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Quasilinear elliptic eigenvalue problems

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Summary. The generalized Palais–Smale condition introduced in [26] is applied to obtain multiple solutions of variational eigen-value problems with quasilinear principal part, thereby extending some well-known existence results for semilinear elliptic problems.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $a=(a^{\alpha\beta})_{1\le\alpha,\beta\le n}$ be a uniformly elliptic, symmetric, and bounded matrix function $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^{n\times n}$. For $u\in H_0^{1,2}(\Omega;\mathbb{R}^N)$ define the energy integral

$$E(u)=\frac{1}{2}\int a^{\alpha\beta}(x,u)\,\partial_\alpha u^i\,\partial_\beta u^i\,dx. \tag{1.1}$$

By convention, repeated Greek indices are summed from 1 to n, Latin indices from 1 to N. Unless otherwise stated all integrals are taken over Ω. Let $G:H_0^{1,2}(\Omega;\mathbb{R}^N)\to\mathbb{R}$ be a sufficiently regular function (cp. conditions (3.4)–(3.7)).

In this note we present existence results for the nonlinear eigen-value problem in $H_0^{1,2}(\Omega;\mathbb{R}^N)$:

$$\nabla E(u)=\mu\nabla G(u),\qquad G(u)=1,\qquad \mu\in\mathbb{R}. \tag{1.2}$$

The results obtained generalize results of Amann [1], [2], Ambrosetti [3], Berger [6], Browder [7], Clark [8], Coffman [9], [10], Hempel [12], [13], Pohožaev [20], Rabinowitz [21], [22] and others for semilinear eigenvalue problems. In the above setting such problems correspond to functionals E with coefficients $a=a(x)$ independent of u.

In contrast to this latter situation under the hypotheses made here the functional E need not be differentiable in $H_0^{1,2}$. Thus the classical Palais–Smale

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condition (cp. [19], [24]) cannot hold in this space and the standard Lusternik–Schnirelman theory of critical points (cp. [16], [23]) cannot be applied to obtain solutions of (1.2). Instead the existence results presented will be deduced from the generalization of the Lusternik–Schnirelman method that was introduced in [26].

In the next section this method will be briefly recalled; in particular, the compactness Criterion A* will be restated which replaces the Palais–Smale condition in applications of Lusternik–Schnirelman theory to irregular functionals. Precise formulations and proofs of existence results for problem (1.2) are given in Section 3.

These results are not presented in the most conceivable generality. Instead they have been selected as models to illustrate the application of Criterion A* in a "non-reflexive" situation. Possible extensions and generalizations are mentioned without proof in Section 4.

Remark that we retain a compactness assumption on the term $\nabla G(u)$ in (1.2), cp. condition (3.7). However, even with coefficients a_{ij} independent of u, the Palais–Smale condition may no longer hold true for (1.2) if ∇G is only continuous, i.e. if in assumption (3.7) $p=2n/(n-2)$ is admitted. In this case (and with coefficients independent of u) important progress has recently been made by Brezis and Nirenberg [28], [29], cp. also [30].

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2. The compactness criterion

Throughout this section we shall make the following *Assumption A*:

B is a reflexive Banach space with norm $\|\cdot\|_B$. $T\subset B$ is a dense subspace of B (in $\|\cdot\|_B$) given by $T=\bigcup_{\iota\in I}T_\iota$, where $\{T_\iota\}$ is family of Banach spaces T_ι with norms $\|\cdot\|_{T_\iota}$. (2.1)

$E:B\to\mathbb{R}\{\pm\infty\}$ for any $\iota\in I$ is continuously Fréchet differentiable with respect to T_ι in the following sense: For any $u\in B$ such that $|E(u)|<\infty$, any $\iota\in I$, the derivative $\nabla E(u)\in T_\iota^*$ exists and the mapping $u\to\nabla E(u)\in T_\iota^*$ is continuous on its domain. (2.2)

Set $T'=\{\xi:T\to\mathbb{R}\mid \xi_{|T_\iota}\in T_\iota^* \text{ for all } \iota\in I\}$. By (2.2) it is meaningful to define a critical point of E as an element $u\in B$ such that $|E(u)|<\infty$ and $\nabla E(u)=0\in T'$. The value $E(u)$ at such a critical point will be called a critical value of E.

DEFINITION. A sequence $\{\xi_m\}$ in T' converges to $0 \in T'$ T_ι-uniformly iff for any $\iota \in I$

$$\|\xi_m\|^*_{T_\iota} \to 0 \quad (m \to \infty).$$

As an illustration of this definition consider the extreme cases:

Ex. 1. Let $B = T = T_\iota$. Then for any $\xi \in T' = B^*$, any $\iota : \|\xi\|^*_{T_\iota} = \|\xi\|_{B^*}$, $\|\cdot\|_{B^*}$ denoting the norm in the dual B^*. Thus, T_ι-uniform convergence is equivalent to norm-convergence in B^*.

Ex. 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $B = H_0^{1,2}(\Omega)$, $T = C_0^\infty(\Omega)$, and $T_\iota = \{\lambda_\iota \mid \lambda \in \mathbb{R}\} \cong \mathbb{R}$, $\iota \in T$. In this case, T_ι-uniform convergence is equivalent to weak convergence in the sense of distributions.

In the application given below, a nontrivial choice of B, T, T_ι will be presented.

Let M be a subset of B such that E is finite on M. Assuming (2.1), (2.2) we may then state our

CRITERION A*. *If $\{u_m\}$ is any sequence in M such that $u_m \rightharpoonup u$ weakly in B and $\nabla E(u_m) \to 0$ T_ι-uniformly as $m \to \infty$, then we may extract a subsequence that converges strongly to a critical point of E in M.*

Remark. In Ex. 1 Criterion A* reduces to a variant of the classical Palais–Smale condition. Similarly, imposing a coerciveness condition on E (with respect to B) and assuming M to be regular (with respect to T, $\{T_\iota\}$), from Criterion A* the existence of saddle-type critical points characterized by "mountain-pass" or "minimax" conditions may be derived for functionals which may be irregular in B (cp. [26]).

Note that since we are exhausting the testing space T by a family of Banach spaces $\{T_\iota\}$, the "limit space" T itself may be very badly behaved; in particular, it need not be a reflexive Banach space.

