Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 58 (1983)

Artikel: Two types of birational models.

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DOI: https://doi.org/10.5169/seals-44606

Comment. Math. Helvetici 58 (1983) 388–415 0010–2571/83/003388–28$01.50 + 0.20/0

Two types of birational models

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1. Introduction

Let V be a quasiprojective irreducible algebraic variety over an algebraically closed field k. Using the terminology of [8] we call a birational model $Y \xrightarrow{\varphi} V$ in which Y is a Cohen–Macaulay (CM) variety a *Macaulayfication* of V. We say that a birational model $Y \xrightarrow{\varphi} V$ *preserves a* given local *property* P if a point p of Y satisfies P whenever its image $\varphi(p)$ in V does. The aim of this paper is to describe two classes of blow-up, which for certain varieties V furnish a very simple way to get Macaulayfications that preserve normality and regularity. In view of the fact that V admits nonsingular models if k is of characteristic 0 [15] or of dimension $\leq$ 3 [1] [22] our construction should satisfy two requirements: It should work in any characteristic and it should be essentially simpler than the processes of desingularization. Our results will show that both requirements are satisfied.

Let W be the closed subset of the non-CM points of V. We only will deal with the case $\dim(W) \leq 1$, as it is done in [8] where a very effective method is given to construct Macaulayfications in this case. Unfortunately the Macaulayfications described in [8] do not preserve normality nor regularity. So using them means losing a lot of information on the basic variety V. Our main results are:

(1.1) THEOREM. *Assume that* $\dim(W)=0$. *Then there is a curve* $C \subseteq V$ *such that the blow-up* $Bl_V(C) \xrightarrow{\pi} V$ *of* V *at* C *is a Macaulayfication which preserves normality and regularity.*

(1.2) THEOREM. *Assume that* $\dim(W)=1$. *Then there is a surface* $S \subseteq V$ *and a two-codimensional closed subvariety* T *of the blow-up* $X := Bl_V(S) \xrightarrow{\varphi} V$ *of* V *at* S *such that the blow-up* $Y := Bl_X(T) \xrightarrow{\psi} X$ *of* X *at* T *is a Macaulayfication and such that* φ *and* ψ *preserve normality and* $\varphi \circ \psi$ *preserves regularity. So* $Y \xrightarrow{\varphi\psi} V$ *is a Macaulayfication which preserves normality and regularity.*

In [8] Macaulayfications also are constructed by one or two consecutive blow-up according to whether $\dim(W)$ equals 0 or 1. (According to [4] it is possible in the case $\dim(W)=1$ to replace the second blow-up by a finite birational covering. This at least gives the preservation of normality and regularity for the second step). But the blow-up are centered at ideals which in non-trivial cases never may occur as ideals of sections vanishing at a subvariety. This is contrary to our construction which makes essentially use of the reducedness of the ideals which define the blow-up. It turns out to be important for the preservation of normality that the occurring centers C, S and T are normally torsion-free (This implies more than the mere preservation of normality, namely that normality is preserved even "arithmetically": e.g. the morphism from the projecting cone of the blow-up to V preserves normality). The essential feature to guarantee the CM-propery of our blow-up is that the centers C, S and T define ideals which locally are the unmixed part of "standard-ideals" (e.g. ideals which are generated by "standard sequences"). Standard-sequences (S-sequences) already have been introduced and studied in [4]. Standard-ideals may be considered as a cohomological analogue to the permissible subvarieties which occur as the centers of blow-up in Hironakas' resolution of singularities [15]. So one of the main properties of blowing-up at a S-ideal is the preservation of the cohomology type of the exceptional fiber [4], [5], [6]. Under the mentioned analogy this corresponds to the 'preservation" of the local Hilbert-functions under a permissible blow-up [2], [14], [20]. The above property of the blow-up at a S-ideal makes it basically useful for Macaulayfication, as this latter is nothing else than an improvement of the local cohomological properties by means of blow-up.

(1.1) and its proof allow to draw the following consequences:

(1.3) COROLLARY. *Let V be a normal and assume that* $\dim(W) = 0$. *Then there is a curve $C \subseteq V$ such that $Bl_V(C) \to V$ is a Macaulayfication which preserves regularity and such that $Bl_V(C)$ is arithmetically normal.*

(1.4) COROLLARY. *Let V be a normal and of dimension* 3. *Then there is a curve $C \subseteq V$ such that $Bl_V(C)$ is arithmetically normal and arithmetically* CM *and such that $Bl_V(C)$ preserves regularity.*

(1.5) *Remark.* The proof of (1.1) will show more. Namely, if $\dim(W) = 0$ we may write $W = \{p_1, \ldots, p_t\}$. Then there are $\mathfrak{m}_{p_i}$-primary ideals $\mathfrak{q}_i \subseteq \mathcal{O}_{V,p_i}$ such that the following holds: The general curve C which is the restriction of a set-theoretic complete intersection in the projective closure of V and whose defining ideal at the point p_i is contained in $\mathfrak{q}_i$ has the property requested in (1.1), (1.3), (1.4). So, embed V into a projective closure V'. Then there are natural numbers $\nu_1, \ldots, \nu_t$ (which may be estimated by the lengths of the local cohomology of $\mathcal{O}_V$ at the points p_i) such that $C := C'|_V$ is of the requested type as soon as C' is a set-theoretic complete intersection which vanishes of order $\geq \nu_i$ at p_i and which is in a sufficiently general position.

As a consequence of (1.2) and its proof we get:

(1.6) COROLLARY. *Let V be normal and assume that* $\dim(W)=1$. *Then there is a surface $S\subseteq V$ and a subvariety $T\subseteq Bl_V(S)=:X$ such that $Bl_V(C)$ and $Bl_X(T)$ are arithmetically normal, such that the morphism $Y:=Bl_X(T)\to V$ preserves regularity and such that T is CM.*

(1.7) COROLLARY. *Let V be normal and of dimension ≤ 4. Then there are $S\subseteq V$ and $T\subseteq Bl_V(S)$ which are as in* (1.6).

(1.8) COROLLARY. *Let V be of dimension ≤ 4. Then there is a Macaulayfication $Y\to V$ of V which preserves regularity and such that Y is normal.*

(1.9) *Remark.* For the varieties S and $T\subseteq X=Bl_V(S)$ it holds a statement similar to (1.5): S may be realized by the restriction of a set-theoretic complete intersection vanishing of sufficiently high order at W and being in a sufficiently general position (defined in a projective closure of V). Once having chosen S, there are finitely many points $p_1,\dots,p_t\in V$ such that T may be obtained as the restriction of a set-theoretic complete intersection hypersurface in a projective closure of the exceptional fiber $\bar{X}$ of $X\to V$, which moreover vanishes of sufficiently high order at the fibers of $p_1,\dots,p_t$ and which is in a sufficiently general position.

The main difficulty of the proofs in fact consists in showing some corresponding local results. The local features needed to proof (1.1) are some results given in [6], where already a local version of (1.1) is given. These results also are needed to show (1.2). They are presented in the second section. Here we also list a number of rather technical results from [6] which will be used currently in the sequel.

(1.2) also needs some additional algebraic background, mainly the concept of "double-standard-sequences". We already used this concept (in a slightly different way) in [4]. The corresponding results are given in section 3.

Section 4 gives the conclusive globalization of the previously local results and so completes all the proofs. Here we mainly will use arguments of Bertini-type of the kind which are found in [9].

We use the following notations:

Concerning graded rings: If $A=R_0\oplus R_1\oplus\cdots$ is a graded ring and if $M=\bigoplus_{h\in\mathbb{Z}}M_h$ is a graded A-module, $M_{>n}$ is the A-submodule $\bigoplus_{h>n}M_h$ of M. If $N=\bigoplus_{h\in\mathbb{Z}}N_h$ is another graded A module we write $\varphi: M\xrightarrow[(d)]{}N$ to express that φ is a homogeneous A-homomorphism of degree d.

Concerning Rees-rings and associated graded rings: If I is an ideal of the ring R, $\mathfrak{R}(I)$ denotes the *Rees-ring* $\oplus_{n\geq 0} I^n$ of I, $\mathrm{Gr}\,(I)$ the corresponding *associated graded ring* $\mathfrak{R}(I)/I\mathfrak{R}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$. If $\mathcal{J}$ is an ideal in the structure sheaf of a scheme, $\mathfrak{R}(\mathcal{J})$ and $\mathrm{Gr}\,(\mathcal{J})$ are defined analogous.

Let $x \in I$. Then x^* denotes the one-form in $\mathfrak{R}(I)$ which is induced by x: $x^* = (0, x, 0, \ldots)$. $\bar{x}$ is the corresponding one-form in $\mathrm{Gr}\,(I)$. $\mathrm{Bl}_R(I)$ stands for the blow-up $\mathrm{Proj}\,(\mathfrak{R}(I))$ of R at I. If $\mathcal{J}$ is an ideal in the structure-sheaf of a scheme X, $\mathrm{Bl}_X\,(\mathcal{J})$ stands for the blow-up of X at $\mathcal{J}$. If $\mathcal{J}$ is the ideal of sections vanishing at a closed subset $Y \subseteq X$, we write $\mathrm{Bl}_X\,(Y)$ for $\mathrm{Bl}_X\,(\mathcal{J})$ and speak of the blow-up X at Y.

Concerning local cohomology (s. [11], [12]): Let R be a noetherian ring and let $J \subseteq R$ be an ideal. Then $\Gamma_J(M)$ stands for the *J-torsion* $\bigcup_{n\geq 0} (0:J^n)_M = \varinjlim_n \mathrm{Hom}_R\,(R/J^n, M)$ of the R-module M. H^i_J stands for the ith *local cohomology functor supported at J* which is the ith right derived functor $R^i\Gamma_J$ of the J-torsion and which also may be written as $\varinjlim_n \mathrm{Ext}^i_R\,(R/J^n, \cdot) \cdot D_J$ stands for the functor of *J-transform* $\varinjlim_n \mathrm{Hom}\,(J^n, \cdot) \cdot \overline{M^J}$ (or $\bar{M}$ if no confusion is possible) stands for the *J-reduction* $M/\Gamma_J(M)$ of the R-module M. We frequently shall use the following well known relations between these functors:

(1.10) $\quad H^0_J(\overline{M^J}) = 0, \qquad H^i_J(\overline{M^J}) = H^i_J(M)\quad$ for all $i > 0$.

(1.11) $\quad D_J(\overline{M^J}) = D_J(M), \qquad R^iD_J = H^{i+1}_J\quad$ for all $i > 0$.

Moreover we may write

(1.12) $\quad D_J(M) = \bigcup_{n\geq 0} (\overline{M^J} : J^n)_{S^{-1}\overline{M^J}}$, where S is any multiplicatively closed set of R which consists of non-zerodivisors with respect to $\overline{M^J}$ and meets $\sqrt{J}$ (such S exist if M is of finite type).

Moreover there is an exact sequence

(1.13) $\quad 0 \to \Gamma_J(M) \to M \to D_J(M) \to H^1_J(M) \to 0$

for each R-module M, which induces in particular

(1.14) $\quad H^i_J(D_J(M)) = 0,\;$ if $\; i \leq 1; \qquad H^i_J(D_J(M)) = H^i_J(M),\;$ if $\; i > 1$.

If X is a locally noetherian scheme and if $Z \subseteq X$ is stable under specialization, the corresponding local cohomology functors supported in Z are denoted by H^i_Z

(These functors are obtained by globalization of the previous one.) The corresponding transform is denoted by D_Z (D_Z is the Z-closure introduced in [10]; if Z is closed and if $\mathcal{F}$ is a quasicoherent sheaf over X, $D_Z(\mathcal{F})$ is the direct image of the restriction $\mathcal{F}|_{X-Z}$).

Concerning loci: Let X be a locally noetherian scheme. Then by Reg(X), Sing(X), Nor(X), Fac(X), CM(X) we respectively denote the set of its regular, singular, normal, factorial and CM-points. If $X = \operatorname{Spec}(R)$, where R is a noetherian ring, these loci are denoted respectively by Reg(R), Sing(R), Nor(R), Fac(R), CM(R).

As for the unexplained notations and terminology see [13] (algebraic geometry) and [17] (commutative algebra).

(1.15) ***Remark***: Our results hold in fact over any infinite field k. The adjustment needed to treat this case is the use of double standard sequences, which have this property universally, e.g. under finite extensions of the base field k. We used this concept in [4]. To keep our arguments less technical we decided only to present the case of an algebraically closed field k.

2. Standard-sequences

In this section we present some notions and results from [4], [5] and [6]. These will furnish the algebraic background of our proofs.

In the sequel let R be a noetherian ring, let $J \subseteq R$ be an ideal and let M be a finitely generated R-module. An element $x \in R$ is said to be *J-filter-regular* (resp. a J-f-regular element or a J-f-element) *with respect to M* if it belongs to the set reg$(\overline{M^J})$ of regular elements (= non-zerodivisors) with respect to $\overline{M^J}$. A sequence $x_1, \ldots, x_r \in R$ is called J-f-regular (or a J-f-sequence) with respect to M if x_i is J-f-regular with respect to $M/(x_1, \ldots, x_{i-1})M$ for all $i \leq r$. These concepts have been introduced in [21] for the special case of a local ring with a maximal ideal J. $x_1, \ldots, x_{i-1}$ is a J-f-sequence with respect M iff it is a regular sequence with respect to $M_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(R) - V(J)$. J-f-regularity with respect to M is equivalent to J-f-regularity with respect to $\overline{M^J}$ and to J-f-regularity with respect to $D_J(M)$. J-f-regularity with respect to M induces JR'-f-regularity with respect to $R' \otimes M$, where R' is a noetherian and flat R-algebra. The following easy statements will be used frequently (grade (J) denotes the common length of all R-regular sequences in J):

(2.1) LEMMA. (I) *Assume that* $\operatorname{Spec}(R) - V(J)$ *is* CM *and let* $x_1, \ldots, x_r \in \varepsilon\sqrt{J}$ *be such that* $ht(x_1, \ldots, x_r) = r$. *Then* $x_1, \ldots, x_r$ *is J-f-regular with respect to* R.

