**Zeitschrift:** Commentarii Mathematici Helvetici

**Herausgeber:** Schweizerische Mathematische Gesellschaft

**Band:** 58 (1983)

**Artikel:** Higher dimensional simple knots and minimal Seifert surfaces.

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**DOI:** https://doi.org/10.5169/seals-44618

Comment. Math. Helvetici 58 (1983) 646–655 0010–2571/83/040646–10$01.50 + 0.20/0


# Higher dimensional simple knots and minimal Seifert surfaces

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## Introduction

A knot $K^{2q-1} \subset S^{2q+1}$, $q \geq 2$, is said to be *simple* if $K^{2q-1}$ has a $(q-1)$-connected Seifert surface. Such a Seifert surface is said to be minimal if the associated Seifert matrix is non-singular. Levine has given an isotopy classification of simple $(2q-1)$-knots and their minimal Seifert surfaces in terms of $S$-equivalence and congruence of Seifert matrices (cf. [8]). Another algebraic classification of simple $(2q-1)$-knots can be obtained via the isometry classification of Blanchfield forms (cf. Trotter [14] or Kearton [7]) which is usually easier to handle than the $S$-equivalence relation.

In the first section of the present paper we define a $(-1)^{q+1}$-hermitian form which gives an isotopy classification of minimal Seifert surfaces. The Blanchfield form can be obtained from this form by an extension of the scalars. This is inspired by Trotter's papers [14] and [15].

The main purpose of this paper is to apply the algebraic results of [1] and [3] to the classification of a special type of simple knots, called Dedekind knots, which are defined as follows. Let $L$ be the knot module of $K^{2q-1}$ and let $\lambda \in \mathbb{Z}[X]$, $\lambda(1) = \pm 1$, be a generator of the annihilator ideal of the $\mathbb{Z}[X, X^{-1}]$-module $L$ (cf. [9], [10] §7). We shall say that $K^{2q-1}$ is a Dedekind knot if $\lambda$ is irreducible and $\mathbb{Z}[X, X^{-1}]/(\lambda)$ is Dedekind.

Non-fibered Dedekind $(2q-1)$-knots, $q \geq 3$, are always easy to classify (see Theorem 3). For fibered Dedekind knots we have two quite different cases: if the Blanchfield form is indefinite, then we have the same kind of classification theorem as for non-fibered knots. On the other hand, the classification of fibered Dedekind knots with definite Blanchfield pairing seems very difficult.

In Sections 2 and 3 we give applications to the cancellation problem, to the number of minimal Seifert surfaces, and to the symmetries of Dedekind knots. For instance we shall give a complete criterion for a Dedekind knot to be $(-1)$-amphicheiral.

I thank Neal W. Stoltzfus for useful conversations.

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* Supported by the "Fonds National de la recherche scientifique" of Switzerland.

## 1. An algebraic classification of the minimal Seifert surfaces of a given simple $(2q-1)$-knot, $q \geq 3$.

Let $\Gamma_0 = \mathbb{Z}[z]$, $\Lambda_0 = \mathbb{Z}[z, z^{-1}, (1-z)^{-1}]$ and let $E_0$ be the field of quotients of $\Lambda_0$. These rings have involutions induced by $\bar{z} = 1 - z$.

Let $N$ be a $\mathbb{Z}$-torsion free, finitely generated $\Gamma_0$-torsion $\Gamma_0$-module. We shall say that an $\varepsilon$-hermitian ($\varepsilon = \pm 1$) form $h: N \times N \to E_0/\Gamma_0$ is *unimodular* if the adjoint map from $N$ to $\mathrm{Hom}_{\Gamma_0}(N, E_0/\Gamma_0)$ which sends $x$ to $f_x$, defined by $f_x(y) = h(y, x)$, is a conjugate-linear isomorphism.

