

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 58 (1983)  
  
**Artikel:** A reciprocity law for  $K_2$ -traces.  
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**DOI:** <https://doi.org/10.5169/seals-44587>

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## A reciprocity law for $K_2$ -traces

SHMUEL ROSSET and JOHN TATE

Suppose  $E \subset F$  is a finite field extension and let

$$\mathrm{Tr}: K_2(F) \rightarrow K_2(E)$$

be the trace map (also called transfer, see [5, §14]). If  $x, y \in F^*$  and  $\{x, y\}$  is the corresponding symbol in  $K_2(F)$  then we know, since  $K_2(E)$  is generated by symbols, that  $\mathrm{Tr}_{F/E}(\{x, y\})$  can be expressed as a sum of symbols. In this paper we give an algorithm for computing such an expression explicitly (cf. the proposition in §3). The algorithm is based on a reciprocity law (§2) and involves repeated polynomial division with remainder, like the Euclidean algorithm. The proof works not only for Milnor's  $K_2$ , but for functors sufficiently like  $K_2$ , which we define in §1 and call Milnor functors. This abstraction is useful for it yields as a corollary (§3) the fact that the canonical map from  $K_2$  to any Milnor functor commutes with traces. Another corollary is that, if  $(F:E) = n$ , then  $\mathrm{Tr}_{F/E}(\{x, y\})$  can be written as a sum of  $n$  symbols (or less). On the other hand this is also the best bound: in §4 we give an example, using division algebras, of a symbol whose trace is not a sum of less than  $n$  symbols.

One of us (S.R.) would like to thank David Saltman for a conversation which helped realize the example in section 4.

### 1. Milnor functors

Let  $K$  be a fixed base field and let  $\mathfrak{C}$  be the category of commutative finite dimensional  $k$ -algebras.

**DEFINITION.** A *Milnor functor over  $k$*  is a functor  $M: \mathfrak{C} \rightarrow (\text{Abelian groups})$  together with

- (i) For each  $A \in \mathfrak{C}$  a bilinear map  $\varphi = \varphi_A: A^* \times A^* \rightarrow M(A)$ ;
- (ii) For each extension  $A \rightarrow B$  in  $\mathfrak{C}$  such that  $B$  is a projective  $A$ -module, a homomorphism  $\mathrm{Tr}_{B/A}: M(B) \rightarrow M(A)$ ; such that the following properties hold.

( $\varphi$ ) The maps  $\varphi$  are functorial, i.e., induce a morphism of functors from the functor  $A \mapsto A^* \times A^*$  to the functor  $A \mapsto M(A)$ , and satisfy

$$\varphi_A(a, 1-a) = 0, \quad \text{if } a \in A^* \text{ and } 1-a \in A^*,$$

$$\varphi_A(a, -a) = 0, \quad \text{if } a \in A^*.$$

(Tr) if  $A \rightarrow B \rightarrow C$  are  $\mathfrak{C}$ -morphisms such that  $C$  is projective over  $B$  and  $B$  over  $A$ , then

$$\text{Tr}_{C/A} = \text{Tr}_{B/A} \circ \text{Tr}_{C/B}$$

(Tr- $\varphi$ ) If  $A \rightarrow B$  is a  $\mathfrak{C}$ -morphism with  $B$  projective as  $A$ -module, and if  $x \in A^*, y \in B^*$  then

$$\text{Tr}_{B/A} \varphi_B(x, y) = \varphi_A(x, N_{B/A} y),$$

where  $N_{B/A} : B^* \rightarrow A^*$  is the usual norm:

$$N_{B/A}(y) = \det \quad (\text{multiplication by } y).$$

EXAMPLE 1. Milnor's  $K_2$ ; see [5] and [6].

EXAMPLE 2. Assume that the characteristic of  $k$  does not divide a given integer  $n$  and let  $\mu_n$  denote the sheaf on  $n$ -th roots of 1 on the étale site over  $\text{Spec } A$ ; here  $A$  is a given element in  $\text{Ob}(\mathfrak{C})$ . By Kummer theory

$$H^1(\text{Spec } A, \mu_n) = A^*/(A^*)^n.$$

The cup product

$$H^1(\text{Spec } A, \mu_n) \times H^1(\text{Spec } A, \mu_n) \rightarrow H^2(\text{Spec } A, \mu_n^{\otimes 2}) = M(A)$$