In the applications given in Section 3 (and in those presented in [26]) Criterion A* can be verified by introducing a "necessary constraint", i.e. by restricting E to a set M of admissible functions given by

$$M = \{u \in B \mid \langle \nabla E(u), \Psi(u) \rangle = 0\} \tag{2.3}$$

for some mapping Ψ. Necessary conditions of this kind seem to have been first introduced by Nehari [18] in 1957 as a means to improve properties of badly behaved functions. Working with the constraint $\langle \nabla E(u), u\rangle = 0$ he and other authors were able to derive existence and multiplicity results for superlinear elliptic boundary value problems (cp. [4], [5], [7], [9], [12], [18], [25]). For such problems the functional in variation is neither bounded from above nor from below on the whole space, whereas it was found to be coercive on the set of functions satisfying the above constraint.

3. Quasilinear eigenvalue problems

In this section we apply Criterion A* to prove the existence of multiple solutions for problem (1.2) mentioned in the introduction. The following assumptions (3.1)–(3.7) will be made throughout this section.

The symmetric matrix function $a = (a^{\alpha\beta}) : \Omega \times \mathbb{R}^N \to \mathbb{R}^{n\times n}$, $a^{\alpha\beta} = a^{\beta\alpha}$, is measurable in $x \in \Omega$, differentiable in $u \in \mathbb{R}^N$, and for a.e. x, $\partial_u a^{\alpha\beta}(x, \cdot)$ is uniformly continuous in u, uniformly in x. Moreover, there is a constant c such that $|a|$, $|\partial_u a|$, $|u \cdot \partial_u a(\cdot, u)| \le c$ a.e. in $\Omega \times \mathbb{R}^N$. (3.1)

There exists $\lambda > 0$ such that

$$a^{\alpha\beta}(x, u)\xi^i_\alpha \xi^i_\beta \ge \lambda\, |\xi|^2 \tag{3.2}$$

for a.e. $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$.

Also we shall need the one-sided condition:

There exists a constant $\lambda^* < \lambda$ such that

$$-u \cdot \partial_u a^{\alpha\beta}(\cdot, u)\xi^i_\alpha \xi^i_\beta \le 2\lambda^*\, |\xi|^2 \tag{3.3}$$

for a.e. $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$.

With respect to the function G we suppose:

$G : H_0^{1,2}(\Omega; \mathbb{R}^N) \to \mathbb{R}$[1] is continuous with respect to weak convergence in $H_0^{1,2}$. $G(u) \ge 0$, $G(0) = 0$. (3.4)

[1] $H_0^{1,2}(\Omega; \mathbb{R}^N)$ is the completion of the set of C^∞-functions $u : \Omega \to \mathbb{R}^N$ having compact support in Ω with respect to the norm $\|u\|_{1,2}^2 = \int_\Omega |\nabla u|^2\, dx$. For brevity in the following we simply write $H_0^{1,2}$.

G is continuously Fréchet differentiable in $H_0^{1,2}$ with compact derivative $\nabla G : H_0^{1,2} \to H^{-1} := (H_0^{1,2})^*$. (3.5)

Moreover, we require the non-degeneracy condition:

At any point $u \in H_0^{1,2}$ such that $G(u) > 0$ we have $\langle \nabla G(u), u \rangle > 0$. (3.6)

Finally, we also need a regularity condition:

For any $u \in H_0^{1,2}$ $\nabla G(u) \in H^{-1}$ is represented by a function $g(x, u)$ such that $\langle \nabla G(u), \varphi \rangle = \int g(u)\varphi \, dx$ for all $\varphi \in H_0^{1,2}$, and $g : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the estimate for some $p < 2^* = \dfrac{2n}{n-2}$[2]: (3.7)

$$|g(x, u)| \leq c(1 + |u|^{p-1}) \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N.$$

EXAMPLE 3.1. The assumptions (3.4)–(3.7) are satisfied with $G(u) = \int |u|^p \, dx$, for some $p \in]1, 2^*[$. The assumptions (3.1)–(3.3) are satisfied with $a^{\alpha\beta} = (\lambda + \operatorname{arctg}(|u|^2))\delta_{\alpha\beta}$, where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta, = 0$ else.

We will show that the general assumptions (2.1), (2.2) are satisfied for the functional E given by (1.1), if we let $B = H_0^{1,2}(\Omega; \mathbb{R}^N)$, $T = H_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$, and for any $L \in \mathbb{N}$ let T_L be the set T equipped with the norm $\|\cdot\|_L = \|\cdot\|_{1,2} + L^{-1} \|\cdot\|_\infty$.

LEMMA 3.1. *Assumption A is satisfied with B, T_L, T, and E as above. Moreover, for any $L \in \mathbb{N}$ the function $\nabla E : H_0^{1,2} \to T_L^*$ is uniformly bounded on $H_0^{1,2}$-bounded sets U, and for any such U the mappings $\nabla E(u + \cdot) : T_L \to T_L^*$ are continuous at $0 \in T_L$, uniformly with respect to $u \in U$.*

Proof. (2.1) is trivially verified. Similarly, it is easy to check that E is finite on $H_0^{1,2}$ and differentiable with respect to T_L, for any $L \in \mathbb{N}$. Indeed, for any $u \in H_0^{1,2}$, any $\varphi \in H_0^{1,2} \cap L^\infty$

$$\langle \nabla E(u), \varphi \rangle = \int a^{\alpha\beta}(x, u) \, \partial_\alpha u^i \, \partial_\beta \varphi^i + \tfrac{1}{2}\varphi^j \, \partial_{u^j} a^{\alpha\beta}(x, u) \, \partial_\alpha u^i \, \partial_\beta u^i \, dx, \tag{3.8}$$

and the stated continuity properties may readily be derived from condition (3.1) and standard convergence theorems. By (3.1) and (3.8) also the uniform boundedness on $H_0^{1,2}$-bounded sets of $\nabla E : H_0^{1,2} \to T_L^*$ is immediate. q.e.d.

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[2] $2^* = \infty$, if $n = 2$.

Thus it is meaningful to define a critical point u of E subject to the constraint $G(u)=1$.

DEFINITION 3.1. $u\in H_0^{1,2}$ is called a critical point of E on the set $\{u\in H_0^{1,2} \mid G(u)=1\}$ iff there exists a number $\mu\in\mathbb{R}$ such that

$$\nabla E(u)+\mu\nabla G(u)=0\in T'.$$

If u is a critical point of E the value $E(u)$ is called critical.

