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Wall's obstructions and Whitehead torsion

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In this note we show that the Wall-type obstruction defined by S. Ferry in [4] is in fact the original Wall's one. As a consequence we obtain the geometric proof of the Product Formula (see [5]) for the Wall finiteness obstructions.

1. Introduction

Let X be a topological space which is homotopy dominated by a finite CW complex. In [9] C. T. C. Wall introduced the obstruction $w(X)$ which is an element of $\tilde{K}_0(Z(\pi_1(X)))$ to decide when X has the homotopy type of some finite CW complex. Alternatively in [4] S. Ferry has found, in a geometric manner, an analogous obstruction $\sigma(X)$ in $Wh(X \times S^1)$. The natural question about the relation between these two obstructions was not considered in [4] (note that this question was explicitly asked by H. J. Munkholm in [10]). The purpose of this note is to fill this gap. We prove a rather expected result that these two obstructions are the same. To be more precise; we prove that $w(X)$ is the image of $\sigma(X)$ under the Bass–Heller–Swan isomorphism, thus answering the question from [10].

As a consequence we obtain the geometric proof of the Product Formula for the Wall finiteness obstructions. Originally the Product Formula was proved by S. Gersten in [5] in a purely algebraic manner. This note does not pretend to the orginality, but we hope that it will a little bit clarify the geometry of the Wall finiteness obstruction.

We will assume some familiarity with the simple homotopy theory. An excellent reference is [3].

2. Wall's obstruction and simple types

In our note we will consider the Whitehead group of an arbitrary topological space following [8].

Let us recall the construction of the obstruction to the finiteness given by S. Ferry in [4].

Let X be a topological space which is homotopy dominated by a finite CW complex K, i.e. there exist maps $g: X \to K$, $f: K \to X$ such that $fg \simeq id_X$. By the theorem of M. Mather (see [6]) $X \times S^1$ has a homotopy type of a finite CW complex. To see it we repeat his beautiful geometric argument. Namely, consider the mapping torus $T(\alpha)$ of the map $\alpha = gf: K \to K$; recall that $T(\alpha)$ is the space obtained from the mapping cylinder $M(\alpha)$ by identifying the top and bottom of $M(\alpha)$ using the identity map. Of course we can assume that up to homotopy type $T(\alpha)$ is a finite CW complex. Now the following picture shows that $X \times S^1 \simeq T(\alpha)$.

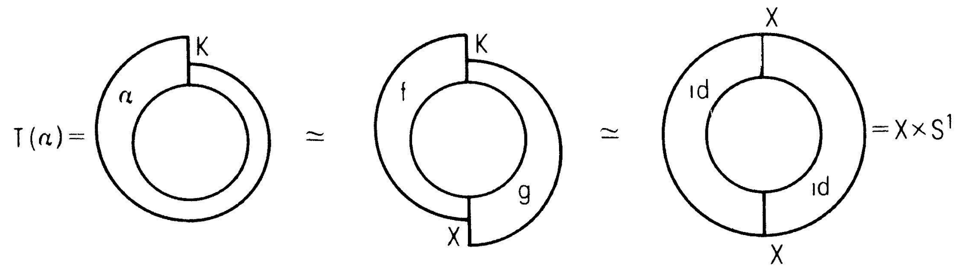

Figure 1

We will denote this homotopy equivalence by $\Phi: T(\alpha) \to X \times S^1$ and its inverse by $\Phi^{-1}: X \times S^1 \to T(\alpha)$.

DEFINITION 2.1 (S. Ferry [4]). Let $\mathrm{T}: \mathrm{X} \times \mathrm{S}^1 \to \mathrm{X} \times \mathrm{S}^1$ be a homeomorphism given by $T(x, \theta) = (x, \bar{\theta})$. We define $\sigma(X) = \Phi_*(\tau(\Phi^{-1}T\Phi)) \in Wh(X \times S^1)$, where $\tau(\Phi^{-1}T\Phi)$ is a torsion of the homotopy equivalence $\Phi^{-1}T\Phi: T(\alpha) \to T(\alpha)$.

