  with  $T'_m \subset T_m$  ( $1 \leq m \leq p$ ) and  $T'_m \perp T'_n$  ( $1 \leq m < n \leq p$ ). Define  $S'_m = h_m^{-1}(T'_m)$  ( $1 \leq m \leq p$ ). ■

Now consider the set  $\mathfrak{F}$  of flags  $F = \{S_1 \supset S_2 \supset \dots\}$  with  $\bigcap S_i = \{0\}$  as defined in section 2. Let  $F = \{S_1 \supset S_2 \supset \dots\}$ ,  $F' = \{S'_1 \supset S'_2 \supset \dots\}$  and  $h \in GL(V)$ . We write  $F' \leq F$  if  $S'_i \subset S_i$  for all  $i$ . If  $S'_i \perp S_i$ , we write  $F' \perp F$ . If in addition  $S_1 \oplus S'_1 \in Gr$ , the spaces  $S_1 \oplus S'_1 \supset S_2 \oplus S'_2 \supset \dots$  form a flag which we call  $F' \oplus F$ . Finally the flag  $\{h(S_1) \supset h(S_2) \supset \dots\}$  is called  $h(F)$ .

We may reformulate lemma 7 for flags.

LEMMA 8. Let  $F_1, \dots, F_p \in \mathfrak{F}$  and let  $h_1, \dots, h_p \in GL(V)$ . There exist  $F'_1, \dots, F'_p \in \mathfrak{F}$  with  $F'_m \leq F_m$  ( $1 \leq m \leq p$ ),  $F'_m \perp F'_n$  and  $h_m(F'_m) \perp h_n(F'_n)$  ( $1 \leq m < n \leq p$ ).

*Proof.* Let  $F_m = \{S_{m,1} \supset S_{m,2} \supset \dots\}$  and write  $T_{m,i} = S_{m,i}^\perp \cap S_{m,i-1}$  where  $S_{m,0} = V$  ( $1 \leq m \leq p$  and  $i \geq 1$ ). Then  $S_{m,i} = \bigoplus_{j=i+1}^\infty T_{m,j}$ . The result now follows by applying lemma 7 to the spaces  $T_{1,j}, \dots, T_{p,j}$  for each  $j \geq 1$ . ■

We review now the Milnor construction for classifying space (see e.g. [Hu], chap. 4, §11). Given any (discrete) group  $\Gamma$ , let  $E\Gamma$  be the simplicial complex whose  $p$ -simplices are the ordered subsets  $(\gamma_0, \dots, \gamma_p)$  of  $\Gamma$ . We denote by  $|E\Gamma|$  the topological space obtained by realizing  $E\Gamma$ . It is well-known and easy to see that  $|E\Gamma|$  is contractible (compare the proof of lemma 10 below). Moreover the group  $\Gamma$  acts freely on  $|E\Gamma|$  by multiplication on the left. Thus the quotient space  $B\Gamma = \Gamma \backslash |E\Gamma|$  is a model (the “infinite join” model) for the classifying space of the group  $\Gamma$ . In particular this means that the groups  $H_i(\Gamma)$  ( $i \in \mathbb{N}$ ) are just the integral homology groups of the space  $B\Gamma$ .

For the rest of this section, we will write  $G$  for  $GL(V)$ ,  $E$  for  $EGL(V)$  and  $B$  for  $BGL(V)$ . For each flag  $F = \{S_1 \supset S_2 \supset \dots\}$  in  $\mathfrak{F}$ , let  $G_F$  be the subgroup of  $G$  containing those operators which agree with the identity on  $S_i$  for  $i$  large enough, and let  $E_F$  be the subcomplex of  $E$  defined as follows: a  $k$ -simplex  $(g_0, \dots, g_k)$  of  $E$  is in  $E_F$  if  $g_0, \dots, g_k$  agree on  $S_i$  for  $i$  large enough. (For short, we will say that  $g_0, \dots, g_k$  agree on  $F$ .) Let  $F, F' \in \mathfrak{F}$ . If  $F' \leq F$ , observe that  $G_F \subset G_{F'}$  and that  $E_F$  is a subcomplex of  $E_{F'}$ . If  $F \perp F'$  and if  $F \oplus F' \in \mathfrak{F}$ , then  $G_{F \oplus F'} = G_F \cap G_{F'}$ .

LEMMA 9. For any  $F \in \mathfrak{F}$ , the complex  $E_F$  is  $G$ -invariant and the quotient  $G \backslash |E_F|$  is naturally isomorphic to  $BG_F$ .

*Proof.* “Naturally” means that, if  $F, F' \in \mathfrak{F}$  with  $F' \leq F$ , then the map  $BG_F \rightarrow BG_{F'}$  induced by  $G_F \hookrightarrow G_{F'}$  is just the inclusion of  $BG_F$  in  $BG_{F'}$  (both are subspaces of  $B$ ).

