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$D(\mathbb{Z}\pi)^+$ and the Artin cokernel

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Let p be an odd prime. If π is a p-group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order containing $\mathbb{Z}\pi$, then we set

$$D(\mathbb{Z}\pi) = \mathrm{Ker}\,[K_0(\mathbb{Z}\pi) \to K_0(\mathfrak{M})]$$

as usual. Since $D(\mathbb{Z}\pi)$ is a p-group [2], the involution $g \mapsto g^{-1}$ induces a natural splitting

$$D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+ \oplus D(\mathbb{Z}\pi)^-;$$

where $D(\mathbb{Z}\pi)^+$ and $D(\mathbb{Z}\pi)^-$ are the $+1$ and -1 eigenspaces, respectively. The involution can be extended to a linear action of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$; and this induces further eigenspace decompositions

$$D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi) \quad \text{and} \quad D(\mathbb{Z}\pi)^- = \sum_{i=0}^{(p-3)/2} {}^{2i+1}D(\mathbb{Z}\pi).$$

The groups $D(\mathbb{Z}\pi)^-$ and ${}^{2i+1}D(\mathbb{Z}\pi)$ have been studied extensively in [3] and [14]; and the order of $D(\mathbb{Z}\pi)^-$ for abelian π is computed in [3]. In this paper, attention is focused on ${}^0D(\mathbb{Z}\pi)$. This is the part of $D(\mathbb{Z}\pi)^+$ which is independent of number theoretic properties of p; and in fact, ${}^0D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+$ whenever p is regular.

For any finite group π, we define the "Artin cokernel" $A_{\mathbb{Q}}(\pi)$ to be the group

$$A_{\mathbb{Q}}(\pi) = \mathrm{Coker}\,[\mathrm{Ind}: \textstyle\sum \{R_{\mathbb{Q}}(\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \to R_{\mathbb{Q}}(\pi)],$$

where $R_{\mathbb{Q}}(\pi)$ denotes the rational representation ring. The main result here is that ${}^0D(\mathbb{Z}\pi) \cong A_{\mathbb{Q}}(\pi)$ for any p-group π (p odd). Among other consequences, this gives new insight into Martin Taylor's result [13] that the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism has order equal to the Artin exponent of $\pi : T(\mathbb{Z}\pi)$ corresponds under this isomorphism to multiples of the identity in $R_{\mathbb{Q}}(\pi)$.

We end by deriving a formula for $|{}^0D(\mathbb{Z}\pi)|$—in fact, a formula for the order of $A_{\mathbb{Q}}(\pi)$ for arbitrary finite π. Also, for the sake of completeness, the formula for $|D(\mathbb{Z}\pi)^-|$ in [3] is generalized to cover arbitrary p-groups; thus giving a complete calculation of $|D(\mathbb{Z}\pi)|$ when π is a p-group and p any odd regular prime.

For $n \geqq 1$, $\mathbb{Q}\zeta_n$ always denotes the field generated by the n-th roots of unity (and similarly for $\mathbb{Z}\zeta_n$, $\hat{\mathbb{Z}}_p\zeta_n$, etc.) Also, $\varphi(n)$ will always mean the Euler φ-function.

Throughout the paper, unless otherwise stated, p will be a fixed odd prime. We start with the following well known description of $\mathbb{Q}\pi$ when π is a p-group.

PROPOSITION 1. *Let π be a p-group. Then $\mathbb{Q}\pi$ is a product of matrix rings over fields $\mathbb{Q}\zeta_{p^s}$ for various $s \geqq 0$. Furthermore, for each $s \geqq 0$, the number of simple summands isomorphic to matrix rings over $\mathbb{Q}\zeta_{p^s}$ is equal to the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ such that*

$$|\sigma/[\sigma, N(\sigma)]| = |\sigma| \cdot |Z(\sigma)|/|N(\sigma)| = p^s.$$

Proof. That $\mathbb{Q}\pi$ is a product of matrix algebras over the $\mathbb{Q}\zeta_{p^s}$ is shown (though not explicitly stated) by Roquette in [11]. In Section 2 of [11] he shows that the division algebra for any irreducible representation M of π is isomorphic to the division algebra of a primitive faithful representation of some subquotient of π; and in Section 3 he shows that the only p-groups with primitive faithful representations are the cyclic groups.

Now, for all $s \geqq 0$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$; and let v_s be the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ with $|\sigma/[\sigma, N(\sigma)]| = p^s$. Let $p^n = |\pi|$. We say that two elements $g, h \in \pi$ are $\mathbb{Q}\zeta_{p^m}$-conjugate (any $m \geqq 0$) if g is conjugate to h^a for some

$$a \in \mathrm{Gal}\,(\mathbb{Q}\zeta_{p^n}/\mathbb{Q}\zeta_{p^m}) \subseteq \mathrm{Gal}\,(\mathbb{Q}\zeta_{p^n}/\mathbb{Z}) = (\mathbb{Z}/p^n)^*.$$

In other words, $a \equiv 1 \pmod{p^m}$ if $m \geqq 1$, or $p \nmid a$ if $m = 0$.

For each cyclic $\sigma \subseteq \pi$, the number of conjugacy classes of generators of σ is just $\varphi(|\sigma/[\sigma, N(\sigma)]|)$. So for any $m \geqq 0$, the number of $\mathbb{Q}\xi_{p^m}$-conjugacy classes in π is

$$\sum_{s \geqq 0} v_s \cdot \min\{\varphi(p^m), \varphi(p^s)\}. \tag{1}$$

On the other hand

$$\mathrm{rk}\, K_0(\mathbb{Q}\xi_{p^m}[\pi]) = \sum_{s \geqq 0} w_s \cdot \min\{\varphi(p^m), \varphi(p^s)\}; \tag{2}$$

and by Theorem 21.5 in [1], the numbers in (1) and (2) are equal for all m. So $v_s = w_s$ for all $s \geqq 0$. □

Thus, if π is a p-group and $L = Z(\mathbb{Q}\pi)$ is the center, then the reduced norm

$$\nu : K_1(\hat{\mathbb{Q}}_p\pi) \xrightarrow{\cong} (\hat{L}_p)^*$$

is just the product of the determinant maps for the simple summands of $\hat{\mathbb{Q}}_p\pi$. If $\mathfrak{M} \subseteq \mathbb{Q}\pi$ and $\mathcal{O} \subseteq L$ are maximal orders, then ν induces an isomorphism of $K_1(\hat{\mathfrak{M}}_p)$ with $(\hat{\mathcal{O}}_p)^*$: by [9, Theorem 21.6], $\hat{\mathfrak{M}}_p$ is a product of matrix algebras over the components of $\hat{\mathcal{O}}_p$. In particular, $K_1(\hat{\mathfrak{M}}_p)$ can be regarded as a subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$. We let $K_1'(\mathfrak{M})$ and $K_1'(\hat{\mathbb{Z}}_p\pi)$ denote the images of $K_1(\mathfrak{M})$ and $K_1(\hat{\mathbb{Z}}_p\pi)$ in $K_1(\hat{\mathbb{Q}}_p\pi)$; and let $K_1'(\mathfrak{M})^\wedge$ denote the p-adic closure of $K_1'(\mathfrak{M})$.

We will use the following description of $D(\mathbb{Z}\pi)$, based on the formulas in [4].