Of course, in the semilinear case we may take $T=B=H_0^{1,2}$ and the above definition reduces to the standard definition of a constrained critical point.
Set

$$M=\{u\in H_0^{1,2} \mid G(u)=1\}. \tag{3.9}$$

Now we can formulate our first result:

THEOREM 3.1. *Suppose conditions* (3.1)–(3.7) *are satisfied and that* $M\neq\emptyset$. *Then there exists a constant* $\rho_1>0$ *such that whenever the condition*

$$|\partial_u a|<\rho_1 \quad \text{a.e. in} \quad \Omega\times\mathbb{R}^N$$

is satisfied there exists a solution $(u,\mu)\in H_0^{1,2}\cap L^\infty(\Omega;\mathbb{R}^N)\times\mathbb{R}$ *of problem* (1.2), *characterized by the condition*

$$u\in M: E(u)=\inf_{v\in M} E(v).$$

In the symmetric case a much stronger existence result may be obtained. Assume that

$$a^{\alpha\beta}(x,u)=a^{\alpha\beta}(x,-u) \quad \text{a.e. in} \quad \Omega\times\mathbb{R}^N \tag{3.10}$$

and

$$G(u)=G(-u) \quad \text{for all} \quad u\in H_0^{1,2}. \tag{3.11}$$

Let

$$\Sigma=\{A\subset H_0^{1,2}\backslash\{0\} \mid A \text{ is closed and symmetric}\}$$

and define the Krasnoselskii "genus" $\gamma : \Sigma \to \mathbb{N}_0 \cup \{\infty\}$ on Σ by letting $\gamma(\emptyset) = 0$, and for $A \neq \emptyset$:

$$\gamma(A) = \min \{m \in \mathbb{N} \mid \exists h : A \to \mathbb{R}^m \setminus \{0\},\ h \text{ is continuous, } h(-u) = -h(u)\}$$

if $\{\cdots\} \neq \emptyset$, $\gamma(A) = \infty$ else (cp. [9], [14]). The genus has the following properties (cp. [21, Lemma 1.1]):

PROPOSITION 3.1. *Let* $A, B \in \Sigma$:

i) *If there exists an odd continuous mapping* $h : A \to B$ *then* $\gamma(A) \leq \gamma(B)$.

ii) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$

iii) *If* A *is compact, then* $\gamma(A) < \infty$ *and there exists a neighborhood* N *of* A *in* $H_0^{1,2}$ *such that* $\bar{N} \in \Sigma$ *and* $\gamma(\bar{N}) = \gamma(A)$.

For any $l \in \mathbb{N}$ set

$$\Sigma_l = \{A \in \Sigma \mid A \subset M,\ \gamma(A) \geq l,\ A \text{ is compact}\},$$

and for any $\beta \in \mathbb{R}$ let

$$K_\beta = \{u \in M \mid E(u) = \beta, \exists \mu : \nabla E(u) + \mu \nabla G(u) = 0\}$$

be the set of constrained critical points of E of energy β. We then obtain:

THEOREM 3.2. *Suppose conditions* (3.1)–(3.7), (3.10), (3.11) *are satisfied and that* $\Sigma_l \neq \emptyset$ *for* $l \leq m$. *Then for any* $k \leq m$ *there is a constant* ρ_k *such that if the condition*

$$|\partial_u a| < \rho_k \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N$$

is satisfied there exist solutions $(u_l, \mu_l) \in H_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N) \times \mathbb{R}$, $1 \leq l \leq k$, *of problem* (1.2) *characterized by the generalized minimax-principle*

$$u_l \in M : E(u_l) = \beta_l := \inf_{A \in \Sigma_l} \sup_{u \in A} E(u).$$

If for some numbers $k, l \in \mathbb{N}$ *we have*

$$\beta = \beta_l = \cdots = \beta_{l+k}$$

then

$$\gamma(K_\beta) \geq k + 1$$

and, in particular, K_β must be infinite. Finally, if $\Sigma_l \neq \emptyset$ for all $l \in \mathbb{N}$

$$\beta_l \to \infty \qquad (l \to \infty).$$

EXAMPLE. If the coefficients $a^{\alpha\beta}$ and the function G are given as in Example 3.1 all the assumptions (3.1)–(3.7), (3.10), (3.11) are satisfied, $M \neq \emptyset$ and $\Sigma_l \neq \emptyset$ for all $l \in \mathbb{N}$. Moreover, for a scalar equation ($N = 1$) or a plane system ($n = 2$) the parameters ρ_k may be chosen as $\rho_k = \infty$ (cp. Section 4).

Remarks. For coefficients $a^{\alpha\beta} = a^{\alpha\beta}(x)$ independent of u as a special case of Theorem 3.2 we obtain the well-known existence results cited in the Introduction.

For general elliptic eigenvalue problems of the type

$$-\Delta u + f(x, u, \nabla u) = 0 \tag{3.12}$$

only the results of Browder [7] seem to have been available. However, by using a standard Palais-Smale type condition Browder was forced to impose the "unnatural" growth condition on the free term in (3.12):

$$|f(x, u, \eta)| \leq c(1 + |u|^p + |\eta|^q)$$

with $p < 2^* - 1$ and $q \leq (n+2)/n$ if $n > 2$, resp. $p < \infty$, $q < 2$ if $n = 2$ (cp. [11]). In the case of quadratic growth ($q = 2$) which naturally arises from variational problems like the problems considered here the question of existence of non-minimum critical points seems to have been completely open.

Bounds for the numbers ρ_k in terms of the structure parameters of the system (1.2) can be obtained from the proof of Lemma 3.4 ii) (cp. (3.16)).

To prove Theorems 3.1, 3.2 we first need an additional information on the set of admissible functions.