(ii) *Let* $x_1, \dots, x_r \in \sqrt{J}$ *be J-f-regular with respect to R. Then, if* $r \leqslant ht(J)$, *it holds* $ht(x_1, \dots, x_r) = r$ if $r \leqslant \operatorname{grade}(J)$ $x_1, \dots, x_r$ *even constitute a regular sequence with respect to R.*

Now, we define the *cohomological J-finiteness-dimension* of M:

(2.2) $e_J(M) := \inf\{i \mid H_J^i(M)$ is not finitely generated over $R\}$.

Using (1.10) we obtain $e_J(M) = e_J(\overline{M^J})$. If $M \neq \Gamma_J(M)$ it is well known that $e_J(M) \leqslant \operatorname{ht}(J \cdot R/\operatorname{ann} M)$. In particular we have $e_J(M) = \infty \Leftrightarrow M = \Gamma_J(M)$. If x is J-f-regular with respect to M, there is a short exact sequence $0 \to \overline{M^J} \xrightarrow{x} \overline{M^J} \to \overline{M^J}/x\overline{M^J} \to 0$, which shows that $e_J(M/xM) \geqslant e_J(M) - 1$.

In [4] we introduced the notion of *standard-sequences*. In [5] we systematically studied a certain class of *truncated standard-sequences* (but only for the case where J was the maximal ideal of a local ring).

(2.3) DEFINITION. A sequence $x_1, \dots, x_r \in R$ is called a J-standard-sequence (J-S-sequence) with with respect to M if

(i) $r \leqslant e_J(M)$,

(ii) $x_1, \dots, x_r$ is a J-f-sequence with respect to m,

(iii) $(x_1, \dots, x_r)H_J^i(M/(x_1, \dots, x_j)M) = 0$, for all i, j with $i + j < r$. $x_1, \dots, x_r$ is called a truncated J-standard-sequence (J-S^+-sequence) with respect to M if $r < e_J(M)$ and if (iii) holds for all pairs i, j for which $i + j < r$. It is the same to say that there is a $y \in R$ such that $x_1, \dots, x_r, y$ is a J-S-sequence with respect to M.

$x_1, \dots, x_r$ is a J-S-sequence (resp. a J-S^+-sequence) with respect to M iff it is with respect to $\overline{M^J}$. If $e_J(M) > 1$ (which implies in particular that $D_J(M)$ is finitely generated), the same statement holds for the pair of modules M and $D_J(M)$. If $x_1, \dots, x_r$ is a J-f-sequence with respect to M, $x_2, \dots, x_r$ is a J-S (resp. a J-S^+)-sequence with respect to M/x_1M if $x_1, \dots, x_r$ has the corresponding property with respect to M.

The following result has been shown in [4]:

(2.4) LEMMA. *Let* $x_1, \dots, x_r \in R$ *be a J-f-sequence with respect to M. Then*:

$$\operatorname{ann}(H_J^i(M/(x_1, \dots, x_r)M \supseteq \prod_{j=0}^{r} [\operatorname{ann}(H_J^{i+j}(M))]^{(?)}.$$

If $H_J^i(M)$ *is finitely generated, it holds* $\sqrt{\operatorname{ann}(H_J^i(M))} \supseteq J$.

So (2.4) induces

(2.5) COROLLARY. *There is an ideal* $\mathfrak{a} \subseteq R$ *such that* $\sqrt{\mathfrak{a}} \supseteq J$ *and such that each* J-f-*sequence* $x_1, \ldots, x_r \in \mathfrak{a}$ *with respect to* M (*with* $r \leqslant e_J(M)$) *is a* J-S-*sequence with respect to* M. *Consequently if* $r < e_J(M)$, $x_1, \ldots, x_r$ *is a* J-S^+-*sequence under these assumptions.* (*It suffices to choose* $\mathfrak{a} = [\prod_{i<e_J(M)} \operatorname{ann}(H^i_J(M))]^{2^{e_J(M)-1}}$.)

If a sequence keeps having one of these properties under all its permutations, we express this in using the prefix p. So we shall speak of J-pf-sequences (= permutable J-f-sequences (= permutable J-standard-sequences) and J-pS^+-sequences (= permutable truncated J-standard-sequences).

(2.6) *Remark.* We later mainly shall use the concept of truncated standard-sequence. The situation in which they will come up is as follows: Let V and W be as in the introduction. Let $\mathcal{J} \subseteq \mathcal{O}_V$ be such that $V(\mathcal{J})$ is of codimension h in V and such that $W \subseteq V(\mathcal{J})$. Then $V - V(\mathcal{J})$ is CM. So if $p \in V - V(\mathcal{J})$ it holds $\operatorname{depth}(\mathcal{O}_{V,p}) + \operatorname{codim}(\overline{\{p\}}, \overline{\{p\}} \cap V(\mathcal{J})) = \operatorname{codim}(V, \overline{\{p\}}) + \operatorname{codim}(\overline{\{p\}}, \overline{\{p\}} \cap V(\mathcal{J})) \geqslant \operatorname{codim}(V, V(\mathcal{J})) = h$, where for a locally noetherian scheme X and a closed subscheme Y $\operatorname{codim}(X, Y)$ denotes the codimension of Y with respect to X. If $h > 0$ and if p is the generic point of V, equality holds in the above estimate. So by Grothendiecks finiteness theorem for local cohomology [11] the sheaves $H^i_{V(\mathcal{J})}(\mathcal{O}_V)$ are coherent for all $i < b$. Now, let $p \in V(\mathcal{J})$ and put $R = \mathcal{O}_{V,p}$ and $J = \mathcal{J}_p$. Then it follows that $e_J(R) = h$ if $h > 0$. Let $\mathfrak{b}$ be the following ideal of $\mathcal{O}_V$: $\mathfrak{b} = [\prod_{i<h} \operatorname{ann}(H^i_{V(\mathcal{J})}(\mathcal{O}_V))]^{2^{h-1}}$ $(h > 0)$ and put $\mathfrak{a} = \mathfrak{b}_p$. Then $\mathfrak{a} \subseteq R$ is as in (2.5). Moreover $\operatorname{Spec}(R) - V(J)$ is CM. So (2.1) (i) implies that each partial system of parameters $x_1, \ldots, x_r \in \mathfrak{a}$ is a J-pS-sequence with respect to R. If $r < h$, $x_1, \ldots, x_r$ is even a J-pS^+-sequence with respect to R.

For the rest of this paragraph we fix the following notations: Let $0 < r < e_J(M)$ and let $x_1, \ldots, x_r \in \sqrt{J}$ be a J-pS^+-sequence with respect to M. Let $L = (x_1, \ldots, x_r)$ and put $\overline{M^J} = \bar{M}$. In [4] we have proved the following results:

(2.7) LEMMA. (i) *The canonical maps* $H^1_J(L^n M) \to H^1_J(L^{n-1}M)$ *vanish for all* $n > 0$.

(ii) $LM \cap \Gamma_J(M) = 0$.

(iii) $D_J(L^n M) = \bigcup_i (L^n \bar{M} : J^i)_{\bar{M}} \subseteq L^{n-1}\bar{M}$ $(n > 0)$.

(iv) $\overline{M/L^n M^J} = \bar{M}/D_J(L^n M)$.

(2.8) LEMMA. $\overline{L^n M/L^{n+1}M^J} = \overline{M/LM}^{J\binom{n+r-1}{r-1}}$.

Now, let $M' = M/x_1 M$, $L' = (x_2, \ldots, x_r)$. Then $x_2, \ldots, x_r$ is a J-pS^+-sequence with respect to M'. According to [4] we have

(2.9) LEMMA. *If x_1^* denotes the maps induced by multiplication with x_1, we have the following canonical exact sequences*

(i) $$0 \to L^{n-2}\bar{M}/D_J(L^{n-1}M) \xrightarrow{x_1^*} L^{n-1}\bar{M}/D_J(L^nM) \to L'^{n-1}\overline{M'}/D_J(L'^nM') \to 0$$

for all $n > 1$. (Thereby this sequences split).

(ii) $$0 \to D_J(L^{n-1}M) \xrightarrow{x_1} D_J(L^nM) \to D_J(L'^nM') \to 0,$$

$$0 \to D_J(L^{n-1}M)/D_J(L^nM) \xrightarrow{x_1} D_J(L^nM)/D_J(L^{n+1}M) \to D_J(L'^nM')/D_J(L'^{n+1}M') \to 0, \textit{ for all } n > 0.$$

(iii) $$0 \to L^{n-1}\bar{M} \xrightarrow{x_1} L^nM \to L'^nM' \to 0,$$

$$0 \to L^{n-1}\bar{M}/L^nM \xrightarrow{x_1} L^nM/L^{n+1}M \to L'^nM'/L'^{n+1}M' \to 0, \textit{ for all } n > 0.$$

We denoted the above injections with x_1^* by the following reason:

$$\bigoplus_n L^n\bar{M}/D_J(L^{n+1}M) = \bigoplus_n \overline{L^nM/L^{n+1}M^J}, \qquad \bigoplus_n D_J(L^nM) = D_J\Big(\bigoplus_n L^nM\Big),$$

$$\bigoplus_n D_J(L^nM)/D_J(L^{n+1}M) = D_J\Big(\bigoplus_n L^nM\Big)\Big/D_J\Big(\bigoplus_n L^{n+1}M\Big),$$

$$\bigoplus_n L^m\bar{M} \quad \text{and} \quad \bigoplus_n L^n\bar{M}/L^{n+1}\bar{M}$$

all are in a canonical way graded modules over the Rees-algebra $\mathfrak{R}(L)$. So the operation of the one-form x_1^* on these modules is exactly the corresponding map x_1^* of (2.9). The purpose of the above sequences is of merely technical nature, as they repeatedly come up to perform different induction arguments. They come close to generalize the fact that the associated graded module with respect to an M-regular sequence $x_1, \ldots, x_r$ is a polynomial extension of $M/(x_1, \ldots, x_r)M$ [18].

The following result was shown in [6] for the special case where R is local and J its maximal ideal. But in fact the proof works in full generality.

(2.10) LEMMA. *For all $n > 0$ it holds $L^nD_J(LM) = D_J(L^{n+1}M)$.*

As a consequence we get:

(2.11) COROLLARY. *Let* $0<r<e_J(R)$. *Put* $\bar{R}=\overline{R^J}$. *Let* $x_1,\ldots,x_r\in\sqrt{J}$ *be a* J*-pS*$^+$*-sequence with respect to* R *and put* $L=(x_1,\ldots,x_r)$, $\tilde{L}=\bigcup_j(L\bar{R}:J^j)_{\bar{R}}$. *Then it holds:*

(i) $\tilde{L}^n=D_J(L^n)$, $(n>0)$,

(ii) $\operatorname{Ass}(\bar{R}/\tilde{L}^n)=\operatorname{Ass}(\bar{R}/\tilde{L})=\operatorname{Ass}(R/L)-V(J)$.

Proof. By (2.7) (iii) we have $\tilde{L}^n\subseteq D_J(L^n)$ and $\tilde{L}=D_J(L)$. (2.10) implies $D_J(L^n)=L^{n-1}D_J(L)\subseteq\tilde{L}^{n-1}\tilde{L}=\tilde{L}^n$. This proves (i).

To show (ii) we first observe that $\overline{L^{n-1}\bar{R}/\tilde{L}^n}=\overline{L^{n-1}/L^n}$ is a free module over $\overline{R/L}=\bar{R}/\tilde{L}$ (use (2.7) (iii), (2.8) and (2.11) (i)). So we have $\operatorname{Ass}(L^{n-1}\bar{R}/\tilde{L}^n)=\operatorname{Ass}(\bar{R}/\tilde{L})$. By the short exact sequence $0\to L^{n-1}\bar{R}/\tilde{L}^n\to\tilde{L}^{n-1}/\tilde{L}^n\to\tilde{L}^{n-1}/L^{n-1}\bar{R}\to 0$ we see that $\operatorname{Ass}(\bar{R}/\tilde{L})=\operatorname{Ass}(L^{n-1}\bar{R}/\tilde{L}^n)\subseteq\operatorname{Ass}(\tilde{L}^{n-1}/\tilde{L}^n)\subseteq\operatorname{Ass}(\bar{R}/\tilde{L})\cup\operatorname{Ass}(\tilde{L}^{n-1}/L^{n-1}\bar{R})$. As $\tilde{L}^{n-1}=\bigcup_j(L^{n-1}\bar{R}:J^j)_{\bar{R}}$ we have $J^\nu\tilde{L}^{n-1}\subseteq L^{n-1}\bar{R}$ for some ν. So we have $\operatorname{Ass}(\tilde{L}^{n-1}/L^{n-1}\bar{R})\subseteq V(J)$. As $\tilde{L}^{n-1}/\tilde{L}^n=\overline{\tilde{L}^{n-1}/\tilde{L}^n}$ has no J-torsion it follows that $\operatorname{Ass}(\tilde{L}^{n-1}/\tilde{L}^n)\cap V(J)=\emptyset$. So we get $\operatorname{Ass}(\tilde{L}^{n-1}/\tilde{L}^n)=\operatorname{Ass}(\bar{R}/\tilde{L})$. Now, using the exact sequences $0\to\tilde{L}^{n-1}/\tilde{L}^n\to\bar{R}/\tilde{L}^n\to\bar{R}/\tilde{L}^{n-1}\to 0$ we conclude by induction on n that $\operatorname{Ass}(\bar{R}/\tilde{L}^n)=\operatorname{Ass}(\bar{R}/\tilde{L})$.

$\operatorname{Ass}(\bar{R}/\tilde{L})=\operatorname{Ass}(\overline{R/L^J})=\operatorname{Ass}(R/L)-V(J)$ is clear by the definition of the functor $\overline{\cdot}^J$.