The space  $|E_F|$  is not connected. Indeed two 0-simplices  $(g)$  and  $(g')$  define points lying in the same connected component if and only if there is a sequence of 1-simplices in  $E_F$  of the form

$$(g, g_1), (g_1, g_2), \dots, (g_m, g').$$

This holds if and only if  $g$  and  $g'$  agree on  $F$ , namely if and only if  $g$  and  $g'$  belong to the same right coset of  $G_F$  in  $G$ . It follows that connected components of  $|E_F|$

are parametrized by  $G/G_F$ . The coset  $G_F$  corresponds to  $|E'_F|$ , where  $E'_F$  is the subcomplex of  $E_F$  consisting of simplices  $(g_0, \dots, g_k)$  where  $g_0, \dots, g_k$  agree with the identity on  $F$ .

It is clear that  $E_F$  is  $G$ -invariant. It follows from the discussion above that  $G \backslash |E_F|$  may be identified with  $G_F \backslash |E'_F|$ , which is nothing but the infinite join model  $BG_F$  for the classifying space of  $G_F$ . ■

Let  $E_*$  be the union of the  $E_F$ 's over  $F \in \mathfrak{F}$ ; it is a subcomplex of  $E$  which is invariant by  $G$ . Let  $B_* = G \backslash |E_*|$ ; it is a subspace of  $B$  which is the union of the  $G \backslash |E_F|$ 's over  $F$  in  $\mathfrak{F}$ .

LEMMA 10. *The space  $E_*$  is contractible.*

*Proof.* Let  $\sigma_1, \dots, \sigma_p$  be simplices in  $E_*$ . Choose

$$F_1 = \{S_{1,1} \supset S_{1,2} \supset \dots\}, \dots, F_p = \{S_{p,1} \supset S_{p,2} \supset \dots\}$$

in  $\mathfrak{F}$  with  $\sigma_m \in E_{F_m}$ . There is an integer  $k$  such that the vertices in  $\sigma_m$  agree on  $S_{m,k}$ ; denote by  $h_m$  their common restriction on  $S_{m,k}$  ( $1 \leq m \leq p$ ). Let  $F'_1, \dots, F'_p$  be as in lemma 8: one has  $\sigma_m \in E_{F'_m}$  ( $1 \leq m \leq p$ ). Then the cone on  $\sigma_1 \cup \dots \cup \sigma_p$  with vertex  $h_0$  is in  $E_*$ .

It follows that, for any finite subcomplex  $K$  of  $E_*$ , there exists a subcomplex  $L$  of  $E_*$  containing  $K$  and contracting to a point. Hence  $|E_*|$  itself is contractible (see e.g. corollary 7.6.24 in [Sp]). ■

LEMMA 11. *The inclusion  $B_* = \bigcup_{F \in \mathfrak{F}} BG_F \rightarrow B = BG$  is a homotopy equivalence.*

*Proof.* Since the quotient maps  $|E| \rightarrow B$  and  $|E_*| \rightarrow B_*$  are covering maps, this follows immediately from the two previous lemmas. ■

The following lemma holds for  $p = 1$  by section 2.

LEMMA 12. *Let  $F_1, \dots, F_p \in \mathfrak{F}$ . Then  $BG_{F_1} \cup \dots \cup BG_{F_p}$  is contained in an acyclic subspace of  $B_*$ .*

*Proof.* Choose any flag  $F_0 \in \mathfrak{F}$ . By Lemma 8 there exist  $F'_0, F'_1, \dots, F'_p \in \mathfrak{F}$  with  $F'_m \leq F_m$  ( $0 \leq m \leq p$ ) and  $F'_m \perp F'_n$  ( $0 \leq m < n \leq p$ ); in particular  $F'_1 \oplus \dots \oplus F'_p$  is a flag in  $\mathfrak{F}$ . As  $BG_{F_m} \subset BG_{F'_m}$  ( $1 \leq m \leq p$ ), it suffices to check that  $BG_{F'_1} \cup \dots \cup BG_{F'_p}$  is acyclic. Hence we may assume without loss of generality that  $F_m \perp F_n$  ( $1 \leq m < n \leq p$ ) and that  $F_1 \oplus \dots \oplus F_p \in \mathfrak{F}$ .



Let us assume as induction hypothesis that, in this situation, both

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_1 \oplus F_{p-1}} \cup \cdots \cup BG_{F_{p-2} \oplus F_{p-1}}$$

are acyclic. (When  $p=2$ , the former works by proposition 1 and the latter is vacuous.)