It turns out (see [4]) that $\sigma(X)$ is well-defined (does not depend from f, g and K) and $\sigma(X) = 0$ if and only if X is a homotopy equivalent to some finite CW complex.

The crucial role in our considerations plays the following Bass-Heller-Swan decomposition of the Wh functor (see [1], [2]).

Let X be a topological space. Then there exists a functorial direct sum decomposition

$$Wh(X \times S^1) = Wh(X) \oplus \mathrm{Nil}\,(X) \oplus \mathrm{Nil}\,(X) \oplus \tilde{K}_0(X)$$

where by $\mathrm{Nil}\,(X)$, $\tilde{K}_0(X)$ we mean $\mathrm{Nil}\,(Z(\pi_1(X)))$, $\tilde{K}_0(Z(\pi_1(X)))$ respectively. Using this we prove:

THEOREM 2.2. *Let X be a topological space which is homotopy dominated by a finite CW complex. Then the Wall finiteness obstruction $w(X)$ is a image of $\sigma(X)$ under the Bass-Heller-Swan decomposition of Wh $(X\times S^1)$.*

Proof. Let K be a finite CW complex and let $g: X\to K$, $f: K\to X$ be maps such that $fg\simeq id_X$. As previous by $T(\alpha)$ we denote the mapping torus of the map $\alpha=gf: K\to K$.

Let $\Phi: T(\alpha)\to X\times S^1$ be a homotopy equivalence. The natural infinite cyclic covering of $X\times S^1$ induces an infinite cyclic covering $I(\alpha)$ of $T(\alpha)$.

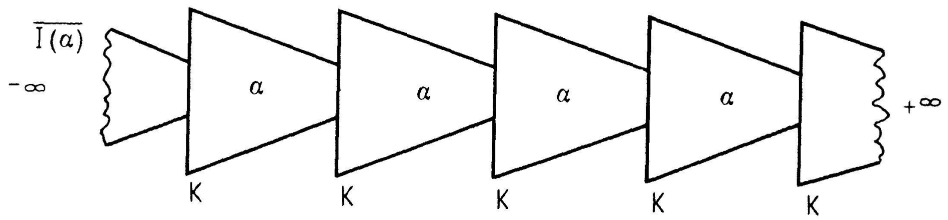

Figure 2

The space $I(\alpha)$ is an infinite CW complex with two ends ϵ_+, ϵ_- which correspond to the two ends of the real line.

Observe that the homotopy equivalence $h=\Phi^{-1}T\Phi: T(\alpha)\to T(\alpha)$ induces a proper homotopy equivalence $\tilde{h}$ between $I(\alpha)$ and its reversed copy $\overline{I(\alpha)}$ (reversed with respect to the ends).

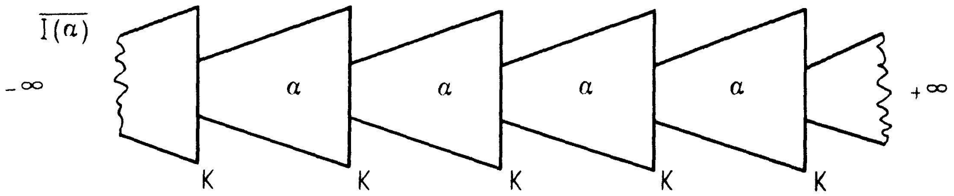

Figure 3

Without loss of generality we can assume that $\tilde{h}$ is a strong proper deformation retraction of $I(\alpha)$.

Now we proceed as in [7]. In $I(\alpha)$ consider a subcomplex L such that L is a neighborhood of ϵ_+ and $(I(\alpha)-L)\cup\overline{I(\alpha)}$ is a neighborhood of ϵ_-. Put $L_1=\overline{I(\alpha)}\cap L$ and consider the pair (L, L_1). It can be easly proved (see Lemma 4.5 in [7]) that the pair (L, L_1) is homotopy dominated by a pair $(L_0\cup L_1, L_1)$, where L_0 is a finite subcomplex of L. Then the cellular chain complex $C_*(\tilde{L}, \tilde{L}_1)$ of the universal covering $p: \tilde{L}\to L$ of the pair (L, L_1), which is a free $Z(\pi_1(I(\alpha))$-complex is chain homotopy dominated by the free $Z(\pi_1(I(\alpha))$-complex $C_*(\tilde{L}_0\cup\tilde{L}_1, \tilde{L}_1)$; we used the notation: for every $B\subset L$, $\tilde{B}=p^{-1}(B)$. Hence we can define

$w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = w(C_*(\tilde{L}, \tilde{L}_1) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$, where $w(C_*(\tilde{L}, \tilde{L}_1))$ is the Wall obstruction. It is not difficult to see that $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+)$ is well-defined i.e. does not depend of the choice of L_1.

