Comment. Math. Helvetici 58 (1983) 379–387 0010-2571/83/003379-09$01.50 + 0.20/0

On the cohomology of groups of p-length 1

THOMAS DIETHELM

1. Introduction

Let G be a finite group, whose order is divisible by the prime p, and let k denote the field of p elements. We consider the cohomology $H^n(G, A)$, where A is a simple kG-module. It is well known that $H^n(G, A) \neq 0$ implies that A lies in the principal block of kG. We ask, if the converse is true, i.e. if to every simple kG-module A in the principal block there is an $n \in \mathbb{N}$ with $H^n(G, A) \neq 0$.

Swan proved that this is true for the trivial module k. Therefore the above question has a positive answer for p-nilpotent groups ($G = O_{p'p}G$). In this paper we show: (Theorem 5.3) if $G = O_{p'pp'p}G$, then there are infinitely many $n \in \mathbb{N}$ with $H^n(G, A) \neq 0$.

In §3 we first consider the case where G is of p-length 1. In order to show the nontriviality of $H^n(G, A)$ we analyze the action of the p'-group $Q = G/O_{p'p}G$ on the cohomology ring $H^*(O_{p'p}G/O_{p'}G, k)$ of the p-group $P = O_{p'p}G/O_{p'}G$. We prove the following result, which is of interest in its own right (Theorem 4.5):

If the p'-group Q acts faithfully on the p-group P, then every simple kQ-module A appears infinitely often in $H^*(P, k)$ as a direct summand.

The proof of this result is by induction on the length of a central series of P with elementary abelian factors. With the aid of this result we can prove Theorem 4.6:

Let G be a group of p-length 1, and let A be a simple kG-module lying in the principal block of kG. Then $H^n(G, A) \neq 0$ for infinitely many $n \in \mathbb{N}$.

In §5 we show how the result for groups G of p-length 1 can be used to treat the case where $G = O_{p'pp'p}G$. We do that by considering the extension

$$O_{p'pp'}G \rightarrowtail G \twoheadrightarrow G/O_{p'pp'}G.$$

Most of the results of this paper first appeared in the author's doctoral thesis (ETH, Zürich, Switzerland, 1981; adviser: U. Stammbach).

2. Techinal lemmas

As a preparation we state the following well known results:

LEMMA 2.1. *Let G be an extension of a p'-group N by a group H, $N \rightarrowtail G \twoheadrightarrow H$. If V is an indecomposable kG-module lying in the principal block, then:*

$$H^n(G, V) \cong H^n(H, V); \qquad n \geqslant 0.$$

Proof. Since N is a p'-group, V is centralised by N and the spectral sequence of the extension $N \rightarrowtail G \twoheadrightarrow H$ collapses.

LEMMA 2.2. *Let G be an extension of a group N by a p'-group H. If V is a kG-module, then $H^n(G, V) \cong H^n(N, V)^H$; $n \geqslant 0$.*

Proof. Since H is a p'-group, the spectral sequence of the extension $N \rightarrowtail G \twoheadrightarrow H$ collapses.

LEMMA 2.3. *Let G be an extension of N by a group H, and let A be a kG-module with $C_G(A) \supseteq N$. Then:*

$$H^n(N, A)^H \cong \mathrm{Hom}_{kH}(H_n(N, k), A); \qquad n \geqslant 0.$$

Proof. Since A is a trivial kN-module, the universal coefficient theorem holds

$$H^n(N, A) \cong \mathrm{Hom}_k(H_n(N, k), A).$$

The above isomorphism is natural and thus H acts diagonally on the right hand side. Hence

$$H^n(N, A)^H = \mathrm{Hom}_{kH}(H_n(N, k), A).$$

3. The cohomology of groups of p-length 1

Let G be a group of p-length 1 ($G = O_{p'pp'}G$), and let A be a simple kG-module lying in the principal block of kG. Then $O_{p'p}G \subseteq C_G(A)$. ([6] p. 164.)