PROPOSITION 2. *Let π be a p-group, and let $\mathfrak{M} \subseteq \mathbb{Z}\pi$ be a maximal order in $\mathbb{Q}\pi$. Then there is an isomorphism*

$$D(\mathbb{Z}\pi) \cong \operatorname{Coker}[K_1'(\hat{\mathbb{Z}}_p\pi) \to K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})^\wedge]$$

which is natural in π.

Proof. Let $L \subseteq \mathbb{Q}\pi$ be its center, and let $\mathcal{O} \subseteq L$ be its maximal order. Since $\hat{\mathbb{Z}}_q[\pi] = \hat{\mathfrak{M}}_q$ for all primes $q \neq p$ [9, Theorem 41.1], Theorems 1 and 2 in [4] reduce to the formula

$$D(\mathbb{Z}\pi) \cong K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M}) \cdot K_1'(\hat{\mathbb{Z}}_p\pi).$$

Furthermore, $K_1(\hat{\mathfrak{M}}_p)/K_1'(\hat{\mathbb{Z}}_p\pi)$ is finite [12, Proposition 8.15]; and so $K_1'(\mathfrak{M})$ can be replaced by its p-adic closure. □

Now let

$$\kappa : \mathbb{F}_p^* \to (\hat{\mathbb{Z}}_p)^*$$

denote the inclusion into the group of $(p-1)$-st roots of unity (with $\kappa(a) = a \pmod p$ for $a \in \mathbb{F}_p^*$). If π is an abelian p-group, then the group homomorphisms $g \to g^{\kappa(a)}$ for $a \in \mathbb{F}_p^*$ and $g \in \pi$ induce actions of $\mathbb{F}_p^*$ on $K_i(\mathbb{Z}\pi)$, $K_i(\hat{\mathbb{Z}}_p\pi)$, $D(\mathbb{Z}\pi)$, etc. The next problem is to find a natural way to do this when π is non-abelian.

Let π be an arbitrary p-group. Then by Proposition 1, the center $Z(\mathbb{Q}\pi)$ is a product of fields $\mathbb{Q}\zeta_{p^s}$ for various $s \geqq 0$. The natural embedding $\mathbb{F}_p^* \subseteq \mathrm{Gal}(\mathbb{Q}\zeta_{p^s}/\mathbb{Q})$ (where $a \in \mathbb{F}_p^*$ sends ζ_{p^n} to $\zeta_{p^n}^{\kappa(a)}$) induces actions of $\mathbb{F}_p^*$ on the centers $Z(\mathbb{Q}\pi)$ and $Z(\hat{\mathbb{Q}}_p\pi)$; and hence on $K_1(\hat{\mathbb{Q}}_p\pi)$. If $\alpha : \pi \to \pi'$ is a homomorphism of p-groups, then

$$\alpha_* : K_1(\hat{\mathbb{Q}}_p\pi) \to K_1(\hat{\mathbb{Q}}_p\pi')$$

is a product of norm, includion, and diagonal maps between the groups of units of the field components of the centers; and is hence $\mathbb{F}_p^*$-linear.

PROPOSITION 3. *Let π be a p-group. Then*
(i) *$K_1'(\hat{\mathbb{Z}}_p\pi)$ is an $\mathbb{F}_p^*$-invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$.*
(ii) *For any $0 \leqq t \leqq p-2$, set*

$$^tK_1'(\hat{\mathbb{Z}}_p\pi) = \{x \in K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)} : \tau_a(x) = \kappa(a)^t \cdot x \text{ for all } a \in \mathbb{F}_p^*\}$$

(here τ_a denotes the action of $a \in \mathbb{F}_p^*$). Then for $t \neq 1$, $^tK_1'(\hat{\mathbb{Z}}_p\pi)$ is generated by induction from cyclic subgroups.

Proof. By [8, Theorem 2], there is an exact sequence

$$0 \to \langle \lambda g : \lambda \in \hat{\mathbb{Z}}_p^*, g \in \pi \rangle \hookrightarrow K_1'(\hat{\mathbb{Z}}_p\pi) \xrightarrow{\Gamma} \overline{I(\hat{\mathbb{Z}}_p\pi)} \xrightarrow{\omega} \pi^{ab} \to 0,$$

natural in π, where

$$\overline{I(\hat{\mathbb{Z}}_p\pi)} = \mathrm{Ker}[\varepsilon : \hat{\mathbb{Z}}_p\pi \to \hat{\mathbb{Z}}_p]/\langle gxg^{-1} - x : g \in \pi, x \in \hat{\mathbb{Z}}_p\pi \rangle$$

and

$$\varepsilon\left(\sum \lambda_i g_i\right) = \sum \lambda_i; \qquad \omega\left(\sum \lambda_i g_i\right) = \prod g_i^{\lambda_i}.$$

For $a \in \mathbb{F}_p^*$, let $\hat{\tau}_a$ be the action of a on $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ given by: $\hat{\tau}_a(\sum \lambda_i g_i) = \sum \lambda_i g_i^{\kappa(a)}$. This clearly leaves $\mathrm{Ker}(\omega) = \mathrm{Im}(\Gamma)$ invariant. By the definition of Γ in [8], Γ is $\mathbb{F}_p^*$-linear when π is abelian.

Now set

$$X(\pi) = \mathrm{Im}\left[\mathrm{Ind}: \sum \{K_1'(\hat{\mathbb{Z}}_p\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \to K_1'(\hat{\mathbb{Z}}_p\pi)\right]$$

and

$$Y(\pi) = \Gamma^{-1}\left({}^1\overline{I(\hat{\mathbb{Z}}_p\pi)}\right) \cap \mathrm{Ker}[\varepsilon_* : K_1'(\hat{\mathbb{Z}}_p\pi) \to \hat{\mathbb{Z}}_p^*]. \tag{1}$$

Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, $\mathrm{Ker}\,(\Gamma)\subseteq X(\pi)$, and

$$ {}^t\overline{I(\hat{\mathbb{Z}}_p\pi)} = {}^t\mathrm{Ker}\,(\omega) = {}^t\mathrm{Im}\,(\Gamma) \tag{2}$$

for $t\neq 1$ $(0\leqq t\leqq p-2)$; it follows that

$$K_1'(\hat{\mathbb{Z}}_p\pi) = X(\pi) + Y(\pi). \tag{3}$$

By naturality, $X(\pi)$ is an $\mathbb{F}_p^*$-invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$, and $\Gamma\,|\,X(\pi)$ is $\mathbb{F}_p^*$-linear. It follows that for n large,

$$Y(\pi)^{p^n} \subseteq {}^1X(\pi). \tag{4}$$

Furthermore,

$$\mathrm{tors}\,(K_1(\hat{\mathbb{Q}}_p\pi))_{(p)} \subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi):$$

the torsion in $K_1(\hat{\mathbb{Q}}_p\pi)$ comes from roots of unity in the center. It follows by (4) that $Y(\pi)\subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi)$; and hence by (3) that $K_1'(\hat{\mathbb{Z}}_p\pi)$ is $\mathbb{F}_p^*$-invariant. Furthermore, by (1), this shows that Γ is $\mathbb{F}_p^*$-linear. Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, ${}^tK_1'(\hat{\mathbb{Z}}_p\pi)$ is generated by cyclic induction for $t\neq 1$ by (2). □

Propositions 2 and 3 now imply the existence of natural actions of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$:

PROPOSITION 4. *For any p-group π, there is a natural linear action of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$ such that the isomorphism of Proposition 2 is $\mathbb{F}_p^*$-linear.* □

In particular, for any p-group π and $0\leqq t\leqq p-2$, set

$${}^tD(\mathbb{Z}\pi) = \{x\in D(\mathbb{Z}\pi) : \tau_a(x) = \kappa(a)^t\cdot x \text{ for all } a\in\mathbb{F}_p^*\}.$$

Since $D(\mathbb{Z}\pi)$ is a p-group [2], and $p\nmid|\mathbb{F}_p^*|$,

$$D(\mathbb{Z}\pi) = \sum_{t=0}^{p-2} {}^tD(\mathbb{Z}\pi) \quad\text{and}\quad D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi).$$

Here, $D(\mathbb{Z}\pi)^+$ is the group of elements invariant under the involution τ_{-1}; induced by complex conjugation on $Z(\mathbb{Q}\pi)$.