The following is a consequence of results of Levine [8] and Trotter [14], [15]:

THEOREM 1. *Let $K^{2q-1}$ be a simple knot, $q \geq 3$, and let $b: L \times L \to E_0/\Lambda_0$ be the associated Blanchfield form. The isotopy classes of minimal Seifert surfaces of $K^{2q-1}$ are in bijection with the isometry classes of unimodular $(-1)^{q+1}$-hermitian forms*

$$h: N \times N \to E_0/\Gamma_0$$

*such that $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$.*

DEFINITION. A form $(N, h)$ as in Theorem 1 will be called a *Trotter form.*

*Proof.* Let $A$ be a non-singular Seifert matrix associated with a minimal Seifert surface of $K^{2q-1}$, and let $M_A = A - z(A + (-1)^q A')$ where $A'$ denotes the transpose of $A$. Let $\Lambda_0^n/M_A\Lambda_0^n$, $N = \Gamma_0^n/M_A\Gamma_0^n$ ($A$ is an $n \times n$-matrix) and let

$$b: L \times L \to E_0/\Lambda_0$$
$$h: N \times N \to E_0/\Gamma_0$$

be the quotient forms associated with $M_A$ (cf. [14], p. 178). The $(-1)^{q+1}$-hermitian form $b$ is the Blanchfield form of $K^{2q-1}$ (cf. [9], 14.3, p. 44). Clearly $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$. It is easy to check that if $A$ and $B$ are congruent Seifert matrices, then the quotient forms associated to $M_A$ and $M_B$ are isometric.

Conversely let $h: N \times N \to E_0/\Gamma_0$ be a unimodular $(-1)^{q+1}$-hermitian form such that $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$. Following Trotter (cf. [14], [15]) let us define a trace function $s: E_0 \to \mathbb{Q}$ by setting $s(f)$ equal to the coefficient of $z^{-1}$ in the Laurent expansion of $f$ at infinity. Set $[a_1, a_2] = s(h(a_1, a_2))$ for $a_1, a_2 \in N$. Then $[\ ]: N \times N \to \mathbb{Z}$ is a unimodular $(-1)^q$-symmetric $\mathbb{Z}$-bilinear form (cf. [14] pp. 292–294). We have $[za_1, a_2] = [a_1, (1-z)a_2]$, i.e. $(N, [\ ], z)$ is an isometric structure. It is easy to check that isometric $(-1)^{q+1}$-hermitian forms give rise to isomorphic isometric structures. Let $S$, $Z$ be the matrices of [ ], $z$ with respect to a $\mathbb{Z}$-basis of $N$. Set $A = ZS^{-1}$. Then $A$ is a Seifert matrix, i.e. $A+(-1)^q A' = S^{-1}$ is unimodular. $A$ is non-singular as $\det(A) = \det(Z)$. By [14], Proposition 2.11, $(L, b)$ is isometric to the quotient form associated to $M_A$. Trotter's main theorem in [14] implies that $A$ is in the $S$-equivalence class determined by $K^{2q-1}$. It is easy to check that if two isometric structures are isomorphic then the corresponding Seifert matrices are congruent so by Levine [8] the associated minimal Seifert surfaces are isotopic.

*Remark.* The existence of at least one minimal Seifert surface follows from Trotter, [13] and Levine, [8].

Let $b$ and $h$ be unimodular $\varepsilon$-hermitian forms as in Theorem 1. Let $\varphi \in \mathbb{Z}[X]$ be the minimal polynomial of $z: L \to L$ and let $\Gamma = \mathbb{Z}[X]/(\varphi)$. Set $\lambda(X) = (1-X)^{\deg \varphi}\varphi(1/1-X) \in \mathbb{Z}[X]$. We have $\lambda L = 0$ and $\lambda(1) = \pm 1$. Notice that $\lambda$ is a generator of $\mathrm{Ann}_{\Lambda_0}(L)$, cf. Levine [11], proof of Theorem 7.1. Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda) = \Lambda_0/(\lambda)$. Then $L$ is a $\Lambda$-module and $b$ takes values in $(1/\lambda)\Lambda_0/\Lambda_0 \simeq \Lambda$. So we can consider $b$ and $h$ as unimodular $\varepsilon$-hermitian forms $b: L \times L \to \Lambda$. $h: N \times N \to \Gamma$.

We shall apply Theorem 1 to give a short proof of a theorem of Trotter, in a special case. Let $F \in \mathbb{Z}[X]$ be the characteristic polynomial of $z: L \to L$.

THEOREM 2 (Trotter, [14] Corollary 4.7). *Let $K^{2q-1} \subset S^{2q+1}$ be a simple knot, $q \geq 3$, such that $F(0) = \pm p$ where $p$ is a prime. Then the knot $K^{2q-1}$ has only one isotopy class of minimal Seifert surfaces.*

Let us assume that $\varphi$ is *irreducible.* As $\varphi$ and $F$ have the same irreductible factors, $F$ is then a power of $\varphi$. If the constant term of $F$ is $\pm p$, where $p$ is a prime number, then we must have $F = \varphi$.