Now let L_-, L_+ be a neighborhoods of ϵ_-, ϵ_+ so that $I(\alpha)-L_+$, $I(\alpha)-L_-$ are again neighborhoods of ϵ_-, ϵ_+ respectively and $L_- \cup L_+ = I(\alpha)$. Then $L_- \cap L_+$ is a finite CW complex and since $I(\alpha)$ is homotopy dominated by a finite CW comples (in fact by K) then from the Mayer-Vietoris sequence

$$0 \to C_*(\tilde{L}_- \cap \tilde{L}_+) \to C_*(\tilde{L}_-) \oplus C_*(\tilde{L}_+) \to C_*(I(\alpha)) \to 0$$

we infer that $C_*(\tilde{L}_+)$ is chain homotopy dominated by a finitely generated free complex. This gives us the well defined element $w(I(\alpha), \epsilon_+) = w(C_*(\tilde{L}_+)) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. Analogously we can define $w(\overline{I(\alpha)}, \epsilon_+) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. An elementary property of the Wall obstructions yields:

$$w(I(\alpha), \epsilon_+) = w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) + w(\overline{I(\alpha)}, \epsilon_+)$$

Observe (see Fig. 4) that in our situation $w(\overline{I(\alpha)}, \epsilon_+) = 0$ and $\tilde{h}_*(w(I(\alpha)) = w(I(\alpha), \epsilon_+)$ by a homotopy type invariance of the Wall obstruction.

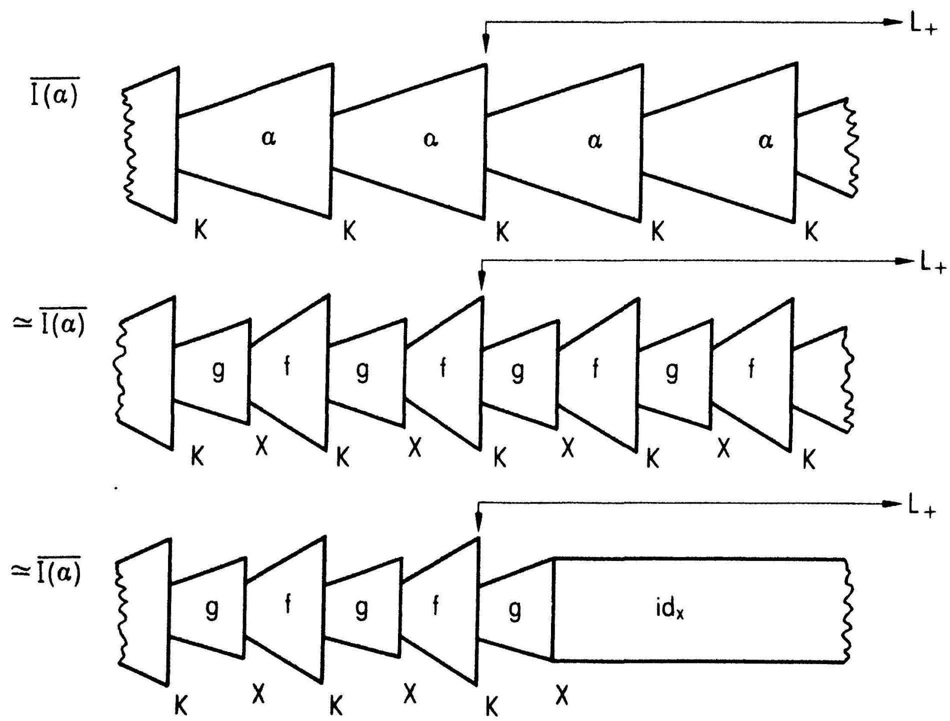

Figure 4

Hence $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = \tilde{h}_*(w(I(\alpha)))$. The Bass–Heller–Swan projection (B–H–S) : $Wh(X \times S^1) \to \tilde{K}_0(X)$ induces a natural projection $p : Wh(T(\alpha)) \to \tilde{K}_0(I(\alpha))$.