If p is regular, then results of Iwasawa (see, *fx*, [7, Theorem 7.5.2]) show that for any $s \geqq 0$ and any even $0 < t \leqq p-3$,

$$^t[(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,] = {}^t(\hat{\mathbb{Z}}_p\zeta_{p^s})^*.$$

So by Proposition 1, if $\mathfrak{M} \supseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then ${}^tK_1'(\mathfrak{M})\hat{} = {}^tK_1(\hat{\mathfrak{M}}_p)$; and hence ${}^tD(\mathbb{Z}\pi) = 0$ for such t. In other words:

PROPOSITION 5. *If p is an odd regular prime and π is a p-group, then*

$$D(\mathbb{Z}\pi)^+ = {}^0D(\mathbb{Z}\pi). \quad \square$$

In order to study the groups ${}^0D(\mathbb{Z}\pi)$, we must first describe

$$^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,]$$

for any $s \geqq 0$. The following result must be well known, but we have been unable to find a reference.

PROPOSITION 6. *For any $s \geqq 0$,*

$$^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,]_{(p)} \cong \hat{\mathbb{Z}}_p.$$

Proof. This is clear if $s=0$. So fix $s>0$, let $K \subseteq \mathbb{Q}\zeta_{p^s}$ be the fixed subfield of $\mathbb{F}_p^*$, and let $R \subseteq K$ be the ring of integers. Then $\Gamma = \mathrm{Gal}\,(K/\mathbb{Q})$ is cyclic of order p^{s-1}. Set $\zeta = \zeta_{p^s}$, and let $\gamma \in \Gamma$ be the generator: $\gamma(\zeta) = \zeta^{p+1}$.

Let

$$\mathfrak{p} = \left\langle z = \prod_{a=1}^{p-1} (1-\zeta^{\kappa(a)}) \right\rangle \subseteq R$$

be the prime ideal over p. Set

$$U' = \mathrm{Ker}\,[N : (\hat{R}_p)^* \to (\hat{\mathbb{Z}}_p)^*],$$

where N is the norm of $\hat{K}_p/\hat{\mathbb{Q}}_p$. Note that $U' \subseteq 1 + \mathfrak{p}\hat{R}_p$.

Fix $u \in U'$. By Hilbert's Theorem 90, there is $x \in \hat{K}_p^*$ such that $u = \gamma(x)/(x)$. Write $x = z^i v$, where $v \in (\hat{R}_p)^*$ and z is the element defined above. Then

$$\gamma(z^i)/z^i = \prod_{a=1}^{p-1} \left(\frac{1-\zeta^{(p+1)\kappa(a)}}{1-\zeta^{\kappa(a)}}\right)^i \in R^* \cap (1+\mathfrak{p}).$$

Furthermore, $N(v) \in \kappa(\mathbb{F}_p^*) \times (1+p^s\hat{\mathbb{Z}}_p)$ (the norm group has index p^{s-1} by local class field theory). So there exists $w \in (\hat{\mathbb{Z}}_p)^*$ such that $N(w) = w^{p^{s-1}} = N(v)$. Then

$$N(vw^{-1}) = 1 \quad \text{and} \quad \gamma(vw^{-1})/(vw^{-1}) = \gamma(v)/v = x \cdot (\gamma(z^i)/z^i)^{-1}.$$

In other words,

$$U' = \{\gamma(v)/v : v \in U'\} \cdot (R^* \cap (1+\mathfrak{p})). \tag{1}$$

But U' is a $\hat{\mathbb{Z}}_p[\Gamma]$-module, and the closure of $R^* \cap (1+\mathfrak{p})$ is a $\hat{\mathbb{Z}}_p[\Gamma]$-submodule. Since $\hat{\mathbb{Z}}[\Gamma]$ is a local ring with maximal ideal generated by p and $\gamma - 1$, no proper submodule of U' can generate

$$U'/\langle \gamma(v)/v, v^p : v \in U' \rangle.$$

So by (1), $R^* \cap (1+\mathfrak{p})$ is dense in $U' = \operatorname{Ker}(N)$; and

$${}^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{\ }]_{(p)} = [(\hat{R}_p)^*/(R^*)\hat{\ }]_{(p)} \cong (\operatorname{Im}(N))_{(p)} \cong \hat{\mathbb{Z}}_p. \quad \square$$

By Proposition 6, if π is a p-group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order, then ${}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)}$ is a sum of one copy of $\hat{\mathbb{Z}}_p$ for each irreducible $\mathbb{Q}\pi$-module; and is thus (abstractly, at least) isomorphic to $\hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi)$. The key remaining step is to construct a natural isomorphism between these groups; once this is done the isomorphism between ${}^0D(\mathbb{Z}\pi)$ and the Artin cokernel will follow easily.

We temporarily allow p to be an arbitrary prime (possibly $p = 2$). If A is a $\hat{\mathbb{Q}}_p$-algebra, and V is an A-module with $\dim_{\hat{\mathbb{Q}}_p}(V) < \infty$; let $\det(u, V)$, for $u \in A$, denote the determinant over $\hat{\mathbb{Q}}_p$ of $u : V \to V$. Define

$$L : (\hat{\mathbb{Z}}_p)^* \to \hat{\mathbb{Z}}_p$$

by setting $L(u) = 1/p \log(u/\kappa(\bar{u}))$ for $u \in (\hat{\mathbb{Z}}_p)^*$ and $\bar{u} \in \mathbb{F}_p^*$ its reduction mod p (note that $u/\kappa(\bar{u}) \in 1 + p\hat{\mathbb{Z}}_p$).