Consider first the Mayer–Vietoris homology sequence of the subcomplexes

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-2} \oplus F_p} \quad \text{and} \quad BG_{F_{p-1} \oplus F_p}$$

of  $B_*$  with intersection

$$BG_{F_1 \oplus (F_{p-1} \oplus F_p)} \cup \cdots \cup BG_{F_{p-2} \oplus (F_{p-1} \oplus F_p)}.$$

By the induction hypothesis, two of any three consecutive terms in this sequence vanish. Hence all terms vanish and

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}$$

is acyclic.

Consider now the Mayer–Vietoris sequence of the subcomplexes

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_p}$$

of  $B_*$  with intersection

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}.$$

From the previous step and from the induction hypothesis it follows that

$$BG_{F_1} \cup \cdots \cup BG_{F_p}$$

is acyclic. ■

**THEOREM 13.** *The group  $G$  is acyclic.*

*Proof.* The homology of a complex is generated by that of its finite subcomplexes. Thus lemma 12 implies that  $B_*$  is an acyclic space, and lemma 11 that  $G$  is acyclic. ■

#### 4. Variations

*Unitary group  $U(V)$  of an infinite dimensional Hilbert space  $V$ .*

The proof that  $U(V)$  is acyclic is much simpler than for  $GL(V)$  since section 2 may be reduced to Lemmas 2 and 3. Section 3 is unchanged.

*Symmetric group  $\Sigma(X)$  of an infinite set  $X$*

Here a flag is a nested sequence  $\{S_1 \supset S_2 \supset \cdots\}$  of subsets of  $X = S_0$  such that  $S_{i-1} - S_i$  is equipotent with  $X$  for each  $i \geq 1$  and such that  $\bigcap S_i = \emptyset$ . Define

$$\Sigma_i = \{g \in \Sigma(X) \mid g = \text{id on } S_i\}$$

for each  $i \geq 0$  (no distinction here between  $\Sigma'_i$  and  $\Sigma_i$ ) and  $\Sigma_\infty = \bigcup_{i=0}^\infty \Sigma_i$ . The argument of Lemma 3 shows that  $\Sigma_\infty$  is a flabby group. Read “disjoint union” instead of “direct sum”, “injection” instead of “isometry”. The adjoint  $\rho^*$  of an injection  $\rho$  is defined only on the image of  $\rho$  by  $\rho^* \rho = \text{id}$ ; then a formula like  $\rho g \rho^* + \sigma h \sigma^*$  is clear because  $\rho g \rho^*$  is a permutation of some subset of  $X$  and  $\sigma h \sigma^*$  is a permutation of its complement. The group  $\Sigma_\infty$  is consequently acyclic.

Let  $Gr$  be the set of those subsets  $S$  of  $X$  equipotent with their complements  $S^\perp = X - S$ . For two subsets  $S_1, S_2$  of  $X$ , read  $S_1 \cap S_2 = \emptyset$  for  $S_1 \perp S_2$ . Lemmas 7 and 8 may then be repeated without change and all of section 3 with minor changes only. It follows that  $\Sigma(X)$  is acyclic.

*Automorphism group  $\mathcal{A}(\Omega)$  of a Lebesgue space  $(\Omega, \mathcal{B}, \mu)$*

Let  $(\Omega, \mathcal{B}, \mu)$  be a Lebesgue space where the measure  $\mu$  is infinite and non atomic. A flag is now a nested sequence  $F = \{S_1 \supset S_2 \supset \cdots\}$  of measurable subsets of  $\Omega = S_0$  such that  $S_{i-1} - S_i$  has infinite measure for each  $i \geq 1$  and such that  $\bigcap S_i$  has measure zero. Comments for  $\Sigma(X)$  above apply to  $\mathcal{A}(\Omega)$ , with the understanding that everything in view is now measurable. Therefore  $\mathcal{A}(\Omega)$  is also acyclic.

Let  $(\tilde{\Omega}, \mathcal{B}, \mu)$  be a Lebesgue measure space. Let  $X$  be the set of atoms in  $\tilde{\Omega}$ , let  $X = \bigsqcup_j X_j$  be the partition of  $X$  according to the masses of the atoms, and let  $\Omega = \tilde{\Omega} - X$ . Then the sequence

$$1 \rightarrow \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\tilde{\Omega}) \rightarrow \prod_j \Sigma(X_j) \rightarrow 1$$

is exact (and splits). Suppose  $\mu(\Omega) = \infty$ , and suppose that  $X$  is not empty. Then  $\mathcal{A}(\tilde{\Omega})$  is clearly acyclic if and only if each  $X_j$  is either one point or an infinite set.

*Automorphisms of an infinite dimensional vector space  $W$  over a (possibly skew) field  $\mathbf{F}$*

Case (i):  $\text{Char } \mathbf{F} = 0$ .