From Lemma 2.1 we obtain

$$H^i(G/O_{p'}G, A) \cong H^i(G, A)$$

and from Lemmas 2.1, 2.2

$$H^i(G,A) \cong H^i(O_{p'p}G/O_{p'}G, A)^{G/O_{p'p}G}.$$

Let Q denote the p'-group $G/O_{p'p}G$, and let P denote the p-group $O_{p'p}G/O_{p'}G$. Then Lemma 2.3 yields

$$H^n(G,A) \cong \mathrm{Hom}_{kQ}(H_n(P,k), A).$$

This preparation allows the proof of the following result.

THEOREM 3.1. *Let G be a group of p-length* 1, *and let A be a simple kG-module lying in the principal block of kG. Then:*

$$H^n(G,A) \neq 0$$

if and only if A is a direct summand of $H_n(P,k)$.

Proof.

"$\Rightarrow$" If $H^n(G,A)$ is nontrivial, then $\mathrm{Hom}_{kQ}(H_n(P,k), A)$ is nontrivial, and the simple kQ-module A is a direct summand of $H_n(P,k)$.

"$\Leftarrow$" By Maschke's theorem $H_n(P,k)$ is semi-simple. If A is a direct summand of $H_n(P,k)$ the projection onto A is a nontrivial kQ-module homomorphism $f: H_n(P,k) \to A$. But the nontriviality of $\mathrm{Hom}_{kQ}(H_n(P,k), A)$ implies the nontriviality of $H^n(G,A)$.

Note 3.1. It follows from Theorem 3.1, that it is necessary to analyze the G/P-module structure of $H_*(P,k)$ induced by conjugation of G in P. Since the cohomology $H^*(P,k)$ is the dual of $H_*(P,k)$, this is equivalent to analyze the G/P-module structure of $H^*(P,k)$. The advantage of working in cohomology is, that we may use its algebra structure which is induced by the cup-product.

Note 3.2. Clearly the p'-group $Q = G/O_{p'p}G$ acts faithfully on the p-group $P = O_{p'p}G/O_{p'}G$.

4. The kQ-module structure of $H^*(P,k)$

By Note 3.2, the p'-group Q acts faithfully on the p-group P. This action induces an action of Q on the cohomology ring $H^*(P,k)$.

Our problem is to determine these kQ-modules which are direct summands of $H^*(P,k)$.

LEMMA 4.1. *Let the p'-group Q act faithfully on the elementary abelian p-group $E=C_p^{(1)}\times\cdots\times C_p^{(m)}$. Then every simple kQ-module A is infinitely often a direct summand of $H^*(E,k)$.*

Proof. It is well known, that the cohomology ring $H^*(E,k)$ contains the polynomial ring $k[x_1,x_2,\ldots,x_m]$; $x_i\in H^2(C_p^{(i)},k)$ as a subring. The generators $x_1,x_2,\ldots,x_m$ correspond to a basis of E, and Q acts faithfully on the subspace $\langle x_1,x_2,\ldots,x_m\rangle$ of $H^2(E,k)$. By the theorem of Steinberg [7], every simple kQ-module A is infinitely often a direct summand of $k[x_1,x_2,\ldots,x_m]$, and $k[x_1,x_2,\ldots,x_m]$ is a direct summand of $H^*(E,k)$.

Note 4.1. The map $\phi_s:k[x_1,\ldots,x_m]\to k[x_1^{p^s},\ldots,x_m^{p^s}]$, $f(x_1,\ldots,x_m)\mapsto f(x_1,\ldots,x_m)^{p^s}=f(x_1^{p^s},\ldots,x_m^{p^s})$; $s=0,1,2,\ldots$ is a kQ-module isomorphism. Therefore $k[x_1,x_2,\ldots,x_m]$ contains infinitely many copies of itself.

LEMMA 4.2. *Let $E=C_p^{(1)}\times C_p^{(2)}\times\cdots\times C_p^{(m)}$ be an elementary abelian central subgroup of the p-group P. Then for some $s\in\mathbb{N}$ the polynomial ring $k[x_1^{p^s},x_2^{p^s},\ldots,x_m^{p^s}]$ lies in the image of the restriction map*

$$\mathrm{res}:H^*(P,k)\to H^*(E,k).$$

Proof. We consider the spectral sequence $E_2^{i,j}\cong H^i(P/E,H^j(E,k))\Rightarrow H^{i+j}(P,k)$ of the extension $E\rightarrowtail P\twoheadrightarrow P/E$. Since E is a central subgroup, we get $E_2^{0,j}=H^j(E,k)^{P/E}=H^j(E,k)$.