Now assume A is a finite dimensional semisimple $\hat{\mathbb{Q}}_p$-algebra, and let $\mathfrak{A} \subseteq A$ be any order. Let $V_1, \ldots V_k$ be the distinct irreducible A-modules, and set

$$n_i = [\operatorname{End}_A(V_i) : \hat{\mathbb{Q}}_p].$$

Define a homomorphism

$$\delta = \delta_{\mathfrak{A}} : K_1(\mathfrak{A}) \to \hat{\mathbb{Q}}_p \otimes_{\mathbb{Z}} K_0(A)$$

by setting, for any matrix $u \in GL_r(\mathfrak{A})$,

$$\delta([u]) = \sum_{i=1}^{k} \frac{1}{n_i} L(\det(u, V_i^r)) \cdot [V_i].$$

PROPOSITION 7. *For any prime p, the maps $\delta_{\mathfrak{A}}$ are natural with respect to homomorphisms between orders in semisimple $\hat{\mathbb{Q}}_p$-algebras.*

Proof. We must show, for any homomorphism $\alpha : A \to B$, orders $\mathfrak{A} \subseteq A$ and $\mathfrak{B} \subseteq B$ such that $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$, and $u \in \mathfrak{A}^*$, that

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \delta_{\mathfrak{B}}(\alpha(u)) \in \hat{\mathbb{Q}}_p \otimes K_0(B).$$

Let $V_1, \ldots, V_s$ be the irreducible A-modules, and $W_1, \ldots, W_t$ the irreducible B-modules. Define $a_{ij}, b_{ij} \in \mathbb{Z}$ by setting

$$\alpha_*(V_i) = \sum_{j=1}^{t} a_{ij} W_j, \qquad \alpha^*(W_j) = \sum_{i=1}^{s} b_{ij} V_i$$

(where $\alpha_*(V_i) = B \otimes_A V_i$, and $\alpha^*(W_j)$ is W_j regarded as an A-module). We also set

$$m_i = [\mathrm{End}_A(V_i) : \hat{\mathbb{Q}}_p], \qquad n_j = [\mathrm{End}_B(W_j) : \hat{\mathbb{Q}}_p],$$

and write $L_i = L(\det(u, V_i))$ for short. Then

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \alpha_*\left(\sum_{i=1}^{s} (L_i/m_i)[V_i]\right) = \sum_{i,j} (a_{ij} L_i/m_i)[W_j]$$

and

$$\delta_{\mathfrak{B}}(\alpha(u)) = \sum_{j=1}^{t} n_j^{-1} L(\det(u, \alpha^*(W_j)))[W_j] = \sum_{i,j} (b_{ij} L_i/n_j)[W_j].$$

It remains to check that $(b_{ij}/n_j) = (a_{ij}/m_i)$ for all i, j. But

$$\dim \mathrm{Hom}_A(V_i, W_j) = m_i b_{ij}, \quad \dim \mathrm{Hom}_B(\alpha_* V_i, W_j) = n_j a_{ij};$$

and these two dimensions are equal by [1, Theorem 2.19]. □

We now again restrict to the case where p is odd.

PROPOSITION 8. *Let π be a p-group, let $\mathfrak{M} \subseteq \mathbb{Q}\pi$ be any maximal order, and set $\delta_\pi = \delta_{\hat{\mathbb{Q}}_p\pi}$. Then*

$$\mathrm{Im}\,[\delta_\pi : K_1(\hat{\mathfrak{M}}_p) \to \hat{\mathbb{Q}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi)] = \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi);$$

and δ_π induces an isomorphism

$$\delta' = \delta'_\pi : {}^0[K_1(\hat{\mathfrak{M}}_p)/K'_1(\mathfrak{M})\hat{}\,]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi) \cong \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi).$$

Proof. Using Proposition 1, it will suffice to show that whenever $A \cong M_r(\mathbb{Q}\zeta_{p^s})$ and $\mathfrak{M} \subseteq A$ is a maximal order, then $\delta = \delta_{\hat{A}_p}$ induces an isomorphism

$$\delta' : {}^0[K_1(\hat{\mathfrak{M}}_p)/K'_1(\mathfrak{M})\hat{}\,]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(A).$$

By [9, Theorem 21.6], we may assume that $\mathfrak{M} = M_r(\mathbb{Z}\zeta_{p^s})$.

Let $V \cong (\hat{\mathbb{Q}}_p\zeta_{p^s})^r$ be the irreducible $\hat{A}_p$-representation. For any $u \in 1 + J(\hat{\mathfrak{M}}_p)$ (where $J(\hat{\mathfrak{M}}_p)$ is the Jacobson radical),

$$\begin{aligned}\delta(u) &= \frac{1}{\varphi(p^s)} L(\det(u, V)) \cdot [V]\\ &= \frac{1}{p\varphi(p^s)} \log\,(N_{\hat{\mathbb{Q}}_p\zeta_{p^s}/\hat{\mathbb{Q}}_p}(\det_{\hat{\mathbb{Q}}_p\zeta_{p^s}}(u)) \cdot [V].\end{aligned}$$

Furthermore, by local class field theory,

$$N \circ \det(1 + J(\hat{\mathfrak{M}}_p)) = 1 + p^s\hat{\mathbb{Z}}_p$$

(or $1 + p\hat{\mathbb{Z}}_p$ if $s = 0$). Since $\log(1 + p^s\hat{\mathbb{Z}}_p) = p^s\hat{\mathbb{Z}}_p$ for $s \geqq 1$, we have

$$\delta(K_1(\hat{\mathfrak{M}}_p)_{(p)} = \delta(1 + J(\hat{\mathfrak{M}}_p)) = \hat{\mathbb{Z}}_p \cdot [V] = \hat{\mathbb{Z}}_p \otimes K_0(A).$$

If u is a global unit, then $N(\det(u)) = \pm 1$, and so $\delta(u) = 0$. Furthermore, δ is $\mathbb{F}_p^*$-linear when $K_0(A)$ is given the trivial action; and so δ induces a surjection

$$\delta' : {}^0[K_1(\hat{\mathfrak{M}}_p)/K'_1(\mathfrak{M})\hat{}\,]_{(p)} \to \hat{\mathbb{Z}}_p \otimes K_0(A) \cong \hat{\mathbb{Z}}_p.$$

But the two groups are isomorphic by Proposition 6, and so δ' is an isomorphism. □

We can now prove the main result. Recall that the Artin cokernel $A_{\mathbb{Q}}(\pi)$ is defined by

$$A_{\mathbb{Q}}(\pi)=\operatorname{Coker}\left[\operatorname{Ind}:\sum\{R_{\mathbb{Q}}(\sigma):\sigma\subseteq\pi,\ \sigma \text{ cyclic}\}\to R_{\mathbb{Q}}(\pi)\right].$$

THEOREM 9. *For any p-group π (p-odd), δ' induces an isomorphism*

$$\delta''_\pi:{}^0D(\mathbb{Z}\pi)\xrightarrow{\cong}A_{\mathbb{Q}}(\pi).$$

Proof. Let C be the set of cyclic subgroups of π. Propositions 2, 7, and 8 combine to give the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\sum_{\sigma\in C}{}^0K_1'(\hat{\mathbb{Z}}_p\sigma) & \xrightarrow{\Sigma\eta_\sigma} & \sum_{\sigma\in C}\hat{\mathbb{Z}}_p\otimes R_{\mathbb{Q}}(\sigma) & \xrightarrow{\Sigma\theta_\sigma} & \sum_{\sigma\in C}{}^0D(\mathbb{Z}\sigma) & \longrightarrow & 0\\
\downarrow{\scriptstyle I_1} & & \downarrow{\scriptstyle I_2} & & \downarrow{\scriptstyle I_3} & & \\
{}^0K_1'(\hat{\mathbb{Z}}_p\pi) & \xrightarrow{\eta_\pi} & \hat{\mathbb{Z}}_p\otimes R_{\mathbb{Q}}(\pi) & \xrightarrow{\theta_\pi} & {}^0D(\mathbb{Z}\pi) & \longrightarrow & 0
\end{array}$$