(2.12) *Remark.* (2.11) (ii) will be of importance in our later applications as it states that L is *normally torsion-free* (An ideal ideal of R is said to be normally torsion-free, if $\operatorname{Gr}(I)$ is torsion-free over R/I or – equivalently – if $\operatorname{Ass}(R/I^n)\subseteq\operatorname{Ass}(R/I)$, $\forall n$.)

Finally we shall make use of the following result, which deals with the special case where R is local and J its maximal ideal.

(2.13) PROPOSITION. *Let* $(R,\mathfrak{m})$ *be a local ring of dimension* $d>1$ *and assume that* $e_{\mathfrak{m}}(R)=d$. *Let* $x_1,\ldots,x_{d-1}\in\mathfrak{m}$ *be a* $\mathfrak{m}$*-pS*$^+$*-sequence with respect to* R *and put* $L=(x_1,\ldots,x_{d-1})$, $\bar{R}=\overline{R^{\mathfrak{m}}}$, $\tilde{L}=\bigcup_j(L\bar{R}:\mathfrak{m}^j)_{\bar{R}}$. *Then:*

(i) $D_{\mathfrak{m}}(\mathfrak{R}(L))$ *is finitely generated as a* $\mathfrak{R}(L)$*-module;*

(ii) $\operatorname{Bl}_{\bar{R}}(\tilde{L})=\operatorname{Proj}(\mathfrak{R}(\tilde{L}))=\operatorname{Proj}(D_{\mathfrak{m}}(\mathfrak{R}(L))$ *is* CM;

(iii) *If* R *is* CM, *we have* $D_{\mathfrak{m}}(R)=R$, $L=\tilde{L}$ *and* $\mathfrak{R}(\tilde{L})$ *is* CM *in this case.*

Proof. Let $\mathfrak{n}$ be the homogeneous maximal ideal $\mathfrak{m}\oplus L\oplus L^2\oplus\cdots$ of $\mathfrak{R}(L)$. Let D be the graded $\mathfrak{R}(L)$-module $D_{\mathfrak{m}}(\mathfrak{R}(L))=\bigoplus_n D_{\mathfrak{m}}(L^n)$. We know by [6] that D is finitely generated over $\mathfrak{R}(L)$ and that its local cohomology supported in $\mathfrak{n}$

satisfies

$$(*)\quad H^i_{\mathfrak{n}}(D)=\begin{cases}0, & \text{if}\quad i\leqslant \min d':=\min(3,d),\\ \bigoplus_{j=-i+3}^{-1}[H^{i-1}_{\mathfrak{m}}(R)]_j, & \text{for}\quad d'<i\leqslant d.\end{cases}$$

Thereby – for an R-module H – $[H]_j$ stands for the graded $\mathfrak{R}(L)$-module whose tth component is 0 or H, according to whether $t\neq j$ or $t=j$.

This shows that the $X:=\operatorname{Proj}(\mathfrak{R}(L))$-sheaf $\tilde{D}$ induced by D is coherent and CM (the latter is a consequence of the weak part of Grothendiecks finiteness theorem [11]). But – as D and $\mathfrak{R}(L)$ differ only in degree 0 – $\tilde{D}$ also is the structure-sheaf of $\mathrm{Bl}_{\bar{R}}(\tilde{L})$. This proves the first statement. If R is CM we have $H^i_{\mathfrak{m}}(R)=0$ for $i\leqslant 1$. This induces $R=D_{\mathfrak{m}}(R)$ (s. (1.13)). Moreover we then have $\overline{R/L^{\mathfrak{m}}}=R/L$, thus $\tilde{L}=L$. This shows that $D=\mathfrak{R}(\tilde{L})$. By (*) we have $H^i_{\mathfrak{n}}(D_{\mathfrak{n}})=0$ for all $i<d+1=\dim D_{\mathfrak{n}}$. So $D_{\mathfrak{n}}$ is CM. So the same is true for $D=\mathfrak{R}(\tilde{L})$ by [16].

(2.14) *Complement. Let R, L and $\tilde{L}$ be as in* (2.13) *and assume moreover that $d\leqslant 3$ and that* depth $(R)\geqslant 2$ (*which latter is the case if R is normal*). *Then $\mathfrak{R}(\tilde{L})$ is* CM.

Proof. Our assumptions imply that $R=D_{\mathfrak{m}}(R)$. So, in the notations of the previous proof we have $\mathfrak{R}(\tilde{L})=D$. Now we conclude as above by (*).

(2.15) *Remark.* We shall use these results mainly in the following context: Let V and W be as in the introduction, assuming that $\dim(W)=0$. Put $d=\dim(V)$ and let $\mathcal{J}$ be the ideal of sections vanishing at W. By (2.6) there is an ideal $\mathfrak{b}\subseteq\mathcal{O}_V$ such that $V(\mathfrak{b})=W$ which has the following property: For each $p\in V$ and each partial system of parameters $x_1,\ldots,x_{d-1}$ of $\mathcal{O}_{V,p}$ contained in $\mathfrak{b}_p$, $x_1,\ldots,x_{d-1}$ is a $\mathfrak{m}_p$-pS^+-sequence with respect to $\mathcal{O}_{V,p}$. So, let $\mathcal{L}\subseteq\mathfrak{b}$ be a locally complete intersection of codimension $d-1$ with respect to V (This means that for any point $p\in V(\mathfrak{b})$ $\mathcal{L}_p$ is generated by a partial system of parameters $x_1,\ldots,x_{d-1}$ with respect to $\mathcal{O}_{V,p}$). Let $\tilde{\mathcal{L}}=\bigcup_j(\mathcal{L}:\mathcal{J}^j)\mathcal{O}_V$. Then, using the previous notations and putting $R=\mathcal{O}_{V,p}$, $\mathfrak{m}=\mathfrak{m}_p$, $L=(x_1,\ldots,x_{d-1})=\mathcal{L}_p$ and $\tilde{L}=\bigcup_j(L:\mathfrak{m}^j)_R$ we have $\tilde{L}=\tilde{\mathcal{L}}_p$. So by (2.13) $\mathcal{L}_p$ is a normally torsion-free ideal of $\mathcal{O}_{V,p}$ for all $p\in V(\mathcal{L})$ and $\mathrm{Bl}_V(\tilde{\mathcal{L}})$ is CM. If $d\leqslant 3$ and if V is normal (It suffices that V is S_2) we see by (2.14) that $Bl_V(\tilde{\mathcal{L}})$ is even arithmetically CM.

3. Double standard-sequences

Let J, $T\subseteq R$ be ideals of the noetherian ring R such that $J\subseteq T$ and such that there is an element $t\in T$ with $\sqrt{T}=\sqrt{(J,t)}$. In this section we consider double standard-sequences and truncated double standard-sequences with respect to J and T. These are J-S-sequences (resp. J-S^+-sequences) subject to another condition, involving local cohomology supported in T. In [4] we developed this concept in a slightly different way, without assuming the existence of an element t as above. Note that the existence of such an element induces $H_T^i(H_J^j(M)) = 0$ for all $i > 1$, all $j \geqslant 0$ and all R-modules M (s. for example [7]).

(3.1) LEMMA. *Let $x_1, \ldots, x_r$ R be a J-f-sequence with respect to M. Then it holds* $\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_r)M^J})) \supseteq \prod_{j=0}^r [\operatorname{ann}(H_T^{i+j}(\overline{M^J}))]^{\binom{r}{j}}$

Proof. (Induction on r). If $r = 0$ or $i = 0$, all is clear. So let $r, i > 0$. As x_r is J-f-regular with respect to $M/(x_1, \ldots, x_{r-1})M$ we have an exact sequence

$$0 \to \overline{M/(x_1, \ldots, x_{r-1})M^J} \xrightarrow{x_r} \overline{M/(x_1, \ldots, x_{r-1})M^J} \to \overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)} \to 0.$$

The corresponding $H_T^{\cdot}$-sequence furnishes the relation

$$\begin{aligned}&\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}))\\ &\quad \supseteq \operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J})) \cdot \operatorname{ann}(H_T^{i+1}(\overline{M/(x_1, \ldots, x_r)M^J})).\end{aligned}$$

By the exact sequence

$$\begin{aligned}0 \to \Gamma_J(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}) &\to \overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}\\ &\to \overline{M/(x_1, \ldots, x_r)M^J} \to 0\end{aligned}$$

we also obtain an exact sequence

$$H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}) \to H_T^i(\overline{M/(x_1, \ldots, x_r)M^J}) \to H_T^{i+1}(\Gamma_J(\cdots)) = 0,$$

which shows that

$$\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_r)M^J})) \supseteq \operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)})).$$

So, using induction, we get

$$\operatorname{ann}(H^i_T(\overline{M/(x_1,\ldots,x_r)M^J})) \supseteq \operatorname{ann}(H^i_T(\overline{M/(x_1,\ldots,x_{r-1})M^J}))$$

$$\times \operatorname{ann}(H^{i+1}_T(\overline{M/(x_1,\ldots,x_{r-1})M^J})) \supseteq \prod_{0\leq j\leq r-1} [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r-1}{j}}$$

$$\times \prod_{0\leq 1\leq r-1} [\operatorname{ann}(H^{i+1+l}_T(\overline{M^J}))]^{\binom{r-1}{l}}$$

$$= \operatorname{ann}(H^i_T(\overline{M^J}))\Big[\prod_{0<j<r} [\operatorname{ann}(H^{i+j}_T)(\overline{M^J}))]^{\binom{r-1}{j}} \cdot [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r-1}{j-1}}\Big]$$

$$\times \operatorname{ann}(H^{i+r}_T(\overline{M^J})) = \prod_{0\leq j\leq r} [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r}{j}}.$$

For a finitely generated R-module M let us introduce the following notations

(3.2) (i) $\lambda_{J,T}(M) = \sup\{n \mid \sqrt{\operatorname{ann}(H^i_T(M))} \supseteq J, \text{ for all } i \leq n\}$,
(ii) $e_{J,T}(M) = \min\{\lambda_{J,T}(M), e_J(M)\}$.

We call $e_{J,T}(M)$ the *J, T-finiteness-dimension* of the R-module M.

(3.3) DEFINITION. Let M be a finitely generated R-module. A sequence $x_1, \ldots, x_r \in R$ is called a *J, T-standard-sequence* ($= J, T$-S-sequence) *with respect to M* if

(i) $r \leq e_{J,T}(M)$

(ii) $x_1, \ldots, x_r$ is a J-S-sequence with respect to M

(iii) $(x_1, \ldots, x_r)H^i_T(\overline{M/(x_1, \ldots, x_j)M^J}) = 0$ for all i, j with $i + j \leq r$.

$x_1, \ldots, x_r$ is said to be a *truncated J, T-standard-sequence* ($= J, T$-S^+-sequence) with respect to M if $r < e_{J,T}(M)$, if $x_1, \ldots, x_r$ is a J-S^+-sequence with respect to M and if (iii) holds for all pairs i, j for which $i + j \leq r + 1$. So $x_1, \ldots, x_r$ is a $J, T - S^+$-sequence with respect to M iff there is an element $y \in R$ such that $x_1, \ldots, x_r, y$ is a J, T-S^+-sequence with respect to M.

This is the concept of the previously announced (truncated) double-standard-sequences.

Using (2.5) and (3.1) we obtain:

(3.4) LEMMA. *Let M be a finitely generated R-module. Then there is an ideal $\mathfrak{a} \subseteq R$ such that $\sqrt{\mathfrak{a}} \supseteq J$ and such that each J-f-sequence $x_1, \ldots, x_r \in \mathfrak{a}$ with respect to M of length $r \leq e_{J,T}(M)$ is a J, T-S-sequence with respect to M. If $r < e_{J,T}(M)$,*

$x_1, \ldots, x_r$ will be a J, T-S^+-sequence with respect to M under the above assumptions. (*It suffices to choose $\mathfrak{a}$ as the ideal*

$$\left[\prod_{j<e_{J,T}(M)} (\operatorname{ann}(H^j_J(M)) \cdot \operatorname{ann}(H^{j+1}_T(\overline{M^J})))\right]^{2^{e_{J,T}(M)-1}}).$$

If a sequence $x_1, \ldots, x_r$ is a J, T-S-sequence under all its permutations we speak again of a J, T-pS-sequence. Similarly we use the notation of J, T-pS^+-sequence in case of a permutable truncated J, T-standard-sequence.