LEMMA 3.2. i) *The mapping $u \mapsto \langle \nabla E(u), u \rangle$ from $H_0^{1,2} \cap L^\infty$ into $\mathbb{R}$ continuously extends to $u \in H_0^{1,2}$.*

ii) *For any $u \in M$ there exists a unique number $\mu = \mu(u)$ such that $\langle \nabla E(u) + \mu \nabla G(u), u \rangle = 0$.*

iii) *The mapping $u \to \mu(u)$ from M into $\mathbb{R}$ is bounded on ($H_0^{1,2}$–) bounded sets, hence continuous.*

Proof. i) Given any $u \in H_0^{1,2}$ and any sequence $\{\varphi_m\}$, $\varphi_m \in H_0^{1,2} \cap L^\infty$, such that $\varphi_m \to u$ in $H_0^{1,2}$ as $m \to \infty$ we show that $\lim_{m\to\infty} \langle \nabla E(\varphi_m), \varphi_m \rangle$ exists and is independent of the approximating sequence $\{\varphi_m\}$. Indeed,

$$\langle \nabla E(\varphi_m), \varphi_m \rangle = \int a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx.$$

By Lusin's theorem, given any number $\delta > 0$ there exists a set $E_\delta \subset \Omega$ such that meas $(E_\delta) < \delta$ and $\varphi_m \to u$ uniformly on $\Omega \setminus E_\delta$ as $m \to \infty$. Hence also $a(\cdot, \varphi_m) \to a(\cdot, u)$ and $\varphi_m \cdot \partial_u a(\cdot, \varphi_m) \to u \cdot \partial_u a(\cdot, u)$ uniformly on $\Omega \setminus E_\delta$ as $m \to \infty$; and since $\varphi_m \to u$ in $H_0^{1,2}(m \to \infty)$ we obtain

$$\int_{\Omega \setminus E_\delta} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx \to$$

$$\int_{\Omega \setminus E_\delta} a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta u^i + \tfrac{1}{2} u^j\, \partial_{u^j} a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta u^i\, dx \quad (m \to \infty).$$

On the remainder set E_δ by uniform boundedness of a, $u \cdot \partial_u a(\cdot, u)$ and uniform absolute continuity of $\int |\nabla \varphi_m|^2\, dx$ we may estimate

$$\left| \int_{E_\delta} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx \right| \le c(\delta),$$

where $c(\delta) \to 0$ $(\delta \to 0)$, uniformly in m. Hence the statement follows on first letting $m \to \infty$ and then passing to the limit $\delta \to 0$.

ii) By condition (3.6) statement ii) is immediate from i).

iii) To show iii) note that the mapping $u \mapsto \langle \nabla E(u), u \rangle$ is bounded on bounded sets. Moreover, by compactness of ∇G and weak continuity of G from condition (3.6) it follows that $|\langle \nabla G(u), u \rangle|$ is uniformly bounded from below by some positive constant if u ranges in any given bounded subset of M. This proves boundedness. From the boundedness continuity follows by uniqueness, part ii). q.e.d.

In conclusion the set M may equivalently be expressed

$$M = \{u \in H_0^{1,2} \setminus \{0\} \mid G(u) = 1, \exists \mu : \langle \nabla E(u) + \mu \nabla G(u), u \rangle = 0\} \tag{3.13}$$

with an additional constraint reminiscent of (2.3). In order to verify Criterion A* for E on M the following regularity result based on a device of Moser [17] will be needed. For this lemma we also assume the following auxiliary condition which will later be removed again:

$$\exists \nu_0 : u \cdot \partial_u a(\cdot, u) \geq 0 \quad \text{for} \quad |u| \geq \nu_0, \text{ a.e. in } \Omega. \tag{3.14}$$

LEMMA 3.3. *Assume in addition to the general assumptions* (3.1)–(3.7) *that condition* (3.14) *is satisfied for some number* ν_0. *Then, if* $\{u_m\}$ *is a sequence in* M *such that* $\|u_m\|_{1,2} \leq M$ *uniformly in* m, *and such that* $u_m \rightharpoonup u$ *weakly in* $H_0^{1,2}$ *while* $\nabla E(u_m) + \mu_m \nabla G(u_m) \to 0$ T_L*-uniformly as* $m \to \infty$ *with* $\mu_m = \mu(u_m)$, *it follows that* $u \in H_0^{1,2} \cap L^\infty$ *and, moreover,* $\|u\|_\infty \leq c = c(\lambda, \lambda^*, n, N, \Omega, p, M)$ *for some function* c *which is non-decreasing in* M.

Proof. First note that by Lemma 3.2 the numbers $\mu_m = \mu(u_m)$ are uniformly bounded by a constant depending only on the parameters of the system and the number M.

For arbitrary numbers $r \geq 0$, $\nu \geq \nu_0$ define test vectors $\varphi_m^{r,\nu} = u_m |u_m|^{-1} \times \min\{|u_m|, \nu\}^{r+1}$. Note that for any r, ν, and L_0 the sequence $\varphi_m^{r,\nu}$ is uniformly bounded in T_L for any $L \geq L_0$. By T_L-uniform convergence we thus obtain, writing $\varphi_m^{r,\nu} = \varphi_m$ for brevity:

$$\begin{aligned} 0 &= \lim_{m\to\infty} \langle \nabla E(u_m) + \mu_m \nabla G(u_m), \varphi_m \rangle \\ &= \lim_{m\to\infty} \int a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta \varphi_m^i + \tfrac{1}{2} \varphi_m^j\, \partial_{u^j} a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i \\ &\qquad\qquad + \mu_m g^i(x, u_m) \varphi_m^i\, dx. \end{aligned}$$

Let

$$A_m^1 = \int a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta \varphi_m^i + \tfrac{1}{2} \varphi_m^j\, \partial_{u^j} a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, dx$$

$$A_m^2 = \int \mu_m g^i(x, u_m) \varphi_m^i\, dx.$$

By conditions (3.2), (3.3), (3.14) we estimate

$$\begin{aligned} A_m^1 &= \int (a^{\alpha\beta}(x, u_m) + \tfrac{1}{2} u_m \cdot \partial_u a^{\alpha\beta}(x, u_m))\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, |u_m|^{-1} \min\{|u_m|, \nu\}^{r+1}\, dx \\ &\quad + r \int_{\{x \mid |u_m(x)| < \nu\}} a^{\alpha\beta}(x, u_m) u_m^i\, \partial_\alpha u_m^i u_m^j\, \partial_\beta u_m^j\, |u_m|^{r-2}\, dx \\ &\quad - \nu^{r+1} \int_{\{x \mid |u_m(x)| \geq \nu\}} a^{\alpha\beta}(x, u_m) u_m^i\, \partial_\alpha u_m^i u_m^j\, \partial_\beta u_m^j\, |u_m|^{-3}\, dx \end{aligned}$$

$$\geq \int_{\{x \mid |u_m(x)|<\nu\}} (a^{\alpha\beta}(x, u_m)+\tfrac{1}{2}u_m \cdot \partial_u a^{\alpha\beta}(x, u_m))\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, |u_m|^r\, dx$$

$$+\nu^{r+1}\int_{\{x \mid |u_m(x)|\geq\nu\}} a^{\alpha\beta}(x, u_m)\left(\delta_{ij}-\frac{u_m^i u_m^j}{|u_m|^2}\right) \partial_\alpha u_m^i\, \partial_\beta u_m^j\, |u_m|^{-1}\, dx$$

$$\geq(\lambda-\lambda^*)\int_{\{x \mid |u_m(x)|<\nu\}} |\nabla u_m|^2\, |u_m|^r\, dx.$$