Here I_1, I_2, and I_3 are the induction maps; and θ_π and η_π are the composites ($\mathfrak{M}\subseteq\mathbb{Q}\pi$ a maximal order):

$$\theta_\pi:\hat{\mathbb{Z}}_p\otimes R_{\mathbb{Q}}(\pi)\xrightarrow[\cong]{(\delta')^{-1}}{}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})^\wedge]_{(p)}\longrightarrow{}^0D(\mathbb{Z}\pi)$$

(the second map being the map of Proposition 2), and

$$\eta_\pi:{}^0K_1'(\hat{\mathbb{Z}}_p\pi)\to{}^0K_1(\hat{\mathfrak{M}}_p)\xrightarrow{\delta_\pi}\hat{\mathbb{Z}}_p\otimes R_{\mathbb{Q}}(\pi).$$

By Proposition 3(ii), I_1 is onto. Assume that ${}^0D(\mathbb{Z}\sigma)=0$ for any cyclic p-group σ. It then follows by diagram chasing that

$${}^0D(\mathbb{Z}\pi)\cong\operatorname{Coker}(I_2)\cong A_{\mathbb{Q}}(\pi).$$

($A_{\mathbb{Q}}(\pi)$ is a p-group by the Artin induction theorem: see, for example, [1, Theorem 15.4]).

It remains to check that ${}^0D(\mathbb{Z}\sigma)=0$ for cyclic σ: This is implicit in [6], [5], and [15]; but doesn't seem to be stated explicitly. If $|\sigma|\leq p$, then $D(\mathbb{Z}\sigma)=0$ by [10, Theorem 6.24].

So assume $|\sigma| = p^n$ for $n \geqq 2$. Let $\rho \subseteq \sigma$ be the order p subgroup, and assume inductively that ${}^0D(\mathbb{Z}[\sigma/\rho]) = 0$. There is a commutative diagram

$$\begin{array}{ccccc} \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\rho) & \xrightarrow{i_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma) & \xrightarrow{j_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho) \\ \downarrow{\scriptstyle \theta_\rho} & & \downarrow{\scriptstyle \theta_\sigma} & & \downarrow{\scriptstyle \theta_{\sigma/\rho}} \\ 0 = {}^0D(\mathbb{Z}\rho) & \longrightarrow & {}^0D(\mathbb{Z}\sigma) & \longrightarrow & {}^0D(\mathbb{Z}[\sigma/\rho]) = 0, \end{array}$$

where i_* and j_* are induced by inclusion and projection. Since $K_1(\hat{\mathbb{Z}}_p\sigma)$ maps onto $K_1(\hat{\mathbb{Z}}_p[\sigma'\rho])$,

$$j_*(\mathrm{Ker}\,(\theta_\sigma)) = \mathrm{Ker}\,(\theta_{\sigma/\rho}) = \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho).$$

In other words, $\theta_\sigma \mid \mathrm{Ker}\,(j_*)$ is onto. Furthermore, $\mathrm{Ker}\,(j_*) \subseteq \mathrm{Im}\,(i_*)$: if $V \cong \mathbb{Q}\zeta_{p^n}$ and $W \cong \mathbb{Q}\zeta_p$ are the faithful irreducible $\mathbb{Q}\sigma$- and $\mathbb{Q}\rho$-representations, then $[V]$ generates $\mathrm{Ker}\,(j_*)$, and $V = \mathrm{Ind}_\rho^\sigma(W)$. We thus get that $\theta_\sigma \mid \mathrm{Im}\,(i_*)$ is onto, but $\theta_\sigma \circ i_* = 0$, and so ${}^0D(\mathbb{Z}\sigma) = 0$.

One easy consequence of Theorem 9 is an alternate proof, for odd p-groups, of Martin Taylor's theorem [13] involving the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism. $T(\mathbb{Z}\pi)$ is the group of all elements

$$[\Sigma, n] - [\mathbb{Z}\pi] \in D(\mathbb{Z}\pi),$$

for $(n, |\pi|) = 1$, where $[\Sigma, n]$ is the projective module

$$[\Sigma, n] = n\mathbb{Z}\pi + \mathbb{Z} \cdot \Big(\sum_{g \in \pi} g\Big) \subseteq \mathbb{Z}\pi.$$

So if π is a p-group and $\mathfrak{M} \subseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then $[\Sigma, n] - [\mathbb{Z}\pi]$ corresponds, under the identification in Proposition 2, to the element of $K_1(\hat{\mathfrak{M}}_p)$ which is $n \in (\hat{\mathbb{Z}}_p)^*$ at the identity component and 1 at all other components (in particular, $T(\mathbb{Z}\pi) \subseteq {}^0D(\mathbb{Z}\pi)$). The isomorphism of Theorem 9 thus sends $T(\mathbb{Z}\pi)$ to the group of multiples of the identity in

$$R_{\mathbb{Q}}(\pi) / \sum \{\mathrm{Ind}_\sigma^\pi(R_{\mathbb{Q}}(\sigma)) : \sigma \subseteq \pi \text{ cyclic}\}.$$

In other words:

THEOREM 10. (*M. Taylor* [13]) *For any p-group π, $T(\mathbb{Z}\pi)$ is cyclic of order equal to the Artin exponent of π.* □

The computation of $|{}^0D(\mathbb{Z}\pi)|$ can now be carried out, using the same idea as for the calculation in [3]: that of comparing discriminants. We first consider the Artin cokernel of an arbitrary finite group.

THEOREM 11. *Let π be any finite group, and write*

$$\mathbb{Q}\pi \cong \prod_{i=1}^{k} M_{r_i}(D_i),$$

where the D_i are division algebras. Let X be a set of conjugacy class representatives for cyclic subgroups $\sigma \subseteq \pi$. Then

$$|A_{\mathbb{Q}}(\pi)| = \left[\left(\prod_{\sigma \in X} \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma|\right) \Big/ \left(\prod_{i=1}^{k} [D_i : \mathbb{Q}]\right)\right]^{1/2}$$

Proof. For convenience, set

$$G = \sum_{\sigma \in X} \mathrm{Ind}_{\sigma}^{\pi}(R_{\mathbb{Q}}(\sigma)) \subseteq R_{\mathbb{Q}}(\pi).$$

Then

$$|R_{\mathbb{Q}}(\pi)/G| = [d(G)/d(R_{\mathbb{Q}}(\pi))]^{1/2}; \tag{1}$$

where $d(-)$ denotes discriminant with respect to the usual inner product

$$\langle [V], [W] \rangle = \frac{1}{|\pi|} \sum_{g \in \pi} \chi_V(g)\chi_W(g).$$

For each i, let V_i denote the irreducible representation of $M_{r_i}(D_i)$; then

$$\begin{aligned} \langle [V_i], [V_j] \rangle = \dim_{\mathbb{Q}} (\mathrm{Hom}_{\mathbb{Q}\pi} (V_i, V_j)) &= 0 && \text{if } i \neq j \\ &= [D_i : \mathbb{Q}] && \text{if } i = j. \end{aligned}$$

