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$D(\mathbb{Z}\pi)^+$ and the Artin cokernel

ROBERT OLIVER

Let p be an odd prime. If π is a p -group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order containing $\mathbb{Z}\pi$, then we set

$$D(\mathbb{Z}\pi) = \text{Ker} [K_0(\mathbb{Z}\pi) \rightarrow K_0(\mathfrak{M})]$$

as usual. Since $D(\mathbb{Z}\pi)$ is a p -group [2], the involution $g \mapsto g^{-1}$ induces a natural splitting

$$D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+ \oplus D(\mathbb{Z}\pi)^-;$$

where $D(\mathbb{Z}\pi)^+$ and $D(\mathbb{Z}\pi)^-$ are the $+1$ and -1 eigenspaces, respectively. The involution can be extended to a linear action of \mathbb{F}_p^* on $D(\mathbb{Z}\pi)$; and this induces further eigenspace decompositions

$$D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi) \quad \text{and} \quad D(\mathbb{Z}\pi)^- = \sum_{i=0}^{(p-3)/2} {}^{2i+1}D(\mathbb{Z}\pi).$$

The groups $D(\mathbb{Z}\pi)^-$ and ${}^{2i+1}D(\mathbb{Z}\pi)$ have been studied extensively in [3] and [14]; and the order of $D(\mathbb{Z}\pi)^-$ for abelian π is computed in [3]. In this paper, attention is focused on ${}^0D(\mathbb{Z}\pi)$. This is the part of $D(\mathbb{Z}\pi)^+$ which is independent of number theoretic properties of p ; and in fact, ${}^0D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+$ whenever p is regular.

For any finite group π , we define the “Artin cokernel” $A_{\mathbb{Q}}(\pi)$ to be the group

$$A_{\mathbb{Q}}(\pi) = \text{Coker} [\text{Ind}: \sum \{R_{\mathbb{Q}}(\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \rightarrow R_{\mathbb{Q}}(\pi)],$$

where $R_{\mathbb{Q}}(\pi)$ denotes the rational representation ring. The main result here is that ${}^0D(\mathbb{Z}\pi) \cong A_{\mathbb{Q}}(\pi)$ for any p -group π (p odd). Among other consequences, this gives new insight into Martin Taylor’s result [13] that the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism has order equal to the Artin exponent of $\pi: T(\mathbb{Z}\pi)$ corresponds under this isomorphism to multiples of the identity in $R_{\mathbb{Q}}(\pi)$.

We end by deriving a formula for $|{}^0D(\mathbb{Z}\pi)|$ —in fact, a formula for the order of $A_{\mathbb{Q}}(\pi)$ for arbitrary finite π . Also, for the sake of completeness, the formula for $|D(\mathbb{Z}\pi)^{-}|$ in [3] is generalized to cover arbitrary p -groups; thus giving a complete calculation of $|D(\mathbb{Z}\pi)|$ when π is a p -group and p any odd regular prime.

For $n \geq 1$, $\mathbb{Q}\zeta_n$ always denotes the field generated by the n -th roots of unity (and similarly for $\mathbb{Z}\zeta_n$, $\hat{\mathbb{Z}}_p\zeta_n$, etc.) Also, $\varphi(n)$ will always mean the Euler φ -function.

Throughout the paper, unless otherwise stated, p will be a fixed odd prime. We start with the following well known description of $\mathbb{Q}\pi$ when π is a p -group.

PROPOSITION 1. *Let π be a p -group. Then $\mathbb{Q}\pi$ is a product of matrix rings over fields $\mathbb{Q}\zeta_{p^s}$ for various $s \geq 0$. Furthermore, for each $s \geq 0$, the number of simple summands isomorphic to matrix rings over $\mathbb{Q}\zeta_{p^s}$ is equal to the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ such that*

$$|\sigma/[\sigma, N(\sigma)]| = |\sigma| \cdot |Z(\sigma)|/|N(\sigma)| = p^s.$$

Proof. That $\mathbb{Q}\pi$ is a product of matrix algebras over the $\mathbb{Q}\zeta_{p^s}$ is shown (though not explicitly stated) by Roquette in [11]. In Section 2 of [11] he shows that the division algebra for any irreducible representation M of π is isomorphic to the division algebra of a primitive faithful representation of some subquotient of π ; and in Section 3 he shows that the only p -groups with primitive faithful representations are the cyclic groups.