There is a cup-product [4]

$$E_r^{i,j}\otimes E_r^{i',j'}\xrightarrow{\cup}E_r^{i+i',j+j'}$$

with the following rules

(i) $a\cdot b=(-1)^{ii'+jj'}b\cdot a$;

(ii) $d_r(a\cdot b)=d_ra\cdot b+(-1)^{i+j}a\cdot d_rb$

$$a\in E_r^{i,j};\qquad b\in E_r^{i',j'}.$$

Suppose $0\neq x\in E_2^{0,2}$. Since $\operatorname{char}k=p$, one easily checks that $d_2(x^p)=pd_2x\cdot x^{p-1}=0$.

Now x^p is a nontrivial cocycle of $E_3^{0,2p}$ and $d_3(x^{p^2})=p\cdot d_3x^p\cdot x^{p(p-1)}=0$. Iteration of this process yields $0\neq x^{p^s}\in E_{s+2}^{0,2p^s}$. By a theorem of Evens [1] the spectral sequence of a finite group extension stops, i.e. there is a $t\in\mathbb{N}$ with $E_t=E_\infty$. Now $s=t-2$ yields $0\neq x^{p^s}\in E_\infty^{0,2p^s}$, but x^{p^s} then lies in the image of the restriction map

$$\mathrm{res}:H^{2p^s}(P,k)\to H^{2p^s}(E,k).$$

It follows that the polynomial ring $k[x_1^{p^s}, \ldots, x_m^{p^s}]$ lies in the image of the restriction map.

Note 4.2. It follows from the naturality of the *LHS*-spectral sequence, that, if the p'-group Q acts on the extension $E \rightarrowtail P \twoheadrightarrow P/E$, then the restriction map

$$\mathrm{res}: H^*(P, k) \to H^*(E, k)$$

is a kQ-module homomorphism.

LEMMA 4.3. *Let the p'-group Q act on the central extension $E \rightarrowtail P \twoheadrightarrow P/E$. Let N denote the centralisator $C_Q(E)$. Then every simple $k(Q/N)$-module A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. The group Q/N acts faithfully on E. By Lemma 4.1 and Note 4.1 every simple $k(Q/N)$-module A is infinitely often a direct summand of $k[x_1^{p^s}, \ldots, x_m^{p^s}]$. By Lemma 4.2 A is infinitely often a direct summand in the image of the restriction map

$$\mathrm{res}: H^*(P, k) \to H^*(E, k),$$

and by Note 4.2 A is infinitely often a direct summand of $H^*(P, k)$.

THEOREM 4.4. *Let the p'-group Q act on the central extension $E \rightarrowtail P \twoheadrightarrow P/E$. If the simple kQ-module A is a direct summand of $H^*(P/E, k)$, then A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. We consider the spectral sequence $E_2^{i,j} \cong H^i(P/E, H^j(E, k)) \Rightarrow H^{i+j}(P, k)$ associated with the extension $E \rightarrowtail P \twoheadrightarrow P/E$. Let $B_1, B_2, \ldots, B_m$ be the simple direct summands of $E_2^{0,*}$ and let $A_1, A_2, \ldots, A_n$ be the simple direct summands of $E_2^{*,0}$.

Since E is a central subgroup, we get

$$H^i(P/E, H^j(E, k)) \cong H^i(P/E, k) \underset{k}{\otimes} H^j(E, k) \cong E_2^{i,0} \otimes E_2^{0,j},$$

and $E_2^{i,j}$ is a direct sum of tensorproducts $A_a \otimes B_b$. If we let the p'-group Q act diagonally on $E_r^{i,0} \otimes E_r^{0,j}$, then the map $E_r^{i,0} \otimes E_r^{0,j} \xrightarrow{\cup} E_r^{i,j}$ is a kQ-module homomorphism.

