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**Autor:** Coray, Daniel / Michel, Françoise  
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## Knot cobordism and amphicheirality

DANIEL CORAY and FRANÇOISE MICHEL

### Introduction

Let  $C_n$  denote the cobordism group of  $n$ -dimensional knots. Cameron Gordon has asked the following question ([Ha], problem 16):

*Can every element of order 2 in  $C_1$  be represented by a  $(-1)$ -amphicheiral knot?*

A knot is called  $(-1)$ -amphicheiral if it is isotopic to its obvious cobordism inverse (see §1 for a precise definition). Hence it is clear that the cobordism class of any  $(-1)$ -amphicheiral knot has order two. Gordon's question is about a partial converse of this statement.

Actually the problem makes sense in any odd dimension. (We recall that  $C_n = 0$  for  $n$  even [K].) But, for  $n = 2q - 1$ , we show:

**STATEMENT 1.** *The answer is negative for every  $q \geq 2$ . More precisely, some Alexander polynomials  $\gamma$  have the following property: the cobordism class of every knot whose Alexander polynomial is  $\gamma$  has order two, but contains no  $(-1)$ -amphicheiral knot.*

**STATEMENT 2.** *For  $q = 1$  the same polynomials provide many examples of algebraic cobordism classes of order 2 which contain no  $(-1)$ -amphicheiral knot. Since they are exceedingly numerous, it seems reasonable to expect that Gordon's question should have a negative answer also in the classical case.*

For the proof we work with the algebraic invariants already used in [T], [Mic] and [Hi]. One of the main features is a new  $(-1)$ -amphicheirality criterion, which is considerably more general than those previously obtained. In particular it is invariant under cobordism and applies to knots of any odd dimension.

We thank J. Hillman, who pointed out the interest of studying Gordon's problem in higher dimensions.

## §1. Statements of the results

We begin with some definitions:

1. An  $n$ -knot  $\Sigma$  is a smooth, oriented submanifold of  $S^{n+2}$  which is homeomorphic to  $S^n$ .

2. Let  $\sigma: S^{n+2} \rightarrow S^{n+2}$  be the reflection in some equatorial plane,  $(\sigma(\Sigma))^-$  the image of  $\Sigma$  with the opposite orientation. By  $-\Sigma$  we shall denote  $(\sigma(\Sigma))^-$  regarded as a submanifold of  $S^{n+2}$ . We call it the *inverse* of  $\Sigma$ . As  $\Sigma \neq -\Sigma$  is null-cobordant, the cobordism class of  $-\Sigma$  is the inverse of the cobordism class of  $\Sigma$ .

3.  $\Sigma$  is said to be  $(-1)$ -*amphicheiral* (“involutory” in the terminology of J. Conway) if it is isotopic to  $-\Sigma$ .

4. For  $\varepsilon = \pm 1$ , let  $C^\varepsilon(\mathbb{Z})$  be the cobordism group of  $\varepsilon$ -forms (cf. [L<sub>1</sub>] or [K]). Associating a Seifert form to a  $(2q-1)$ -knot induces a homomorphism  $\varphi_{2q-1}$  from  $C_{2q-1}$  to  $C^{(-1)^q}(\mathbb{Z})$ . The *algebraic cobordism class* of a  $(2q-1)$ -knot  $\Sigma$  is the image by  $\varphi_{2q-1}$  of the cobordism class of  $\Sigma$ . We recall that  $\varphi_{2q-1}$  is injective if and only if  $q \geq 2$  ([L<sub>1</sub>] and [C-G]). It is the reason why our results do not answer Gordon’s question when  $q = 1$ .

5. For any polynomial  $\Delta \in \mathbb{Z}[X]$ , of degree  $d$  (say), we define  $\Delta^* \in \mathbb{Z}[X]$  by the formula:

$$\Delta^*(X) = X^d \Delta(X^{-1}).$$

We recall that  $\Delta$  is *reciprocal* if  $\Delta = \Delta^*$ .

6. Given an irreducible reciprocal polynomial  $\gamma \in \mathbb{Z}[X]$ , we define  $K$  to be the number field  $\mathbb{Q}[X]/(\gamma)$ , and  $\mathcal{O}_K$  its ring of algebraic integers. As  $\gamma$  is reciprocal, mapping  $X$  into  $X^{-1}$  induces an involution on  $K$ . We write  $\bar{\alpha}$  for the image of  $\alpha \in K$  under this involution. Finally we set  $a = \gamma(0)$  and adopt the following terminology:

$\gamma$  has property  $P_1$  if  $\alpha \bar{\alpha} = -1$  for some  $\alpha$  in  $K$ ;

$\gamma$  has property  $P_2$  if  $\alpha \bar{\alpha} = -1$  for some  $\alpha$  in the ring  $\mathcal{O}_K[1/a]$ ;

$\gamma$  has property  $P_3$  if  $\eta \bar{\eta} = -1$  for some unit  $\eta$  in  $\mathcal{O}_K$ .

We are now in a position to give the precise statements that we shall prove. In what follows,  $q$  is any positive integer, and  $\Sigma$  a  $(2q-1)$ -knot. If  $\Delta$  is the Alexander polynomial of  $\Sigma$ , we have  $\Delta = \Delta^*$ . Hence we can write:

$$\Delta = \delta \delta^* \prod_{i=1}^l \gamma_i,$$

where the  $\gamma_i$  are distinct irreducible reciprocal polynomials. (The  $\gamma_i$  are those

reciprocal polynomials which appear with odd multiplicity among the irreducible factors of  $\Delta$ .)

**THEOREM 1.** *If  $\Sigma$  is  $(-1)$ -amphicheiral then  $\gamma_i$  has property  $P_2$ , for every  $i \leq l$ .*

This  $(-1)$ -amphicheirality criterion is proved in §2. In practice, property  $P_3$  is a lot more convenient to work with than  $P_2$ . This makes the interest of the following two propositions, where  $\gamma$  is assumed to be an irreducible reciprocal polynomial such that  $\gamma(1) = \pm 1$ .