Let $\Gamma = \mathbb{Z}[\alpha]$. Then $\Lambda = [\alpha^{-1}, \bar{\alpha}^{-1}]$ where $\bar{\alpha} = 1-\alpha$. We have $\varphi(0) = \pm p$, therefore $\Gamma/(\alpha) \simeq \mathbb{F}_p$, so $(\alpha)$ is a maximal ideal.

In the special case where $\varphi$ is irreducible, Theorem 2 is a consequence of the following lemma:

LEMMA. *Let $(I, h_1)$ and $(J, h_2)$ be two unimodular $\varepsilon$-hermitian forms where $I$ and $J$ are $\Gamma$-ideals, such that $(I, h_1) \otimes \Lambda \simeq (J, h_2) \otimes \Lambda$. Then $(I, h_1) \simeq (J, h_2)$.*

*Proof of Lemma.* We want to show that if $I$ and $J$ are $\Gamma$-ideals such that $I\Lambda = J\Lambda$ then $\alpha^k \bar{\alpha}^m I = J$ for some integers $k$, $m$. As $\Gamma$ is noetherian we can write $I = I_1 \cap I_2$, $J = J_1 \cap J_2$ where the $I_i$'s, $J_j$'s are the intersection of a finite number of primary ideals (cf. [17] Chap. IV §4 Theorem 4). We can assume that the radicals of $I_1$, $J_1$ are prime to $P=(\alpha)$ and to $\bar{P}$ and that the radical of the primary components of $I_2$ and $J_2$ is $P$ or $\bar{P}$. By [17] Chap. IV §10 Theorem 17 the hypothesis $I\Lambda=J\Lambda$ implies that $I_1=J_1$. Let $Q$ be a $P$-primary component of $I_2$. Then there exists an integer $n$ such that $\alpha^n \in Q$. Let us assume that $n$ is minimal with this property. If $n=0$ then we have finished. We have $Q\subset P$ therefore $Q'=\alpha^{-1}Q\subset \Gamma$. Then either $Q'=\Gamma$ or $Q'$ is $P$-primary so we can repeat the above procedure. We finally obtain $\alpha^{-n+1}Q=\Gamma$. Therefore $I_2=(\alpha^k\bar{\alpha}^m)$, and a similar result holds for $J_2$.

Let $h_1: I\times I\to\Gamma$, $h_2: I\times I\to\Gamma$ be two unimodular $\varepsilon$-hermitian forms such that $(I, h_1)\otimes_\Gamma \Lambda=(I, h_2)\otimes_\Gamma\Lambda$. We have $h_i(x, y)=a_i x\bar{y}$, $i=1, 2$. As $h_1$ and $h_2$ are unimodular, $a_1a_2^{-1}=u$ is a unit of $\Gamma$. There exists $x\in\Lambda$ such that $x\bar{x}=u$. We have $x\Lambda=\Lambda$, therefore $x=v\alpha^k\bar{\alpha}^m$ where $v$ is a unit of $\Gamma$. So $x\bar{x}=\alpha^{k+m}\bar{\alpha}^{k+m}v\bar{v}=u$. This implies that $k=-m$, so $x\bar{x}=v\bar{v}=u$, therefore $h_1$ and $h_2$ are isometric.

## 2. Dedekind knots

Let $K^{2q-1}\subset S^{2q+1}$ be a simple knot, $q\geq 2$, and let $b: L\times L\to\Lambda$ be the associated Blanchfield form, $\Lambda=\mathbb{Z}[X, X^{-1}]/(\lambda)$ as above. We shall say that $K^{2q-1}$ is a *Dedekind knot* if $\lambda$ is irreducible and $\Lambda$ is Dedekind. We shall now apply the results of [1] and [3] to the classification of Dedekind knots and of their minimal Seifert surfaces.