This gives the following commutative diagram:

$$\begin{array}{ccc} Wh(T(\alpha)) & \xrightarrow{\Phi_*} & Wh(X\times S^1) \\ \downarrow p & & \downarrow \text{(B–H–S)} \\ \tilde{K}_0(I(\alpha)) & \xrightarrow{\tilde{\Phi}_*} & \tilde{K}_0(X) \end{array}$$

where the map $\tilde{\Phi} : I(\alpha) \to X \simeq X\times R$ is induced by Φ. So we have:

$$\tilde{\Phi}_* p(\tau(\Phi^{-1}T\Phi) = \text{(B–H–S)}\Phi_*(\tau(\Phi^{-1}T\Phi) = \text{(B–H–S)}(\sigma(X)).$$

But $\tilde{\Phi}_* p(\tau(\Phi^{-1}T\Phi) = \tilde{\Phi}_*(w(I(\alpha), \overline{I(\alpha)}, \epsilon_+))$ by the Proposition 4.7 in [7], hence:

$$\text{(B–H–S)}(\sigma(X)) = \tilde{\Phi}\tilde{h}_*(w(I(\alpha))) = w(X)$$

by the homotopy type invariance of the Wall obstruction.

COROLLARY 2.3 (*Product Formula*). *Let X be a topological space which is homotopy dominated by a finite CW complex, and let L be a finite CW complex. Then*:

$$w(L\times X) = \chi(L)\cdot i_*(w(X))$$

where $i : X \to L\times X$ is given by $i(x) = (1_0, x)$ for some $1_0 \varepsilon L$, and $\chi(L)$ denotes the Euler characteristic of L.

Proof. Let K be a finite CW complex and $g : X \to K$, $f : K \to X$ be maps such that $fg \simeq id_X$. Let $T(\alpha)$ be the mapping torus of the map $\alpha = gf : K \to K$ and let $\Phi : T(\alpha) \to X\times S^1$ be a homotopy equivalence. The space $L\times X$ is a homotopy dominated by the finite CW complex $L\times K$ using the maps $id\times g : L\times X \to L\times K$, $id\times f : L\times K \to L\times X$. Hence we have the homotopy equivalence $\bar{\Phi} : T(id\times\alpha) \to L\times X\times S^1$. But $T(id\times\alpha) = L\times T(\alpha)$ and without loss of the generality we can write $\bar{\Phi} = id\times\Phi : L\times T(\alpha) \to L\times X\times S^1$. Now our finiteness obstruction is given by:

$$\sigma(L\times X) = (id\times\Phi)_*(\tau(id\times\Phi^{-1}T\Phi)) \in Wh(L\times X\times S^1).$$

By the product theorem for Whitehead torsion (see [3] for the nice and short geometric proof) we have:

$$\tau(id\times\Phi^{-1}T\Phi)=\chi(L)\cdot j_*(\tau(\Phi^{-1}T\Phi)$$

where $j: T(\alpha)\to L\times T(\alpha)$ is given by $j(t)=(1_0, t)$, for $t\in T(\alpha)$. Hence $\sigma(L\times X)=\chi(L)\cdot i_*(\sigma(X))$, where $i_*: Wh(X\times S^1)\to Wh(L\times X\times S^1)$. Now the formula $w(L\times X)=\chi(L)\cdot i_*(w(X))$ follows from the naturality of the Bass–Heller–Swan decomposition of $Wh(X\times S^1)$.

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Note added in proof:

In fact $\sigma(X)\in\tilde{K}_0(X)$. This can be deduced from T. Chapman, Approximation results in Hilbert cube manifolds, Trans. Amer. Math. Soc. 262 (1980), 303–334, in particular, see p. 321 of this paper.

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