Now, for all $s \geq 0$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$; and let v_s be the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ with $|\sigma/[\sigma, N(\sigma)]| = p^s$. Let $p^n = |\pi|$. We say that two elements $g, h \in \pi$ are $\mathbb{Q}\zeta_{p^m}$ -conjugate (any $m \geq 0$) if g is conjugate to h^a for some

$$a \in \text{Gal}(\mathbb{Q}\zeta_{p^n}/\mathbb{Q}\zeta_{p^m}) \subseteq \text{Gal}(\mathbb{Q}\zeta_{p^n}/\mathbb{Z}) = (\mathbb{Z}/p^n)^*.$$

In other words, $a = 1 \pmod{p^m}$ if $m \geq 1$, or $p \nmid a$ if $m = 0$.

For each cyclic $\sigma \subseteq \pi$, the number of conjugacy classes of generators of σ is just $\varphi(|\sigma/[\sigma, N(\sigma)]|)$. So for any $m \geq 0$, the number of $\mathbb{Q}\zeta_{p^m}$ -conjugacy classes in π is

$$\sum_{s \geq 0} v_s \cdot \min \{ \varphi(p^m), \varphi(p^s) \}. \quad (1)$$

On the other hand

$$\text{rk } K_0(\mathbb{Q}\zeta_{p^m}[\pi]) = \sum_{s \geq 0} w_s \cdot \min \{ \varphi(p^m), \varphi(p^s) \}; \quad (2)$$

and by Theorem 21.5 in [1], the numbers in (1) and (2) are equal for all m . So $v_s = w_s$ for all $s \geq 0$. \square

Thus, if π is a p -group and $L = Z(\mathbb{Q}\pi)$ is the center, then the reduced norm

$$\nu : K_1(\hat{\mathbb{Q}}_p\pi) \xrightarrow{\cong} (\hat{L}_p)^*$$

is just the product of the determinant maps for the simple summands of $\hat{\mathbb{Q}}_p\pi$. If $\mathfrak{M} \subseteq \mathbb{Q}\pi$ and $\mathcal{O} \subseteq L$ are maximal orders, then ν induces an isomorphism of $K_1(\mathfrak{M}_p)$ with $(\hat{\mathcal{O}}_p)^*$: by [9, Theorem 21.6], \mathfrak{M}_p is a product of matrix algebras over the components of $\hat{\mathcal{O}}_p$. In particular, $K_1(\mathfrak{M}_p)$ can be regarded as a subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$. We let $K'_1(\mathfrak{M})$ and $K'_1(\hat{\mathbb{Z}}_p\pi)$ denote the images of $K_1(\mathfrak{M})$ and $K_1(\hat{\mathbb{Z}}_p\pi)$ in $K_1(\hat{\mathbb{Q}}_p\pi)$; and let $K'_1(\mathfrak{M})^\wedge$ denote the p -adic closure of $K'_1(\mathfrak{M})$.

We will use the following description of $D(\mathbb{Z}\pi)$, based on the formulas in [4].

PROPOSITION 2. *Let π be a p -group, and let $\mathfrak{M} \subseteq \mathbb{Z}\pi$ be a maximal order in $\mathbb{Q}\pi$. Then there is an isomorphism*

$$D(\mathbb{Z}\pi) \cong \text{Coker} [K'_1(\hat{\mathbb{Z}}_p\pi) \rightarrow K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})^\wedge]$$

which is natural in π .

Proof. Let $L \subseteq \mathbb{Q}\pi$ be its center, and let $\mathcal{O} \subseteq L$ be its maximal order. Since $\hat{\mathbb{Z}}_q[\pi] = \mathfrak{M}_q$ for all primes $q \neq p$ [9, Theorem 41.1], Theorems 1 and 2 in [4] reduce to the formula

$$D(\mathbb{Z}\pi) \cong K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M}) \cdot K'_1(\hat{\mathbb{Z}}_p\pi).$$

Furthermore, $K_1(\mathfrak{M}_p)/K'_1(\hat{\mathbb{Z}}_p\pi)$ is finite [12, Proposition 8.15]; and so $K'_1(\mathfrak{M})$ can be replaced by its p -adic closure. \square