Let us denote $E$ the field of quotients of $\Lambda$ and $F$ the fixed field of the involution. For every real embedding of $F$ which extends to an imaginary embedding of $E$ we have a *signature* invariant of $b: L\times L\to\Lambda$. We shall say that $b$ is *definite* if $F$ is totally real, $E$ is totally imaginary and if every signature is maximal. Otherwise we say that $b$ is *indefinite*. The determinant of $(L, b)$ is the rank one form

$$\det(b): \Lambda^n L\times\Lambda^n L\to\Lambda$$
$$(x_1\Lambda\cdots\Lambda x_n, y_1\Lambda\cdots\Lambda y_n)\to\det(b(x_i, y_j)_{ij})$$

where $n=\operatorname{rank}_\Lambda(L)$.

If $\varepsilon=-1$ and $\operatorname{rank}_\Lambda(L)$ is even, we also need a finite number of pfaffians. Let $\Lambda'=\Lambda\cap F$ and let $p$ be a prime $\Lambda'$-ideal such that $p\Lambda=P^2$. The involution on $\Lambda/P$ is trivial (cf. [6], §5), and the skew-hermitian form $b$ induces a non-singular skew-symmetric form $\tilde{b}$ on $\tilde{L}=L/PL$. Let us denote by $\mathrm{Pf}_p(b)$ a *pfaffian* of this form. If $(M, b)$ is another lattice such that $\varphi: (\tilde{L}, \tilde{b})\to(\tilde{M}, \tilde{b})$ is an isometry, then $\mathrm{Pf}_p(L, b)\cdot\det(\varphi)=\mathrm{Pf}_p(M, b)$.

Let us recall the classification theorem of [3]. We have the following hypothesis:
(*) Either $\Lambda \neq \Gamma$ (or equivalently $\lambda(0) \neq \pm 1$) or the $\varepsilon$-hermitian forms $b_i : L_i \times L_i \to \Lambda$ are indefinite.

THEOREM 3. *Assume that the hypothesis* (*) *is satisfied. Then two unimodular $\varepsilon$-hermitian forms $b_1 : L_1 \times L_1 \to \Lambda$ and $b_2 : L_2 \times L_2 \to \Lambda$ are isometric if and only if they have the same rank, same signatures and isometric determinants, and if moreover $\varepsilon = -1$ and the forms have even rank, there exists an isometry $f$ between* $\det(b_1)$ *and* $\det(b_2)$ *such that* $\det(f)\, \mathrm{Pf}_p(b_1) \equiv \mathrm{Pf}_p(b_2) \bmod P$ *if $p\Lambda = P^2$.*

*Proof.* This is a consequence of [3], Theorem 2 and Remark 1. Notice that if $p$ is a prime of $\Gamma' = \Gamma \cap F$ such that $p\Lambda' = \Lambda'$ then $p\Gamma = P\bar{P}$ with $P \neq \bar{P}$. Indeed, $p\Lambda' = \Lambda'$ implies that $p$ contains $\alpha\bar{\alpha}$ (see the proof of Theorem 2). The minimal polynomial of $\alpha$ over $\Gamma'$ is $X^2 - X + \alpha\bar{\alpha}$. Therefore $\Gamma/p\Gamma = \Gamma'/p[X]/(X^2 - X) = \Gamma'/p \times \Gamma'/p$.

The isotopy classes of simple $(2q-1)$-knots, $q \geq 2$, are in bijection with the isometry classes of Blanchfield forms (cf. Kearton [7] or Levine [8] and Trotter [14].) Therefore the above theorem gives an isotopy classification of Dedekind knots satisfying (*). Notice that all non-fibered $(2q-1)$-knots, $q \geqslant 3$, satisfy (*). Indeed, an easy application of the $h$-cobordism theorem shows that a simple $(2q-1)$-knot, $q \geq 3$, is fibered if and only if $\lambda(0) = 1$.

COROLLARY 1. *Let $K_1$ and $K_2$ be Dedekind $(2q-1)$-knots such that the associated Blanchfield forms satisfy* (*), *and let $K$ be any $(2q-1)$-knot. If the connected sum $K_1 + K$ is isotopic to $K_2 + K$ then $K_1$ and $K_2$ are isotopic.*

*In particular, cancellation holds for non-fibered $(2q-1)$-Dedekind knots if $q \geq 3$.*

*Proof.* Let $b_1$, $b_2$ and $b$ be the Blanchfield forms of $K_1$, $K_2$ and $K$. We have an isometry between $b_1 \perp b$ and $b_2 \perp b$ where $\perp$ denotes orthogonal sum. The knot modules of $K_1$ and $K_2$ clearly have the same annihilator $\lambda \in \mathbb{Z}[X]$, $\lambda(1) = 1$. Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$. Taking tensor product over $\mathbb{Z}[X, X^{-1}]$ with $\Lambda$ and then taking the $\mathbb{Z}$-torsion free part we may assume that $b : L \times L \to \Lambda$, where $L$ is a projective $\Lambda$-module of finite rank. Now Theorem 3 implies that $b_1$ and $b_2$ are isometric.