**PROPOSITION 1.** *Suppose  $|\gamma(0)|$  is a prime  $p$  and  $\mathbb{Z}[X, X^{-1}]/(\gamma) = \mathcal{O}_K[1/p]$ . Then  $\gamma$  has property  $P_2$  if and only if it has  $P_3$ .*

**PROPOSITION 2.** *Suppose  $|\gamma(0)|$  is a prime  $p$  and  $K$  is a Galois extension of  $\mathbb{Q}$ . Then  $\gamma$  has property  $P_2$  if and only if it has  $P_3$ .*

For Proposition 1 we give a topological argument, while Proposition 2 is established by purely algebraic means.

*Remark.* Proposition 2 will be used in §4 in constructing the appropriate examples.

In §3 we prove:

**THEOREM 2.** *Let  $\Sigma$  be a  $(2q-1)$ -knot whose Alexander polynomial  $\gamma$  is irreducible. Then:*

- (1)  $\gamma$  has property  $P_1$  if and only if the algebraic cobordism class of  $\Sigma$  has order two;
- (2) if  $\Sigma$  is cobordant to some  $(-1)$ -amphicheiral knot then  $\gamma$  has property  $P_2$ .

**SCHOLIUM.** *If  $q \geq 2$  and  $\gamma$  has property  $P_1$  then the geometric cobordism class of  $\Sigma$  is also of order two, as follows from  $[\mathbf{L}_1]$ .*

**COROLLARY.** *To prove statements 1 and 2 of the introduction it is enough to produce some irreducible, reciprocal polynomials  $\gamma$  with the following properties:*

- (1)  $\gamma$  is the Alexander polynomial of some  $(2q-1)$ -knot;
- (2)  $\gamma$  has property  $P_1$ ;
- (3)  $\gamma$  fails to have property  $P_2$ .

In §4 we show how to construct infinitely many irreducible Alexander polynomials having property  $P_1$  but not  $P_2$ . As a matter of fact there are some examples already in degree 2, but recall:



*Levine's criterion* [ $L_1$ ]. A reciprocal polynomial  $\gamma \in \mathbb{Z}[X]$ , with degree  $d$ , is the Alexander polynomial of some  $(2q-1)$ -knot if and only if  $\gamma(1) = \varepsilon^{d/2}$  and  $\gamma(\varepsilon)$  is a perfect square, where  $\varepsilon = (-1)^{q+1}$ .

Now, if  $\gamma$  is any reciprocal polynomial of degree 2 such that  $\gamma(1) = -1$ , we observe that  $\gamma(-1)$ , being the discriminant of  $\gamma$ , can be a perfect square only if  $\gamma$  is reducible! Therefore, by Levine's criterion, no example with  $q$  even can be obtained in degree 2. That is why we shall give two series of examples:

I. *The quadratic case (which occurs only for odd values of  $q$ )*

$$\gamma(X) = -pX^2 + (2p+1)X - p,$$

where  $p$  runs through a certain set of primes:  $p = 367, 379, 461, 751, 991, \dots$  (61 examples for  $p < 10,000$ ).

*Remark.* In [T], H. F. Trotter already observed that the knots with Alexander polynomial  $\gamma(X) = -367X^2 + 735X - 367$  are not  $(-1)$ -amphicheiral.

II. *The biquadratic case (which occurs for any  $q$ )*

In §4 we prove the following theorem:

**THEOREM 3.** *Let  $p$  be an odd prime and*

$$\gamma(X) = -pX^4 + (2p+1)X^2 - p.$$

*Then  $\gamma$  is irreducible. Moreover:*

- (1)  $\gamma$  has property  $P_1$ ;
- (2)  $\gamma$  fails to have property  $P_2$  if and only if  $p$  is congruent to 3 modulo 4 and the fundamental unit of  $\mathbb{Q}(\sqrt{4p+1})$  has norm  $+1$ .

*Remark.* This yields infinitely many examples. Indeed the fundamental unit of  $\mathbb{Q}(\sqrt{4p+1})$  has norm  $+1$  whenever  $4p+1$  has a prime factor with odd multiplicity which is congruent to 3 modulo 4 (e.g.  $p = 19, 23$ , etc.), and also in certain other cases, like  $p = 367, 379, 751, 991, \dots$  etc.

*Other examples.* The following polynomials:

$$\gamma(X) = X^4 - 2\lambda X^3 + (4\lambda - 1)X^2 - 2\lambda X + 1,$$

with  $\lambda = 36, 45, \dots$  (an infinity of examples), satisfy all three properties of the

above corollary. This is proved in [C] with the techniques that we use in the proof of Theorem 3. The particular interest of these examples is that they can be realized as Alexander polynomials of some *fibred* knots.

## §2. An amphicheirality criterion

*Proof of Theorem 1.* We recall that  $\Sigma$  is a  $(-1)$ -amphicheiral  $(2q-1)$ -knot with Alexander polynomial  $\Delta$ . To prove Theorem 1, we must show: if  $\Delta = \gamma^{2l+1} \cdot \mu$ , where  $\gamma$  is reciprocal, irreducible and prime to  $\mu$ , then  $\gamma$  has property  $P_2$ .

Let  $\tilde{X}$  be the infinite cyclic covering of the complement of  $\Sigma$ . Put  $M = H_q(\tilde{X})$ ; this is a torsion module over  $\mathbb{Z}[X, X^{-1}]$ . Let  $B: M \times M \rightarrow \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}]$  be the Blanchfield pairing associated with  $\Sigma$  (cf. [L<sub>3</sub>], p. 15). If we write  $\varepsilon$  for  $(-1)^{q+1}$ , the Blanchfield pairing is  $\varepsilon$ -hermitian and unimodular (i.e. the adjoint of  $B$  yields a  $\mathbb{Z}[X, X^{-1}]$ -isomorphism between  $M$  and  $\text{Hom}_{\mathbb{Z}[X, X^{-1}]}(M, \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}])$ ) (cf. [L<sub>3</sub>]). We recall that  $B$  can be constructed as follows ([L<sub>3</sub>], Proposition 14.3, p. 44):

Let  $A$  be an  $r \times r$  matrix which represents a Seifert pairing of  $\Sigma$  (see, for example, [K]). We denote by  $A^t$  the transpose of  $A$ . Now  $M$  is isomorphic to

$$(\mathbb{Z}[X, X^{-1}])' / (AX - \varepsilon A^t)$$

and, with this presentation of  $M$ , the form  $B$  corresponds to  $(1-X)(AX - \varepsilon A^t)^{-1}$ .