provides us with a context satisfying (i) and (ii). We refer to Milne's book [4] for details. The existence of a trace can probably be extracted from [7, exp. XVII]. However, this Milnor functor can be expressed entirely in terms of Galois cohomology and the trace in terms of corestriction, as follows. For  $A \in \mathfrak{C}$ ,  $\alpha \in M(A)$ , and  $x \in \text{Spec } A$ , let  $\alpha(x) \in M(A/x)$  be the image of  $\alpha$  under the residue class map  $A \rightarrow A/x$ . then the map

$$\alpha \mapsto (\alpha(x))_{x \in \text{Spec } A}$$

gives an isomorphism

$$M(A) \xrightarrow{\sim} \prod_{x \in \text{Spec } A} M(A/x). \quad (*)$$

For each  $x \in \text{Spec } A$ ,  $A/x$  is a finite extension field of  $k$ . If  $E$  is a finite extension field of  $k$ , then

$$M(E) = H^2(\text{Gal}(E_s/E), \mu_n(E_s) \otimes \mu_n(E_s)),$$

where  $E_s$  is a separable algebraic closure of  $E$ . The map  $\varphi_A$  is characterized in terms of the isomorphism (\*) by

$$(\varphi_A(a, b))(x) = \varphi_{A/x}(a(x), b(x))$$

for each  $x \in \text{Spec } A$ , where  $a(x)$  (resp.  $b(x)$ ) is the residue mod  $x$  of  $a$  (resp.  $b$ ), and for a field  $E$  the map

$$\varphi_E : E^* \times E^* \rightarrow M(E)$$

is the Galois cohomology symbol (cf. [8]) characterized by  $\varphi(a, b) = da \cup db$ , where  $d : E^* \rightarrow H^1(\text{Gal}(E_s/E), \mu_n(E_s))$  is the connecting homomorphism in the exact cohomology sequence associated with

$$0 \rightarrow \mu_n(E_s) \rightarrow E_s^* \xrightarrow{n} E_s^* \rightarrow 0.$$

Let  $A \rightarrow B$  be an extension in  $\mathfrak{C}$  such that  $B$  is a projective  $A$ -module. Then for each  $x \in \text{Spec } A$  and each  $y \in \text{Spec } B$  lying over  $x$ , the local ring  $B_y$  is a free  $A_x$ -module; let  $r(y/x)$  denote its rank. Let  $E_x = A/x$  and let  $F_y$  be the field between  $E_x$  and  $B/y$  such that  $F_y/E_x$  is separable and  $(B/y)/F_y$  purely inseparable. Then the ratio

$$q(y/x) \stackrel{\text{defn}}{=} \frac{r(y/x)}{[F_y : E_x]}$$

is an integer, and the  $M$ -trace from  $B$  to  $A$  is characterized in terms of the isomorphism (\*) by

$$(\text{Tr}_{B/A}\beta)(x) = \sum_{y|x} q(y/x) \text{cor}_{F_y/E_x}(\beta(y)),$$

where  $\text{cor}$  is the corestriction in Galois cohomology, and we identify  $M(B/y)$  with  $M(F_y)$  via the isomorphism induced by the inclusion  $F_y \hookrightarrow B/y$ .



In case  $E \in \mathfrak{C}$  is a field containing a primitive  $n$ -th root of unity  $\zeta$ , we can identify  $M(E)$  with the group  $\text{Br}_n(E)$  of elements of order  $n$  in the Brauer group of  $E$  in such a way that

$$(a, b)_M = \text{the Brauer class of } A_\zeta(a, b)$$

where  $A_\zeta(a, b)$  denote the cyclic algebra generated over  $E$  by elements  $X$  and  $Y$  subject to the relations

$$X^n = a, \quad Y^n = b, \quad XY = \zeta YX;$$

(cf. [5], p. 143).

**EXAMPLE 3.** The  $\text{dlog}$  symbol, see [1]. If  $A$  is a  $k$  algebra in  $\mathfrak{C}$  let  $\Omega_{A/k}^1$  be the  $A$ -module of Kähler differentials of  $A$  over  $k$ , and let  $\Omega_{A/k}^2$  be its second exterior power. Define

$$\text{dlog}: A^* \rightarrow \Omega_{A/k}^1$$

by  $\text{dlog}(f) = f^{-1} \cdot df$ . It is simple to verify that  $\Omega^2$  and  $\text{dlog} \wedge \text{dlog}$  satisfy axioms (i), (ii) above. The existence of a good trace is a non-trivial fact [2].