Now let

$$\kappa : \mathbb{F}_p^* \rightarrow (\hat{\mathbb{Z}}_p)^*$$

denote the inclusion into the group of $(p-1)$ -st roots of unity (with $\kappa(a) = a \pmod{p}$ for $a \in \mathbb{F}_p^*$). If π is an abelian p -group, then the group homomorphisms $g \rightarrow g^{\kappa(a)}$ for $a \in \mathbb{F}_p^*$ and $g \in \pi$ induce actions of \mathbb{F}_p^* on $K_i(\mathbb{Z}\pi)$, $K_i(\hat{\mathbb{Z}}_p\pi)$, $D(\mathbb{Z}\pi)$, etc. The next problem is to find a natural way to do this when π is non-abelian.

Let π be an arbitrary p -group. Then by Proposition 1, the center $Z(\mathbb{Q}\pi)$ is a product of fields $\mathbb{Q}\zeta_p^s$ for various $s \geq 0$. The natural embedding $\mathbb{F}_p^* \subseteq \text{Gal}(\mathbb{Q}\zeta_p/\mathbb{Q})$ (where $a \in \mathbb{F}_p^*$ sends ζ_p to $\zeta_p^{\kappa(a)}$) induces actions of \mathbb{F}_p^* on the centers $Z(\mathbb{Q}\pi)$ and $Z(\hat{\mathbb{Q}}_p\pi)$; and hence on $K_1(\hat{\mathbb{Q}}_p\pi)$. If $\alpha: \pi \rightarrow \pi'$ is a homomorphism of p -groups, then

$$\alpha_*: K_1(\hat{\mathbb{Q}}_p\pi) \rightarrow K_1(\hat{\mathbb{Q}}_p\pi')$$

is a product of norm, inclusion, and diagonal maps between the groups of units of the field components of the centers; and is hence \mathbb{F}_p^* -linear.

PROPOSITION 3. *Let π be a p -group. Then*

- (i) $K'_1(\hat{\mathbb{Z}}_p\pi)$ is an \mathbb{F}_p^* -invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$.
- (ii) For any $0 \leq t \leq p-2$, set

$$'K'_1(\hat{\mathbb{Z}}_p\pi) = \{x \in K'_1(\hat{\mathbb{Z}}_p\pi)_{(p)} : \tau_a(x) = \kappa(a)^t \cdot x \text{ for all } a \in \mathbb{F}_p^*\}$$

(here τ_a denotes the action of $a \in \mathbb{F}_p^*$). Then for $t \neq 1$, $'K'_1(\hat{\mathbb{Z}}_p\pi)$ is generated by induction from cyclic subgroups.

Proof. By [8, Theorem 2], there is an exact sequence

$$0 \rightarrow \langle \lambda g : \lambda \in \hat{\mathbb{Z}}_p^*, g \in \pi \rangle \hookrightarrow K'_1(\hat{\mathbb{Z}}_p\pi) \xrightarrow{\Gamma} \overline{I(\hat{\mathbb{Z}}_p\pi)} \xrightarrow{\omega} \pi^{ab} \rightarrow 0,$$

natural in π , where

$$\overline{I(\hat{\mathbb{Z}}_p\pi)} = \text{Ker} [\varepsilon: \hat{\mathbb{Z}}_p\pi \rightarrow \hat{\mathbb{Z}}_p] / \langle gxg^{-1} - x : g \in \pi, x \in \hat{\mathbb{Z}}_p\pi \rangle$$

and

$$\varepsilon(\sum \lambda_i g_i) = \sum \lambda_i; \quad \omega(\sum \lambda_i g_i) = \prod g_i^{\lambda_i}.$$

For $a \in \mathbb{F}_p^*$, let $\hat{\tau}_a$ be the action of a on $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ given by: $\hat{\tau}_a(\sum \lambda_i g_i) = \sum \lambda_i g_i^{\kappa(a)}$. This clearly leaves $\text{Ker}(\omega) = \text{Im}(\Gamma)$ invariant. By the definition of Γ in [8], Γ is \mathbb{F}_p^* -linear when π is abelian.