As  $-A$  is a Seifert matrix for  $-\Sigma$ , it follows from the above that  $(M, -B)$  is the Blanchfield pairing of  $-\Sigma$ . Now the isomorphism class of  $(M, B)$  is an invariant of the isotopy class of  $\Sigma$ . Hence the  $(-1)$ -amphicheirality of  $\Sigma$  yields a  $\mathbb{Z}[X, X^{-1}]$ -automorphism  $F$  of  $M$  such that  $B(F(\alpha), F(\beta)) = -B(\alpha, \beta)$  for all  $\alpha$  and  $\beta$  in  $M$ .

Let  $A_0$  be any non-degenerate Seifert matrix in the  $S$ -equivalence class of  $A$  (cf. [T]). Then  $\Delta(X) = \det(A_0 X - \varepsilon A_0^t)$  is independent of the choice of  $A_0$ , and  $\Delta(0) = \det(A_0) \neq 0$ .

By assumption,  $\Delta = \gamma^{2l+1} \cdot \mu$ , with coprime  $\gamma$  and  $\mu$ . Let us define:

$$M_\gamma = \mu(X)M \subset \text{Ker } \gamma(X)^{2l+1} \quad \text{and} \quad M_\mu = \gamma^{2l+1}(X)M \subset \text{Ker } \mu(X).$$

Clearly  $M_\gamma \cap M_\mu = 0$ . Since  $\mu$  and  $\gamma$  are both reciprocal, the Blanchfield pairing  $B$  splits orthogonally on  $M_\gamma \oplus M_\mu$ . Moreover, the index of  $M_\gamma \oplus M_\mu$  in  $M$  is finite; therefore the restriction,  $B_\gamma$ , of  $B$  to  $M_\gamma$  is non-degenerate. Furthermore the restriction,  $F_\gamma$ , of  $F$  to  $M_\gamma$  yields an isomorphism from  $(M_\gamma, B_\gamma)$  to  $(M_\gamma, -B_\gamma)$ .

We now define:

$$M^i = \{\alpha \in M_\gamma \mid \gamma^i(X)\alpha = 0\}$$

$$H^i = M^i / (M^{i-1} + \gamma(X)M^{i+1}).$$

Put  $R = \mathbb{Z}[X, X^{-1}]/(\gamma)$ . Then  $H^i$  is an  $R$ -module of finite rank  $e_i$  (say) and the  $\mathbb{Q}[X, X^{-1}]$ -module  $M_\gamma \otimes_{\mathbb{Z}} \mathbb{Q}$  is isomorphic to:

$$\bigoplus_{i=1}^{\infty} (\mathbb{Q}[X, X^{-1}]/(\gamma)^i)^{e_i}.$$

As  $\sum_{i=1}^{\infty} i \cdot e_i = 2l + 1$ , there is only a finite number of non-zero  $e_i$ ; and one of them, say  $e_{i_0}$ , must be odd. We write:  $n = e_{i_0}$ ,  $H = H^{i_0}$ , and denote by  $[\alpha]$  the class in  $H$  of an element  $\alpha \in M^{i_0}$ . One can define a non-degenerate,  $\varepsilon$ -hermitian form  $b: H \times H \rightarrow \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}]$  by setting, for any  $\alpha$  and  $\beta$  in  $M^{i_0}$ :

$$b([\alpha], [\beta]) = B_\gamma(\gamma^{i_0-1}(X)\alpha, \beta).$$

That the form  $b$  is well-defined is proved in [Mil], where it is also shown that  $b$  is non-degenerate provided  $B_\gamma$  is.

As  $\gamma(X)b(\alpha, \beta)$  is in  $\mathbb{Z}[X, X^{-1}]$  for all  $\alpha$  and  $\beta$  in  $H$ , it follows that  $b(\alpha, \beta) = P(X)/\gamma(X)$ , where  $P(X)$  is some polynomial in  $\mathbb{Z}[X, X^{-1}]$ . Setting  $b'(\alpha, \beta) = P(X)$  defines a non-degenerate  $\varepsilon$ -hermitian form  $b': H \times H \rightarrow R$ , and  $F_\gamma$  induces an  $R$ -isomorphism from  $(H, b')$  to  $(H, -b')$ . Since  $H$  is of rank  $n$  over  $R$ , we see that  $\Lambda^n H$ , the  $n$ -th exterior power of  $H$ , can be identified with an  $R$ -ideal  $I$ . In [B] (§1, no 9, p. 31) the  $n$ -th exterior power of  $b'$  is defined, and it is shown that  $\Lambda^n b'$  is non-degenerate provided  $b'$  is. Let  $f$  be the isomorphism from  $(I, \Lambda^n b')$  to  $(I, \Lambda^n (-b'))$  which is induced by  $F_\gamma$ . We write  $R_I$  for the ring of coefficients of the  $R$ -ideal  $I$ , i.e.  $R_I = \{\alpha \in K \mid \alpha I \subset I\}$ . We recall that  $a = \gamma(0)$ ; so  $R \subset \mathcal{O}_K[1/a]$ .

LEMMA 1.  $R_I \subset \mathcal{O}_K[1/a]$ .

*Proof.* Put  $S = \mathcal{O}_K[1/a]$  and  $J = I \cdot S$ . Clearly the ring of coefficients,  $S_J$ , of  $J$  contains  $R_I$ . Hence it is enough to show that  $S_J \subset S$ . But  $S$  is a Dedekind ring; hence the ring of coefficients of any non-zero  $S$ -ideal is  $S$  itself.  $\square$

As  $f$  is an  $R$ -automorphism of  $I$ , there exists  $u$  in  $R_I$ , hence in  $\mathcal{O}_K[1/a]$  (by the lemma), such that  $f(\alpha) = u\alpha$  for all  $\alpha$  in  $I$ . Now  $n$  is odd, hence  $\Lambda^n (-b') = -\Lambda^n b'$ .