## 2. Reciprocity

Let  $M$  be a Milnor functor over  $k$ . In this section we shall write the  $M$ -symbol  $\varphi_E(x, y)$  by

$$(x, y)_E, \quad \text{or} \quad (x, y)$$

if  $E$  is evident.

Let  $K$  be a field of finite degree over  $k$ . For relatively prime non-zero polynomials  $f(T), g(T)$  in  $K[T]$  we define a new kind of symbol  $(f/g)$ . Its values are in the group  $M(K)$  and it is defined by the following requirements.

1) It is additive in  $g$ , i.e. if  $g_1, g_2$  are both prime to  $f$  then

$$\left( \frac{f}{g_1 g_2} \right) = \left( \frac{f}{g_1} \right) + \left( \frac{f}{g_2} \right)$$

2) It is 0 if  $g$  is a constant or  $g = T$ .

3) If  $g$  is *monic irreducible*  $\neq T$  and  $x$  is a root of  $g(T)$  then

$$\left(\frac{f}{g}\right) = \text{Tr}_{K(x)/K}(x, f(x))_{K(x)}.$$

It is clear that, thus defined, the symbol  $(f/g)$  is additive in  $f$ , as well as in  $g$ , and it depends only on the residue class of  $f$  modulo  $(g)$ . As function of  $g$  it depends only on the ideal generated by  $g$  in the ring  $K[T, T^{-1}]$ .

To formulate the reciprocity law satisfied by  $(f/g)$  we introduce some notation: if

$$p(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_m T^m$$

with  $a_m a_n \neq 0$ . let

$$p^*(T) = (a_m T^m)^{-1} p(T)$$

$$c(p) = (-1)^n a_n.$$

*Reciprocity law*

$$\left(\frac{f}{g}\right) = \left(\frac{g^*}{f}\right) - (c(g^*), c(f)). \quad (**)$$

*Proof.* We first dispose of a few trivial cases. If  $g$  is a constant or  $T$  it is easily checked that both sides are 0, so we assume henceforth that  $g(T)$  is monic irreducible  $\neq T$ . let  $x$  be a root of  $g(T)$ . If  $f(T)$  is a constant  $c$  then the left side of  $(**)$  is

$$\begin{aligned} \text{Tr}_{K(x)/K}(x, c)_{K(x)} &= (N_{K(x)/K} x, c)_K \\ &= ((-1)^{\deg(g)} \cdot g(0), c) = -((-1)^{\deg(g)} \cdot g(0)^{-1}, c) \\ &= -(c(g^*), c(f)) \end{aligned}$$

which is equal to the right hand side since  $(g^*/f) = 0$ , by definition.

A similar computation using  $(x, -x) = 0$  works when  $f(T) = T$  so we now assume that both  $f$  and  $g$  are monic irreducible, and not  $T$ .

Let  $x$  be a root of  $g$  and  $y$  a root of  $f$ . Let

$$A = K(x) \otimes_K K(y).$$

$K(x)$  and  $K(y)$  are naturally imbedded in  $A$  and we identify them as such. Then

the elements  $x, y, x - y$  are invertible in  $A$ , indeed the norm

$$N_{A/K(x)}(x - y) = f(x)$$

is invertible, so  $x - y$  is.

The identity

$$(x, x - y) = \left(y, \frac{y - x}{-x}\right) + (x, -1)$$

follows from a little computation with the relations  $(u, 1 - u) = (u, -u) = 0$ . We use it to compute the same thing in two ways

$$\begin{aligned} \mathrm{Tr}_{A/K}(x, x - y) &= \mathrm{Tr}_{K(x)/K} \mathrm{Tr}_{A/K(x)}(x, x - y) \\ &= \mathrm{Tr}_{K(x)/K}(x, N_{A/K(x)}(x - y)) \\ &= \mathrm{Tr}_{K(x)/K}(x, f(x)) = \left(\frac{f}{g}\right). \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_{A/K}\left(y, \frac{y - x}{-x}\right) &= \mathrm{Tr}_{K(y)/K} \mathrm{Tr}_{A/K(y)}\left(y, \frac{y - x}{-x}\right) \\ &= \mathrm{Tr}_{K(y)/K}\left(y, \frac{N_{A/K(y)}(y - x)}{N_{A/K(y)}(-x)}\right) \\ &= \mathrm{Tr}_{K(y)/K}\left(y, \frac{g(y)}{g(0)}\right) \\ &= \mathrm{Tr}_{K(y)/K}(y, g^*(y)) = \left(\frac{g^*}{f}\right). \end{aligned}$$

Finally

$$\begin{aligned} \mathrm{Tr}_{A/K}(x, -1) &= \mathrm{Tr}_{K(y)/K} \mathrm{Tr}_{A/K(y)}(x, -1)_A \\ &= \mathrm{Tr}_{K(y)/K}(N_{A/K(y)}x, -1)_{K(y)} \\ &= \mathrm{Tr}_{K(y)/K}(N_{K(x)/K}x, -1)_{K(y)} \\ &= (c(g^*)^{-1}, (-1)^{\deg(f)}) = -(c(g^*), c(f)). \end{aligned}$$

Here we used the obvious fact that

$$N_{A/K(y)}(x) = N_{K(x)/K}(x).$$

This completes the proof of the reciprocity law.