A flag is in this case a nested sequence  $\{S_1 \supset S_2 \supset \cdots\}$  of subspaces of  $W = S_0$  such that  $S_{i-1}/S_i$  is isomorphic to  $W$  for each  $i \geq 1$  and such that  $\bigcap S_i = \{0\}$ . As in Lemma 3 we may identify  $W$  with  $\bigoplus_k \bigoplus_j T_j^k$ , where each  $T_j^k \cong W$ , in such a way that  $S_i = \bigoplus_k \bigoplus_{j=i}^\infty T_j^k$  for all  $i$ . Then the subspace  $R_i = \bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$  complements  $S_i$ .

Define

$$G_i^W = \{g \in GL(W) \mid g = \text{id on } S_i\},$$

$$G_i^{W'} = \{g \in G_i^W \mid g(R_i) = R_i\}.$$

One checks as in Lemma 3 that  $G_\infty^{W'}$  is flabby. When  $\text{Char } \mathbf{F} = 0$ , Lemma 4 and 5 show that  $G_\infty^W$  is acyclic.

In Lemmas 6 to 8, understand  $S'_m \perp S'_n$  as  $S'_m \cap S'_n = \{0\}$ , and  $v \in S \cap \{v_1, \dots, v_m\}^\perp$  as  $v \in S$  with  $v$  not in the linear span of  $\{v_1, \dots, v_m\}$ . Then section 3 holds for  $GL(W)$ , which is consequently an acyclic group. All our arguments allow the field  $\mathbf{F}$  to be non-commutative.

Case (ii):  $\text{Char } \mathbf{F} = p > 0$ .

The arguments of section 2 show that  $\tilde{H}_*(G_\infty^W; \mathbf{K}) = 0$  if  $\text{Char } \mathbf{K} \neq \text{Char } \mathbf{F}$  (where  $\tilde{H}_*$  denotes reduced homology). It follows that  $\tilde{H}_*(GL(W); \mathbf{K}) = 0$  when  $\text{Char } \mathbf{K} \neq \text{Char } \mathbf{F}$ . Therefore, in order to show that  $GL(W)$  is acyclic, it will suffice to prove that  $\tilde{H}_*(GL(W); \mathbf{K}) = 0$  when  $\mathbf{K}$  is the algebraic closure  $\bar{\mathbf{k}}$  of the finite field  $\mathbf{k}$  with  $p$  elements. To do this we need

LEMMA 14. *For each flag  $F$  and integer  $d > 0$  there is a subgroup  $G_F^d$  of  $GL(W)$  which contains  $G_F$  and is such that  $H_j(G_F^d; \bar{\mathbf{k}}) = 0$  for  $0 < j < d$ .*

*Proof.* Quillen proves the following lemma in [Q2] §9.

LEMMA. *Let  $\bar{\mathbf{k}}$  be an algebraically closed field and  $d$  an integer  $> 0$ . Then there exists an order  $D$  in a number field of degree  $d$  over  $\mathbf{Q}$  with the following properties: Given any  $D$ -module  $N$ , let the group of units  $D^*$  act on it by multiplication, and let the group homology  $H_*(N, \bar{\mathbf{k}})$  be endowed with the induced action of  $D^*$ . Then for each  $t$ ,  $H_t(N, \bar{\mathbf{k}})$  is a direct sum of one-dimensional representations of  $D^*$  over  $\bar{\mathbf{k}}$ . Furthermore,  $H_t(N, \bar{\mathbf{k}})$  does not contain the trivial representation for  $0 < t < d$ .*

Let  $D$  be as in this lemma. The choice of a basis over  $\mathbf{Z}$  for  $D$  gives rise to a ring homomorphism

$$\rho_0: D \rightarrow M_d(\mathbf{Z}) \rightarrow M_d(\mathbf{F})$$

where  $M_d(A)$  is the ring of  $d$ -by- $d$  matrices over  $A$  and where  $M_d(\mathbf{Z}) \rightarrow M_d(\mathbf{F})$  is reduction mod  $p$ . Let  $F$  be the flag  $\{S_1 \supset S_2 \supset \cdots\}$ . For each pair  $(j, k)$  of positive integers, let now  $T_j^k$  be a copy of  $\mathbf{F}^d$ . We identify  $W$  and  $T = \bigoplus_k \bigoplus_j T_j^k$  in such a way that  $S_i = \bigoplus_k \bigoplus_{j=i}^\infty T_j^k$ , and we denote by  $R_i$  "the" complement  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$  of  $S_i$ . Define a ring homomorphism  $\rho_i: D \rightarrow GL(W)$  by setting

$$\rho_i(\lambda) = \begin{cases} \rho_0(\lambda) & \text{in } T_j^k \text{ for } j \geq i, \text{ all } k \\ \text{id} & \text{in the other } T_j^k. \end{cases}$$

Now put

$$G_i^d = \{g \in GL(W) \mid g = \rho_i(\lambda) \text{ in } S_i \text{ for some } \lambda \in D^*\}$$

and let  $G_F^d = \bigcup_{i \geq 1} G_i^d$ . Clearly  $G_F \subset G_F^d$ . We must show that  $H_j(G_F^d; \bar{\mathbf{k}}) = 0$  for  $0 < j < d$ .