(3.5) *Remark.* We shall apply the above concept in the following context: Let V and W be as in the introduction and put $\dim(V) = d$. Assume that $\dim(W) \leq 1$ and let $\mathscr{J} \subseteq \mathcal{O}$ be an ideal such that $V(\mathscr{J}) \supseteq W$. Let $Z \subseteq V$ be the set of all closed points of V. Z is stable under specialization. So the local cohomology functors H^i_Z are defined in the category of quasicoherent $\mathcal{O}_V$-sheaves. As $V - V(\mathscr{J})$ is CM [7] guarantees that $\sqrt{\operatorname{ann}}(H^{j+1}_Z(\mathcal{O}_V)) \supseteq \mathscr{J}$ and $\sqrt{\operatorname{ann}}(H^j_{V(\mathscr{J})}(\mathcal{O}_V)) \supseteq \mathscr{J}$ for all $j < d-1$ (The second statement also follows by (2.6)). Now, put

$$\mathfrak{b} = \left[\prod_{j<d-1} \operatorname{ann}(H^j_{V(\mathscr{J})}(\mathcal{O}_V)) \cdot \operatorname{ann}(H^{j+1}_Z(\mathcal{O}_V))\right]^{2^{d-2}}.$$

Let $p \in Z$, $R = \mathcal{O}_{V,p}$, $J = \mathscr{J}_p$, $\mathfrak{a} = \mathfrak{b}_p$, $\mathfrak{m} = \mathfrak{m}_{V,p}$. Then we have $H^i_{V(\mathscr{J})}(\mathcal{O}_V)_p = H^i_J(R)$, $H^i_Z(\mathcal{O}_V)_p = H^i_{\mathfrak{m}}(R)$, $e_{J,\mathfrak{m}}(R) = d-1$ and $\mathfrak{a} = [\prod_{j<d-1} \operatorname{ann}(H^j_J(R)) \cdot \operatorname{ann}(H^{j+1}_{\mathfrak{m}}(R))]^{2^{d-2}}$. By (2.1) (i) each partial system of parameters $x_1, \ldots, x_{d-2} \in \mathfrak{a}$ for the local ring R is a J-f-sequence with respect to R, thus a $J, \mathfrak{m}$-pS^+-sequence with respect to R (3.4). So, let $\mathscr{L} \subseteq \mathfrak{b}$ be a locally complete intersection-ideal of codimension $d-2$. This means in particular, that $\mathscr{L}_p$ is generated by a partial system of parameters $x_1, \ldots, x_{d-2}$ for the local ring $\mathcal{O}_{V,p}$ for each closed $p \in V(\mathscr{J})$. If $p \in V(\mathscr{J})$ this system is a $\mathscr{J}_p, \mathfrak{m}_p$-$pS^+$-sequence by the previous remark. If $p \notin V(\mathscr{J})$ the same is true as $x_1, \ldots, x_{d-2}$ is a regular sequence in the CM-ring $\mathcal{O}_{V,p}$. So $\mathscr{L}_p$ is generated by a $\mathscr{J}_p, \mathfrak{m}_p$-$pS^+$-sequence of length $d-2$ for all closed $p \in V(\mathscr{L})$. Let $\tilde{\mathscr{L}} = \bigcup_j (\mathscr{L} : \mathscr{J}^j)_{\mathcal{O}_V}$. Then, by (2.13) $\tilde{\mathscr{L}}_p$ is a normally torsion-free ideal of the local ring $\mathcal{O}_{V,p}$ for each closed $p \in V(\mathscr{L})$.

The following result is an extension of (2.7)(i) to double-standard-sequences. Similar to (2.7)(i), which is of fundamental significance for the treatment of standard-sequences, the result to come is a strong tool to treat double-standard-sequences.

(3.6) LEMMA. *Let* $x_1, \ldots, x_r$ $(r < e_{J,T}(M))$ *be a* J, T-pS^+*-sequence with respect to* M *(which is assumed to be finitely generated). Put* $L = (x_1, \ldots, x_r)R$. *Then the canonical maps*

$$H^2_T(D_J(L^nM)) \to H^2_T(L^{n-1}\overline{M^J}) \qquad (n > 0)$$

(induced by the injections $D_J(L^nM) \hookrightarrow L^{n-1}\overline{M^J}$ (2.7) (iii)) *vanish.*

Proof. The case $r=0$ is trivial. First we treat the case $r=1$. Clearly we may assume $\Gamma_J(M)=0$. By the commutative diagram

$$\begin{array}{ccc} L^nM & \hookrightarrow & L^{n-1}M \\ \wr\Vert\uparrow & & \wr\Vert\uparrow \\ M & \xrightarrow{x_1} & M \end{array} \quad \text{we get the situation} \quad \begin{array}{ccc} H^2_T(L^nM) & \xrightarrow{\alpha} & H^2_T(L^{n-1}M) \\ \Vert\wr & & \Vert\wr \\ H^2_T(M) & \xrightarrow{x_1} & H^2_T(M) \end{array}$$

As x_1 forms a J, T-pS^+-sequence with respect to M we have $x_1H^2_T(M)=0$. This shows that α vanishes. By the sequence $0 \to L^nM \to D_J(L^nM) \to H^1_J(L^nM) \to 0$ we obtain exact sequence $H^2_T(L^nM) \xrightarrow{\beta} H^2_T(D_J(L^nM)) \to H^2_T(H^1_J(L^nM)) = 0$, which shows that β is onto. Now the diagram

$$\begin{tikzcd} H^2_T(D_J(L^nM)) \arrow[rr] & & H^2_T(L^{n-1}M) \\ & H^2_T(L^nM) \arrow[ul, "\beta"] \arrow[ur, "\alpha=0"'] & \end{tikzcd}$$

allows to conclude.

Next we treat the case $n=1$. By the above we know the result if $r=1$. Let $r>1$. Using (2.9) (ii) (iii) and putting $L' = (x_2, \ldots, x_r)R$ and $M' = M/x_1M$ we have the following diagram with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & D_J(M) \arrow[r, "\cong"'] \arrow[d, "x_1"] & D_J(x_1M) \arrow[d] & & \\ 0 \arrow[r] & D_J(LM) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \overline{M/LM^J} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & D_J(L'M') \arrow[r] \arrow[d] & \overline{M'^J} \arrow[r] \arrow[d] & \overline{\overline{M'^J}/L'\overline{M'^{J}}^J} \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

Applying $H_T(\cdot)$ to this diagram we obtain the following situation

$$\begin{tikzcd}
H^2_T(D_J(M)) \arrow[r] \arrow[d, "\cong"] & H^2_T(D_J(LM)) \arrow[r] \arrow[d] & H^2_T(D_J(L'M')) \arrow[d] \\
H^2_T(D_J(x_1M)) \arrow[r] & H^2_T(M) \arrow[r] & H^2_T(\overline{M'^J})
\end{tikzcd} \qquad \text{(rows exact!)}$$

By the case $r=1$ the first map in the second row vanishes, so that the second map in this row is mono. By induction the last vertical map vanishes. So the vertical arrow in the middle is trivial. But this proves our claim.

Finally we treat the case $r, n>1$ by double induction, starting with the following commutative diagram with exact rows and columns and splitting last column (s. (2.9))

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D_J(L^{n-1}M) \arrow[r] \arrow[d, "x_1^*"] & L^{n-2}M \arrow[r] \arrow[d, "x_1^*"] & L^{n-2}M/D_J(L^{n-1}M) \arrow[r] \arrow[d, "x_1^*"] & 0 \\
0 \arrow[r] & D_J(L^nM) \arrow[r] \arrow[d] & L^{n-1}M \arrow[r] \arrow[d] & L^{n-1}M/D_J(L^nM) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_J(L'^nM') \arrow[r] \arrow[d] & L'^{n-1}M' \arrow[r] \arrow[d] & L'^{n-1}M'/D_J(L'^nM') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \qquad (*)$$

Applying $H_T(\cdot)$ to (*) we get the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
H^2_T(D_J(L^{n-1}M)) \arrow[r] \arrow[d] & H^2_T(L^{n-2}M) \arrow[r] \arrow[d] & H^2_T(L^{n-2}M/D_J(L^{n-1}M)) \arrow[d] \\
H^2_T(D_J(L^nM)) \arrow[r] \arrow[d] & H^2_T(L^{n-1}M) \arrow[r] \arrow[d] & H^2_T(L^{n-1}M/D_J(L^nM)) \arrow[d] \\
H^2_T(D_J(L'^nM')) \arrow[r] & H^2_T(L'^{n-1}M') \arrow[r] & H^2_T(L'^{n-1}M'/D_J(L'^nM')) \arrow[d] \\
 & & 0
\end{tikzcd}$$

By induction the second map in the first row and the second map in the last row are injective. So the second map in the middle row is injective too, which proves our claim.

(3.7) COROLLARY: *In the notations and under the hypotheses of* (3.6) (*and setting* $L'=(x_2,\ldots,x_r)$, $M'=M/x_1M$) *the natural maps* $D_T(D_J(L^nM)/D_J(L^{n+h}M))\to D_T(D_J(L'^nM')/D_J(L'^{n+h}M'))$ *are onto for all* $n, h>0$ *and all* $r>0$.

Proof. By the sequence (1.13) it suffices to show that the maps $H^1_T(D_J(L^nM)/D_J(L^{n+h}M))\xrightarrow{\pi_n}H^1_T(D_J(L'^nM')/D_J(L'^{n+h}M'))$ are onto. For $r=1$ the right hand term vanishes. So we also may assume that $r>1$. Clearly we only have to consider the case $\Gamma_J(M)=0$.

Consider the diagram (*) of the proof of (3.6) (whose last column splits) and apply H_T. So, making use of (3.6) we obtain the following diagram (for each $n>1$)

$$\begin{tikzcd}
H^1_T(L^{n-1}M/D_J(L^nM)) \arrow[r] \arrow[d] & H^2_T(D_J(L^nM)) \arrow[r] \arrow[d] & 0 \\
H^1_T(L'^{n-1}M'/D_J(L'^nM')) \arrow[r] \arrow[d] & H^2_T(D_J(L'^nM')) \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

These diagrams show that the induced map $H^2_T(D_J(L^{n+h}M))\xrightarrow{\varepsilon}H^2_T(D_J(L'^{n+h}M'))$ is surjective for all $n, h>0$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D_J(L^{n+h}M) \arrow[r, "\iota"] \arrow[d] & D_J(L^nM) \arrow[r] \arrow[d] & D_J(L^nM)/D_J(L^{n+h}M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_J(L'^{n+h}M') \arrow[r, "\iota'"] & D_J(L'^nM') \arrow[r] & D_J(L'^nM')/D_J(L'^{n+h}M') \arrow[r] & 0
\end{tikzcd} \tag{**}$$

Making use of the diagram

$$\begin{tikzcd}
D_J(L^{n+h}M) \arrow[rr, "\iota"] \arrow[dr] & & D_J(L^nM) \\
& L^{n+h-1}\overline{M^J} \arrow[ur] &
\end{tikzcd}$$

and of (3.6) we see that $H_T^2(\iota)=0$. Clearly it also holds $H_T^2(\iota')=0$. By (1.12) it holds $D_T(D_J(L^nM))=D_J(L^nM)$, so that (1.13) induces $H_T^1(D_J(L^nM))=0$. By the same argument we have $H_T^1(D_J(L'^nM'))=0$. So, applying H_T to (**) we obtain finally the situation:

$$
(3.8)\quad \begin{array}{c} H_T^1(D_J(L^nM)/(D_J(L^{n+h}M)) \xrightarrow{\cong} H_T^2(D_J(L^{n+h}M)) \\ H_T^1(D_J(L'^nM')/D_J(L'^{n+h}M')) \xrightarrow{\cong} H_T^2(D_J(L'^{n+h}M')) \end{array}
$$

So π_n is an epimorphism.

(3.9) COROLLARY. *In the notations and under the hypotheses of* (3.6) *it holds*

(i) $H_T^1(D_J(L^nM))=0$ *for all* $n>0$,
(ii) $\operatorname{ann}(H_T^1(\overline{M/LM^J}))\cdot H_T^2(D_J(L^nM))=0$ *for all* $n>0$,
(iii) $D_J(L)\cdot H_T^2(D_J(L^nM))=0$ *for all* $n>0$.

Proof. Using the sequences $0\to D_J(L^nM)\to L^{n-1}\overline{M^J}\to L^{n-1}M/D_J(L^nM)\to 0$ and (3.6) we get epimorphism $H_T^1(D_J(L^nM))\to H_T^2(D_J(L^nM))\to 0$. So (2.8) shows that $H_T^2(D_J(L^nM))$ is a homomorphic image of copies of $H_T^1(\overline{M/LM^J})$. This proves (ii). (iii) is implied by the obvious fact $D_J(L)\cdot\overline{M/LM^J}=0$. (i) has been observed in the proof of (3.7).

(3.10) PROPOSITION. *Let* $(R,\mathfrak{m})$ *be local, universally catenary, of pure dimension* $d\geq 2$ *and such that* $R/\mathfrak{m}$ *is algebraically closed. Let* $J\subseteq R$ *be an ideal such that* $\dim(R/J)=1$ *and such that* $\operatorname{Spec}(R)-V(J)$ *is* CM. *Assume that* $e_{J,\mathfrak{m}}(R)=d-1$ *and choose* $\mathfrak{a}\subseteq R$ *according to* (3.4). *Let* $x_1,\ldots,x_{d-2}\in\mathfrak{a}$ *be a partial system of parameters and put* $L=(x_1,\ldots,x_{d-2})R$ *and* $\tilde{L}=D_J(L)$ $(=\bigcup_i(L\overline{R^J}:J^i)_{\overline{R^J}})$. *Then* $D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L}))$ *is finitely generated as a* $\operatorname{Gr}(L)$*-module and* $\operatorname{Proj}(D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L})))$ *is a* CM*-scheme.*

Proof. By our hypotheses $x_1,\ldots,x_{d-2}$ form a $J,\mathfrak{m}$-pS^+-sequence with respect to R (s. (2.1) (i) and (3.4)). Clearly we may replace R by $\overline{R^J}$ thus assuming that $\Gamma_J(R)=0$. $\operatorname{Gr}(\tilde{L})$ is given by $\mathfrak{R}(\tilde{L})/D_J(L\mathfrak{R}(L))$ (see (2.11)(i)). $\mathfrak{R}(\tilde{L})$ is a $\mathfrak{R}(L)$-submodule of $D_J(\mathfrak{R}(L))=\oplus_n D_J(L^n)$. But $D_J(\mathfrak{R}(L))$ is known to be a finitely generated module over $\mathfrak{R}(L)$ in our situation [4]. So $\operatorname{Gr}(\tilde{L})$ is finite over $\operatorname{Gr}(L)$. Therefore, to show that $D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L}))$ is a finitely generated module over $\operatorname{Gr}(L)$ it suffices to find an element $a\in R$, which is regular with respect to $Gr(L)$ (thus with respect to $D_{\mathfrak{m}}(Gr(\tilde{L}))$) and which satisfies $a\cdot D_{\mathfrak{m}}(Gr(\tilde{L}))\subseteq Gr(\tilde{L})$. As $x_1,\ldots,x_{d-2}$ form a $J,\mathfrak{m}$-pS^+-sequence with respect to R we have $\sqrt{\operatorname{ann}}(H_{\mathfrak{m}}^1(\overline{R/L^J})\supseteq J$. Moreover it holds $\operatorname{Ass}(\overline{R/L^J})\cap V(J)=\emptyset$. These facts allow to find an element

$a \in R$ which is regular with respect to $\overline{R/L^J}$ (thus with respect to $\mathrm{Gr}(\tilde{L})$) and which belongs to $\mathrm{ann}(H^1_{\mathfrak{m}}(\overline{R/L^J}))$. By (3.9)(ii) we therefore $aH^2_{\mathfrak{m}}(D_J(L^{n+1}))=0$ for all $n>0$. By the isomorphism $H^1_{\mathfrak{m}}(D_J(L^n)/D_J(L^{n+1}))=H^2_{\mathfrak{m}}(D_J(L^{n+1}))$, (3.8), we now may conclude that $a \cdot H^1_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L}))=a \cdot [H^1_{\mathfrak{m}}(\overline{R/L^J}) \oplus_{n\geqslant} H^1_{\mathfrak{m}}(D_J(L^n)/D_J(L^{n+1}))]=0$. By (1.13) we see that a is of the requested type. It remains to show that $\mathrm{Proj}(D_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L})))$ is a CM-scheme. We do this by induction on d. First let $d=2$. Then (as $\tilde{L}=0$) we have $\mathrm{Proj}(D_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L})))=\mathrm{Spec}(D_{\mathfrak{m}}(R))$. By the above arguments the morphism $\mathrm{Spec}(D_{\mathfrak{m}}(R)) \to \mathrm{Spec}(R)=\mathrm{Proj}(\mathrm{Gr}(L))$ is finite. So it suffices to proof that $D_{\mathfrak{m}}(R)$ (which is a finitely generated R-module) is a CM-module over R. But this is clear as $H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(R))=0$ for $i<2$ (1.14).