### 3. Consequences

Let  $E \subset F$  be a finite extension of fields finite over  $k$ , and let  $x, y \in F^*$ . Then

$$\mathrm{Tr}_{F/E}(x, y) = \left( \frac{f}{g} \right)$$

where  $g(T) \in E[T]$  is the monic irreducible polynomial with root  $x$  and  $f(T) \in E[T]$  is the polynomial of smallest degree such that  $N_{F/E(x)}y = f(x)$ .

**PROPOSITION.** *Let  $g_0, g_1, \dots, g_m \neq 0, g_{m+1} = 0$  be the sequence of polynomials defined by:*

$$g_0 = g, \quad g_1 = f,$$

and for  $i \geq 1$

$$g_{i+1} = \text{the remainder of the division of } g_{i-1}^* \text{ by } g_i,$$

as long as  $g_i \neq 0$ . We have then

$$1 \leq m \leq \deg g = [E(x) : E] \leq [F : E]$$

and

$$\mathrm{Tr}_{F/E}(x, y) = - \sum_{i=1}^m (c(g_{i-1}^*), c(g_i)).$$

By the reciprocity law, we find by induction on  $j$ , using  $(g_{i-1}^*/g_i) = (g_{i+1}/g_i)$ :

$$\left( \frac{g_1}{g_0} \right) = - \sum_{i=1}^j (c(g_{i-1}^*), c(g_i)) + \left( \frac{g_{j-1}^*}{g_j} \right)$$

for  $1 \leq j \leq m$ . But the last non-zero polynomial  $g_m$  is a constant because it divides the relatively prime polynomials  $g_0$  and  $g_1$ . Hence  $(g_{m-1}^*/g_m) = 0$ , and the proposition follows on putting  $j = m$ ; We have  $m \leq \deg g$  because the degrees of the polynomials in the sequence are strictly decreasing, and  $m \geq 1$  because  $f \neq 0$ .

**COROLLARY 1.** *If  $[F : E] = r$  and  $x, y \in F^*$ , then  $\mathrm{Tr}_{F/E}(x, y)$  is a sum of at most  $r$  symbols.*

The sequence of polynomials in the proposition depends only on  $F, E, x$ , and  $y$ , not on the Milnor functor  $M$ . Thus the trace of a symbol  $(x, y)_M$  has an expression as a sum of symbols which is *independent of the Milnor functor  $M$* ; on symbols, the trace is uniquely determined. Any morphism  $M_1 \rightarrow M_2$  of Milnor functors which carries each symbol  $(a, b) \in M_1(A)$  to the “same” symbol  $(a, b) \in M_2(A)$  must therefore commute with  $\text{Tr}_{F/E}$  on symbols. In particular, letting  $R_F: K_2(F) \rightarrow M(F)$  be the homomorphism (whose existence and unicity are guaranteed by Matsumoto’s theorem) such that  $R_F(\{a, b\}) = (a, b)_M$  for  $a, b \in F^*$ , and similarly  $R_E$ , we have

**COROLLARY 2.** *The diagram*

$$\begin{array}{ccc} K_2 & \xrightarrow{R_F} & M(F) \\ \text{Tr}_{F/E} \downarrow & & \downarrow \text{Tr}_{F/E} \\ K_2(E) & \xrightarrow{R_E} & M(E) \end{array}$$

*is commutative.*

#### 4. An example

We have just proved that if  $[F:E] = r$  and  $x, y \in F^*$  then  $\text{Tr}_{F/E}(x, y)$  is a sum of  $r$  symbols. Yet it is known that in some cases, e.g. global or local fields, every element of  $K_2$  (say) is a symbol [8, 3], so it is well to give an example where  $\text{Tr}(x, y)$  cannot be written as a sum of fewer than  $r$  symbols. For this it will suffice to work with the functor of Example of Section 1.