Let

$$G_i^{d'} = \{g \in G_i^d \mid g(R_i) = R_i\}.$$

and consider the induced  $D^*$ -action on the spectral sequence of the extension  $0 \rightarrow N \rightarrow G_i^d \rightarrow G_i^{d'} \rightarrow 1$ . It follows from the lemma that each  $E_{st}^r$ ,  $2 \leq r \leq \infty$ , breaks up into a sum of one dimensional representations preserved by the differentials. Since  $D^*$  acts trivially on the abutment, the subspaces on which  $D^*$  acts trivially form a spectral sequence which converges to  $H_*(G_i^d; \bar{\mathbf{k}})$ . By the lemma, the terms  $E_{st}^2$  of this sequence vanish when  $0 < t < d$ . Hence  $H_j(G_i^d; \bar{\mathbf{k}}) \cong H_j(G_i^{d'}; \bar{\mathbf{k}})$  for  $0 < j < d$ .

Now note that  $G_i^{d'}$  is the product of  $G_i'$  with  $\rho_i(D^*)$ . But  $\rho_i(D^*)$  is isomorphic to a subgroup of the group of units of  $D/pD \cong \mathbf{k}_d$ , where  $\mathbf{k}_d$  is the field of order  $p^d$ . Hence  $\rho_i(D^*)$  has order prime to  $p$ . Therefore  $\tilde{H}_*(\rho_i(D^*); \bar{\mathbf{k}}) = 0$  which implies that  $H_*(G_i^{d'}; \bar{\mathbf{k}}) \cong H_*(G_i'; \bar{\mathbf{k}})$ . Now consider the diagram

$$\begin{array}{ccc} G_i^d & \xrightarrow{\alpha_3} & G_{i+1}^d \\ \alpha_2 \uparrow & & \nearrow \alpha_4 \\ G_i^{d'} & & \\ \alpha_1 \uparrow & & \\ G_i' & & \end{array}$$

We have seen that the inclusions  $\alpha_1$  and  $\alpha_2$  induce an isomorphism on  $H_j(-; \bar{\mathbf{k}})$ ,  $0 < j < d$ . Since  $\alpha_4$  factors through a group isomorphic to  $G'_\infty$ , it induces the zero map on  $\tilde{H}_j(-; \bar{\mathbf{k}})$ . Hence  $\alpha_3$  must induce the zero map on  $H_j(-; \bar{\mathbf{k}})$ ,  $0 < j < d$ . This implies that

$$H_j(G_F^d; \bar{\mathbf{k}}) = \lim_i H_j(G_i^d; \bar{\mathbf{k}}) = 0, \quad 0 < j < d. \quad \blacksquare$$

To finish the proof of the theorem we must find an appropriate substitute for Lemma 12. If  $F_1, \dots, F_n$  are disjoint flags such that  $F_1 \oplus \dots \oplus F_n$  is also a flag, choose groups  $G_{F_i}^d$  as above and, for each subset  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, n\}$ , set

$$G_{F_{i_1} \oplus \dots \oplus F_{i_k}}^d = G_{F_{i_1}}^d \cap \dots \cap G_{F_{i_k}}^d.$$

The proof of Lemma 14 shows that these groups  $G_F^d$ , for  $F = F_{i_1} \oplus \dots \oplus F_{i_k}$ , are acyclic. The inductive argument of Lemma 12 then readily shows that

$$H_j(BG_{F_1}^d \cup \dots \cup BG_{F_n}^d; \bar{\mathbf{k}}) = 0 \quad 0 < j < d - 2n.$$

Clearly, this suffices to show that the inclusion  $B_* \hookrightarrow B$  annihilates  $\tilde{H}_*(-; \bar{\mathbf{k}})$ .

### *Properly infinite von Neumann algebras*

Let  $M$  be a properly infinite von Neumann algebra, faithfully represented in  $L(V)$  for some complex Hilbert space  $V$ . A flag is a nested sequence  $\{S_1 \supset S_2 \supset \dots\}$  of closed subspaces of  $V = S_0$  with  $\bigcap S_i = \{0\}$  such that the orthogonal projection  $P_i$  from  $V$  onto  $S_i$  is in  $M$  and such that  $P_{i-1} - P_i$  is equivalent to the identity for each  $i \geq 1$ . It is easy to choose every operator appearing in sections 2 and 3 in the algebra  $M$ . Therefore the appropriately defined groups  $G'_\infty$  and  $G_\infty$  are acyclic, as well as  $U(M)$  and  $GL(M)$ .