Let us take  $\alpha$  and  $\beta$  in  $I$ , both non-zero; then  $(\Lambda^n b')(\alpha, \beta) \neq 0$ ; so the relations

$$(\Lambda^n b')(f(\alpha), f(\beta)) = u\bar{u}(\Lambda^n b')(\alpha, \beta) = -(\Lambda^n b')(\alpha, \beta)$$

imply  $u\bar{u} = -1$ . This completes the proof of Theorem 1.  $\square$

*Proof of Proposition 1.* Suppose  $\gamma$  has property  $P_2$ . Under the assumptions of Proposition 1, we show that  $\gamma$  also has property  $P_3$ . Let  $M = \mathbb{Z}[X, X^{-1}]/(\gamma)$ ; we define a unimodular hermitian form  $B: M \times M \rightarrow \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}]$  by setting  $B(\alpha, \beta) = \alpha\bar{\beta}/\gamma(X)$  for any  $\alpha$  and  $\beta$  in  $M$ .

As  $\gamma$  has property  $P_2$  and  $\mathcal{O}_K[1/\gamma(0)] = \mathbb{Z}[X, X^{-1}]/(\gamma)$ , multiplication by an element  $u$  in  $\mathcal{O}_K[1/\gamma(0)]$  such that  $u\bar{u} = -1$  yields an isomorphism from  $(M, B)$  to  $(M, -B)$ . Now the form  $(M, B)$  is always the Blanchfield pairing of some  $(2q-1)$ -knot, provided we choose  $q$  odd (see Theorem 12.1 in [L<sub>3</sub>]). Let  $A$  be a non-degenerate Seifert matrix associated with such a knot. Assuming  $|\gamma(0)|$  is a prime number  $p$ , Trotter [T] (Corollary 4.7, p. 196) shows that  $(M, B)$  is isomorphic to  $(M, -B)$  if and only if  $A$  is isomorphic to  $-A$ . (A word of caution: Trotter calls Seifert form what is usually called Blanchfield pairing, as here.)

On the other hand, since  $\gamma$  is irreducible, the isomorphism between  $A$  and  $-A$  implies the existence of  $u$  in  $\mathcal{O}_K$  such that  $u\bar{u} = -1$  (cf. [Mic]). This completes the proof of Proposition 1.  $\square$

*Proof of Proposition 2.* We begin by showing that Proposition 2 can be deduced from the following lemma:

**LEMMA 2.** *Let  $F$  be a number field,  $\mathcal{O}_F$  its ring of algebraic integers and  $a \in \mathbb{N}^*$ . Suppose there exists a Galois automorphism  $\sigma: F \rightarrow F$  such that  $\sigma^2 = \text{id}$  ( $\sigma$  is an involution of  $F$ ), and an element  $\alpha$  in  $\mathcal{O}_F[1/a]$  such that  $\alpha \cdot \sigma(\alpha) = -1$ . If, for some odd integer  $\lambda \in \mathbb{N}^*$ , every prime ideal  $\mathfrak{p} \subset \mathcal{O}_F$  dividing  $a$  and distinct from  $\sigma(\mathfrak{p})$  is such that  $\mathfrak{p}^\lambda$  is principal, then there exists  $\eta$  in  $\mathcal{O}_F$  such that  $\eta\sigma(\eta) = -1$ .*

*Lemma 2 implies Proposition 2:*

We recall that  $p = |\gamma(0)|$  is prime. Consider the following polynomial:

$$\varphi(X) = \gamma(1)X^d \gamma\left(1 - \frac{1}{X}\right) = \gamma(1)(\gamma(1)X^d + \cdots + (-1)^d \gamma(0)),$$

where  $d$  is the degree of  $\gamma$ . As  $\gamma(1) = \pm 1$ , the polynomial  $\varphi$  is monic, and  $\varphi(0) = \pm p$ .

Let  $\xi_1, \dots, \xi_d$  be the roots of  $\varphi$ . Since  $\varphi$  is irreducible, they are all distinct. Moreover,  $K$  is Galois; hence they all lie in  $K$ . For every  $i$ , the ideal  $\mathfrak{p}_i = (\xi_i)$  is

prime (with degree one), since  $N_{K/\mathbb{Q}}(\xi_i) = \pm p$ . By construction it is principal and

$$(p) = \prod_{i=1}^d \mathfrak{p}_i.$$

(We do not claim that the  $\mathfrak{p}_i$  are all distinct!) Therefore all prime ideals dividing  $p$  are principal. Thus Proposition 2 is a consequence of Lemma 2 (with  $F = K$ ,  $\sigma(\alpha) = \bar{\alpha}$ ,  $a = p$  and  $\lambda = 1$ ).

*Proof of Lemma 2.* Suppose  $\alpha \cdot \sigma(\alpha) = -1$  for some  $\alpha$  in  $\mathcal{O}_F[1/a]$ . We may write the fractional ideal  $(\alpha)$  as a product of prime ideals:

$$(\alpha) = \prod \mathfrak{p}^{v_{\mathfrak{p}}(\alpha)} \quad (2.1)$$

Since  $\alpha\sigma(\alpha) = -1$ , we have:

$$v_{\mathfrak{p}}(\alpha) + v_{\sigma(\mathfrak{p})}(\alpha) = 0 \quad \forall \mathfrak{p}. \quad (2.2)$$

If  $v_{\mathfrak{p}}(\alpha) \neq 0$ , it follows from (2.2) that either  $v_{\mathfrak{p}}(\alpha)$  or  $v_{\sigma(\mathfrak{p})}(\alpha)$  is negative; hence  $\mathfrak{p}$  divides  $a$ . The relation (2.2) shows also that  $v_{\mathfrak{p}}(\alpha) = 0$  if  $\mathfrak{p} = \sigma(\mathfrak{p})$ .

Let us now consider the prime ideals  $\mathfrak{p}_i \neq \sigma(\mathfrak{p}_i)$  which divide  $a$ . By assumption, we may write  $\mathfrak{p}_i^{\lambda} = (\pi_i)$  for some  $\pi_i \in F$ . Then the relations (2.1) and (2.2) imply:

$$\alpha^{\lambda} = \eta \prod_i \left( \frac{\sigma(\pi_i)}{\pi_i} \right)^{\mu_i}, \quad (2.3)$$

with  $\mu_i \in \mathbb{Z}$  and  $\eta$  a unit in  $\mathcal{O}_F$ . We see that  $\alpha^{\lambda} \sigma(\alpha)^{\lambda} = \eta \cdot \sigma(\eta)$ . Now  $\lambda$  is *odd*; hence  $\eta \cdot \sigma(\eta) = -1$ . This completes the proof of Lemma 2.  $\square$

### §3. Knot cobordism classes of order two

*Proof of the first assertion of Theorem 2.* Let  $\Sigma$  be a  $(2q-1)$ -knot whose Alexander polynomial  $\gamma$  is irreducible. Put  $\varepsilon = (-1)^q$ . We recall some definitions and basic facts about algebraic cobordism (for more details see [K]).

