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Wall's obstructions and Whitehead torsion

SŁAWOMIR KWASIK

In this note we show that the Wall-type obstruction defined by S. Ferry in [4] is in fact the original Wall's one. As a consequence we obtain the geometric proof of the Product Formula (see [5]) for the Wall finiteness obstructions.

1. Introduction

Let X be a topological space which is homotopy dominated by a finite CW complex. In [9] C. T. C. Wall introduced the obstruction $w(X)$ which is an element of $\tilde{K}_0(Z(\pi_1(X)))$ to decide when X has the homotopy type of some finite CW complex. Alternatively in [4] S. Ferry has found, in a geometric manner, an analogous obstruction $\sigma(X)$ in $Wh(X \times S^1)$. The natural question about the relation between these two obstructions was not considered in [4] (note that this question was explicitly asked by H. J. Munkholm in [10]). The purpose of this note is to fill this gap. We prove a rather expected result that these two obstructions are the same. To be more precise; we prove that $w(X)$ is the image of $\sigma(X)$ under the Bass–Heller–Swan isomorphism, thus answering the question from [10].

As a consequence we obtain the geometric proof of the Product Formula for the Wall finiteness obstructions. Originally the Product Formula was proved by S. Gersten in [5] in a purely algebraic manner. This note does not pretend to the originality, but we hope that it will a little bit clarify the geometry of the Wall finiteness obstruction.

We will assume some familiarity with the simple homotopy theory. An excellent reference is [3].

2. Wall's obstruction and simple types

In our note we will consider the Whitehead group of an arbitrary topological space following [8].

Let us recall the construction of the obstruction to the finiteness given by S. Ferry in [4].

Let X be a topological space which is homotopy dominated by a finite CW complex K , i.e. there exist maps $g : X \rightarrow K$, $f : K \rightarrow X$ such that $fg \simeq id_X$. By the theorem of M. Mather (see [6]) $X \times S^1$ has a homotopy type of a finite CW complex. To see it we repeat his beautiful geometric argument. Namely, consider the mapping torus $T(\alpha)$ of the map $\alpha = gf : K \rightarrow K$; recall that $T(\alpha)$ is the space obtained from the mapping cylinder $M(\alpha)$ by identifying the top and bottom of $M(\alpha)$ using the identity map. Of course we can assume that up to homotopy type $T(\alpha)$ is a finite CW complex. Now the following picture shows that $X \times S^1 \simeq T(\alpha)$.

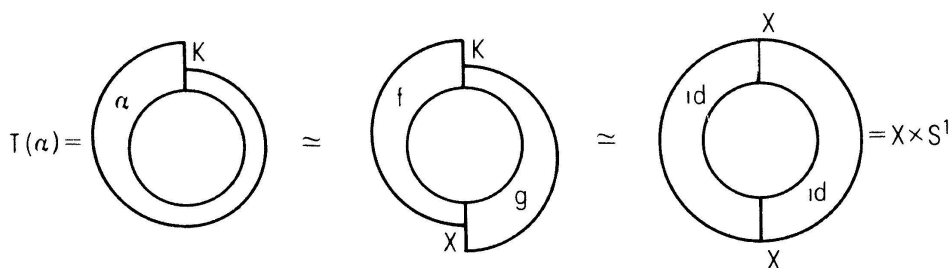


Figure 1

We will denote this homotopy equivalence by $\Phi : T(\alpha) \rightarrow X \times S^1$ and its inverse by $\Phi^{-1} : X \times S^1 \rightarrow T(\alpha)$.

DEFINITION 2.1 (S. Ferry [4]). Let $T : X \times S^1 \rightarrow X \times S^1$ be a homeomorphism given by $T(x, \theta) = (x, \bar{\theta})$. We define $\sigma(X) = \Phi_*(\tau(\Phi^{-1}T\Phi)) \in Wh(X \times S^1)$, where $\tau(\Phi^{-1}T\Phi)$ is a torsion of the homotopy equivalence $\Phi^{-1}T\Phi : T(\alpha) \rightarrow T(\alpha)$.

It turns out (see [4]) that $\sigma(X)$ is well-defined (does not depend from f , g and K) and $\sigma(X) = 0$ if and only if X is a homotopy equivalent to some finite CW complex.