First we prove that there is a simple kQ-module A_s depending on A such that A_s is a direct summand of $E_\infty^{i',0}$. Secondly we show that A is infinitely often a direct summand in the image of the map

$$E_\infty^{i',0} \otimes E_\infty^{0,*} \xrightarrow{\cup} E_\infty^{i',*}.$$

(1) Let i' be the smallest i such that A is a direct summand in some tensorproduct $A_s \otimes B_u$ with $A_s \subseteq E_2^{i',0}$ and $B_u \subseteq E_2^{0,*}$. We show that A_s is a direct summand in $E_\infty^{i',0}$:

If A_s lies in the image of the differential $d_r : E_r^{i'-r-1,r} \to E_r^{i',0}$, then A_s is a direct summand in some tensorproduct $A_t \otimes B_v$ with

$$A_t \subseteq E_2^{i'-r-1,0} \quad \text{and} \quad B_v \subseteq E_2^{0,r}.$$

The module A is then a direct summand in the tensorporoduct $(A_t \otimes B_v) \otimes B_u = A_t \otimes (B_v \otimes B_u)$.

But $B_v \otimes B_u = \bigoplus_w B_w$ and therefore A is a direct summand in $A_t \otimes B_w$. Since B_w is a direct summand of $E_2^{0,*}$ it follows that A is a direct summand of $E_2^{i'-r-1,*}$. This contradicts the minimality of i'. Hence A_s is a direct summand of $E_\infty^{i',0}$.

(2) By Lemma 4.2 and Lemma 4.3 B_u is infinitely often a direct summand in the image of the restriction map

$$\text{res}: H^*(P, k) \to H^*(E, k) \quad \text{i.e.}$$

B_u is infinitely often a direct summand of $E_\infty^{0,*}$. Hence A is infinitely often a direct summand in $E_\infty^{i',0} \otimes E_\infty^{0,*}$.

If A is contained in the kernel of the map $E_\infty^{i',0} \otimes E_\infty^{0,*} \xrightarrow{\cup} E_\infty^{i',*}$, then A lies in the image of some differential $d_r : E_r^{i'-r-1,*} \to E_r^{i',*}$.

This contradicts the minimality of i'. It thus follows that A is infinitely often a direct summand of $E_\infty^{i',*}$.

THEOREM 4.5. *If the p'-group Q acts faithfully on the p-group P, then every simple kQ-module A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. Let us consider the lower central series of P

$$P = P^{(0)} \trianglerighteq P^{(1)} \trianglerighteq \cdots \trianglerighteq P^{(m)} = e.$$

We obviously can refine this series to a central series

$$P = \tilde{P}^{(0)} \trianglerighteq \tilde{P}^{(1)} \trianglerighteq \cdots \trianglerighteq \tilde{P}^{(n)} = e,$$

with elementary abelian factor groups $\tilde{P}^{(i)}/\tilde{P}^{(i+1)}$ and $\tilde{P}^{(0)}/\tilde{P}^{(1)} = P/\Phi(P)$.

If Q acts faithfully on P, Q acts faithfully on $P/\Phi(P)$, see for example [5] p. 102.

By Lemma 4.1 every simple kQ-module A is infinitely often a direct summand of $H^*(P/\tilde{P}^{(1)}, k)$. By Theorem 4.4 A is infinitely often a direct summand of $H^*(P/\tilde{P}^{(2)}, k)$. Iterating this step for the factor groups $P/\tilde{P}^{(i)}$ yields the result that A is infinitely often a direct summand of $H^*(P, k)$.

THEOREM 4.6. *Let G be a group of p-length* 1, *and let A be a simple kG-module lying in the principal block of kG. Then*

$$H^n(G, A) \neq 0 \quad \textit{for infinitely many} \quad n \in \mathbb{N}.$$

Proof. Let A^* denote the dual of A. By Theorem 4.5 A^* is infinitely often a direct summand of $H^*(P, k)$. Dualisation yields the fact that A is infinitely often a direct summand of $H_*(P, k)$. By Theorem 3.1 $H^n(G, A)$ is nontrivial for infinitely many $n \in \mathbb{N}$.