Now let $d>2$. Put $X=\mathrm{Proj}(D_{\mathfrak{m}}(\mathrm{Gr}(L)))$ and let $q \in X$ be a closed point. We want to show that $\mathcal{O}_{X,q}$ is a CM-ring. Let $\mathfrak{q} \in \mathrm{Spec}(D_{\mathfrak{m}}(\mathrm{Gr}(L)))$ be the (homogeneous) prime ideal which corresponds to the point q ($\mathfrak{q}$ is essential and satisfied $\dim(D_{\mathfrak{m}}(\mathrm{Gr}(L))/\mathfrak{q})=1$). Let $\mathfrak{p} \subseteq \mathrm{Gr}(L)$ be the retraction of $\mathfrak{q}$. By the previously proved finiteness of $X \to \mathrm{Proj}(\mathrm{Gr}(L))$, $\mathfrak{p}$ corresponds to a closed point p of the latter scheme. As $\mathfrak{p} \cap R=\mathfrak{m}$ and as $R/\mathfrak{m}$ is algebraically closed, the homogeneous Nullstellensatz guarantees the existence of an element y of $L-\mathfrak{m}L$ such that the induced one-form $\bar{y} \in \mathrm{Gr}(L)$ is contained in $\mathfrak{p}$. y clearly is a parameter. So we may assume without loss of generality that $y=x_1$. Let $R'=R/x_1R$ and $L'=(x_2,\ldots,x_{d-2})R'$. Then R' and $x_1'=x_1 \cdot 1_{R'},\ldots,x_{d-2}'=x_{d-2} \cdot 1_{R'}$ satisfy again our hypotheses with $d-1$ instead of d. Let $\widetilde{L'}=D_J(L')$. We may write $\widetilde{L'}=\bigcup_j (L'\overline{R'^J}:J^j)_{\overline{R'^J}}$. According to (2.9) the canonical map $\tilde{L}=D_J(L) \to D_J(L')=\widetilde{L'}$ is onto. So $\widetilde{L'}$ is the image of $\tilde{L}$ under the canonical map $R \to \overline{R'^J}$. So we have a canonical projection $\psi:\mathrm{Gr}(\tilde{L}) \to \mathrm{Gr}(\widetilde{L'})$ which is given in positive degrees by the maps $D_J(L^n)/D_J(L^{n+1}) \to D_J(L'^n)/D_J(L'^{n+1})$ which occur in (2.9)(ii). Using (2.9)(ii), (3.7) and the left-exactness of $D_{\mathfrak{m}}$ we obtain an exact sequence of graded $\mathfrak{R}(L)$-modules: $0 \to D_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L}))_{>0} \xrightarrow{x_1} D_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L}))_{>1} \to D_{\mathfrak{m}}(\mathrm{Gr}(\widetilde{L'}))_{>1} \to 0$. So $X':=\mathrm{Proj}(D_{\mathfrak{m}}(\mathrm{Gr}(L')))$ is a closed subscheme of X, which contains q and whose ideal of vanishing sections in $\mathcal{O}_X$ is the invertible ideal defined by the (non-degenerate) one-form $\overline{x_1}$. By induction $\mathcal{O}_{X',q}$ is a CM-ring. So the same holds for $\mathcal{O}_{X,q}$.

(3.11) COROLLARY. *Keep the notations and hypotheses of* (3.10), *assuming moreover that* $d \geqslant 3$. *Let* $X_0=\mathrm{Proj}(\mathfrak{R}(\tilde{L}))$ *and let* $y \in \mathfrak{m}$ *be such that* $ht(L,y)=d-1$ *and* $yH^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0})=0$. *Let* $\mathcal{K}_0 \subseteq \mathcal{O}_{X_0}$ *be the ideal* $\bigcup_j [(\tilde{L},y)\mathcal{O}_{X_0}:\mathfrak{m}^j]_{\mathcal{O}X_0}$. *Then* (i) $\mathcal{K}_0$ *is normally torsion-free and without embedded component.* (ii) *The blow up* $Y_0=\mathrm{Proj}(\mathfrak{R}(\mathcal{K}_0))=\mathrm{Bl}_{X_0}(\mathcal{K}_0)$ *of* X_0 *at* $\mathcal{K}_0$ *is a* CM-scheme.

Proof. Let $p \in X_0$ be a closed point and put $B=\mathcal{O}_{X_0,p}$. Then $\mathfrak{m}B \neq B$. As $\tilde{L}\mathcal{O}_{X_0}$ is invertible (being the exceptional divisor of the blow-up $X_0 \to \mathrm{Spec}(R)$), there is a regular element $t \in mB$ such that $\tilde{L}\mathcal{O}_{X_0,p} = tB$. Gr $(\tilde{L})$ is a finite and torsion-free extension of the ring Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$, which is canonically isomorphic to the polynomial ring $\bar{R}/\tilde{L}[X_1, \ldots, X_{d-2}]$ (for the finiteness statement see (3.10), for the shape of Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$ see (2.7) and (2.8)). As L is generated by a partial system of parameters the embedded members of Ass (R/L) may not belong to CM (R) and so must belong to $V(J)$. Therefore we get by (2.11)(ii) that Ass $(R/\tilde{L}) = \mathrm{Min}\,(R/\tilde{L})$. This shows that y is regular with respect to $R/\tilde{L}$, thus with respect to Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$, thus with respect to Gr $(\tilde{L})$. This shows in particular that t, y form a regular sequence with respect to B. Moreover it holds $(\mathcal{K}_0)_p = \bigcup_i ((t, y)B : \mathfrak{m}^i)_B := \tilde{K} \subseteq B$.

We have to show that $\tilde{K}$ is normally torsion-free, without embedded component and that $Y_1 := \mathrm{Proj}\,(\mathfrak{R}(\tilde{K}))$ is a CM-scheme.

To prove the first claim, observe that $H^0_{\mathfrak{m}B}(B) = H^1_{\mathfrak{m}B}(B) = 0$, as t, y is a regular sequence contained in $\mathfrak{m}B$. Note also, that $D_\mathfrak{m}(B/tB)$ is the semilocal ring of the (finitely many and closed) preimage points of p under the finite morphism Proj $(D_\mathfrak{m}(\mathrm{Gr}(\tilde{L})) \to \tilde{X}_0 := \mathrm{Proj}\,(\mathrm{Gr}(\tilde{L}))$. So – by (3.10) – $D_\mathfrak{m}(B/tB)$ is a CM-module over B/tB which is of finite type. In particular we see that $H^1_{\mathfrak{m}B}(B/tB) = H^1_\mathfrak{m}(B/tB)$ is finitely generated (1.13). On the other side the isomorphisms $H^1_\mathfrak{m}(\tilde{L}^n/\tilde{L}^{n+1}) \cong H^2_\mathfrak{m}(\tilde{L}^{n+1})$ (s. (3.8)) give rise to an isomorphism $H^1_\mathfrak{m}(\mathfrak{R}(\tilde{L})/\tilde{L}\mathfrak{R}(\tilde{L})) \cong H^2_\mathfrak{m}(\tilde{L}\mathfrak{R}(\tilde{L}))$, thus to an isomorphism $H^1_{\mathfrak{m}B}(B/tB) \cong H^2_{\mathfrak{m}B}(tB) \cong H^2_{\mathfrak{m}B}(B)$. So $H^2_{\mathfrak{m}B}(B)$ is finitely generated and we obtain $e_{\mathfrak{m}B}(B) \geq 3$. Moreover, by our choice of y, we have $yH^1_{\mathfrak{m}B}(B/tB) = 0$. So t, y form a $\mathfrak{m}B$-S^+-sequence with respect to B. As $tH^1_{\mathfrak{m}B}(B/tB) = 0$ we get by the above isomorphism, that $tH^2_{\mathfrak{m}B}(B) = 0$. Applying $H^\bullet_{\mathfrak{m}B}$ to the exact sequence $0 \to B \xrightarrow{y} B \to B/yB \to 0$ we moreover obtain an injection $H^1_{\mathfrak{m}B}(B/yB) \hookrightarrow H^2_{\mathfrak{m}B}(B)$ which induces $tH^1_{\mathfrak{m}B}(B/yB) = 0$. So y, t form a $\mathfrak{m}B$-S^+-sequence with respect to B. This shows that t, y forms a $\mathfrak{m}B$-pS^+-sequence with respect to B. By (2.12) we now get that $\tilde{K}$ is normally torsion-free. Note that there is a canonical embedding $B/\tilde{K} = \overline{(B/tB)/y(B/tB)}^\mathfrak{m} = (B/tB)/(B/tB) \cap yD_\mathfrak{m}(B/tB) \subseteq D_\mathfrak{m}(B/tB)/yD_\mathfrak{m}(B/tB) =: U$. As $D_\mathfrak{m}(B/tB)$ is a CM-module, Ass (U) has no embedded members. So the same is true for Ass $(B/\tilde{K})$.

It remains to show that Y_1 is CM. This is done by induction on d. If $d = 3$, we have $\tilde{L} = x_1DJ(R)$, hence $X_0 = \mathrm{Proj}\,(\mathfrak{R}(x_1D_j(R)) = \mathrm{Proj}\,(\bigoplus_{n \geq 0} x_1^n D_J(R)) = \mathrm{Proj}\,(D_J(R)[X]) = \mathrm{Spec}\,(D_J(R))$. As $D_J(R)$ is a finitely generated R-module (observe that $H^1_J(R)$ is, finitely generated and use (1.13)) and as B is a localization of $D_JJ(R)$ in one of its maximal ideals, $\mathfrak{m}B$ is primary to the maximal ideal $\mathfrak{n}$ of B. So t, y is a $\mathfrak{n}$-pS^+-sequence with respect to the 3-dimensional ring B. Therefore we see by (2.13) that Y_1 is CM.

Finally let $d > 3$. Using the homogeneous Nullstellensatz we may assume as in the proof of (3.10) that the one-form $x_1^* \in \mathfrak{R}(L)$ defines a closed subscheme of