**DEFINITION.** An  $n \times n$  integral matrix  $B$  represents an  $\varepsilon$ -form if the matrix  $B + \varepsilon B^t$  is invertible over  $\mathbb{Z}$ .

If  $A$  is a Seifert matrix associated with  $\Sigma$ , then  $A + \varepsilon A^t$  is the matrix of the

intersection form on a Seifert surface of  $\Sigma$ . Since  $\Sigma$  is a sphere, this intersection form is unimodular [K]. Hence  $A$  represents an  $\varepsilon$ -form.

**DEFINITION.** An  $\varepsilon$ -form is *null-cobordant* if it is represented by a matrix of the form  $\begin{pmatrix} 0 & A_1 \\ A_2 & A_3 \end{pmatrix}$ , where the  $A_i$  are all square integral matrices.

Let  $C^\varepsilon(\mathbb{Z})$  be the group of cobordism classes of  $\varepsilon$ -forms. On tensoring with  $\mathbb{Q}$ , we obtain an injective map from  $C^\varepsilon(\mathbb{Z})$  to the group of cobordism classes of rational  $\varepsilon$ -forms, say,  $C^\varepsilon(\mathbb{Q})$ .

The first assertion of Theorem 2 can therefore be stated as follows: *Given a Seifert matrix  $A$  of  $\Sigma$ , then  $A \oplus A$  is null-cobordant if and only if  $\gamma$  has property  $P_1$ .* This fact can be deduced from Levine's description of  $C^\varepsilon(\mathbb{Q})$  [L<sub>2</sub>] or from Stoltzfus's computation of  $C^\varepsilon(\mathbb{Z})$  [St], but we shall give here a direct and elementary proof.

As in §1,  $K$  is the number field  $\mathbb{Q}[X]/(\gamma)$ . Let  $H^\varepsilon(K)$  be the Witt group of non-degenerate  $\varepsilon$ -hermitian forms  $B: M \times M \rightarrow K$ , where  $M$  runs through the finite-dimensional vector spaces over  $K$ .

**LEMMA 3.** *Suppose  $M$  is a one-dimensional vector space over  $K$ . Then the class of  $B$  in  $H^\varepsilon(K)$  has order two if and only if  $\gamma$  has property  $P_1$ .*

*Proof.* If  $\gamma$  has property  $P_1$ , then  $\alpha\bar{\alpha} = -1$  for some  $\alpha$  in  $K$ . Multiplication by  $\alpha$  yields an automorphism of  $M$  that carries  $B$  into  $-B$ . Thus  $B \oplus B$  is isomorphic to  $B \oplus (-B)$ ; therefore its Witt class is zero.

As  $B$  has rank one, if the Witt class of  $B \oplus B$  is zero, this form is represented by a matrix of the form  $\begin{pmatrix} 0 & \beta_1 \\ \varepsilon\bar{\beta}_1 & \beta_2 \end{pmatrix}$  with  $\beta_i \in K$ . If  $\beta \in K^*$  is the determinant of  $B$ , then:

$$\beta = \varepsilon\bar{\beta}. \quad (3.1)$$

As the determinant is defined up to an element of  $K^*$  of the form  $\eta \cdot \bar{\eta}$ , we obtain the relation:

$$\det(B \oplus B) = \beta^2 = -\varepsilon\beta_1\bar{\beta}_1\eta\bar{\eta}. \quad (3.2)$$

If we write  $\alpha = \beta^{-1}\beta_1\eta$ , the relations (3.1) and (3.2) show that  $\alpha \cdot \bar{\alpha} = -1$ . This completes the proof of Lemma 3.  $\square$

In the  $S$ -equivalence class of Seifert matrices corresponding to  $\Sigma$ , we can choose one which is non-degenerate [T]. Call it  $A$ . Then the rank of  $A$  is equal to the degree,  $d$ , of  $\gamma$ . Let  $M$  be a  $d$ -dimensional vector space over  $\mathbb{Q}$ . The matrix  $T = -\varepsilon A^{-1} A^t$  represents an automorphism of  $M$ . Put  $X \cdot \alpha = T(\alpha)$ . This action of  $\mathbb{Q}[X]$  induces on  $M$  the structure of a one-dimensional  $K$ -vector space. There exists an  $\varepsilon$ -hermitian form  $B : M \times M \rightarrow K$  such that the relation:

$$(a\alpha)^t(A + \varepsilon A^t)(\beta) = \text{trace}_{K/\mathbb{Q}} aB(\alpha, \beta) \quad (3.3)$$

is satisfied for all  $a$  in  $K$  and  $\alpha, \beta$  in  $M$  (see [Mil]). Now, using (3.3), a direct computation shows that  $A \oplus A$  is null-cobordant if and only if the Witt class of  $B$  has order two. By Lemma 3 this completes the proof of assertion (1).

*Proof of the second assertion of Theorem 2.* Suppose  $\Sigma$  is cobordant to  $\Sigma'$ . Then the Fox–Milnor relation shows that the Alexander polynomial of  $\Sigma'$  is of the form  $\delta \cdot \delta^* \cdot \gamma$  for some integral polynomial  $\delta$  (for a proof see [L<sub>1</sub>], p. 237). If, moreover,  $\Sigma$  is  $(-1)$ -amphicheiral, it follows from Theorem 1 that  $\gamma$  has property  $P_2$ . This completes the proof of Theorem 2.  $\square$

#### §4. Explicit examples

In this section, which is purely number-theoretical, we show that there exist infinitely many irreducible Alexander polynomials of low degree having property  $P_1$  but not  $P_2$ .

##### I. The quadratic case

**PROPOSITION 3.** *Let  $p$  be an odd prime,  $D$  the square-free part of  $4p+1$ , and*

$$\gamma(X) = -pX^2 + (2p+1)X - p.$$

*Then  $\gamma$  is irreducible. Moreover:*

- (1)  $\gamma$  has property  $P_1$  if and only if all prime factors of  $D$  are congruent to 1 modulo 4;
- (2)  $\gamma$  fails to have property  $P_2$  if and only if the fundamental unit of  $\mathbb{Q}(\sqrt{D})$  has norm  $+1$ .