Let  $n \geq 2$  and  $r \geq 1$  be integers. Let  $k_0$  be a field containing a primitive  $n$ -th root of unity,  $\zeta$ . Let  $u_1, v_1, \dots, u_r, v_r$  be  $2r$  independent variable over  $k_0$  and let

$$F = k_0(u_1, v_1; u_2, v_2; \dots; u_r, v_r)$$

be the field they generate. Let  $M$  be the Milnor functor of Example 2.

**LEMMA.** *The element  $\beta = \sum_{i=1}^r (u_i, v_i)$  in  $M(F)$  is not a sum of fewer than  $r$  symbols.*

*Proof.* We use the identification  $M(F) \xrightarrow{\sim} \text{Br}_n(F)$  discussed at the end of

Example 2. For  $1 \leq i \leq r$  let  $B_i$  be the cyclic algebra over  $F$  generated by elements  $X_i$  and  $Y_i$  subject to the relations

$$X_i^n = u_i, \quad Y_i^n = v_i, \quad X_i Y_i = \zeta Y_i X_i,$$

so that  $(u_i, v_i)$  is the Brauer class of  $B_i$ . Then  $\beta$  is the Brauer class of  $B = \bigotimes_{i=1}^r B_i$ , an algebra of dimension  $n^{2r}$  over  $F$ . We will show  $B$  is a division algebra. This will prove the lemma, for it shows that  $\beta$  cannot be the Brauer class of an algebra of dimension less than  $n^{2r}$ , and consequently cannot be a sum of fewer than  $r$  symbols.

If  $B$  were not a division algebra it would have zero divisors, and multiplying these zero divisors by a common denominator of their coefficients in  $F$  relative to the basis

$$\{X_1^{l_1} Y_1^{m_1} \cdots X_r^{l_r} Y_r^{m_r}\} \quad (0 \leq l_i, m_i < n)$$

for  $B$  over  $F$ , we would find zero divisors in the ring

$$R = k_0[u_1, v_1, \dots, u_r, v_r][X_1, Y_1, \dots, X_r, Y_r] = k_0[X_1, Y_1, \dots, X_r, Y_r].$$

But this ring has no zero divisors, for it has a basis over  $k_0$  consisting of the monomials

$$X_1^{l_1} Y_1^{m_1} \cdots X_r^{l_r} Y_r^{m_r}$$

with  $l_i, m_i$  integers  $\geq 0$ , and the product of two such monomials is a power of  $\zeta$  times the monomial obtained by adding exponents. Hence, if we order the monomials by the lexicographical order of their exponent sequences, the product of two non-zero polynomials will contain the product of the highest terms in the two factors with a non-zero coefficient, so will not be 0. This proves the lemma.

Let  $\sigma$  be the automorphism of  $F$  which is identity on  $k_0$  and acts on the variables by

$$\begin{aligned} \sigma u_i &= u_{i+1}, & 1 \leq i \leq r; & & u_{r+1} &= u_1, \\ \sigma v_i &= v_{i+1}, & 1 \leq i \leq r; & & v_{r+1} &= v_1. \end{aligned}$$

Let  $G$  be the cyclic group of order  $r$  generated by  $\sigma$ , and let  $E = F^G$ .

**PROPOSITION.** *The image of  $\{u_1, v_1\}$  under  $\text{Tr}_{F/E} : K_2 F \rightarrow K_2 E$  is not a sum of fewer than  $r$  symbols.*

*Proof.* We use the commutativity of

$$\begin{array}{ccc} K_2(F)/nK_2(F) & \xrightarrow{R_F} & \text{Br}_n(F) \\ \text{Tr} \downarrow & & \downarrow \text{Tr} \\ K_2(E)/nK_2(E) & \xrightarrow{R_E} & \text{Br}_n(E). \end{array}$$

and the rule

$$\text{res}_{E/F} \text{Tr}_{F/E} \alpha = \sum_{\tau \in G} \tau \alpha$$

for  $\alpha \in \text{Br } F$ . If  $\text{Tr} \{u_1, v_1\}$  were a sum of  $s < r$  symbols so also would be

$$\begin{aligned} \text{res } R_E \text{Tr} \{u_1, v_1\} &= \text{res } \text{Tr } R_F \{u_1, v_1\} = \text{res } \text{Tr} (u_1, v_1) \\ &= \sum_{\tau \in G} \tau(u_1, v_1) = \sum_{i=1}^r (u_i, v_i) = \beta, \end{aligned}$$

contradicting the lemma.

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Received August 18, 1982