In the fibered definite case there are counter-examples to cancellation (cf. [2]).

*Minimal Seifert surfaces*

The isotopy classes of the minimal Seifert surfaces of a given simple $(2q-1)$-knot $K$, $q \geq 3$, are classified by the isometry classes of the Trotter forms associated to $K$ (cf. Theorem 1). Therefore Theorem 3 implies the following

COROLLARY 2. *Let $K^{2q-1}$ be a Dedekind knot such that the associated Blanchfield form is indefinite and that $\Gamma$ is Dedekind. Let $S_1$ and $S_2$ be two minimal Seifert surfaces of K and let $(N_1, h_1)$ and $(N_2, h_2)$ be the associated Trotter forms. Then $S_1$ and $S_2$ are isotopic if and only if there exists an isometry $f: \det(N_1, h_1) \rightarrow \det(N_2, h_2)$ such that*

$$\mathrm{Pf}_p(N_1, h_1) \det(f) \equiv \mathrm{Pf}_p(N_2, h_2) \bmod P \textit{ if } p\Gamma = P^2.$$

*Remark.* We have $\Lambda = \Gamma[\alpha^{-1}, \bar{\alpha}^{-1}]$ so if $\Gamma$ is Dedekind then $\Lambda$ is Dedekind too. But the converse is not true. I thank Jonathan Hillman for the following example: let $\lambda(X) = 9X^4 - 3X^3 - 11X^2 - 3X + 9$, then $\varphi(X) = X^4 - 2X^3 + 34X^2 - 33X + 9$, $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$, $\Gamma = \mathbb{Z}[X]/(\varphi)$.

Then $\Lambda$ is Dedekind by Levine's criterion (cf. [10], §28). On the other hand $\varphi(X) \in (3, X)^2$, so $\Gamma$ is not Dedekind by Uchida's criterion (cf. [16]).

COROLLARY 3. *If $K^{2q-1}$ is a Dedekind knot, $q \geq 3$, such that the associated Blanchfield form is indefinite and that $\Gamma$ is Dedekind, the number of isotopy classes of minimal Seifert surfaces of K only depends on $\Lambda$.*

*If moreover $\lambda(0) = \pm p$ where p is a prime number, then K has only one isotopy class of minimal Seifert surfaces. (This is a generalization of Theorem* 1, *in the case of Dedekind knots.)*

*Remark.* The above corollary is no longer true if the Blanchfield form is definite. For instance let $\lambda(X) = aX^2 + (1-2a)X + a$, $\varphi(X) = X^2 - X + a$, where $a$ is a positive integer, $a \neq 1$, and $1-4a$ is square free. Then $E = \mathbb{Q}[X]/(\lambda) = \mathbb{Q}(\sqrt{1-4a})$ is an imaginary quadratic field. Let $p(n)$ be the number of partitions of $n$ into the sum of positive integers. There are at least $p(n)$ unimodular forms $h: N \times N \rightarrow \Gamma$, $\mathrm{rank}(N) = 4n$ such that $(N, h) \otimes_\Gamma \Lambda$ is isomorphic to $\langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$ (cf. [2], Remark 2). On the other hand the number of unimodular forms $h: N \times N \rightarrow \Gamma$ such that $(N, h) \otimes_\Gamma \Lambda$ is isomorphic to $\langle 1 \rangle \perp \langle -1 \rangle \perp \cdots \perp \langle 1 \rangle$ does not depend on $n$.