So

$$d(R_{\mathbb{Q}}(\pi)) = \prod_{i=1}^{k} [D_i : \mathbb{Q}]. \tag{2}$$

To compute $d(G)$, consider first the set

$$S = \{[\mathbb{Q}(\pi/\sigma)] : \sigma \in X\} \subseteq R_{\mathbb{Q}}(\pi);$$

where $\mathbb{Q}(\pi/\sigma)$ denotes the permutation representation with $\mathbb{Q}$-basis π/σ. These elements generate G: since

$$\mathbb{Q}(\pi/\sigma) = \mathrm{Ind}_\sigma^\pi(\mathbb{Q}) \in G,$$

and $R_{\mathbb{Q}}(\sigma)$ $(\sigma \in X)$ is generated by the elements

$$\{[\mathbb{Q}(\sigma/\tau)] = \mathrm{Ind}_\tau^\sigma([\mathbb{Q}]) : \tau \subseteq \sigma\}.$$

Also, $\mathrm{rk}\,(R_{\mathbb{Q}}(\pi)) = |X|$ (see [1, Theorem 21.5]); and so S is a basis for G.
It follows that

$$d(G) = \det(M) \tag{3}$$

where $M = (M_{\sigma\tau})_{\sigma,\tau \in X}$ is the matrix defined by

$$M_{\sigma\tau} = \langle [\mathbb{Q}(\pi/\sigma)], [\mathbb{Q}(\pi/\tau)] \rangle.$$

For $\sigma \in X$, let χ_σ denote the character of $\mathbb{Q}(\pi/\sigma)$. For any $x \in \pi$,

$$\chi_\sigma(x) = \frac{1}{|\sigma|} \cdot \#\{g \in \pi : xg\sigma = g\sigma\} = \frac{1}{|\sigma|} \cdot \#\{g \in \pi : x \in g\sigma g^{-1}\}.$$

Hence, for σ, $\tau \in X$,

$$\begin{aligned} M_{\sigma\tau} &= \frac{1}{|\pi|} \sum_{x \in \pi} \chi_\sigma(x)\chi_\tau(x) = \frac{1}{|\sigma| \cdot |\tau|} \cdot \frac{1}{|\pi|} \sum_{g,h \in \pi} |g\sigma g^{-1} \cap h\tau h^{-1}| \\ &= \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} |\sigma \cap g\tau g^{-1}|. \end{aligned} \tag{4}$$

To simplify what follows, define, for $n \geqq 1$ and $m \geqq 1$,

$$\varphi_m(n) = n - \sum_{\substack{d \mid n \\ d < m}} \varphi(d).$$

Note in particular that $\varphi_1(n) = n$, $\varphi_n(n) = \varphi(n)$, and $\varphi_m(n) = 0$ for $m > n$. Let $N = \max\{|\sigma| : \sigma \in X\}$; and define, for $1 \leqq m \leqq N$:

$$X_m = \{\sigma \in X : |\sigma| = m\} \qquad Y_m = \{\sigma \in X : |\sigma| \geqq m\} = \bigcup_{i \geqq m} X_i.$$

For all $0 \leqq m \leqq N$, define a matrix $M^{(m)} = (M^{(m)}_{\sigma\tau})_{\sigma,\tau \in Y_m}$, by setting

$$M^{(m)}_{\sigma\tau} = \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|).$$

In particular, $M^{(1)} = M$.

Fix $1 \leqq m \leqq N$. For $\sigma, \tau \in X_m$ (i.e., $|\sigma| = |\tau| = m$),

$$\begin{aligned} M^{(m)}_{\sigma\tau} &= m^{-2} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) \\ &= (\varphi(m)/m^2) \cdot \#\{g \in \pi : \sigma = g\tau g^{-1}\} = 0 \qquad \text{if } \sigma \neq \tau \\ &= \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \qquad \text{if } \sigma = \tau \end{aligned}$$

In particular,

$$\det(M^{(N)}) = \prod_{\sigma \in X_N} \left[\frac{\varphi(|\sigma|)}{|\sigma|} |N(\sigma)/\sigma| \right]. \tag{5}$$

If $1 \leqq m < N$ and $\sigma, \tau \in Y_{m+1}$ (i.e., $|\sigma|, |\tau| > m$), consider the entries $M^{(m)}_{\sigma\rho}$ for $\rho \in X_m$ ($|\rho| = m$). By definition, $M^{(m)}_{\sigma\rho} = 0$ unless $|\sigma \cap g\rho g^{-1}| \geqq m$ for some g; i.e., unless $g\rho g^{-1} \subseteq \sigma$. If $m \nmid |\sigma|$, then these $M^{(m)}_{\sigma\rho}$ all vanish; and also $M^{(m)}_{\sigma\tau} = M^{(m+1)}_{\sigma\tau}$ ($\varphi_m(n) = \varphi_{m+1}(n)$ if $m \nmid n$). If $m \mid |\sigma|$, let $\rho \in X_m$ be the unique element conjugate to a subgroup of σ; then

$$\begin{aligned} M^{(m)}_{\sigma\tau} - (M^{(m)}_{\sigma\rho}/M^{(m)}_{\rho\rho}) \cdot (M^{(m)}_{\rho\tau}) &= \frac{1}{|\sigma| \cdot |\tau|} \left[\sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) - \sum_{g \in \pi} \varphi_m(|\rho \cap g\tau g^{-1}|) \right] \\ &= \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_{m+1}(|\sigma \cap g\tau g^{-1}|) = M^{(m)}_{\sigma\tau}. \end{aligned}$$

In other words, for all $\sigma, \tau \in Y_{m+1}$,

$$M^{(m+1)}_{\sigma\tau} = M^{(m)}_{\sigma\tau} - \sum_{\rho \in X_m} (M^{(m)}_{\sigma\rho}/M^{(m)}_{\rho\rho}) \cdot M^{(m+1)}_{\rho\tau};$$

and $M^{(m+1)}$ is obtained from $M^{(m)}$ by elementary operations which eliminate all entries $M^{(m)}_{\rho\tau}$ for $\rho \in X_m$, $\tau \in Y_{m+1}$. It follows that

$$\begin{aligned} \det(M^{(m)}) &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} M^{(m)}_{\sigma\sigma} \\ &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} \left[\frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \right]. \end{aligned}$$

Combining this with (5) gives

$$d(G)=\det(M)=\det(M^{(1)})=\prod_{\sigma\in X}\left[\frac{\varphi(|\sigma|)}{|\sigma|}\cdot|N(\sigma)/\sigma|\right].$$

Finally, combined with (1) and (2), this gives the desired formula for $|R_{\mathbb{Q}}(\pi)/G|$. □

When π is a p-group (recall that p is always odd), the above formula can be reformulated solely in terms of cyclic subgroups:

THEOREM 12. *Let π be a p-group, and let X be a set of conjugacy class representatives for cyclic subgroups $\sigma\subseteq\pi$. Then*

$$|^0D(\mathbb{Z}\pi)|=\left[\prod_{\sigma\in X}\frac{|N(\sigma)/\sigma|^2}{|Z(\sigma)|}\right]^{1/2}.$$

Proof. By Proposition 1, $\mathbb{Q}\pi\cong\prod_{i=1}^k M_{r_i}(D_i)$, when the D_i are fields, and

$$\prod_{i=1}^k[D_i:\mathbb{Q}]=\prod_{\sigma\in X}\varphi(|\sigma|\cdot|Z(\sigma)|/|N(\sigma)|)$$
$$=\prod_{\sigma\in X}[\varphi(|\sigma|)\cdot|Z(\sigma)|/|N(\sigma)|].$$

The result now follows by substitution into the formula of Theorem 11. □

Finally, for the sake of completeness, we extend Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ in [3] to arbitrary (not necessarily abelian) p-groups π. For any such π, $\overline{\hat{\mathbb{Q}}_p\pi}$ will denote the group ring modulo conjugation:

$$\overline{\hat{\mathbb{Q}}_p\pi}=\hat{\mathbb{Q}}_p\pi/\langle x-gxg^{-1}:x\in\hat{\mathbb{Q}}_p\pi,\ g\in\pi\rangle=\hat{\mathbb{Q}}_p\pi/\langle xy-yx:x,\ y\in\hat{\mathbb{Q}}_p\pi\rangle.$$

This can be regarded as the $\hat{\mathbb{Q}}_p$-vector space with basis the set of conjugacy classes in π. Let $\overline{\hat{\mathbb{Z}}_p\pi}\subseteq\overline{\hat{\mathbb{Q}}_p\pi}$ be the image of $\hat{\mathbb{Z}}_p\pi$; and let $\overline{\mathfrak{M}}\subseteq\overline{\hat{\mathbb{Q}}_p\pi}$ denote the image of any maximal order $\mathfrak{M}\subseteq\hat{\mathbb{Q}}_p\pi$.