Proj $(\mathfrak{R}(L))$, which contains the image point of p. Put $R' = R/x_1R$, $x_i' = x_i \cdot 1_{R'}$, $y' = y \cdot 1_{R'}$, $L' = (x_2', \ldots, x_{d-2}')R'$, $\widetilde{L'} = D_J(L')$ and $X_0' = \operatorname{Proj}(\mathfrak{R}(L'))$ and $\mathcal{K}_0' = \bigcup_j ((L', y')\mathcal{O}_{X_0} : \mathfrak{m}^j)\mathcal{O}_{X_0}$. The epimorphisms π_n in the proof of (3.7) give rise to an epimorphism $H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) \to H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/L'\mathcal{O}_{X_0})$, which gives $y' \cdot H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\widetilde{L'}\mathcal{O}_{X_0})$. This shows that the dashed objects satisfy again our hypotheses with $d-1$ instead of d. The exact sequence $0 \to \mathfrak{R}(\tilde{L})_{>0} \xrightarrow{x^*} \mathfrak{R}(\tilde{L})_{>1} \to \mathfrak{R}(\tilde{L})_{>1} \to 0$ (2.9) (ii) shows that X_0' is a closed subscheme of X_0 which contains p and that the ideal sections of vanishing at X_0 is the invertible ideal of $\mathcal{O}_{X_0}$ induced by x_1^*. Let $w \in B$ be a generator of this latter ideal in the point p (wB = stalk at p) and set $B' = \mathcal{O}_{X_0', p}$. Then the above sequence gives rise to an exact sequence $0 \to B \xrightarrow{w} B \to B' \to 0$ and shows that $\widetilde{L'}\mathcal{O}_{X_0,p} = tB'$. Clearly we have $(\mathcal{K}_0')_p = \bigcup_j ((t, y)B' : \mathfrak{m}^j)_{B'} := \widetilde{K'}$. We now want to verify two properties of $\tilde{K}$. The first of them is that $\widetilde{K'} = \tilde{K} \cdot B'$. As t, y form a B-pS^+-sequence with respect to B we may write $\tilde{K} = D_{\mathfrak{m}B}((t, y)B)$ by (2.7) (iii). Similarly we may write $\widetilde{K'} = D_{\mathfrak{m}B'}((t, y)B')$. By (2.9)(ii) there is a canonical epimorphism $\tilde{K} = D_{\mathfrak{m}B}((t, y)B) \overset{\pi}{\twoheadrightarrow} D_{\mathfrak{m}B}(y \cdot (B/tB)) = yD_{\mathfrak{m}}(B/tB)$. Similarly there is a canonical epimorphism $\widetilde{K'} \to yD_{\mathfrak{m}}(B'/tB')$. The epimorphisms $D_{\mathfrak{m}}(\tilde{L}^n/\tilde{L}^{n+1}) \to D_{\mathfrak{m}}(\tilde{L}'^n/\tilde{L}'^{n+1})$ (3.7) give rise to an epimorphism $D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L})) \twoheadrightarrow D_{\mathfrak{m}}(\operatorname{Gr}(L'))$ hence to an epimorphism $D_{\mathfrak{m}}(B/tB) \twoheadrightarrow D_{\mathfrak{m}}(B'/tB')$, thus finally to a surjection $\tilde{K}/tB \twoheadrightarrow \widetilde{K'}/tB'$. As $tB' \subset \tilde{K} \cdot B'$, the canonical map $B \to B'$ gives rise to an epimorphism $\tilde{K} \to \widetilde{K'}$, which is our claim. The second property of $\tilde{K}$ in which we are interested is that w is regular with respect to $B/\tilde{K}^n$ for all $n > 0$. We have seen above that t, y form a regular sequence with respect to B. By the same argument applied to the dashed objects we see that they are a regular sequence for B'. As w is regular this shows that w is regular with respect to $B/(t, y)$, thus with respect to $B/\tilde{K}$. As $\tilde{K}$ is normally torsion-free, this gives our claim. The second property of $\tilde{K}$ means that $wB \cap \tilde{K}^n = w\tilde{K}^n$ for all $n > 0$. So, together with the first property we obtain an exact sequence $0 \to \mathfrak{R}(\tilde{K})_{>0} \xrightarrow{w} \mathfrak{R}(\tilde{K})_{>0} \to \mathfrak{R}(\widetilde{K'})_{>0} \to 0$, which induces another exact sequence: $0 \to \mathcal{O}_{Y_1} \xrightarrow{w} \mathcal{O}_{Y_1} \to \mathcal{O}_{Y_1'} \to 0$, where $Y_1' = \operatorname{Proj}(\mathfrak{R}(K'))$. $\mathcal{O}_{Y_1'}$ is CM. So the same holds for $\mathcal{O}_{Y_1}$. This completes our proof.

4. The Globalization

In this section we shall use the following notations: If $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded algebra over a field k, we write $\operatorname{Reg}_+(A)$, $\operatorname{Nor}_+(A)$, $\operatorname{CM}_+(A)$, $\operatorname{Fac}_+(A)$, $\operatorname{Sing}_+(A)$ for the corresponding loci of $\operatorname{Proj}(A)$. If R is noetherian and if $\mathfrak{a} \subseteq R$ is an ideal $V(\mathfrak{a})$ denotes the closed subset $\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, $U(\mathfrak{a})$ the open set $\operatorname{Spec}(R) - V(\mathfrak{a})$. If A is as above and if $\mathfrak{a} \subseteq A$ is a homogeneous ideal, $V_+(\mathfrak{a})$ and

$U_+(\mathfrak{a})$ stand for the closed respectively open subset of Proj (A) which are induced by $V(\mathfrak{a})$ resp. $U(\mathfrak{a})$.

We begin with the following result, which is of Bertini-type:

(4.1) LEMMA. *Let k be an infinite field and let $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded k-algebra which is generated over k by finitely many of its one-forms (as an algebra). Let $\mathfrak{a} \subseteq A$ be a homogeneous ideal and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in U(\mathfrak{a})$ Then there is a form $f \in \mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$ such that* (i) $\mathrm{Reg}_1(A/fA) \supseteq \mathrm{Proj}(A/fA) \cap \mathrm{Reg}_+(A) \cap U_+(\mathfrak{a})$, (ii) $\mathrm{Nor}_+(A/fA) \supseteq \mathrm{Proj}(A/fA) \cap \mathrm{Nor}_+(A) \cap U_+(\mathfrak{a})$.

Proof. We find forms $f_0, \ldots, f_n \in \mathfrak{a}$ of some degree N such that $\sqrt{\mathfrak{a}} = \sqrt{(f_0, \ldots, f_n)}$. So we may replace $\mathfrak{a}$ by $(f_0, \ldots, f_n)$. Assume first that k is of characteristic 0. Then for general $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^{n+1}$ we have $f_\alpha := \sum_{i=0}^{n} \alpha_i f_i \notin \mathfrak{p}_j$ $(j = 1, \ldots, s)$. Here, general means for all α outside the union of finitely many proper linear subspaces. Combining this with [9, (5.4)] we get the requested statement. So let k be of characteristic $p > 0$. Then we have $\sqrt{\mathfrak{a}} = \sqrt{(f_0^p, \ldots, f_n^p)}$. Therefore we may replace $\mathfrak{a}$ by $(f_0^p, \ldots, f_n^p)$, thus assuming additionally that the generators f_i are pth powers of the same degree. Now consider the local ring $B := A_{A_+}$, where A_+ denotes the homogeneous maximal ideal $A_{>0}$ of A. Embed k into the completion $\hat{B}$ of B and let $\hat{B} \xrightarrow{\hat{d}} \hat{\Omega}_{\hat{B}/k} =: \hat{\Omega}$ be the corresponding universal finite differential [19]. Let $x_0, \ldots, x_t$ be a k-basis of A_1. Then, as $\hat{d}(x_0), \ldots, \hat{d}(x_t)$ generate $\hat{\Omega}$, $\hat{d}(f_0), \ldots, \hat{d}(f_n)$, $\hat{d}(f_0x_0)$, $\hat{d}(f_0x_1), \ldots, \hat{d}(f_ix_j), \ldots, \hat{d}(f_nx_t)$ generate $\hat{\Omega}_{\hat{\mathfrak{p}}}$ for each $\hat{\mathfrak{p}} \in U(\mathfrak{a}\hat{B})$. As f_i is a pth power, we have $\hat{d}(f_i) = 0$ for $i = 0, \ldots, n$. So $\hat{d}(f_ix_j)$ $(0 \leq i \leq n;\ 0 \leq j \leq t)$ generate $\hat{\Omega}_{\hat{\mathfrak{p}}}$ for all $\hat{\mathfrak{p}}$ as above.

Let $\hat{\mathfrak{p}}_i \in \mathrm{Spec}(\hat{B})$ such that $\hat{\mathfrak{p}}_i \cap A = \mathfrak{p}_i$. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_h\}$ be the set of those primes $\mathfrak{p}$ of $U(\mathfrak{a}B)$ for which $B_\mathfrak{p}$ satisfies the second Serre-property S_2 and for which $\mathrm{depth}(B_\mathfrak{p}) = 2 < \dim(B_\mathfrak{p})$ (as B is excellent, we have in fact only finitely many such primes [9, (3.2)]). Let $\hat{\mathfrak{q}}_i \in \mathrm{Spec}(\hat{B})$ be such that $\hat{\mathfrak{q}}_i \cap B = \mathfrak{q}_i$. Finally let $\mathfrak{r}_1, \ldots, \mathfrak{r}_l$ be the minimal primes of $\mathrm{Nor}(B) \cap U(\mathfrak{a}B) \cap \mathrm{Sing}(B)$ (which is closed in $\mathrm{Nor}(B) \cap U(\mathfrak{a}B)$ as B is excellent). Let $\hat{\mathfrak{r}}_i \in \mathrm{Spec}(\hat{B})$ such that $\hat{\mathfrak{r}}_i \cap B = \mathfrak{r}_i$. Applying [9, (1.5)] to $\hat{B}$ with $S = k$, $M = \hat{\Omega}$, $U = U(\mathfrak{a}\hat{B})$ and observing the complement to the quoted result we find elements $\alpha_{ij} \in k$ $(0 \leq i \leq n;\ 0 \leq j \leq t)$ such that $f := \sum_{i,j} \alpha_{ij} f_i x_j$ does not belong to the symbolic square $\hat{\mathfrak{p}}^{(2)}$ for any of the primes $\hat{\mathfrak{p}} \in U(\mathfrak{a}\hat{B})$ and such that $f \notin \hat{\mathfrak{p}}_1, \ldots, \hat{\mathfrak{p}}_s, \hat{\mathfrak{q}}_1, \ldots, \hat{\mathfrak{q}}_h, \hat{\mathfrak{r}}_1, \ldots, \hat{\mathfrak{r}}_l$. By our choice of the elements f_i, f clearly is a form. According to [9] (pg. 103, proof of (2.1) and pg. 105, proof of (3.3)) f satisfies the properties: $\mathrm{Reg}(B/fB) \supseteq \mathrm{Spec}(B/fB) \cap \mathrm{Reg}(B) \cap U(\mathfrak{a}B)$, $\mathrm{Nor}(B/fB) \supseteq \mathrm{Spec}(B/fB) \cap \mathrm{Nor}(B) \cap U(\mathfrak{a}B)$. Noticing that the canonical morphism $\mathrm{Spec}(B) - \{\mathfrak{m} := A_+ \cdot B\} \to \mathrm{Proj}(A)$ transforms Reg (Spec

$(B)-\{\mathfrak{m}\})$ to $\mathrm{Reg}_+(A)$, Nor (Spec $(B)-\{\mathfrak{m}\})$ to $\mathrm{Nor}_+(A)$ etc. [9, (5.1)], we are done.

(4.2) COROLLARY. *Let X be a projective variety over the algebraically closed field k. Let $Z \subseteq X$ be a closed subset of codimension $h > 0$. Let $\mathcal{J} \subseteq \mathcal{O}_X$ be any coherent ideal such that $Z = V(\mathcal{J})$. Then there is a complete intersection ideal $\mathcal{L} \subseteq \mathcal{J}$ of codimension $h-1$ such that:*

(i) *$\mathcal{L}_p \subseteq \mathcal{O}_{X,p}$ is reduced for all $p \in (X-Z) \cap \mathrm{CM}(X)$,*

(ii) $\mathrm{Reg}(V(\mathcal{L})) \supseteq \mathrm{Reg}(X) \cap V(\mathcal{L}) \cap (X-Z)$,

(iii) $\mathrm{Nor}(V(\mathcal{L})) \supseteq \mathrm{Nor}(X) \cap V(\mathcal{L}) \cap (X-Z)$.

Proof. Let A be the homogeneous coordinate ring of X and let $\mathfrak{a} \subseteq A$ be the homogeneous ideal which corresponds to $\mathcal{J}$. We have to find forms $f_1, \ldots, f_{h-1} \in \mathfrak{a}$ such that:

(i) $(f_1, \ldots, f_{h-1})A$ is reduced for any homogeneous prime $\mathfrak{p} \in U(\mathfrak{a})$ for which $A_\mathfrak{p}$ is CM,

(ii) $\mathrm{Reg}_+(A/(f_1, \ldots, f_{h-1})) \supseteq \mathrm{Reg}_+(A) \cap \mathrm{Proj}(A/(f_1, \ldots, f_{h-1})) \cap U_+(\mathfrak{a})$,

(iii) $\mathrm{Nor}_+(A/(f_1, \ldots, f_{h-1})) \supseteq \mathrm{Nor}_+(A) \cap \mathrm{Proj}(A/(f_1, \ldots, f_{h-1})) \cap U_+(\mathfrak{a})$.

(iv) $ht(f_1, \ldots, f_{h-1}) = h-1$.

We construct these forms by induction on h. Thereby we only assume that $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded finitely generated algebra over an infinite field k, that $A = k[A_1]$ and that $A_\mathfrak{p}$ is reduced whenver $\mathfrak{p}$ belongs to $\mathrm{CM}(A) \cap U(\mathfrak{a})$.

The case $h = 1$ is trivial. So let $h > 1$. Sing (A) is closed. So there is an ideal $\mathfrak{b} \subseteq A$ such that $\mathrm{Reg}(A) = U(\mathfrak{b})$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of A. As $ht(\mathfrak{a}) = h > 0$, they all belong to $U(\mathfrak{a})$. Clearly they also belong to $\mathrm{CM}(A)$. So $A_\mathfrak{p}$ is reduced for $i = 1, \ldots, t$. This shows that $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ belong to $\mathrm{Reg}(A) = U(\mathfrak{b})$. So there is an element $c \in \mathfrak{b} - \bigcup_{i=1}^t \mathfrak{p}_i$. In particular $U(cA)$ belong to $\mathrm{Reg}(A)$. Now let $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_s$ be those minimal prime divisors of cA which are homogeneous (We do not exclude the case $s \leq t$ in which there are no such primes). Clearly $ht(\mathfrak{p}_i) = 1$ for $t < i \leq s$. This shows that $\mathfrak{p}_{t+1}, \ldots, \mathfrak{p}_s$ belong to $U(\mathfrak{a})$. So, according to (4.1) there is a form $f \in \mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$ such that:

$$(*) \quad \begin{cases} \mathrm{Reg}_+(A/fA) \supseteq \mathrm{Reg}_+(A) \cap \mathrm{Proj}(A/fA) \cap U_+(\mathfrak{a}), \\ \mathrm{Nor}_+(A/fA) \supseteq \mathrm{Nor}_+(A) \cap \mathrm{Proj}(A/fA) \cap U_+(\mathfrak{a}). \end{cases}$$

Let $\mathfrak{q}$ be a minimal prime divisor of fA. $\mathfrak{q}$ is homogeneous and of height one. This implies that $\mathfrak{q} \in U(\mathfrak{a})$ and that $\mathfrak{q} \in U(cA)$ ($as \notin U(cA)$ would imply $\mathfrak{q} = \mathfrak{p}_i$ for some $i \in \{t+1, \ldots, s\}$). So $\mathfrak{q}$ belongs to $\mathrm{Reg}(A) \cap V(fA) \cap U(\mathfrak{a})$, thus corresponds to a generic point p of Proj (A/fA) which belongs to $\mathrm{Reg}_+(A) \cap U_+(\mathfrak{a})$. By (*) we have $p \in \mathrm{Reg}_+(A/fA)$, thus $\mathfrak{q} \in \mathrm{Reg}(A/fA)$. This shows that the minimal primes of A/fA belong to $\mathrm{Reg}(A/fA)$.