## 3. Symmetries of knots

If $X$ is an oriented manifold, let us denote $X^-$ the same manifold with the opposite orientation. We shall say that a knot $K^{2q-1} \subset S^{2q+1}$ is *invertible* if it is isotopic to $(K^{2q-1})^- \subset S^{2q+1}$ $(+1)$-*amphicheiral* if it is isotopic to $K^{2q-1} \subset (S^{2q+1})^-$ and $(-1)$-*amphicheiral* if it is isotopic to $(K^{2q-1})^- \subset (S^{2q+1})^-$. F. Michel [11] has translated these conditions into algebraic conditions on the Blanchfield form $(L, b)$ associated to $K^{2q-1}$, $q \geq 2$. Let us define $(\bar{L}, \bar{b})$ as follows: $\bar{L}$ is equal to $L$ as $\mathbb{Z}$-modules, and the $\Lambda$-module structure of $L$ is given by $\lambda^* x = \bar{\lambda} x$. Let $\bar{b}(x, y) = \overline{b(x, y)}$. Then $K^{2q-1}$ is invertible if $(L, b) \cong (\bar{L}, \bar{b})$, $(+1)$-amphicheiral if $(L, b) \cong (\bar{L}, -\bar{b})$ and $(-1)$-amphicheiral if $(L, b) \cong (L, -b)$ (see [11], [5]).

In this section we shall apply Theorem 3 to determine the symmetries of Dedekind knots.

COROLLARY 4. *Let $K^{2q-1}$ be a Dedekind knot, $q \geq 2$, and let $(L, b)$ be the corresponding Blanchfield form. Then $K^{2q-1}$ is $(-1)$-amphicheiral if and only if*

a) (F. Michel [11]) rank $(L)$ *is odd and there exists a unit $u$ of $\Lambda$ such that* $u\bar{u} = -1$

b) $\operatorname{rank}_\Lambda (L)$ *is even and every signature of $b$ is zero.*

*Proof.* It is easy to see that the conditions are necessary. Let us prove that they are also sufficient:

a) an isometry is given by multiplication with $u$

b) As rank $(L)$ is even, $\det(-b) = \det(b)$, and we have $\mathrm{Pf}_p(-b) = (-1)^n \, \mathrm{Pf}_p(b)$ where $2n = \operatorname{rank}_\Lambda (L)$. Therefore $f(x) = (-1)^n x$ gives an isometry between $\det(b)$ and $\det(-b)$ such that $\det(f)\, \mathrm{Pf}_p(b) \equiv \mathrm{Pf}_p(-b) \bmod P$ if $p\Lambda = P^2$. As $b$ and $-b$ are indefinite and have same signatures, they are isometric by Theorem 3.

The following is a consequence of Corollary 4:

COROLLARY 5. *Let $K^{2q-1}$ be a Dedekind knot, $q \geq 2$, which has order two in the knot cobordism group (i.e. $K^{2q-1} + K^{2q-1}$ is nullcobordant where $+$ denotes connected sum). Assume that the associated Blanchfield form has even rank. Then $K^{2q-1}$ is $(-1)$-amphicheiral.*

In the case of odd rank, D. Coray and F. Michel have given counter-examples to the above statement in [4].

Let $C_\Lambda$ be the group of isomorphism classes of $\Lambda$-ideals and let $C_\varepsilon = \{c \in C_\Lambda$ such that if $I \in c$ then $\bar{I} = xI$ with $x\bar{x} = \varepsilon\}$ (notice that if $c \in C_\Lambda$ contains an ideal $I$ such that $\bar{I} = xI$, $x\bar{x} = \varepsilon$, then every $J \in c$ has this property. Indeed, let $J = aI$ then $\bar{J} = (\bar{a}/a) xJ$).

The following is a generalization of results of F. Michel, (cf. [11], Propositions 2 and 3):

COROLLARY 6. *Let $K^{2q-1}$ be a Dedekind knot, $q \geq 2$, such that the associated Blanchfield form $(L, b)$ satisfies $(*)$. Let $c$ be the ideal class of the $\Lambda$-module $L$. Then $K^{2q-1}$ is invertible (resp. $(+1)$-amphicheiral) if and only if the signatures of $b$ and $\bar{b}$ (resp. $-\bar{b}$) are equal and $c \in C_\varepsilon$ with $\varepsilon = (-1)^{n(q+1)}$ (resp. $\varepsilon = (-1)^{nq}$), where $n = \operatorname{rank}_\Lambda(L)$.*