Also note that by (3.2) for a.e. $(x, u, \xi)\in \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}$

$$a^{\alpha\beta}(x, u)\left(\delta_{ij}-\frac{u^i u^j}{|u|^2}\right)\xi_\alpha^i\xi_\beta^j\geq 0,$$

as may be verified by transforming the positively semidefinite and symmetric matrices $a^{\alpha\beta}$, $\left(\delta_{ij}-\frac{u^i u^j}{|u|^2}\right)$, resp. into diagonal form through orthogonal transformations of the independent and dependent variables $x\in\mathbb{R}^n$, $u\in\mathbb{R}^N$, resp. Thus, letting $u_m^\nu=\min\{|u_m|, \nu\}$ and noting that by a well-known theorem of Stampacchia $\nabla u_m^\nu=0$ a.e. on $\{x \mid u_m^\nu(x)=\nu\}$ we obtain

$$A_m^1\geq 4(\lambda-\lambda^*)(r+2)^{-2}\,\|(u_m^\nu)^{r+2/2}\|_{1,2}^2.$$

On the other hand

$$A_m^2\leq c\mu_m\int (1+|u_m|^{p-1})(u_m^\nu)^{r+1}\, dx.$$

Passing to the limit $m\to\infty$, from the Sobolev embedding theorem we thus obtain with constants $c=c(\lambda, \lambda^*, n, N, \Omega, p)$

$$\|u^\nu\|_{(r+2)n/(n-2)}^{r+2}\leq c\,\|(u^\nu)^{r+2/2}\|_{1,2}^2\leq c(r+2)^2\lim_{m\to\infty} A_m^1$$

$$\leq c(r+2)^2\lim_{m\to\infty}|A_m^2|\leq c(r+2)^2\int (1+|u|^{p-1})\,|u^\nu|^{r+1}\, dx. \tag{3.15}$$

Choosing $r_0=0$ and letting $\nu\to\infty$ in (3.15) we obtain

$$\|u\|_{2^*}^2\leq c\int 1+|u|^p\, dx\leq c\,\|u\|_p^p.^{(1)}$$

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[1] Note that by condition (3.6) there exists a constant $c>0$ such that $\|u\|_p\geq c$ for $\|u\|_{1,2}\leq M$.

Similarly, letting $r_{i+1}=\dfrac{n}{n-2}(r_i+2)-p$ for $i\in\mathbb{N}$ by induction it results from (3.15) that

$$\begin{aligned}\|u\|_{r_{i+1}+p}&\leq c^{1/(r_i+2)}(r_i+2)^{2/(r_i+2)}\|u\|_{r_i+p}^{(r_i+p)/(r_i+2)}\\&\leq c^{\Sigma_{j\leq i}\sigma_j^i}\prod_{j\leq i} r_j^{2\sigma_j^i}\|u\|_p^{\Pi_{j\leq i}\tau_j},\end{aligned}$$

with

$$\sigma_j^i=\frac{1}{r_j+2}\prod_{j\leq k\leq i}\tau_k;\qquad \tau_k=\frac{r_k+p}{r_k+2}.$$

From the asymptotic behavior $r_{i+1}/r_i\sim n/n-2$ as $i\to\infty$ it is elementary to verify that the products converge. By a standard estimate we thus obtain

$$\|u\|_\infty\leq\limsup_{i\to\infty}\|u\|_{r_i}\leq c\,\|u\|_p^c\leq cM^c=:c(\lambda,\lambda^*,n,N,\Omega,p,M)\quad\text{q.e.d.}$$

For $\beta\in\mathbb{R}$, $L\in\mathbb{N}$ define

$$\begin{aligned}M_\beta&=\{u\in M\mid E(u)<\beta\},\\N_{\beta,L}&=\{u\in M\mid |E(u)-\beta|<L^{-1},\|\nabla E(u)+\mu(u)\nabla G(u)\|_{T_L^*}^2<5L^{-1}\}.\end{aligned}$$

LEMMA 3.4. *i) The functional $E:M\to\mathbb{R}$ is coercive with respect to $\|\cdot\|_{1,2}$: $E(u)\to\infty$ if $\|u\|_{1,2}\to\infty$.*

ii) Assume that condition (3.14) *holds for some $\nu_0\geq 0$. Then for any $\beta\geq 0$ there exists a number $\rho>0$ such that the functional E satisfies Criterion A^* on the set M_β provided that $|\partial_u a|<\rho$ a.e. on $\Omega\times\mathbb{R}^N$.*

iii) Under the assumptions of ii) for any $\beta\geq 0$ there exists a number ρ such that whenever $|\partial_u a|<\rho$ a.e. on $\Omega\times\mathbb{R}^N$ then the following is true:

If $K_\beta=\emptyset$ there exists $L\in\mathbb{N}$ such that $N_{\beta,L}=\emptyset$.

Proof. i) By assumption (3.2) the first statement is immediate.

ii) Let $\{u_m\}$ be a sequence in M_β such that $u_m\rightharpoonup u$ weakly in $H_0^{1,2}$ and $\nabla E\,|_M(u_m)=\nabla E(u_m)+\mu(u_m)\nabla G(u_m)\to 0$ T_L-uniformly as $m\to\infty$. By i) of this proof there exists a constant $M=M(\beta)$ such that $\|u_m\|_{1,2}\leq M$, uniformly in m. Applying Lemma 3.3 we obtain that $u\in H_0^{1,2}\cap L^\infty$ and that $\|u\|_\infty<c(\lambda,\lambda^*,n,N,\Omega,p,M)$. Fix $\rho>0$ such that

$$\|u\|_\infty\,\rho<\lambda-\lambda^*. \tag{3.16}$$

For $m \in \mathbb{N}$ let A_m be the quadratic form $A_m(\varphi)=\frac{1}{2}\int a^{\alpha\beta}(x, u_m)\, \partial_\alpha\varphi^i\, \partial_\beta\varphi^i\, dy$. By convexity of A_m we then obtain for every $m \in \mathbb{N}$:

$$\begin{aligned} A_m(u_m)-A_m(u) &\leq \langle \nabla A_m(u_m), u_m-u\rangle \\ &= \langle \nabla E(u_m)+\mu(u_m)\nabla G(u_m), u_m-u\rangle - \langle \mu(u_m)\nabla G(u_m), u_m-u\rangle \\ &\quad -\frac{1}{2}\int (u_m-u)\, \partial_u a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, dx. \end{aligned} \tag{3.17}$$