The crucial role in our considerations plays the following Bass-Heller-Swan decomposition of the Wh functor (see [1], [2]).

Let X be a topological space. Then there exists a functorial direct sum decomposition

$$Wh(X \times S^1) = Wh(X) \oplus Nil(X) \oplus Nil(X) \oplus \tilde{K}_0(X)$$

where by $Nil(X)$, $\tilde{K}_0(X)$ we mean $Nil(Z(\pi_1(X)))$, $\tilde{K}_0(Z(\pi_1(X)))$ respectively. Using this we prove:

THEOREM 2.2. *Let X be a topological space which is homotopy dominated by a finite CW complex. Then the Wall finiteness obstruction $w(X)$ is a image of $\sigma(X)$ under the Bass-Heller-Swan decomposition of $Wh(X \times S^1)$.*

Proof. Let K be a finite CW complex and let $g : X \rightarrow K$, $f : K \rightarrow X$ be maps such that $fg \simeq id_X$. As previous by $T(\alpha)$ we denote the mapping torus of the map $\alpha = gf : K \rightarrow K$.

Let $\Phi : T(\alpha) \rightarrow X \times S^1$ be a homotopy equivalence. The natural infinite cyclic covering of $X \times S^1$ induces an infinite cyclic covering $I(\alpha)$ of $T(\alpha)$.

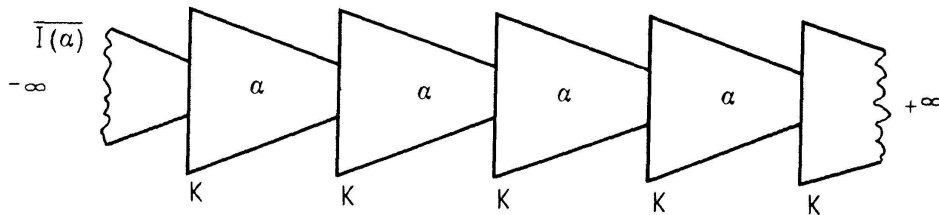


Figure 2

The space $I(\alpha)$ is an infinite CW complex with two ends ϵ_+ , ϵ_- which correspond to the two ends of the real line.

Observe that the homotopy equivalence $h = \Phi^{-1}T\Phi : T(\alpha) \rightarrow T(\alpha)$ induces a proper homotopy equivalence \tilde{h} between $I(\alpha)$ and its reversed copy $\overline{I(\alpha)}$ (reversed with respect to the ends).

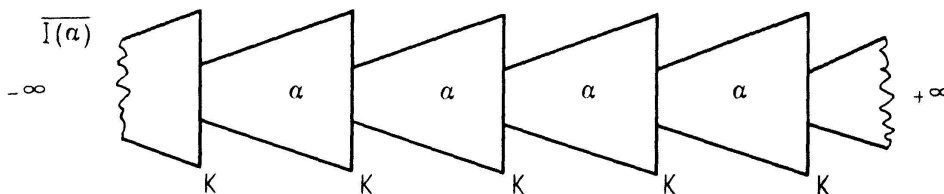


Figure 3

Without loss of generality we can assume that \tilde{h} is a strong proper deformation retraction of $I(\alpha)$.

Now we proceed as in [7]. In $I(\alpha)$ consider a subcomplex L such that L is a neighborhood of ϵ_+ and $(I(\alpha) - L) \cup \overline{I(\alpha)}$ is a neighborhood of ϵ_- . Put $L_1 = \overline{I(\alpha)} \cap L$ and consider the pair (L, L_1) . It can be easily proved (see Lemma 4.5 in [7]) that the pair (L, L_1) is homotopy dominated by a pair $(L_0 \cup L_1, L_1)$, where L_0 is a finite subcomplex of L . Then the cellular chain complex $C_*(\tilde{L}, \tilde{L}_1)$ of the universal covering $p : \tilde{L} \rightarrow L$ of the pair (L, L_1) , which is a free $Z(\pi_1(I(\alpha)))$ -complex is chain homotopy dominated by the free $Z(\pi_1(I(\alpha)))$ -complex $C_*(\tilde{L}_0 \cup \tilde{L}_1, \tilde{L}_1)$; we used the notation: for every $B \subset L$, $\tilde{B} = p^{-1}(B)$. Hence we can define

$w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = w(C_*(\tilde{L}, \tilde{L}_1) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$, where $w(C_*(\tilde{L}, \tilde{L}_1))$ is the Wall obstruction. It is not difficult to see that $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+)$ is well-defined i.e. does not depend of the choice of L_1 .