If F is any field and $r\geqq 1$, it is easy to check that

$$\langle xy-yx:x,\ y\in M_r(F)\rangle=\mathrm{Ker}\,[\mathrm{tr}:M_r(F)\to F].$$

Thus, if $\hat{\mathbb{Q}}_p\pi\cong\prod M_{r_i}(F_i)$, then $\overline{\hat{\mathbb{Q}}_p\pi}\cong\prod F_i$, and the projection $\hat{\mathbb{Q}}_p\pi\to\overline{\hat{\mathbb{Q}}_p\pi}$ is the product of the trace maps. If $R_i\subseteq F_i$ is the ring of integers, then any maximal order $\mathfrak{M}_i \subseteq M_{r_i}(F_i)$ is conjugate to $M_{r_i}(R_i)$ [9, Theorem 21.6]; and so $\mathrm{tr}(\mathfrak{M}_i) = R_i$. In particular, $\hat{\mathfrak{M}} = \prod R_i$ under the above identification (and is thus independent of the choice of maximal order).

PROPOSITION 13. *Let π be a p-group, and let $\overline{\hat{\mathbb{Z}}_p\pi}$, $\hat{\mathfrak{M}} \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$ be as above. Then, for any odd $1 \leqq t \leqq p-2$,*

$$\begin{aligned}|{}^tD(\mathbb{Z}\pi)| &= |{}^t(\hat{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p\pi})| && \text{if } t \neq 1\\ &= |{}^1(\hat{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p\pi})| \cdot \frac{|\pi^{ab}|}{|\mathrm{tors}_p\,(\mathbb{Q}\pi)^*|} && \text{if } t=1.\end{aligned}$$

Proof. Let $\mathfrak{M} \supseteq \mathbb{Z}\pi$ be a maximal order, and write

$$\hat{\mathbb{Q}}_p\pi = \prod_{i=1}^{k} A_i; \qquad A_i \cong M_{r_i}(F_i); \qquad \hat{\mathfrak{M}}_p \cong \prod_{i=1}^{k} \mathfrak{M}_i;$$

where F_i are fields and $\mathfrak{M}_i \subseteq A_i$ is a maximal order for all i. Given any $x \in \mathfrak{M}_i$ which is topologically nilpotent (i.e., $p \mid x^n$ for some n), the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

converges in A_i. We claim that for such x,

$$\mathrm{tr}(\log(1+x)) = \log(\det(1+x)) \in F_i. \tag{1}$$

To see (1), choose n such that $p \mid x^{p^n}$. Then for any $m \geqq 0$, $(1+x)^{p^{m+n}} = 1 + p^{m+1}y$ for some $y \in \mathfrak{M}$, and

$$\begin{aligned}\log(\det(1+x)^{p^{n+m}}) &= \log(\det(1+p^{m+1}y)) \equiv p^{m+1}\cdot \mathrm{tr}(y)\\ &\equiv \mathrm{tr}(\log(1+p^{m+1}y)) = \mathrm{tr}(\log(1+x)^{p^{n+m}}) \pmod{p^{2m+2}}.\end{aligned}$$

So for all $m \geqq 0$,

$$\log(\det(1+x)) \equiv \mathrm{tr}(\log(1+x)) \pmod{p^{m-n+2}};$$

and (1) holds.

In particular, $\log(1+x)$ for $x \in J(\hat{\mathbb{Z}}_p\pi)$ or $x \in J(\hat{\mathfrak{M}}_p)$ induces homomorphisms

$$L_1: K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)} \to \overline{\hat{\mathbb{Q}}_p\pi}; \qquad L_2: K_1(\hat{\mathfrak{M}}_p)_{(p)} \to \overline{\hat{\mathbb{Q}}_p\pi}$$

such that $L_1 = L_2 \,|\, K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)}$. (Here J means Jacobson radical; note that $J(\hat{\mathbb{Z}}_p\pi) \not\subseteq J(\mathfrak{M}_p)$ in general.) Furthermore,

$$\operatorname{Ker}(L_2) = \operatorname{tors}_p(\mathfrak{M}_p^*) = \operatorname{tors}_p(\mathbb{Q}\pi)^*; \qquad \operatorname{Ker}(L_1) \cong \pi^{ab};$$

and so by Proposition 2, for all odd t:

$$|{}^tD(\mathbb{Z}\pi)| = [{}^t\operatorname{Im}(L_2) : {}^t\operatorname{Im}(L_1)]. \tag{2}$$

For any $\hat{\mathbb{Z}}_p$-lattices $M_1, M_2 \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$, we write for short

$$[M_1 : M_2] = [M_1 : M_1 \cap M_2]/[M_2 : M_1 \cap M_2].$$

By Theorem 2 in [8], for any $1 \leqq t \leqq p-2$,

$$\begin{aligned} \left(1 - \frac{1}{p}\Phi\right)({}^t\operatorname{Im}(L_1)) &= {}^t\overline{\hat{\mathbb{Z}}_p\pi} && \text{if} \quad t \neq 1 \\ &= \operatorname{Ker}[{}^1\overline{\hat{\mathbb{Z}}_p\pi} \to \pi^{ab}] && \text{if} \quad t = 1 \end{aligned}$$

Here, $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$; and Φ is nilpotent (${}^t\overline{\hat{\mathbb{Q}}_p\pi}$ lies in the augmentation ideal, since $t \neq 0$). So

$$\det\left(1 - \frac{1}{p}\Phi\right) = 1,$$

and hence

$$\begin{aligned} [{}^t\overline{\hat{\mathbb{Z}}_p\pi} : {}^t\operatorname{Im}(\mathbb{L}_1)] &= 1 && \text{if} \quad t \neq 1 \\ &= |\pi^{ab}| && \text{if} \quad t = 1. \end{aligned} \tag{3}$$

Finally, note that for $s \geqq 0$,

$$\begin{aligned} [{}^t\hat{\mathbb{Z}}_p\zeta_{p^s} : \log({}^t(\hat{\mathbb{Z}}_p\zeta_{p^s})^*)] &= 1 && \text{if} \quad t \neq 1 \\ &= p^s && \text{if} \quad t = 1: \end{aligned} \tag{4}$$

this follows by noting that $\log(1 + p\hat{\mathbb{Z}}_p\zeta_{p^s}) = p\hat{\mathbb{Z}}_p\zeta_{p^s}$, and then counting orders of the quotients. Since by (1),

$$\operatorname{Im}(L_2) = \prod_{i=1}^{k} \log(R_i^*)_{(p)} \subseteq \prod_{i=1}^{k} F_i \cong \overline{\hat{\mathbb{Q}}_p\pi}$$