It is likely that the argument applies to a large class of infinite  $C^*$ -algebras. Let  $B$  be such an algebra, let  $M(B)$  be its multiplier algebra, let  $U(B)$  be the subgroup of the unitary group  $U(M(B))$  consisting of those elements  $g$  for which  $g - 1 \in B$ , and let  $U(B)_0$  be the connected component of  $U(B)$  with respect to the norm topology. There are many cases in which  $U(B)_0$  is known to be contractible for the norm topology [Mi]; in these cases,  $U(B)_0$  and the similarly defined “general linear group”  $GL(B)_0$  should “often” be acyclic.

### *Finite von Neumann algebras*

Let  $M$  be a finite continuous factor, and let  $U(M)$  be the group of unitaries in  $M$ . When given the norm topology,  $U(M)$  has a fundamental group isomorphic to

the additive group of the real numbers: this was first proved in [AS], but it follows also essentially from Bott periodicity as formulated in theorem 1.11 of chapter III of [Ka]. Indeed

$$\pi_i(U(M)_{\text{norm}}) \approx \begin{cases} \mathbf{R} & \text{if } i \text{ is odd, } i \geq 0 \\ 0 & \text{if } i \text{ is even, } i > 0 \end{cases}$$

(See III.7.7 in [Ka], or theorem 5 in [Br]; both state the analogous “stable fact”, but the isomorphism holds also as above.) Let

$$0 \rightarrow \mathbf{R} \rightarrow \tilde{U}(M) \rightarrow U(M) \rightarrow 1$$

be the (topological) universal covering of  $U(M)$ . It is known that  $U(M)$  is perfect (indeed simple up to centre [FH]). One may conjecture that  $\tilde{U}(M)$  is also perfect, namely that the short exact sequence above is still a covering in the algebraic sense of [Ker], and thus that there exists a surjective homomorphism of  $H_2(U(M))$  onto  $\mathbf{R}$ . In any event it seems very unlikely that the group  $U(M)$  is acyclic.

## Appendix 1. About normal subgroups

If  $X$  is an infinite countable set,  $\Sigma(X)$  has exactly two non trivial normal subgroups: the group  $\Sigma_f(X)$  of permutations of  $X$  with finite support and its derived group  $A_f(X)$  of even permutations [SU]. If  $X$  is any infinite set, normal subgroups of  $\Sigma(X)$  which are neither trivial nor  $A_f(X)$  are in bijection (via supports) with infinite cardinals smaller than the cardinal of  $X$  [B].

If  $(\Omega, \mathcal{B}, \mu)$  is a Lebesgue measure space with  $\mu$  infinite and non atomic,  $\mathcal{A}(\Omega)$  has exactly one non trivial normal subgroup consisting of those bi-measurable transformations  $\alpha$  with support  $\{\omega \in \Omega \mid \alpha(\omega) \neq \omega\}$  of finite measure [F1], [Ei].

If  $W$  is an infinite dimensional vector space over a field  $\mathbf{F}$ , normal subgroups of  $GL(W)$  have been studied in [R]; we present hereafter part of these results with different proofs inspired by [And], [Ep] and [Hi].

**LEMMA A1.** *The group  $GL(W)$  is perfect.*

*Proof.* If  $I$  is a set and if  $(W_i)_{i \in I}$  is a family of copies of  $W$ , we write any element in  $GL(\bigoplus W_i)$  as an  $(I \times I)$ -matrix with coefficients in  $\text{End}(W)$ . If  $I$  is countable, we may identify  $\bigoplus W_i$  and  $W$ .

In  $GL(W \oplus W \oplus W)$  one has

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for each  $x \in \text{End}(W)$ . It follows that any element of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$  is a product of two commutators. In  $GL(\bigoplus_{i \in N} W_i)$ , one may apply the infinite repetition argument used in section 2. We write  $\gamma_1 \sim \gamma_2$  if two elements  $\gamma_1, \gamma_2$  in a group  $\Gamma$  are conjugate. For any  $x \in GL(W)$  one has

$$\begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ & x & & \\ & & 1 & \\ & & & x \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix} \sim \begin{pmatrix} 1 & & & \\ & x & & \\ & & 1 & \\ & & & x \\ & & & & \ddots \\ & & & & & \ddots \end{pmatrix}$$

in  $GL(\bigoplus_{i \in N} W_i)$ . It follows that any element of the form  $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$  is a commutator.