*Proof.* The discriminant  $4p+1$  of  $\gamma$  is not a square, since it is congruent to 5 modulo 8. Hence  $\gamma$  is irreducible.

(1) As  $K = \mathbb{Q}(\sqrt{D})$ , it is clear that  $P_1$  holds if and only if the equation

$$x^2 - Dy^2 = -1 \quad (4.1)$$

can be solved with  $x, y \in \mathbb{Q}$ . A local calculation and the Hasse–Minkowski theorem show that this is the case if and only if all prime factors of  $D$  are congruent to 1 modulo 4. (In fact this is a well-known result on sums of two squares.)

(2) This is an immediate consequence of Proposition 2, since  $|\gamma(0)| = p$  and  $K/\mathbb{Q}$  is Galois.  $\square$

**EXAMPLES.** As is well-known, the fundamental unit of  $\mathbb{Q}(\sqrt{D})$  has norm  $+1$  if and only if the period of the continued fraction expansion of  $\sqrt{D}$  is even.

There is a very efficient algorithm for determining that period (see [Si], p. 296; and [P], §26, pp. 102–103, for a useful refinement). In point of fact the fundamental unit itself is detected by this procedure, which involves a computer calculation whose only difficulty is the number of digits to be handled (for  $D = 991$ , already thirty digits are required!). The two smallest examples<sup>(1)</sup> illustrating Proposition 3 are:

$$p = 367; \quad D = 13 \cdot 113; \quad \eta = 56 + 3\delta$$

$$p = 379; \quad D = 37 \cdot 41; \quad \eta = 19 + \delta$$

(We denote by  $\eta$  the fundamental unit of  $\mathbb{Q}(\sqrt{D})$ , and  $\delta = (1 + \sqrt{D})/2$ .)

*Remark.* In these examples,  $D$  is never a prime. This follows from an elementary result, which will be used again later:

**LEMMA 4.** *Suppose  $D$  is a prime congruent to 1 modulo 4. Then equation (4.1) can be solved with  $x, y \in \mathbb{Z}$ . Hence the fundamental unit of  $\mathbb{Q}(\sqrt{D})$  has norm  $-1$ .*

A proof can be found in [Mo], Chap. 8. The idea is to start from the fundamental solution of the Pell equation  $t^2 - Du^2 = 1$ . The assumptions on  $D$  enable one to write

$$\frac{t-1}{2} = x^2 \quad \text{and} \quad \frac{t+1}{2} = Dy^2,$$

with  $x, y \in \mathbb{Z}$ . Then  $(x, y)$  is a solution of (4.1).

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<sup>(1)</sup> A complete list with  $p \leq 50,000$  is available on request.

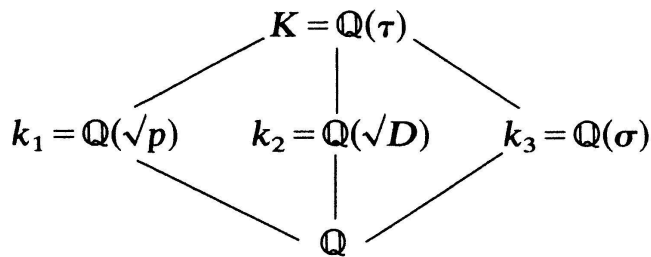


## II. Proof of Theorem 3

Let  $\tau$  be any root of the polynomial

$$\gamma(X) = -pX^4 + (2p+1)X^2 - p.$$

As the other roots are  $-\tau$  and  $\pm 1/\tau$ , we see that  $K = \mathbb{Q}(\tau)$  is a Galois extension of  $\mathbb{Q}$ . Moreover,  $K$  contains  $\mathbb{Q}(\tau^2) = \mathbb{Q}(\sqrt{D})$ , where as above we denote by  $D$  the square-free part of  $4p+1$ . Since  $p$  is odd,  $4p+1$  is congruent to 5 modulo 8, hence  $D \neq 1$ . the fixed field of the involution  $\tau \mapsto 1/\tau$  is the field  $\mathbb{Q}(\sigma)$ , with  $\sigma = \tau + 1/\tau = \sqrt{(4p+1)/p}$ . From this we see that  $K/\mathbb{Q}$  is an extension of degree 4 (whence  $\gamma$  is irreducible), with Galois group  $\mathbb{Z}/2 \times \mathbb{Z}/2$ . Therefore  $K$  contains three quadratic subfields:



It is useful to observe that the involution  $\tau \mapsto 1/\tau$  induces the ordinary conjugation on  $k_1$  and on  $k_2$ :  $\sqrt{p} \mapsto -\sqrt{p}$ , resp.  $\sqrt{D} \mapsto -\sqrt{D}$ .

(1) We wish to prove that  $\gamma$  has property  $P_1$ . Now an element  $\alpha \in K$  can be written in the form  $\alpha = x + y\sqrt{p}$  with  $x, y \in \mathbb{Q}(\sigma)$ . Therefore we have to show that the equation

$$x^2 - py^2 = -1 \tag{4.2}$$

can be solved with  $x, y \in \mathbb{Q}(\sigma)$ . Equivalently, we are reduced to showing that the homogeneous equation

$$x^2 - py^2 + z^2 = 0 \tag{4.3}$$

has a non-trivial solution in  $\mathbb{Q}(\sigma)$ .

By the Hasse–Minkowski theorem for the number field  $\mathbb{Q}(\sigma)$  (see for example [C–F], ex. 4.8, p. 359), it will suffice to show that (4.3) can be solved non-trivially in all completions of  $\mathbb{Q}(\sigma)$ . Since the quadratic form in (4.3) is defined over  $\mathbb{Q}$ , it is indefinite for each of the two real embeddings of  $\mathbb{Q}(\sigma)$ . Therefore it suffices to consider the non-archimedean valuations.

If  $p \equiv 1 \pmod{4}$ , we know that  $p$  is a sum of two squares; hence (4.3) is

already solvable over  $\mathbb{Q}$ . (By Lemma 4 we know that (4.2) is even solvable over  $\mathbb{Z}$ .) Thus we may assume without loss of generality that  $p \equiv 3 \pmod{4}$ . Then  $pD \equiv 3 \pmod{4}$ ; hence the ideal (2) ramifies in  $\mathbb{Q}(\sigma) = \mathbb{Q}(\sqrt{pD})$ . *Therefore it is enough to prove that (4.3) has a non-trivial solution in all non-archimedean completions of  $\mathbb{Q}(\sigma)$  whose residue field is of characteristic  $\neq 2$ .* Indeed, by the product formula ([C-F], ex. 4.5, p. 358), the number of places where a quadratic form in three variables does not represent zero is even; but there is only one prime ideal above (2).