Now let L_-, L_+ be a neighborhoods of ϵ_-, ϵ_+ so that $I(\alpha) - L_+, I(\alpha) - L_-$ are again neighborhoods of ϵ_-, ϵ_+ respectively and $L_- \cup L_+ = I(\alpha)$. Then $L_- \cap L_+$ is a finite CW complex and since $I(\alpha)$ is homotopy dominated by a finite CW complex (in fact by K) then from the Mayer-Vietoris sequence

$$0 \rightarrow C_*(\tilde{L}_- \cap \tilde{L}_+) \rightarrow C_*(\tilde{L}_-) \oplus C_*(\tilde{L}_+) \rightarrow C_*(I(\alpha)) \rightarrow 0$$

we infer that $C_*(\tilde{L}_+)$ is chain homotopy dominated by a finitely generated free complex. This gives us the well defined element $w(I(\alpha), \epsilon_+) = w(C_*(\tilde{L}_+)) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. Analogously we can define $w(\overline{I(\alpha)}, \epsilon_+) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. An elementary property of the Wall obstructions yields:

$$w(I(\alpha), \epsilon_+) = w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) + w(\overline{I(\alpha)}, \epsilon_+)$$

Observe (see Fig. 4) that in our situation $w(\overline{I(\alpha)}, \epsilon_+) = 0$ and $\tilde{h}_*(w(I(\alpha))) = w(I(\alpha), \epsilon_+)$ by a homotopy type invariance of the Wall obstruction.

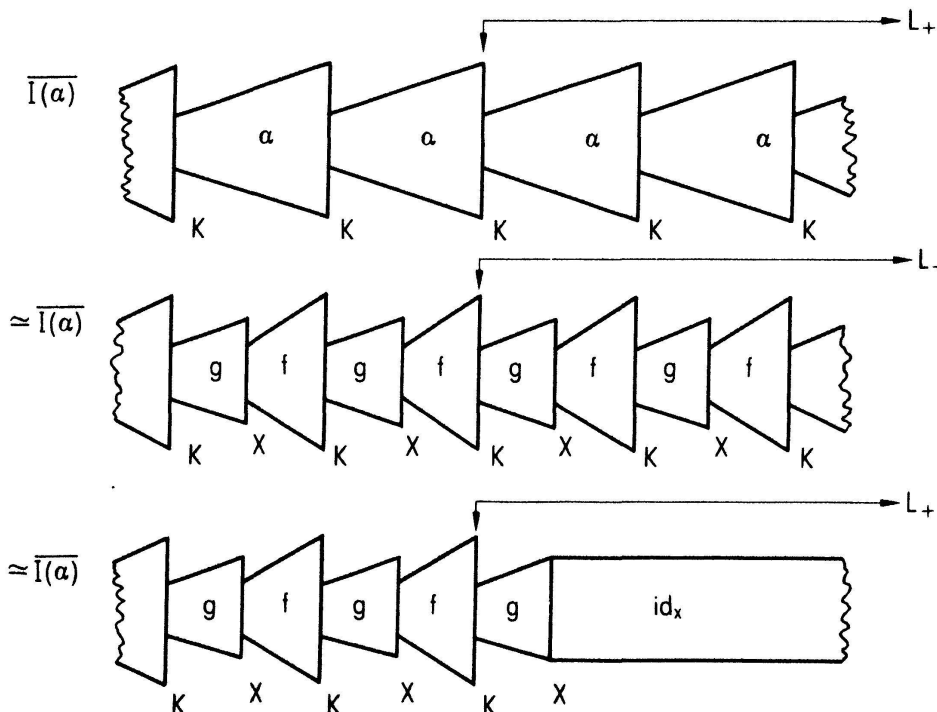


Figure 4

Hence $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = \tilde{h}_*(w(I(\alpha)))$. The Bass–Heller–Swan projection (B–H–S) : $Wh(X \times S^1) \rightarrow \tilde{K}_0(X)$ induces a natural projection $p : Wh(T(\alpha)) \rightarrow \tilde{K}_0(I(\alpha))$.