By weak convergence $u_m \to u$ $(m \to \infty)$

$$\limsup_{m\to\infty} A_m(u_m-u)=\limsup_{m\to\infty}\,[A_m(u_m)-A_m(u)],$$

and by compactness of ∇G and boundedness of $\mu(u_m)$

$$\langle \mu(u_m)\nabla G(u_m), u_m-u\rangle \to 0 \quad (m\to\infty).$$

By Lusin's theorem for any $\delta>0$ there exists $E_\delta \subset \Omega$, meas $(E_\delta)<\delta$, such that $u_m \to u$ uniformly on $\Omega\backslash E_\delta$. Denoting by $c(\delta)$ any constant such that $c(\delta)\to 0$ $(\delta \to 0)$ we hence obtain

$$\begin{aligned} &\limsup_{m\to\infty} -\frac{1}{2}\int (u_m-u)\, \partial_u a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, dx \\ &= \limsup_{m\to\infty} -\frac{1}{2}\int_{E_\delta} (u_m-u)\, \partial_u a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, dx \\ &= \limsup_{m\to\infty} -\frac{1}{2}\int_{E_\delta} (u_m-u)\, \partial_u a^{\alpha\beta}(x, u_m)\, \partial_\alpha (u_m-u)^i\, \partial_\beta (u_m-u)^i\, dx + c(\delta) \\ &\leq \limsup_{m\to\infty}\, (\lambda^*+\|u\|_\infty\, \rho)\int |\nabla(u_m-u)|^2\, dx + c(\delta). \end{aligned}$$

Letting $\delta\to 0$ and inserting our estimates into (3.17) we thus find that

$$\begin{aligned} \lambda\cdot \limsup_{m\to\infty}\int |\nabla(u_m-u)|^2\, dx &\leq \limsup_{m\to\infty} A_m(u_m-u) \\ &\leq (\lambda^*+\|u\|_\infty\, \rho)\limsup_{m\to\infty}\int |\nabla(u_m-u)|^2\, dx. \end{aligned}$$

By our choice of ρ this implies that $u_m \to u$ strongly in $H_0^{1,2}$ as $m\to\infty$, and hence ii).

iii) Given an arbitrary number $\beta \in \mathbb{R}$ let ρ be chosen such that Criterion A* is satisfied for the functional E on the set $M_{\beta+1}$. Suppose $K_\beta = \emptyset$ and assume by contradiction that $N_{\beta,L} \neq \emptyset$ for all $L \in \mathbb{N}$. Select a "diagonal" sequence $u_m \in N_{\beta,m}$, $m \in \mathbb{N}$. Note that $u_m \in M_{\beta+1}$ for all m, hence by part i) of this proof $\{u_m\}$ is bounded in $H_0^{1,2}$ and by the Banach-Saks theorem we may assume that $u_m \rightharpoonup u$ weakly in $H_0^{1,2}$ as $m \to \infty$. Moreover, by continuous embedding $T_L \hookrightarrow T_N$, $\|u\|_N \leq \|u\|_L$ for all $u \in T$, $L \leq N$, we have that for any $L \in \mathbb{N}$

$$\limsup_{m\to\infty} \|\nabla E\,|_M(u_m)\|_{T_L^*} \leq \limsup_{m\to\infty} \|\nabla E\,|_M(u_m)\|_{T_m^*} = 0.$$

Thus $\nabla E(u_m) + \mu(u_m)\nabla G(u_m) \to 0$ T_L-uniformly as $m \to \infty$. By part ii) now we may select a strongly convergent subsequence $u_m \to u$ in $H_0^{1,2}$ $(m \to \infty)$, whence $u \in K_\beta$, contrary to the assumption that $K_\beta = \emptyset$. This concludes the proof. q.e.d.

LEMMA 3.5. *For any numbers $\beta \in \mathbb{R}$, $L \in \mathbb{N}$ there exists a number $\delta > 0$ and a continuous mapping $\phi_L : M \to M$ such that*

$$\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta} \cup N_{\beta,L}.$$

If conditions (3.10), (3.11) *are satisfied, ϕ_L may be chosen to be odd, i.e. $\phi_L(-u) = -\phi_L(u)$.*

Proof. Fix an arbitrary number $\beta \in \mathbb{R}$ and any $L \in \mathbb{N}$. The mapping ϕ_L will be constructed from a "gradient-line" deformation in direction of a gradient-like vector field related to $\nabla E\,|_M \in T_L^*$. (For brevity we again write $\nabla E\,|_M(u) = \nabla E(u) + \mu(u)\nabla G(u)$.)

i) For any $u \in M$ let $\xi_L(u) \in T_L$ be a vector satisfying

$$\|\xi_L(u)\|_{T_L} = \|\nabla E\,|_M(u)\|_{T_L^*}, \qquad \langle \nabla E\,|_M(u), \xi_L(u)\rangle \geq \|\nabla E\,|_M(u)\|_{T_L^*}^2 - L^{-1}.$$

By continuity of $\nabla E : H_0^{1,2} \to T_L^*$ for any $u \in M$ there exists a neighborhood $V(u)$ such that for all $v \in V(u)$

$$\langle \nabla E\,|_M(v), \xi_L(u)\rangle \geq \|\nabla E\,|_M(v)\|_{T_L^*}^2 - 2L^{-1}.$$

Since M is a subset of a Hilbert space there exists a locally finite refinement $\{V(u_\iota)\}_{\iota\in I}$ of the covering $\{V(u)\}$. Letting $\{\psi_\iota\}_{\iota\in I}$ be a partition of unity subordinate to $\{\tilde V(u_\iota)\}$ with continuous functions ψ_ι having support in $\tilde V(u_\iota)$ and such that

$$0 \leq \psi_i \leq 1, \qquad \sum_{\iota\in I} \psi_\iota = 1 \quad \text{on } M,$$

we define

$$\tilde{e}_L(u)=\sum_{\iota\in I}\psi_\iota(u)\xi_L(u_\iota).$$

In the general case we let $e_L(u)=\tilde{e}_L(u)$. In the symmetric case $E(u)=E(-u)$, $G(u)=G(-u)$, corresponding to assumptions (3.10), (3.11), we define

$$e_L(u)=\tfrac{1}{2}(\tilde{e}_L(u)-\tilde{e}_L(-u))=-e_L(-u).$$

In both cases $e_L:M\to T_L$ is a continuous vector field with the property that for any $u\in M$ there holds

$$\langle\nabla E\,|_M(u),\, e_L(u)\rangle\geq\|\nabla E\,|_M(u)\|^2_{T_L^*}-2L^{-1}.$$