Let  $g \in GL(W)$ . Choose sequences  $(u_i)$  and  $(v_i)$  of vectors in  $W$  as follows:

$$u_1 \in W - \{0\} \quad u'_1 = g(u_1) \quad v_1 \in W - \text{span}(u_1, u'_1)$$

and in general

$$u_{i+1} \in W - \text{span} \begin{pmatrix} u_1 & v_1 & g^{-1}(v_1) \\ \cdot & \cdot & \cdot \\ u_i & v_i & g^{-1}(v_i) \end{pmatrix} \quad u'_{i+1} = g(u_{i+1})$$

$$v_{i+1} \in W - \text{span} \begin{pmatrix} u_1 & u'_1 & v_1 \\ \cdot & \cdot & \cdot \\ u_i & u'_i & v_i \\ u_{i+1} & u'_{i+1} & \cdot \end{pmatrix}.$$

(The index  $i$  runs over  $N^*$  if the dimension of  $W$  is countable and over some

suitable set otherwise.) Define

$$\begin{aligned} U &= \text{span}(u_1, u_2, \dots) & V_1 &= \text{span}(v_1, v_3, \dots) \\ V_2 &= \text{span}(v_2, v_4, \dots) & V &= V_1 \oplus V_2. \end{aligned}$$

It is easy to check that  $U \cap V = \{0\}$  and  $g(U) \cap V = \{0\}$ . Thus there exists  $t \in GL(W)$  with  $tu'_i = u_i$  and  $tv_{2i} = v_{2i}$  for each  $i$ . As  $t = \text{id}$  on  $V_2$  one has  $t \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$ ; as  $tg = \text{id}$  on  $U$  one has  $tg \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$ . It follows from the beginning of the proof that  $g$  is a product of commutators in  $GL(W)$ . ■

The proof above shows also the following *fragmentation lemma*: any element in  $GL(W)$  may be written as a product of finitely many elements similar to  $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$ . Indeed, it remains to be checked that  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  has this property, and this is clear if one looks at

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in  $GL(W \oplus W \oplus W)$ .

Let  $N_{\max}$  be the normal subgroup of  $GL(W)$  containing those elements of the form  $\lambda + X$  with  $\lambda$  a homothety and  $X$  an endomorphism of  $W$  with rank strictly smaller than the dimension of  $W$ . Let  $g \in GL(W)$  with  $g \notin N_{\max}$ . Let us check that there exists a subspace  $V$  of  $W$  with  $V$  isomorphic to  $W/V$  and with  $V \cap g(V) = \{0\}$ .

One may choose a sequence  $(v_i)$  of vectors in  $W$  as follows:

$$v_1 \in W - \{0\} \quad \text{with} \quad g(v_1) \in W - \text{span}(v_1)$$

and in general

$$v_{i+1} \in W - \text{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \end{pmatrix} \quad \text{with} \quad g(v_{i+1}) \in W - \text{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \\ v_{i+1} \end{pmatrix}$$



Indeed, suppose one cannot find  $v_{i+1}$ . Let

$$F = \text{span} \begin{pmatrix} v_1 \cdots v_i \\ g(v_1) \cdots g(v_i) \end{pmatrix}.$$

Then  $v \in W - F$  implies  $g(v) \in \text{span}(F, v)$ ; for any  $u \in F$ , one has also  $g(v + u) \in \text{span}(F, v)$ ; hence  $g(u) \in \text{span}(F, v)$ . It follows that  $F$  is invariant by  $g$  and that  $g$  induces a homothety on  $W/F$ . But this is ruled out by hypothesis.

Then  $V = \text{span}(v_1, v_2, \dots)$  has the desired properties.

**PROPOSITION A2.** *Any non trivial normal subgroup of  $GL(W)$  is contained in  $N_{\max}$ .*

*Proof.* Let  $N$  be a normal subgroup of  $GL(W)$  and assume that  $N \not\subseteq N_{\max}$ . There exist  $f \in N$  and a subspace  $V$  of  $W$  with  $V$  isomorphic to  $W/V$  and with  $f(V) \cap V = \{0\}$ . We may thus view  $N$  as a normal subgroup of  $GL(W \oplus W)$  containing an element  $f$  of the form  $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ .

By the fragmentation lemma, it is enough to check that  $N$  contains any element of the form  $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ . Consider  $r, s \in GL(W)$  and define  $g = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ ,  $h = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ . As  $N$  is normal,  $N$  contains  $\hat{h} = hfh^{-1}f^{-1}$  and  $g\hat{h}g^{-1}\hat{h}^{-1}$ . By a straightforward matrix computation, the latter is of the form

$$g\hat{h}g^{-1}\hat{h}^{-1} = \begin{pmatrix} 1 & * \\ 0 & rsr^{-1}s^{-1} \end{pmatrix}.$$

As  $GL(W)$  is perfect, it follows that, for any  $k \in GL(W)$ , there exists  $z \in \text{End}(W)$  with  $\begin{pmatrix} 1 & z \\ 0 & k \end{pmatrix} \in N$ .