5. The case $G = O_{p'pp'p}G$

LEMMA 5.1. *Let $N \rightarrowtail G \twoheadrightarrow P$ be a group extension with $|P| = p^a$; $a \in \mathbb{N}$, and let A be a kG-module. Then*

$$H^n(N, A) \neq 0 \Rightarrow H^n(G, A) \neq 0.$$

Proof. We consider the long exact sequence [3] p. 224

$$\rightarrow H^n(G, A) \rightarrow H^n(N, A) \rightarrow \mathrm{Ext}^n_G(IP, A) \rightarrow$$

where IP denotes the augmentation ideal of the factor group P. Let IP^* denote the dual of IP^* then there is a natural isomorphism

$$\mathrm{Ext}^n_G(IP, A) \cong H^n(G, IP^* \underset{k}{\otimes} A).$$

Since P is a p-group, all composition factors of IP^* are isomorphic to the trivial module k. A composition series of IP^* induces a composition series of $IP^* \otimes_k A$, of which all composition factors are isomorphic to A.

If $H^n(G, IP^* \otimes_k A)$ is nontrivial, it follows by induction, that $H^n(G, A)$ is nontrivial.

From $H^n(N, A) \neq 0$ and from the above sequence we may conclude that $H^n(G, A)$ or $H^n(G, IP^* \otimes_k A)$ and hence again $H^n(G, A)$ is nontrivial.

LEMMA 5.2. *Let G be a p-solvable group with normal subgroup N, and let A be a simple kG-module lying in the principal block of kG. Then:*

(i) *$A = \bigoplus_{i=1}^m B_i$ as a kN-module and all B_i are simple kN-modules.*

(ii) *The simple kN-modules B_i lie in the principal block of kN.*

Proof. (i) is a consequence of Clifford's theorem.

(ii) For p-solvable groups the following holds [2] p. 279

$$C_G(A) \supseteq O_{p'p}G \Leftrightarrow A \quad \text{lies in the principal block of } kG$$

Since $O_{p'p}G$ is the maximal p-nilpotent normal subgroup of G, $O_{p'p}N$ is a subgroup of $O_{p'p}G$. Therefore we get $O_{p'p}N \subseteq O_{p'p}G \subseteq C_G(A)$, and thus all B_i are simple kN-modules lying in the principal block of kN.

THEOREM 5.3. *Let $G = O_{p'pp'p}G$, and let A be a simple kG-module lying in the principal block of kG. Then*

$$H^n(G, A) \neq 0 \quad \textit{for infinitely many} \quad n \in \mathbb{N}.$$

Proof. We consider the extension

$$O_{p'pp'}G \rightarrowtail G \twoheadrightarrow G/O_{p'pp'}G.$$

The factor groups $G/O_{p'pp'}G$ is a p-group, and the normal subgroup $O_{p'pp'}G$ has p-length 1.

By Lemma 5.2 A is a direct sum of simple $k(O_{p'pp'}G)$-modules B_i lying in the principal block of $k(O_{p'pp'}G)$. By Theorem 4.6 $H^n(O_{p'pp'}G, B_i)$ is nontrivial for infinitely many $n \in \mathbb{N}$, and Lemma 5.1 yields

$$H^n(G, A) \neq 0 \quad \text{for infinitely many} \quad n \in \mathbb{N}.$$

REFERENCES

[1] Evens, L., *The spectral sequence of a finite group extension stops.* Trans. amer. math. soc., vol. *210* (1975), p. 269–277.

[2] Fong, P., *On the characters of p-solvable groups.* Trans. amer. math. soc., vol. *98*, p. 263–284.

[3] Hilton, P. J. and Stammbach, U., *A course in Homological Algebra.* Springer 1971.

[4] HOCHSCHILD, G. P. and SERRE, J. P., *Cohomology of group extensions*. Trans. amer. math. soc., vol. *74* (1953), p. 110–135.
[5] KURZWEIL, H., *Endliche Gruppen*. Springer 1977.
[6] PUTTASWAMAIAH, B. M. and DIXON, J. D., *Modular Representations of Finite Groups*. Academic Press 1977.
[7] STEINBERG, R., *Complete sets of representations of algebras*. Proc. amer. math. soc. Vol. *13*, p. 746–747 (1962).

Eidgenössische Technische Hochschule
Abt. Mathematik
Zentrum
CH-8092 Zürich/Switzerland

Received July 26, 1982