Moreover, since $\nabla E\,|_M:M\to T_L^*$ is bounded on $H_0^{1,2}$-bounded subsets of M the same is true for e_L.

ii) Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $0\leq\alpha\leq 1$, $\alpha(t)=1$ if $t\leq\beta+1$, $\alpha(t)=0$ if $t\geq\beta+2$. For $\varepsilon>0$, $u\in M$ let

$$u^\varepsilon=u-\varepsilon\alpha(E(u))e_L(u).$$

Note that $u^\varepsilon\equiv u$ for $u\in M\backslash M_{\beta+2}$. Also, by part i) and Lemma 3.4 i) the vectors $\alpha(E(u))e_L(u)\in T_L$ are uniformly bounded on $M_{\beta+2}$.

By weak compactness of $M_{\beta+2}$ and assumptions (3.4), (3.6) there exists a constant $c>0$ such that $\langle\nabla G(u),u\rangle\geq c$ on $M_{\beta+2}$. Hence by the implicit function theorem there exists a number $\varepsilon_0>0$ and a function $\tau(\varepsilon,u)$ of class C^1 in ε and depending continuously on u such that

$$u_\varepsilon=\tau(\varepsilon,u)u^\varepsilon\in M$$

for all $u\in M$, $0\leq\varepsilon<\varepsilon_0$. Calculating $E(u_\varepsilon)$ by Lemma 3.2 we obtain

$$\begin{aligned}E(u_\varepsilon)-E(u)&=E(u_\varepsilon)+\mu(u)G(u_\varepsilon)-(E(u)+\mu(u)G(u))\\&=\int_0^\varepsilon\frac{d}{d\varepsilon}(E(u_\varepsilon)+\mu(u)G(u_\varepsilon)\,d\varepsilon\\&=-\int_0^\varepsilon\langle\nabla E(u_\varepsilon)+\mu(u)\nabla G(u_\varepsilon),\,\tau(\varepsilon,u)\alpha(E(u))e_L(u)\rangle\,d\varepsilon\\&\quad+\int_0^\varepsilon\langle\nabla E(u_\varepsilon)+\mu(u)\nabla G(u_\varepsilon),\,u_\varepsilon\rangle\frac{\partial}{\partial\varepsilon}\ln\tau(\varepsilon,u)\,d\varepsilon\\&\leq-\varepsilon\langle\nabla E(u)+\mu(u)\nabla G(u),\,\alpha(E(u))e_L(u)\rangle+o(\varepsilon),\end{aligned}\qquad(3.18)$$

with a Landau function $o(\varepsilon)$ such that $o(\varepsilon)/\varepsilon \to 0$ $(\varepsilon \to 0)$ uniformly with respect to $u \in M$. In particular, choosing ε_0 sufficiently small we may assume that $o(\varepsilon) \le \varepsilon L^{-1}$. Moreover, by locally uniform continuity of $\nabla E|_M \in T_L^*$ we may assume that $\|\nabla E|_M(u_\varepsilon)\|_{T_L^*}^2 < 5L^{-1}$ whenever $\varepsilon \le \varepsilon_0$ and $\|\nabla E|_M(u)\|_{T_L^*}^2 < 4L^{-1}$.

Fix such a number $0 < \varepsilon \le \frac{1}{4}$ and let $\phi_L(u) = u_\varepsilon$. Set $\delta = \varepsilon/2L \le 1/L$. Then, if $u \in M_{\beta+\delta}$, either $\|\nabla E|_M(u)\|_{T_L^*}^2 \ge 4L^{-1}$ and hence by (3.18)

$$E(\phi_L(u)) \le E(u) - 2\varepsilon L^{-1} + \varepsilon L^{-1} = E(u) - 2\delta,$$

i.e. $\phi_L(u) \in M_{\beta-\delta}$; or $\|\nabla E|_M(u)\|_{T_L^*}^2 < 4L^{-1}$ whence $\|\nabla E|_M(\phi_L(u))\|_{T_L^*}^2 < 5L^{-1}$ while

$$E(\phi_L(u)) \le E(u) + 2\varepsilon L^{-1} + \varepsilon L^{-1} < \beta + 4\varepsilon L^{-1} \le \beta + L^{-1},$$

i.e. $\phi_L(u) \in M_{\beta-\delta} \cup N_{\beta,L}$.

Finally, in the case of assumptions (3.10), (3.11) $\phi_L(-u) = -\phi_L(u)$. The proof is complete. q.e.d.

Proof of Theorems 3.1 *and* 3.2 *for coefficients satisfying* (3.14).

i) Let $\beta = \inf_{u \in M} E(u)$. Choose $\rho_1 > 0$ corresponding to Lemma 3.4 iii) and assume that $|\partial_u a| < \rho_1$ a.e. in $\Omega \times \mathbb{R}^N$. Then if $K_\beta = \varnothing$ by Lemma 3.4 iii) there exists L such that $N_{\beta,L} = \varnothing$, and by Lemma 3.5 we can find $\delta > 0$ and a continuous mapping $\phi_L : M \to M$, such that $\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta}$. But $M_{\beta-\delta} = \varnothing$ and a contradiction results, proving Theorem 3.1.

ii) In the case of Theorem 3.2 fix a number $l \le m$, and let $\beta = \beta_l$. Choose $\rho_l > 0$ corresponding to Lemma 3.4 iii) and assume that $|\partial_u a| < \rho_l$ a.e. in $\Omega \times \mathbb{R}^N$. Then, if $K_\beta = \varnothing$ as in part i) of this proof there exists a continuous mapping $\phi_L : M \to M$ such that $\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta}$ for some number $\delta > 0$. Moreover, by assumptions (3.10), (3.11) ϕ_L can be chosen to be odd. Letting $A \in \Sigma_l$ be such that $A \subset M_{\beta+\delta}$, we thus obtain that $\phi_L(A) \in \Sigma_l$ (cp. Proposition 3.1 i)) and satisfies $\phi_L(A) \subset M_{\beta-\delta}$. But this contradicts the definition of $\beta = \beta_l$, concluding the existence proof.