Put $\bar{A} = A/fA$, $\bar{\mathfrak{a}} = \mathfrak{a} \cdot \bar{A}$. As $f \in \mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$ we clearly have $ht(\bar{\mathfrak{a}}) = h - 1$. Let $\bar{\mathfrak{p}} \in U(\bar{\mathfrak{p}})$ be a homogeneous prime such that $\bar{A}_{\bar{\mathfrak{p}}}$ is CM. By the previous remark the minimal primary components of $\bar{A}_{\bar{\mathfrak{p}}}$ are in fact all prime. As $\bar{A}_{\bar{\mathfrak{p}}}$ is CM, it is unmixed. So $\bar{A}_{\bar{\mathfrak{p}}}$ is reduced. Therefore we may apply induction to the pair $\bar{A}, \bar{\mathfrak{a}}$ to find form $\bar{f}_2, \ldots, \bar{f}_{h-1} \in \bar{A}$ such that:

(i) $(\bar{f}_2, \ldots, \bar{f}_{h-1})$ is reduced for any homogeneous prime $\bar{\mathfrak{p}} \in U(\bar{\mathfrak{a}})$ such that $\bar{A}_{\bar{\mathfrak{p}}}$ is CM.

(ii) $\mathrm{Reg}_+(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \supseteq \mathrm{Reg}_+(\bar{A}) \cap \mathrm{Proj}(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \cap U_+(\bar{\mathfrak{a}})$,

(iii) $\mathrm{Nor}_+(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \supseteq \mathrm{Nor}_+(\bar{A}) \cap \mathrm{Proj}(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \cap U_+(\bar{\mathfrak{a}})$.

(iv) $ht(\bar{f}_2, \ldots, \bar{f}_{h-1}) = h - 2$.

Now, put $f_1 = f$ and let $f_2, \ldots, f_{h-1} \in A$ be forms which respectively lift $\bar{f}_2, \ldots, \bar{f}_{h-1}$. Let $\mathfrak{p} \in \mathrm{U}(\mathfrak{a})$ be a homogeneous prime such that $A_{\mathfrak{p}}$ is CM. If $\bar{f}_1 \notin \mathfrak{p}$, we have $(f_1, \ldots, f_{h-1})A_{\mathfrak{p}} = A_{\mathfrak{p}}$, thus statement (i). If $f_1 \in \mathfrak{p}$, $\bar{\mathfrak{p}} = \mathfrak{p}\bar{A}$ belongs to $U(\bar{\mathfrak{a}})$. As $A_{\mathfrak{p}}$ is reduced $f_1 \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$ implies that f_1 is regular with respect to A. So $\bar{A}_{\bar{\mathfrak{p}}} = A_{\mathfrak{p}}/fA_{\mathfrak{p}}$ is a CM-ring. Therefore by (i) $(f_1, \ldots, f_{h-1})A_{\mathfrak{p}}/f_1A_{\mathfrak{p}} = (\bar{f}_2, \ldots, \bar{f}_{h-1})\bar{A}_{\bar{\mathfrak{p}}}$ is reduced. So $(f_1, \ldots, f_{h-1})A$ is reduced, which proves (i). (iv) follows by (iv), whereas (ii) and (iii) are induced by (ii), (iii) and (*).

(4.3) COROLLARY. *Let X be a quasiprojective variety over the algebraically closed field k. Let $Z \subseteq X$ be a closed subset of codimension $h > 0$ and such that $X - Z$ is CM. Let $\tilde{\mathscr{J}} \subseteq \mathcal{O}_X$ be a coherent ideal such that $Z = V(\tilde{\mathscr{J}})$. Then there is a locally complete intersection ideal $\mathscr{L} \subseteq \tilde{\mathscr{J}}$ of codimension $h - 1$ such that*

(i) *$\mathscr{L}_p \subseteq \mathcal{O}_{X,p}$ is reduced for all $p \in X - Z$,*

(ii) $\mathrm{Reg}(V(\mathscr{L})) \supseteq \mathrm{Reg}(X) \cap V(\mathscr{L}) \cap (X - Z)$,

(iii) $\mathrm{Nor}(V(\mathscr{L})) \supseteq \mathrm{Nor}(X) \cap V(\mathscr{L}) \cap (X - Z)$.

Proof. X may be written as an open dense subset of a projective variety $\bar{X}$. Let $\bar{Z} \subseteq \bar{X}$ be the closure of Z. Then $\bar{Z}$ also is of codimension h with respect to $\bar{X}$ (as X is dense in $\bar{X}$). As $\bar{Z} \cap X = Z$ there is a coherent ideal $\bar{\tilde{\mathscr{J}}} \subseteq \mathcal{O}_{\bar{X}}$ such that $V(\bar{\tilde{\mathscr{J}}}) = \bar{Z}$ and such that $\bar{\tilde{\mathscr{J}}}|_X = \tilde{\mathscr{J}}$. Now apply (4.2) to $\bar{X}$, $\bar{\tilde{\mathscr{J}}}$ and $\bar{Z}$ to get a complete intersection $\bar{\mathscr{L}} \subseteq \bar{\tilde{\mathscr{J}}}$ of codimension $h - 1$ such that:

(i) $\bar{\mathscr{L}}_{\bar{p}}$ is reduced for all $\bar{p} \in (\bar{X} - \bar{Z}) \cap \mathrm{CM}(\bar{X})$,

(ii) $\mathrm{Reg}(V(\bar{\mathscr{L}})) \supseteq \mathrm{Reg}(\bar{X}) \cap V(\bar{\mathscr{L}}) \cap (\bar{X} - \bar{Z})$,

(iii) $\mathrm{Nor}(V(\bar{\mathscr{L}})) \supseteq \mathrm{Nor}(\bar{X}) \cap V(\bar{\mathscr{L}})\ (\bar{X} - \bar{Z})$.

Setting $\mathscr{L} = \bar{\mathscr{L}}|_X$ our statement follows as $(\bar{X} - \bar{Z}) \cap X = X - Z$, $\mathrm{CM}(\bar{X}) \cap X = \mathrm{CM}(X) \supseteq X - Z$.

(4.4) COMPLEMENT. *If in* (4.3) $X - Z = \mathrm{CM}(X)$, (ii) *may be replaced by*
(ii)′ $\mathrm{Reg}(V(\mathscr{L})) \supseteq \mathrm{Reg}(X) \cap V(\mathscr{L})$.

Proof. In this case we clearly have $\operatorname{Reg}(X) \subseteq X - Z$.

Let Y be a closed subset of a locally noetherian scheme X and let $\mathcal{J} \subseteq \mathcal{O}_X$ be the ideal of sections vanishing at Y. We say that Y is *normally torsion-free with respect to* X if $\mathcal{J}$ is normally torsion-free in $\mathcal{O}_X$.

Let $\mathcal{M} \subseteq \mathcal{O}_X$ be a coherent ideal. We say that $\mathcal{M}$ is *generically a complete intersection ideal* if $\mathcal{M}_p \subseteq \mathcal{O}_{X,p}$ is a complete intersection-ideal for each generic point p of $V(\mathcal{M})$ (which means that $\mathcal{M}_p$ may be generated by h elements, where $h = ht(\mathcal{M}_p)$). We say that Y is generically a complete intersection with respect to X if its ideal $\mathcal{J}$ of vanishing sections is generically a complete intersection. If X is an algebraic variety (or generally an excellent scheme) this is equivalent to saying that each irreducible component of Y leaves the singular locus $\operatorname{Sing}(X)$ of X.

(4.5) LEMMA. *Let X be a noetherian scheme and let $Y \subseteq X$ be a closed subset of pure codimension h which generically is a complete intersection with respect to X. Consider the blow-up* $\operatorname{Bl}_X(Y) = \operatorname{Proj}(\mathfrak{R}(\mathcal{J})) \xrightarrow{\pi} X$, *where $\mathcal{J} \subseteq \mathcal{O}_X$ is the ideal of sections vanishing at Y. Then it holds*

(i) $\operatorname{Reg}(\operatorname{Bl}_X(Y)) \supseteq \pi^{-1}((\operatorname{Reg}(X) \cap \operatorname{Reg}(Y)) \cup \operatorname{Reg}(X - Y))$.

If Y is normally torsion-free with respect to X it holds

(ii) $\operatorname{Nor}(\operatorname{Bl}_X(Y)) \supseteq \pi^{-1}(\operatorname{Nor}(X))$.

(iii) *If Y is moreover irreducible we also have*

$\operatorname{Fac}(\operatorname{Bl}_X(Y)) \supseteq \pi^{-1}\operatorname{Fac}(X))$.

Proof. The result is of local nature. So let $X = \operatorname{Spec}(R)$, $I \subseteq R$, where R is a noetherian local ring and where I is a reduced ideal, which is of pure codimension and generically a complete intersection. Now (i) is clear, as $\operatorname{Proj}(\mathfrak{R}(I))$ is regular if R and R/I are. To show (ii) assume that R is normal and that I is normally torsion-free. It suffices to show that $\operatorname{Proj}(\mathfrak{R}(I))$ is normal under these assumptions. In fact it holds even more, namely $\mathfrak{R}(I)$ is normal [6, (6.10)], so that $\operatorname{Proj}(\mathfrak{R}(I))$ is arithmetically normal. To prove (iii) we may assume that R is factorial, that I is prime and restrict ourselves to show that $\operatorname{Proj}(\mathfrak{R}(I))$ is locally factorial. Let $x_1, \ldots, x_t$ be a system of generators of I, which are $\neq 0$. Then $\operatorname{Proj}(\mathfrak{R}(I))$ has an affine open covering by the sets $\operatorname{Spec}\left(A_i := R\left[\frac{x_1}{x_i}, \ldots, \frac{x_t}{x_i}\right]\right)$ $(i = 1, \ldots, t)$. So it suffices to verify that the rings A_i are factorial. As I is normally torsion-free the prime divisors of the ideal x_iA_i retract all to the prime I in R (observe that $x_iA_i = \Gamma(\operatorname{Spec}(A_i), I\mathcal{O}_{\operatorname{Proj}(\mathfrak{R}(I))})$). As R_I is regular $\operatorname{Proj}(\mathfrak{R}(I \cdot R_I)/I\mathfrak{R}(I \cdot R_I))$ is a projective space over the field $K(I) := R_I/I \cdot R_I$. So $x_i(A_i)_I$ is a prime ideal of $(A_i)_I$. In view of the above remark on the prime divisors of x_iA_i this latter ideal is prime. Clearly $A_i\left[\frac{1}{x_i}\right]=R_{x_i}$. So $A_i\left[\frac{1}{x_i}\right]$ is factorial. Thus it is well known that A_i is factorial, too, [17].

(4.6) *Remark*. The previous proof shows that (ii) may be sharpened to $\mathrm{Nor}\,(\mathfrak{R}(\mathscr{J}))\supseteq\lambda^{-1}\,\mathrm{Nor}\,(X)$, where $\lambda:\mathrm{Spec}\,(\mathfrak{R}(\mathscr{J}))\to X$ is defined canonically. So, if $Y\subseteq X$ is normally torsion-free, of pure codimension and generically a complete intersection, $\mathrm{Bl}_X\,(Y)$ is arithemtically normal, if X is normal.

Proof of Theorem (1.1). We use the notations of the introduction and of (2.6), (2.15), assuming that $\dim(W)\leqslant 0$. Choose $\mathscr{J},\mathfrak{b}\subseteq\mathcal{O}_V$ according to (2.15). Observe that $V(\mathfrak{b})=W$ is of pure codimension d (or empty). According to (4.3) there is a locally complete intersection ideal $\mathscr{L}\subseteq\mathfrak{b}$ of codimension $d-1$ such that $\mathscr{L}_p\subseteq\mathcal{O}_{V,p}$ is reduced for all $p\in V-W$ and such that $\mathrm{Reg}\,(V(\mathscr{L})\supseteq\mathrm{Reg}\,(V)\cap V(\mathscr{L})$ (choose $\tilde{\mathscr{J}}=\mathfrak{b}$ and observe (4.4)). In particular $C:=V(\mathscr{L})$ is a (reduced) curve. Define $\tilde{\mathscr{L}}$ as in (2.15). We claim that $\tilde{\mathscr{L}}$ is the ideal of sections vanishing at C, thus that $\sqrt{\mathscr{L}}=\tilde{\mathscr{L}}$. As C is of pure dimension one (as $\mathscr{L}$ is of pure codimension $d-1$) we have $\mathscr{L}_p=\tilde{\mathscr{L}}_p=\sqrt{\mathscr{L}_p}$, for all generic points p of C ((4.3)(*i*)), for these points are not closed, thus outside W. As $\tilde{\mathscr{L}}\subseteq\sqrt{\mathscr{L}}$ this shows that $\sqrt{\mathscr{L}}=\tilde{\mathscr{L}}$ (as both of these ideals have no embedded component and are of pure dimension 1). So our claim is shown. According to (2.15) it follows that C is normally torsion-free with respect to V and that $\mathrm{Bl}_V\,(C)=\mathrm{Proj}\,(\mathfrak{R}(\tilde{\mathscr{L}}))$ is CM. As $\tilde{\mathscr{L}}_p=\mathscr{L}_p$ for all generic points we also see that C is a generic complete intersection. Let p any point in $C\cap\mathrm{Reg}\,(V)$. Then we have $\mathscr{L}_p=\tilde{\mathscr{L}}_p$ (as $\mathscr{L}_p$ is reduced), thus $\mathcal{O}_{C,p}=\mathcal{O}_{V,p}/\mathscr{L}_p$. This shows that $p\in\mathrm{Reg}\,(C)$, and it follows $\mathrm{Reg}\,(C)\supseteq\mathrm{Reg}\,(V)\cap C$. So the canonical map $\pi:\mathrm{Bl}_V\,(C)\to V$ preserves regularity and normality (this latter even arithmetically) (4.5), (4.6). This proves (1.1).