By a well-known result (a special case of the Chevalley–Warning theorem), (4.3) has non-trivial solutions in every finite field. By Hensel’s lemma (cf. [C-F], p. 83), these solutions can be lifted over the corresponding completions, provided the characteristic of the residue field is not equal to 2 or  $p$ . In addition, the ideal  $(p)$  ramifies in  $\mathbb{Q}(\sigma) = \mathbb{Q}(\sqrt{pD})$ ; therefore all we have to show is that (4.3) can be solved  $\mathfrak{p}$ -adically, where  $\mathfrak{p}$  denotes the unique ideal above  $(p)$ .

Since  $(p) = \mathfrak{p}^2$ , locally we can write  $p = \pi^2 \eta$ , where  $\pi$  is a uniformizing element and  $\eta$  a  $\mathfrak{p}$ -adic unit. Now, if we write  $Y = \pi y$ , we are reduced to showing that

$$x^2 - \eta Y^2 + z^2 = 0$$

has a non-trivial  $\mathfrak{p}$ -adic solution. Since now  $\eta$  is a unit, the above argument with Hensel’s lemma applies. This completes the proof that  $\gamma$  has property  $P_1$ .

(2) *Let us examine under what conditions  $\gamma$  has property  $P_2$ .* In each quadratic subfield  $k_i$  of  $K$  there is a fundamental unit  $\varepsilon_i$ . Now since the involution  $\tau \mapsto 1/\tau$  acts as the ordinary conjugation on  $k_1$  and  $k_2$ , it is clear that  $\gamma$  has property  $P_3$  (a fortiori  $P_2$ ) if either  $\varepsilon_1$  or  $\varepsilon_2$  has norm  $-1$ . As we saw in Lemma 4,  $\varepsilon_1$  has norm  $-1$  if  $p \equiv 1 \pmod{4}$ ; and only then, since obviously (4.2) has no rational solution for  $p \equiv 3 \pmod{4}$ . This proves one of the implications in the second assertion of Theorem 3. In order to establish the converse, we first note that the two properties  $P_2$  and  $P_3$  are in fact equivalent in our case, as follows from Proposition 2. Therefore we are reduced to proving the following lemma:

**LEMMA 5.** *Suppose  $\varepsilon_1$  and  $\varepsilon_2$  have norm  $+1$ . Then  $\gamma$  fails to have property  $P_3$ .*

*Proof.* The general theory of units in biquadratic fields is fairly well understood (cf. [Kur], [Kub], [N]); but the special shape of the polynomial  $\gamma$  yields some further information, which will be needed. Let  $U_K$  be the group of units in the ring of integers  $\mathcal{O}_K$ . As  $K$  is totally real, we choose once for all a real embedding and denote by  $U_K^+$  the free  $\mathbb{Z}$ -module of rank 3 consisting of all positive units. Correspondingly, we agree that  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_3$  are elements of  $U_K^+$ .

(a) A classical argument [Kur] shows that the sub- $\mathbb{Z}$ -module  $R \subset U_K^+$  generated by  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_3$  is of finite index in  $U_K^+$ . For if  $\eta \in U_K$  is any unit then  $\eta^2 \in R$ . Indeed, let  $\eta'$  be the conjugate of  $\eta$  above  $k_1$ . Then

$$N_{K/k_1}(\eta) = \eta\eta' = \pm\varepsilon_1^a,$$

and similarly:

$$\eta\bar{\eta}' = \pm\varepsilon_2^b, \quad \eta\bar{\eta} = \pm\varepsilon_3^c \quad (a, b, c \in \mathbb{Z}).$$

Hence  $\eta^2 = \pm \eta^2(\eta\eta'\bar{\eta}\bar{\eta}') = \varepsilon_1^a \varepsilon_2^b \varepsilon_3^c \in R$ .

*Remark.* This argument shows that the index  $J = [U_K^+ : R]$  is in fact a divisor of 8. Kuroda [Kur] has shown that, in the general case, there are seven essentially distinct possibilities and that every divisor of 8 can occur. In our present case, however,  $J$  is always equal to 2, since we prove below that  $U_K^+$  is generated by  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\sqrt{\varepsilon_3}$ .

(b) Suppose now  $\eta\bar{\eta} = \pm 1$  for some unit  $\eta$ ; then  $\eta^2$  belongs to the submodule  $R' \subset R$  which is generated by  $\varepsilon_1$  and  $\varepsilon_2$ . Indeed, we have just seen that  $\eta^2 = \varepsilon_1^a \varepsilon_2^b \varepsilon_3^c$ ; in addition  $\varepsilon_3 = \bar{\varepsilon}_3$ . From the assumption  $\eta\bar{\eta} = \pm 1$  we therefore get:  $1 = \eta^2 \bar{\eta}^2 = \varepsilon_3^{2c}$ , which is possible only if  $c = 0$ .

(c) Suppose  $\gamma$  has property  $P_3$ , i.e. there exists a unit  $\eta \in U_K^+$  such that  $\eta\bar{\eta} = -1$ . Then  $\eta \notin R'$ , since by assumption  $\varepsilon_1 \bar{\varepsilon}_1 = \varepsilon_2 \bar{\varepsilon}_2 = +1$ . Further we know, by (b) above, that  $\eta^2$  is of the form  $\varepsilon_1^a \varepsilon_2^b$  with  $a, b \in \mathbb{Z}$ . Since  $\eta \notin R'$ , we see that  $a$  and  $b$  are not both even. This implies that at least one of the numbers  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_1 \varepsilon_2$  is a square in  $K$ . Therefore the lemma will be proved once we show that none of the numbers  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\varepsilon_1 \varepsilon_2$  is a square in  $K$ .