*Proof.* It is easy to check that there conditions are necessary, let us prove that they are also sufficient. Let us choose a basis $e_1, \ldots, e_n$ of $V$ such that $L = Ie_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$ with $\bar{I} = xI$, $x\bar{x} = \varepsilon$. We can identify $V$ and $\bar{V}$ using the isomorphism $f: V \to \bar{V}$, $f(\lambda e_i) = \lambda^* e_i$. We have $\bar{L} = \bar{I}e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$, and multiplication by $\pm x$ gives an isometry between $\det(L, b)$ and $\varepsilon(-1)^{n(q+1)} \det(\bar{L}, b)$. If $n$ is even and $b$ is skew-hermitian, we see that $\mathrm{Pf}_p(L, b) = \mathrm{Pf}_p(\bar{L}, b)$ for $p\Lambda = P^2$. Theorem 3 now gives the desired result.

In the fibered definite case there are counter-examples to the above corollary. Let for instance $\Lambda = \mathbb{Z}[\xi]$ where $\xi$ is a 52th root of unity. Then there exists a non-trivial $\Lambda$-ideal $I$ such that $I^3$ is principal and that $I$ supports a rank one form $b$ (cf. [12], [1] §1). Notice that $I$ is not isomorphic to $\bar{I}$, therefore $(I, b)$ cannot be isometric to $(\bar{I}, \bar{b})$. So by unique factorisation of definite forms (cf. [2]) $b \perp b \perp b$ cannot be isometric to $\bar{b} \perp \bar{b} \perp \bar{b}$.

EXAMPLE. Let $I$ be a $\Lambda$-ideal which supports a rank one form $b$. Let $(L, b') = (I, b) \perp (\bar{I}, -\bar{b})$. The simple $(2q-1)$-knot, $q \geq 2$, which has Blanchfield pairing $(L, b')$ is clearly $(+1)$-amphicheiral, but it also has the two other symmetries by Corollary 4 and Corollary 6. This answers a question of J. Hillman in [5], for the special case $\Lambda = \mathbb{Z}[w, \frac{1}{53}]$, $I = (5, w+1)$, with $w = 1 + \sqrt{-211}/2$.

## 4. Rank one forms

Theorem 3 essentially reduces the classification of non-fibered Dedekind $(4q+1)$-knots, $q \geqslant 1$, to the classification of rank one hermitian forms. These have been studied in [1], §1 and §2. Let $C_\Lambda$, $C_{\Lambda'}$, denote the ideal class groups (recall $\Lambda' = \{x \in \Lambda$ such that $\bar{x} = x\}$) and let $N: C_\Lambda \to C_{\Lambda'}$ be the norm homomorphism. Let $U_\Lambda$ be the group of units of $\Lambda$, and $N(u) = u\bar{u}$. Let $I(\Lambda)$ be the set of isomorphism classes of rank one forms, which is a group under tensor product. The following diagram summarizes the relation between $\Gamma$-lattices and $\Lambda$-lattices. The rows and columns are exact.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Ker}\,(f) & \longrightarrow & \mathrm{Ker}\,(g) & \longrightarrow & \mathrm{Ker}\,(h) & \longrightarrow & Y \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & U_{\Gamma'}/N(U_\Gamma) & \longrightarrow & I(\Gamma) & \longrightarrow & \mathrm{Ker}\,(N_\Gamma) & \longrightarrow & 1 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\
1 & \longrightarrow & U_{\Lambda'}/N(U_\Lambda) & \longrightarrow & I(\Lambda) & \longrightarrow & \mathrm{Ker}\,(N_\Lambda) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & X & & 1 & & 1 & & \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}$$

EXAMPLE. Let $\varphi(X) = X^2 - X + 122$, $\lambda(X) = 112X^2 - 223X + 112$ $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$, $\Gamma = \mathbb{Z}[X]/(\varphi)$. Then we have the following diagram:

$$\begin{array}{ccccccccccc}
 & & 1 & \longrightarrow & \mathbb{Z}/7 & \longrightarrow & \mathbb{Z}/14 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2\times\mathbb{Z}/14 & \longrightarrow & \mathbb{Z}/14 & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Z}/2\times\mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2\times\mathbb{Z}/2 & \longrightarrow & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & \mathbb{Z}/2 & & 1 & & & & & & \\
 & & \downarrow & & & & & & & & \\
 & & 1 & & & & & & & &
\end{array}$$

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Received April 21, 1983