In the case of degeneracy $\beta = \beta_l = \cdots = \beta_{l+k}$ for some numbers $k, l \in \mathbb{N}$ we estimate $\gamma(K_\beta)$ as follows. By Criterion A* the set K_β is compact. Hence from Proposition 3.1 iii) we can find a symmetric neighborhood N of K_β, $N \subset H_0^{1,2} \backslash \{0\}$, such that the closure $\bar{N} \in \Sigma$ and $\gamma(\bar{N}) = \gamma(K_\beta)$. Again by Criterion A* there exists L such that $N_{\beta,L} \subset \bar{N}$. Let $\delta > 0$, ϕ_L be chosen according to Lemma 3.5 corresponding to this number L and β and let $A \in \Sigma_{l+k}$ be such that $A \subset M_{\beta+\delta}$. Then by

Proposition 3.1.

$$l+k \le \gamma(A) \le \gamma(\phi_L(A)) \le \gamma(\phi_L(A)\backslash N_{\beta,L}) + \gamma(\bar{N})$$
$$= \gamma(\phi_L(A)\backslash N_{\beta,L}) + \gamma(K_\beta).$$

But $\phi_L(A)\backslash N_{\beta,L} \subset M_{\beta-\delta}$, whence $\gamma(\phi_L(A)\backslash N_{\beta,L}) < l$, and $\gamma(K_\beta) \ge k+1$.

(The proof of the asymptotic behavior of the sequence β_l will be postponed to the full proof of Theorems 3.1, 3.2 without assumption (3.14).)

Proof of Theorems 3.1, 3.2 (completed). To remove assumption (3.14) approximate the functional E by functions E_ν with coefficients $a_\nu^{\alpha\beta}$ satisfying condition (3.14) and coinciding with $a^{\alpha\beta}$ for $|u| \le \nu$. To construct these coefficients let $\varphi: \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \le \varphi \le 1$, $\varphi' \le 0$, $\varphi(0) = 1$, $\varphi(1) = 0$. Then given any number $\nu \in \mathbb{N}$, for $u \in \mathbb{R}^N$, $|u| = 1$, define

$$a_\nu^{\alpha\beta}(x, u \cdot t) = \varphi(t-\nu)a^{\alpha\beta}(x, u \cdot t) + (1 - \varphi(t-\nu))\lambda\delta_{\alpha\beta}.$$

Clearly, assumptions (3.1), (3.2) are satisfied for the coefficients $a_\nu^{\alpha\beta}$ with the uniform ellipticity constant λ. Moreover, for any u

$$-u \cdot \partial_u a_\nu^{\alpha\beta}(x, u) = -\varphi(|u|-\nu)u \cdot \partial_u a^{\alpha\beta}(x, u) - |u|\,|\varphi'(|u|-\nu)|(a^{\alpha\beta}(x, u) - \lambda \cdot \partial_{\alpha\beta}).$$

Hence, by conditions (3.2), (3.3) on $a^{\alpha\beta}$, condition (3.3) is satisfied by the coefficients $a_\nu^{\alpha\beta}$ with the uniform constant $\lambda^* < \lambda$.

By the preceding proof there exist solutions (u_l, μ_l) of (1.2) for the functionals E_ν characterized by the condition $E_\nu(u_l) = \beta_{l,\nu} = \inf_{A \in \Sigma_l} \sup_{u \in A} E_\nu(u)$. Now, for any $u \in M$ $E_\nu(u) \to E(u)$ $(\nu \to \infty)$, and similarly, for any $A \in \Sigma_l$ $\sup_{u \in A} E_\nu(u) \to \sup_{u \in A} E(u)$ $(\nu \to \infty)$. Hence $\beta_{l,\nu} \to \beta_l(\nu \to \infty)$, for any $l \le m$. By Lemma 3.3 therefore $\|u_l\|_\infty \le c(\lambda, \lambda^*, n, N, \Omega, p, \beta_l + 1)$ for ν sufficiently large, and u_l is in fact a critical point of E with energy $E(u_l) = \beta_l = \beta_{l,\nu}$ for such ν. (This also justifies writing u_l instead of $u_{l,\nu}$.) This shows existence and by the preceding proof also the assertion about K_β in the case of degeneracy now follows.

To show that $\beta_l \to \infty$ $(l \to \infty)$ assume by contradiction that $\beta_l \le c$ for all $l \in \mathbb{N}$ with a uniform constant c. Then there exist sets $A_l \in \Sigma_l$, $l \in \mathbb{N}$, such that $A = \bigcup_l A_l \subset M_{c+1}$. By [21, proof of Lemma 2.21] or [25, proof of Lemma 10] A contains a sequence of mutually orthogonal vectors $u_m \in A$, and by weak compactness of M_{c+1} we may assume that $u_m \rightharpoonup u$ weakly as $m \to \infty$. But by mutual orthogonality of $\{u_m\}$, $u=0$. Hence from $0=G(u)=\lim_{m\to\infty} G(u_m)=1$ a contradiction results. q.e.d.

4. Extensions and generalizations

Without proof we remark that in the subquadratic case corresponding e.g. to functionals of the type

$$E_\delta(u)=\int |\nabla u|^2\,dx+\int (a^{\alpha\beta}\,\partial_\alpha u^i\,\partial_\beta u^i)^{2-\delta/2}\,dx, \qquad \delta>0 \tag{4.1}$$

existence results similar to Theorems 3.1, 3.2 hold without any smallness assumption on $|\partial_u a|$ in addition to (3.3).[1] By a different kind of argument than used in Section 3 also in case of quadratic growth such a smallness assumption can be removed if either $N=1$ (scalar equation) or $n=2$ (plane system). This may be shown by approximating the functional E by functionals E_δ of the form (4.1) and using the regularity results of Ladyshenskaya–Ural'tseva [15] for example, resp. the results of Wiegner [28] for plane diagonal systems to pass to the limit $\delta\to 0$ in the Euler equations which are satisfied at the critical points of E_δ.

By the same techniques as presented above also existence results for the boundary value problem

$$u\in H_0^{1,2}:\nabla E(u)-\nabla G(u)=0\in T'$$

can be given under hypotheses similar to those of Theorems 3.1 and 3.2 plus some additional hypotheses to ensure the regularity (with respect to T_L, for any $L\in\mathbb{N}$) of the set of admissible functions

$$M=\{u\in H_0^{1,2}\mid \langle\nabla E(u)-\nabla G(u),u\rangle=0\}.$$

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