Let now  $a, b \in GL(W)$  with  $a + b = 1$ . (One may define  $a$  as an infinite direct sum of automorphisms of a vector space of dimension two, each represented by  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ , and similarly for  $b$  with  $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ .) There exist  $x, y \in \text{End}(W)$  with

$$\begin{pmatrix} 1 & x \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & y \\ 0 & b^{-1} \end{pmatrix}$$

in  $N$ . Then

$$\begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -xa \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & z(a-1) \\ 0 & 1 \end{pmatrix} \in N$$

and

$$\begin{pmatrix} 1 & z(a-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z(b-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} \in N.$$

It follows that

$$\begin{pmatrix} 1 & z \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in N$$

the proof is complete. ■

It would be easy to prove by similar arguments all of theorem B (and thus also theorem A) in [R].

Let now  $V$  be an infinite dimensional Hilbert space over the reals, complexes or quaternions and  $GL(V)$  be as in the introduction. Let  $GE(V, C)$  be the normal subgroup of  $GL(V)$  containing those elements of the form  $\lambda + x$  with  $\lambda$  a homothety and  $X$  a compact operator (we assume  $V$  to be separable). It is quite easy to check that  $GL(V)$  is perfect (see problems 191 and 192 in [Hal]). There is a fragmentation lemma which follows straightforwardly from polar decomposition and spectral theorem. Any  $g \in GL(V)$  with  $g \notin GE(V, C)$  is similar to an element of the form  $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$  in  $GL(V \oplus V)$ : this is corollary 3.4 in [BP] or theorem 1 in [AnS]. Hence the proof above applies, and is very much simpler than that of [H1]. The subgroup of  $GL(V)$  containing all bijective isometries of  $V$  can be handled either as in [H1] or as suggested in [H3], and we have proved the following result.

**PROPOSITION A3.** *Any non trivial normal subgroup of  $GL(V)$  is contained in  $GE(V, C)$ . Any non trivial normal subgroup of  $U(V)$  is contained in  $UE(V, C) = U(V) \cap GE(V, C)$ .*

For normal subgroups of  $GL(M)$  and  $U(M)$ , when  $M$  is a properly infinite von Neumann algebra, see [H3] and papers reviewed there.

**COROLLARY A4.** *Let  $G$  be one of the groups described in the introduction and let  $N$  be a non trivial normal subgroup of  $G$ . Then  $N$  is of uncountable index in  $G$ .*

Let  $G$  be as above and let  $N_{\max}$  be the maximal normal subgroup of  $G$ . There are cases for which we have information about the homology of  $N_{\max}$ : see works

by Nakaoka and Priddy [P] if  $G = \Sigma(X)$  and  $N_{\max} = \Sigma_f(X)$  with  $X$  infinite countable, the papers on group cohomology in [E] if  $G = GL(W)$ , or [BHS] if  $G = GL(V)$ . In each case our main theorem provides corresponding information about the homology of the quotient  $G/N_{\max}$ .

## Appendix 2. About monoids of monomorphisms

Each of the acyclic groups of automorphisms considered above is the group of units in a corresponding monoid (or semigroup) of monomorphisms. For example,  $\Sigma(X)$  is the group of units in the monoid  $M(X)$  formed by all injective maps from  $X$  to  $X$ . One can form the classifying space  $BM$  of a monoid in exactly the same way as that of a group; see [Se]. In particular, the Eilenberg–MacLane homology groups  $H_i(M; \mathbf{Z})$  are just the integral homology groups of the space  $BM$ . Quillen pointed out in an unpublished version of [Q1] that the classifying spaces of monoids such as  $M(X)$  are contractible. Of course, this implies that the monoids are acyclic.

Here is a sketch of his argument. Say two homomorphisms  $f, g: M \rightarrow M$  are semi-conjugate if there is  $m \in M$  such that  $mf(n) = g(n)m$  for all  $n \in M$ . The argument is based on the fact that two homomorphisms which are semi-conjugate induce homotopic maps on  $BM$ ; see [Q1] §1. Choose  $p \in M(X)$  so that the image  $p(X)$  of  $X$  under  $p$  is in  $Gr$ . Define  $f: M(X) \rightarrow M(X)$  by  $f(n)(x) = pnp^{-1}(x)$  if  $x \in p(X)$  and by  $f(n)(x) = x$  otherwise. Then  $f$  is semi-conjugate both to the identity homomorphism and to the trivial homomorphism which takes every  $n \in M(X)$  to the identity element. It follows that  $BM(X)$  is contractible.

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*Section de mathématiques,  
C.P. 124, 1211 Genève 24*

*Department of Mathematics  
SUNY, Stony Brook, NY 11794*

Received May 24, 1982