($R_i \subseteq F_i$ the ring of integers), (4) implies that

$$\begin{aligned}[{}^t\mathfrak{M} : {}^t\mathrm{Im}\,(L_2)] &= 1 && \text{if} \quad t \neq 1 \\ &= \prod |\mathrm{tors}_p\,(R_i^*)| = |\mathrm{tors}_p\,(\mathbb{Q}\pi)^*| && \text{if} \quad t = 1.\end{aligned} \tag{5}$$

So (2), (3), and (5) combine to prove the proposition. □

Generalizing Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ is now straightforward:

THEOREM 14. *Let π be a p-group (p odd). Let $S \subseteq \pi$ be a set of conjugacy class representatives for all $1 \neq g \in \pi$. Set*

$$p^n = |\pi^{ab}| \quad \textit{and} \quad p^k = \prod_{g \in S} |Z(g)|.$$

For $s \geqq 1$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$. Then $|D(\mathbb{Z}\pi)^-| = p^N$, where

$$N = \frac{1}{4}\left[k + 4n - \sum_{s \geqq 1} w_s(sp^s - (s+1)p^{s-1} + 4s + 1)\right].$$

Proof. Let $\overline{\hat{\mathbb{Z}}_p\pi} \subseteq \mathfrak{M} \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$ be as above. By Proposition 13,

$$|D(\mathbb{Z}\pi)^-| = |(\mathfrak{M}/\overline{\hat{\mathbb{Z}}_p\pi})^-| \cdot p^n \cdot \left[\prod_{s \geqq 1} p^{sw_s}\right]^{-1}. \tag{1}$$

Write $\hat{\mathbb{Q}}_p\pi = \prod_{i=1}^k A_i$, where $A_i \cong M_{r_i}(F_i)$ and the F_i are fields. As before, the trace maps $\mathrm{tr}_i : A_i \to F_i$ induce an identification of $\overline{\hat{\mathbb{Q}}_p\pi}$ with $\prod F_i$. Let $\mathrm{pr}_i : \hat{\mathbb{Q}}_p\pi \to A_i$ be the projection; and define an inner product on $\hat{\mathbb{Q}}_p\pi$ by setting

$$\langle x, y \rangle = \sum_{i=1}^k \mathrm{tr}_{F_i/\hat{\mathbb{Q}}_p}(\mathrm{tr}_i \circ \mathrm{pr}_i\,(x) \cdot \mathrm{tr}_i \circ \mathrm{pr}_i\,(y)) \quad (x, y \in \overline{\hat{\mathbb{Q}}_p\pi}).$$

Since $\mathfrak{M} \subseteq \prod F_i$ is the product of the rings of integers, we have by definition discriminants

$$d(\mathfrak{M}) = \prod_i \Delta(F_i) \quad \text{and} \quad d(\mathfrak{M}^+) = 2^{rk(\mathfrak{M})-1} \cdot \prod_i \Delta(F_i \cap \mathbb{R}).$$

Here $\Delta(F_i)$, $\Delta(F_i \cap \mathbb{R})$ denote the discriminants over $\mathbb{Q}$; and the power of 2 arises due to using the trace over F_i instead of $F_i \cap \mathbb{R}$.

By [16, Proposition 7-5-7], for $s \geqq 1$,

$$\Delta(\mathbb{Q}\zeta_{p^s}) = p^{p^s(ps-s-1)}.$$

By the same proof, or by the composition formula applied to the fields $\mathbb{Q}\zeta_{p^s}/\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}$ [16, Corollary 3-7-20]:

$$\Delta(\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}) = p^{\frac{1}{2}[sp^s-(s+1)p^{s-1}-1]}.$$

Hence, $d(\bar{\mathfrak{M}}^-) = p^{N_0}$, where

$$N_0 = \tfrac{1}{2} \sum_{s \geqq 1} (sp^s - (s+1)p^{s-1} + 1). \tag{2}$$

Now fix $g, h \in \pi$. For any given $1 \leqq i \leqq k$,

$$\mathrm{tr}_{F_i/\hat{\mathbb{Q}}_p}(\mathrm{tr}_i \circ \mathrm{pr}_i\,(g) \cdot \mathrm{tr}_i \circ \mathrm{pr}_i\,(h)) = \sum_{j=1}^{t} \chi_j(g)\chi_j(h),$$

where $\chi_1, \ldots, \chi_t$ are the distinct irreducible (complex) characters contained in the summand A_i. Let π^* denote the set of all irreducible complex characters. Then

$$\begin{aligned} \langle g, h\rangle = \sum_{\chi \in \pi^*} \chi(g)\chi(h) &= 0 \qquad \text{if} \quad g \text{ not conjugate to } h^{-1} \\ &= |Z(g)| \quad \text{if} \quad g \text{ is conjugate to } h^{-1} \end{aligned}$$

by the second orthogonality relation [1, Proposition 9.26]. Hence, eliminating factors prime to p,

$$d(\overline{\hat{\mathbb{Z}}_p\pi}) = \left[\prod_{g \in S} |Z(g)|\right]^{1/2} = p^{k/2}. \tag{3}$$

By (2) and (3), $|(\mathfrak{M}'/\overline{\hat{\mathbb{Z}}_p\pi})^-| = p^{N_1}$, where

$$N_1 = \frac{1}{4}\left[k - \sum_{s \geqq 1} (sp^s - (s+1)p^{s-1} + 1)\right].$$

Together with (1), this proves the theorem. □

This can also be reformulated solely in terms of cyclic subgroups of π:

THEOREM 15. *Let π be any p-group, and let X_0 be a set of conjugacy class representatives for cyclic subgroups $1 \neq \sigma \subseteq \pi$. For each $\sigma \in X_0$, set*

$$a_\sigma = \mathrm{ord}_p\,|N(\sigma)/\sigma|; \qquad b_\sigma = \mathrm{ord}_p\,(|\sigma| \cdot |Z(\sigma)|/|N(\sigma)|).$$

Then

$$\mathrm{ord}_p\,|D(\mathbb{Z}\pi)^-| = \mathrm{ord}_p\,|\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1].$$

Proof. Let $w_s (s \geqq 1)$ and k be as in Theorem 14. Then each $\sigma \in X_0$ has $\varphi(p^{b_\sigma})$ conjugacy classes of generators, and so

$$k = \sum_{\sigma \in X_0} \varphi(p^{b_\sigma}) \cdot (a_\sigma + b_\sigma).$$

By Proposition 1, w_s is the number of $\sigma \in X_0$ such that $b_\sigma = s$. So Theorem 14 takes the form

$$\begin{aligned} \mathrm{ord}_p\,|D(\mathbb{Z}\pi)^-| &= \mathrm{ord}_p\,|\pi^{ab}| \\ &\quad + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma + b_\sigma)\varphi(p^{b_\sigma}) - b_\sigma p^{b_\sigma} + (b_\sigma + 1)p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \mathrm{ord}_p\,|\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [a_\sigma \varphi(p^{b_\sigma}) + p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \mathrm{ord}_p\,|\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1]. \quad \square \end{aligned}$$

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