(d) We consider first  $\varepsilon_2$ . If it were a square in  $K$ , there would exist  $\alpha, \beta \in k_2$  such that  $\varepsilon_2 = (\alpha + \beta\sqrt{p})^2 = (\alpha^2 + p\beta^2) + 2\alpha\beta\sqrt{p}$ . Of course  $\beta \neq 0$ , since  $\varepsilon_2$  is not a square in  $k_2$ . But the coefficient of  $\sqrt{p}$  must vanish, hence  $\alpha = 0$ . Thus we get:  $\varepsilon_2 = p\beta^2$ , with  $\beta \in k_2$ . This is impossible, since  $p$  does not ramify in  $k_2 = \mathbb{Q}(\sqrt{4p+1})$ . (One can also proceed as in (g) below:  $\beta$  is in fact an element of  $\mathcal{O}_{k_2}$ , and  $p$  does not divide the unit  $\varepsilon_2$ .) We have shown that  $\varepsilon_2$  is not a square in  $K$ .

(e) Let us examine  $\varepsilon_1$ . As  $p \equiv 3 \pmod{4}$ , we can write  $\varepsilon_1 = a_1 + b_1\sqrt{p}$  with  $a_1, b_1 \in \mathbb{Z}$ . We claim that  $b_1$  is odd. To see that, it suffices to repeat the argument by which one proves Lemma 4: if  $b_1$  were even, the equality  $a_1^2 - pb_1^2 = 1$  would imply

$$uv = p \left( \frac{b_1}{2} \right)^2,$$

where

$$u = \frac{|a_1| - 1}{2} \quad \text{and} \quad v = \frac{|a_1| + 1}{2}$$

are coprime integers. Since  $p$  is a prime, either  $u$  or  $v$  is a square. In either case we get a contradiction: if  $u = s^2$  and  $v = pt^2$ , then  $s^2 - pt^2 = u - v = -1$ ; hence  $s + t\sqrt{p} \in k_1$  would be a unit of norm  $-1$ . If  $v = s^2$  and  $u = pt^2$ , then  $s^2 - pt^2 = v - u = 1$ , and  $1 < |s| < |a_1|$ . This is impossible, for the fundamental unit  $\varepsilon_1$  corresponds to a solution of the Pell equation  $s^2 - pt^2 = 1$  for which  $|s| > 1$  is minimal.

(f) We put  $\rho = \sqrt{p}$ ,  $\delta = (1 + \sqrt{D})/2$ . As  $D \equiv 1 \pmod{4}$ , one checks easily that  $\mathcal{O}_K$  is the free  $\mathbb{Z}$ -module with basis  $\{1, \rho, \delta, \rho\delta\}$ . (This follows also from [L], chap. 3, §3, prop. 17.) Thus any element  $\xi \in \mathcal{O}_K$  can be written in the form  $\xi = \alpha + \beta\rho$  with  $\alpha, \beta \in \mathcal{O}_{k_2}$ . Then  $\xi^2$  takes the form  $(\alpha^2 + p\beta^2) + 2\alpha\beta\rho$ . On writing  $\alpha\beta = a + b\delta$  with  $a, b \in \mathbb{Z}$ , we reach the following conclusion: *when  $\xi^2$  is expressed in the  $\mathbb{Z}$ -base  $\{1, \rho, \delta, \rho\delta\}$ , the coefficients of  $\rho$  and  $\rho\delta$  are even.*

(g) Putting (e) and (f) together, it is immediate that  $\varepsilon_1 = a_1 + b_1\rho$  is not a square in  $K$ , since  $b_1$  is odd. Finally, let us write  $\varepsilon_2 = a_2 + b_2\delta$ , with  $a_2, b_2 \in \mathbb{Z}$ ; the coefficients of  $\rho$  and  $\rho\delta$  in the product  $\varepsilon_1\varepsilon_2$  are then respectively  $b_1a_2$  and  $b_1b_2$ . If  $\varepsilon_1\varepsilon_2$  were a square in  $K$ , these integers would have to be even. But  $b_1$  is odd; hence both  $a_2$  and  $b_2$  should be even. This is clearly not the case, since  $\varepsilon_2$  is not divisible by 2. This shows that  $\varepsilon_1\varepsilon_2$  is not a square in  $K$  and completes the proof of the lemma.  $\square$

*Remark.* In (a) above it is claimed that  $U_K^+$  is generated by  $\varepsilon_1$ ,  $\varepsilon_2$  and  $\sqrt{\varepsilon_3}$ . In view of the results gathered so far, it is enough to prove that  $\varepsilon_2$  is a square. Now the situation we are in is quite exceptional in that the fundamental unit  $\varepsilon_3$  is given by an explicit formula! Indeed let

$$\eta_3 = (8p + 1) + 4\sqrt{p(4p + 1)}. \quad (4.4)$$

It is a simple exercise to show that  $(8p + 1, 4)$  is the fundamental solution of the Pell equation  $x^2 - p(4p + 1)y^2 = 1$ . Hence  $\eta_3 = \varepsilon_3$  if  $4p + 1$  is square-free; otherwise  $\eta_3 = \varepsilon_3^\nu$  for some  $\nu \in \mathbb{N}$ . Moreover,  $\eta_3$  is the square of

$$\sqrt{\eta_3} = 2\sqrt{p} + \sqrt{4p + 1} \in K \quad (4.5)$$

Furthermore,  $\nu$  is necessarily odd, since  $\sqrt{\eta_3}$  does not lie in  $k_3$ . Hence in all cases  $\varepsilon_3$  is a square, and  $J = 2$ .

It is a firmly established tradition that unit computations in a number field culminate in the determination of the class number. As  $J = 2$ , one has the following formula, ([Kub], Satz 5, p. 80):

$$H = \frac{1}{2}h_1h_2h_3, \quad (4.6)$$

in which  $h_i$  (resp.  $H$ ) denotes the class number of the field  $k_1$  (resp.  $K$ ). We see that the product  $h_1 h_2 h_3$  is always even. This is not surprising; indeed, using (4.4) or (4.5), one shows easily that every prime factor of  $D$  is the square of a non-principal ideal of  $k_3$ , and therefore accounts for a factor 2 in  $h_3$ .

*Note.* The proof of Theorem 3 shows that there exist infinitely many polynomials of degree four having the required properties. For the quadratic case we do not know whether the constructed family of polynomials is infinite (but we believe so).

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Université de Genève  
 Section de Mathématiques  
 Case Postale 124  
 CH-1211 Genève 24

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