This gives the following commutative diagram:

$$\begin{array}{ccc} Wh(T(\alpha)) & \xrightarrow{\Phi_*} & Wh(X \times S^1) \\ \downarrow p & & \downarrow (B-H-S) \\ \tilde{K}_0(I(\alpha)) & \xrightarrow{\tilde{\Phi}_*} & \tilde{K}_0(X) \end{array}$$

where the map $\tilde{\Phi} : I(\alpha) \rightarrow X \simeq X \times R$ is induced by Φ . So we have:

$$\tilde{\Phi}_* p(\tau(\Phi^{-1} T \Phi)) = (B-H-S) \Phi_*(\tau(\Phi^{-1} T \Phi)) = (B-H-S)(\sigma(X)).$$

But $\tilde{\Phi}_* p(\tau(\Phi^{-1} T \Phi)) = \tilde{\Phi}_*(w(I(\alpha), \overline{I(\alpha)}, \epsilon_+))$ by the Proposition 4.7 in [7], hence:

$$(B-H-S)(\sigma(X)) = \tilde{\Phi} \tilde{h}_*(w(I(\alpha))) = w(X)$$

by the homotopy type invariance of the Wall obstruction.

COROLLARY 2.3 (Product Formula). *Let X be a topological space which is homotopy dominated by a finite CW complex, and let L be a finite CW complex. Then:*

$$w(L \times X) = \chi(L) \cdot i_*(w(X))$$

where $i : X \rightarrow L \times X$ is given by $i(x) = (1_0, x)$ for some $1_0 \in L$, and $\chi(L)$ denotes the Euler characteristic of L .

Proof. Let K be a finite CW complex and $g : X \rightarrow K$, $f : K \rightarrow X$ be maps such that $fg \simeq id_X$. Let $T(\alpha)$ be the mapping torus of the map $\alpha = gf : K \rightarrow K$ and let $\Phi : T(\alpha) \rightarrow X \times S^1$ be a homotopy equivalence. The space $L \times X$ is a homotopy dominated by the finite CW complex $L \times K$ using the maps $id \times g : L \times X \rightarrow L \times K$, $id \times f : L \times K \rightarrow L \times X$. Hence we have the homotopy equivalence $\bar{\Phi} : T(id \times \alpha) \rightarrow L \times X \times S^1$. But $T(id \times \alpha) = L \times T(\alpha)$ and without loss of the generality we can write $\bar{\Phi} = id \times \Phi : L \times T(\alpha) \rightarrow L \times X \times S^1$. Now our finiteness obstruction is given by:

$$\sigma(L \times X) = (id \times \Phi)_*(\tau(id \times \Phi^{-1} T \Phi)) \in Wh(L \times X \times S^1).$$

By the product theorem for Whitehead torsion (see [3] for the nice and short

geometric proof) we have:

$$\tau(id \times \Phi^{-1}T\Phi) = \chi(L) \cdot j_*(\tau(\Phi^{-1}T\Phi))$$

where $j : T(\alpha) \rightarrow L \times T(\alpha)$ is given by $j(t) = (1_0, t)$, for $t \in T(\alpha)$. Hence $\sigma(L \times X) = \chi(L) \cdot i_*(\sigma(X))$, where $i_* : Wh(X \times S^1) \rightarrow Wh(L \times X \times S^1)$. Now the formula $w(L \times X) = \chi(L) \cdot i_*(w(X))$ follows from the naturality of the Bass–Heller–Swan decomposition of $Wh(X \times S^1)$.

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Note added in proof:

In fact $\sigma(X) \in \tilde{K}_0(X)$. This can be deduced from T. Chapman, Approximation results in Hilbert cube manifolds, Trans. Amer. Math. Soc. 262 (1980), 303–334, in particular, see p. 321 of this paper.

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