(4.7) *Remark*. (i) If we may choose C irreducible in addition, it follows by (4.5) (iii) that π even preserves local factoriality. (ii) If $\dim(V)\leqslant 3$ and if V is normal we know that $\mathrm{Bl}_V\,(C)$ is arithemtically CM (see (2.15)). We already have remarked above (4.6) that $\mathrm{Bl}_V(C)$ is arithemtically normal. This proves (1.4). (1.3) is also clear by the previous remark.

Proof of Theorem (1.2). We use the notations of the introduction and of (3.5), assuming moreover that $\dim(W)=1$. This latter implies in particular that $d=\dim(V)\geqslant 3$. Let $\mathscr{J}$ be the ideal of sections vanishing at W. Choose $\mathfrak{b}\subseteq\mathcal{O}_V$ according to (3.5). $\mathfrak{b}$ is of codimension $d-1$. Now apply (4.3) (with $\mathfrak{b}=\tilde{\mathscr{J}}$) to obtain a locally complete intersection ideal $\mathscr{L}\subseteq\mathfrak{b}$ of codimension $d-2$ such that $\mathscr{L}_p\subseteq\mathcal{O}_{V,p}$ is reduced for all $p\in V-W$ and such that $\mathrm{Reg}\,(V(\mathscr{L}))\supseteq\mathrm{Reg}\,(V)\cap V(\mathscr{L})$. Define $\tilde{\mathscr{L}}$ as in (3.5). Clearly $S:=V(\mathscr{L})$ is a pure surface. In literally the same way as in the proof of (1.1) we may verify that $\tilde{\mathscr{L}}$ is the ideal of sections vanishing at S and that S is a generic complete intersection. So, observing (3.5) and arguing as in the proof of (1.1) we see that the blow-up $X := Bl_V(S) = \mathrm{Proj}\,(\Re(\tilde{\mathscr{L}})) \xrightarrow{\varphi} V$ preserves regularity and normality. So it remains to define $T \subseteq X$ in the appropriate way. Let Z be the set of closed points of S. $\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X$ is the structure-sheaf of the exceptional fiber $\varphi^{-1}(S)$ of φ. Let $p \in Z$. We claim that the ideal $\mathfrak{m}_p \cdot \mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X$ is of codimension 2. As $\varphi^{-1}(S)$ is of pure dimension $d-1$, it suffices to show that $\varphi^{-1}(p)$ is of dimension $d-3$. But this is a local result, which follows by (2.10) (applied with $M = \mathcal{O}_{V,p}$, $L = \mathscr{L}_p$), as this latter shows that $\mathscr{L}_p$ is a minimal reduction of $\tilde{\mathscr{L}}_p$ (in the sense of Northcott-Rees). This proves our claim.

Our next claim is that $\mathrm{Ass}\,(\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X)$ has no embedded member. So let q be any of the points associated to $\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X$. As T is normally torsion-free we must have $\varphi(q) \in \mathrm{Ass}\,(\mathcal{O}_V/\tilde{\mathscr{L}})$. So $q' := \varphi(q)$ must be a generic point of S. As S is a generic complete intersection we see that $q' \in \mathrm{Reg}\,(V)$. As φ preserves regularity we have $q \in \mathrm{Reg}\,(X)$. As $(\tilde{\mathscr{L}}\mathcal{O}_X)_q$ is a principal ideal in $\mathcal{O}_{X,q}$ ($\tilde{\mathscr{L}}\mathcal{O}_X$ is invertible), q may not be embedded. This proves the second claim.

Both claims together imply that $H^1_Z(\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X)$ is a coherent sheaf over $\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X$ (s. for example [3, (3.1)] and use an affine open covering of $\varphi^{-1}(S)$). This shows that $\dim(V(\mathfrak{d})) \leq 0$, where $\mathfrak{d} = \mathrm{ann}_{\mathcal{O}_S}(H^1_Z(\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X))$. Therefore clearly $\mathfrak{d}$ is of codimension ≥ 2 in $\mathcal{O}_S$. Applying (4.3) to S and $\mathfrak{d}$ we obtain an invertible $\mathscr{M} \subseteq \mathfrak{d}$ such that $\mathscr{M}_p \subseteq \mathcal{O}_{S,p}$ is reduced for all $p \in S - V(\mathfrak{d})$ and such that $\mathrm{Reg}\,(V(\mathscr{M})) \supseteq \mathrm{Reg}\,(S) \cap V(\mathscr{M})$. Finally let $T = V(\mathscr{M} \cdot \mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X = \mathscr{M} \cdot \mathcal{O}_{\varphi^{-1}(S)}) \subseteq \varphi^{-1}(S)$. We claim that the closed subset $T \subseteq X$ is of the requested type. As $\varphi^{-1}(S)$ is of pure dimension $d-1$ and as T is a hypersurface in $\varphi^{-1}(S)$, T is of codimension 2 with respect to X. It remains to show that the blow-up $\psi : \mathrm{Bl}_X(T) \to X$ is a Macaulayfication which preserves normality and that $\varphi \circ \psi$ preserves regularity. To prove this we choose a point $p \in Z$ and put $R = \mathcal{O}_{V,p}$, $\mathfrak{m} = \mathfrak{m}_p$, $L = \mathscr{L}_p$, $\tilde{L} = \tilde{\mathscr{L}}_p$, $X_0 = \mathrm{Proj}\,(\Re(\tilde{L}))$ and define $\mathscr{N}_0 \subseteq \mathcal{O}_{X_0}$ as the ideal of sections vanishing at the closed set $T_0 := T \cap X_0 \subseteq X_0 = \varphi^{-1}\,(\mathrm{Spec}\,(R)) = \{q \in X \mid p \in \overline{\{\varphi(q)\}}\}$. Consider the canonical morphisms $\varphi_0 : X_0 \to \mathrm{Spec}\,(R)$ and $Y_0 = \mathrm{Bl}_{X_0}(T_0) = \mathrm{Proj}\,(\Re(\mathscr{N}_0)) \xrightarrow{\psi_0} X_0$. It suffices to show that Y_0 is CM, that ψ_0 preserves normality and that $\varphi_0 \circ \psi_0$ preserves regularity.

Note that $R, \mathfrak{m}, L, \tilde{L}$ satisfy the hypotheses of (3.10) and that φ_0 preserves normality and regularity. Assume first that $p \notin V(\mathscr{M})$. Then clearly $T \cap X_0 = \emptyset$, so that $Y_0 = X_0$. As φ_0 preserves regularity and normality, it remains to prove that X_0 is CM in this case. By our choice of p we have $\mathfrak{d}_p = \mathcal{O}_{S,p}$ so that $H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = (H^1_Z(\mathcal{O}_X/\tilde{\mathscr{L}}\mathcal{O}_X))_p = 0$. So we have $\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0} = D_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0})$ (observe that $H^0_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = 0$ by the normal torsion-freeness of $\tilde{L}$). But the righthand term equals $\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L})))$, which latter is CM by (3.10). Therefore the exceptional fiber $\varphi^{-1}(S)$ is CM, which induces that X_0 is CM.

Finally let $p \in V(\mathscr{M})$. Then we find an element $y \in \mathfrak{m}$ such that $\mathscr{M}_p = y \cdot R/\tilde{L}$ (note that $\mathcal{O}_{S,p} = R(\tilde{L})$. By our choice of $\mathscr{M}$ we have $yH^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = 0$, $ht(\tilde{L}, y) =$

$d-1$. Clearly it holds $\mathcal{N}_0=\sqrt{(L, y)\mathcal{O}_{X_0}}$ ($\mathcal{N}_0 \subseteq \mathcal{O}_{X_0}$ is the ideal of sections vanishing at T_0). We claim that $\mathcal{N}_0$ also may be written as $\bigcup_j ((\tilde{L}, y)\mathcal{O}_{X_0} : \mathfrak{m}^j)\mathcal{O}_{X_0} =: \mathcal{K}_0$: As $\varphi^{-1}(p)$ is of dimension $d-3$ and as T is of dimension $d-2$, no generic point of T is mapped to p. This shows that $\mathcal{K}_0 \subseteq \mathcal{N}_0$, thus $\sqrt{\mathcal{K}_0}=\mathcal{N}_0$. So it remains to show that $\mathcal{K}_0$ is reduced. By (3.11) $\mathcal{K}_0$ has no embedded component. So we only have to proof that $(\mathcal{K}_0)_q \subseteq \mathcal{O}_{X_0,q}$ is reduced for any generic point q of T_0. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be the image of such a point q. As $s \neq \mathfrak{p}$, $s \in V(y \cdot R/\tilde{L})$, $\dim(R/\tilde{L})=2$, $\mathfrak{p}$ corresponds to a point of codimension one in S. In particular $\mathfrak{p}$ corresponds to a point $s \in S - V(\mathfrak{b})$ so that $\mathcal{M}_s$ is reduced. This means that $(\tilde{L}, y)R_\mathfrak{p}=\mathfrak{p}R_\mathfrak{p}$. In particular we see that $(R/\tilde{L})_\mathfrak{p}$ is a discrete valuation ring. Clearly $x_1, \ldots, x_{d-2}$ (minimal system of generators of L) form a $\mathfrak{p}R_\mathfrak{p}$-$S^+$-sequence with respect to $R_\mathfrak{p}$ and it holds $\tilde{L}_\mathfrak{p}=D_{\mathfrak{p}R_\mathfrak{p}}(L_\mathfrak{p})$. So $\overline{(\operatorname{Gr}(L)^J)_\mathfrak{p}}=\overline{\operatorname{Gr}(LR_\mathfrak{p})\mathfrak{p}R_\mathfrak{p}}$ is a polynomial algebra over $R_\mathfrak{p}/\tilde{L}_\mathfrak{p}=(R/\tilde{L})_\mathfrak{p}$, thus regular. Moreover $\operatorname{Gr}(\tilde{L})_\mathfrak{p}=\operatorname{Gr}(\tilde{L}R_\mathfrak{p})$ is a finite birational extension ring of $\overline{\operatorname{Gr}(L_\mathfrak{p})^{\mathfrak{p}R_\mathfrak{p}}}$ (use (2.13)(i)). So both rings coincide. This shows that $\operatorname{Gr}(\tilde{L})_\mathfrak{p}$ is a polynomial algebra over $(R/\tilde{L})_\mathfrak{p}$. Therefore $y \cdot \operatorname{Gr}(\tilde{L})_\mathfrak{p}= \mathfrak{p}(R/\tilde{L})_\mathfrak{p} \cdot \operatorname{Gr}(\tilde{L})_\mathfrak{p}$ is a prime ideal of $\operatorname{Gr}(\tilde{L})_\mathfrak{p} \cdot \mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}$ is the structure sheaf of $\operatorname{Proj}(\operatorname{Gr}(\tilde{L}))$; this shows that $((\tilde{L}, y)\mathcal{O}_{X_0})_q$ is a reduced ideal of $\mathcal{O}_{X_0,q}$. This clearly proves our claim.

The last argument shows that $(\mathcal{K}_0)_q=((\tilde{L}, y)\mathcal{O}_{X_0})_q$ for all generic points q of T_0. As $\tilde{L}\mathcal{O}_{X_0}$ is invertible and as T is of pure codimension 2, it follows that T is a generic complete intersection of pure codimension. $\mathcal{K}_0$ is the ideal of sections vanishing at T. So – by (3.11) – T is also normally torsion-free with respect to X_0. Applying (3.11)(ii) and (4.5)(ii) we see that $Y_0=\operatorname{Proj}(\mathfrak{R}(\mathcal{K}_0)) \xrightarrow{\psi_0} X_0$ is a Macaulayfication which preserves normality (even arithmetically).

It remains to show that $\varphi_0\psi_0$ preserves regularity. So let $r \in Y_0$ be such that $\mathfrak{p} := \varphi_0 \cdot \psi_0(r) \in \operatorname{Reg}(R)$. $\mathfrak{p}$ corresponds to a regular point s of V, and therefore specializes to a closed point p' of V which is regular. Replace p by p'. This allows to assume that R is regular. Then, by our choice of $\mathcal{L}$ we have $R/L=R/\tilde{L}$ and this ring is regular too. In particular $\operatorname{Gr}(\tilde{L})$ is a polynomial ring over $R/\tilde{L}$ and X_0 is regular, as φ_0 preserves regularity. Moreover – by our choice of $\mathcal{M}$ – we either have $p \notin V(\mathcal{M})$ or we may assume that y is a regular parameter with respect to $R/\tilde{L}$. In the first case we are done by the above. In the second case $\operatorname{Gr}(\tilde{L})/y \operatorname{Gr}(\tilde{L})$ is regular. So T_0 is regular (observe that in particular $\mathcal{K}_0= (\tilde{L}, y)\mathcal{O}_{X_0}$, as this latter is a complete intersection in the regular scheme X_0 and so has no embedded component). Now we see by (4.5)(i) that $Y_0=\operatorname{Bl}_{X_0}(T_0)$ is regular.

Conclusive remark. The Macaulayfications we give in (1.1) and (1.2) clearly also preserve the property of being a complete intersection point or a Gorenstein point. To see this notice that the occurring centers C, S and T of our blow up are scheme-theoretically complete intersections in those points of the ambient variety which are complete intersection points or of Gorenstein type. But blowing up a local scheme at a complete intersection ideal clearly preserves the two properties in question. More generally, any propery P (of local nature) is preserved under our birational models if only it satisfies the following axioms:

If a local ring satisfies P, it is CM.

If a local ring R satisfies P, R/xR does for any regular $x \in R$.

If, for a local ring R and a regular element $x \in R$, R/xR satisfies P then R does too.

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Received August 11, 1982