Now set

$$X(\pi) = \text{Im} [\text{Ind}: \sum \{K'_1(\hat{\mathbb{Z}}_p\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \rightarrow K'_1(\hat{\mathbb{Z}}_p\pi)]$$

and

$$Y(\pi) = \Gamma^{-1}(\overline{I(\hat{\mathbb{Z}}_p\pi)}) \cap \text{Ker} [\varepsilon_*: K'_1(\hat{\mathbb{Z}}_p\pi) \rightarrow \hat{\mathbb{Z}}_p^*]. \quad (1)$$

Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, $\text{Ker}(\Gamma) \subseteq X(\pi)$, and

$$\overline{{}^tI(\hat{\mathbb{Z}}_p\pi)} = {}^t\text{Ker}(\omega) = {}^t\text{Im}(\Gamma) \quad (2)$$

for $t \neq 1$ ($0 \leq t \leq p-2$); it follows that

$$K'_1(\hat{\mathbb{Z}}_p\pi) = X(\pi) + Y(\pi). \quad (3)$$

By naturality, $X(\pi)$ is an \mathbb{F}_p^* -invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$, and $\Gamma|X(\pi)$ is \mathbb{F}_p^* -linear. It follows that for n large,

$$Y(\pi)^{p^n} \subseteq {}^1X(\pi). \quad (4)$$

Furthermore,

$$\text{tors}(K_1(\hat{\mathbb{Q}}_p\pi))_{(p)} \subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi):$$

the torsion in $K_1(\hat{\mathbb{Q}}_p\pi)$ comes from roots of unity in the center. It follows by (4) that $Y(\pi) \subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi)$; and hence by (3) that $K'_1(\hat{\mathbb{Z}}_p\pi)$ is \mathbb{F}_p^* -invariant. Furthermore, by (1), this shows that Γ is \mathbb{F}_p^* -linear. Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, $\overline{{}^tK'_1(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction for $t \neq 1$ by (2). \square

Propositions 2 and 3 now imply the existence of natural actions of \mathbb{F}_p^* on $D(\mathbb{Z}\pi)$:

PROPOSITION 4. *For any p -group π , there is a natural linear action of \mathbb{F}_p^* on $D(\mathbb{Z}\pi)$ such that the isomorphism of Proposition 2 is \mathbb{F}_p^* -linear. \square*

In particular, for any p -group π and $0 \leq t \leq p-2$, set

$${}^tD(\mathbb{Z}\pi) = \{x \in D(\mathbb{Z}\pi) : \tau_a(x) = \kappa(a)^t \cdot x \text{ for all } a \in \mathbb{F}_p^*\}.$$

Since $D(\mathbb{Z}\pi)$ is a p -group [2], and $p \nmid |\mathbb{F}_p^*|$,

$$D(\mathbb{Z}\pi) = \sum_{t=0}^{p-2} {}^tD(\mathbb{Z}\pi) \quad \text{and} \quad D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi).$$

Here, $D(\mathbb{Z}\pi)^+$ is the group of elements invariant under the involution τ_{-1} ; induced by complex conjugation on $Z(\mathbb{Q}\pi)$.

If p is regular, then results of Iwasawa (see, *fx*, [7, Theorem 7.5.2]) show that

for any $s \geq 0$ and any even $0 < t \leq p-3$,

$${}^t[(\mathbb{Z}\zeta_p^*)^\wedge] = {}^t(\hat{\mathbb{Z}}_p\zeta_p^*)^*.$$

So by Proposition 1, if $\mathfrak{M} \supseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then $'K_1'(\mathfrak{M})^\wedge = 'K_1'(\mathfrak{M}_p)$; and hence $'D(\mathbb{Z}\pi) = 0$ for such t . In other words:

PROPOSITION 5. *If p is an odd regular prime and π is a p -group, then*

$$D(\mathbb{Z}\pi)^+ = {}^0D(\mathbb{Z}\pi). \quad \square$$

In order to study the groups ${}^0D(\mathbb{Z}\pi)$, we must first describe

$${}^0[(\hat{\mathbb{Z}}_p\zeta_p^*)^*/(\mathbb{Z}\zeta_p^*)^\wedge]$$

for any $s \geq 0$. The following result must be well known, but we have been unable to find a reference.

PROPOSITION 6. *For any $s \geq 0$,*

$${}^0[(\hat{\mathbb{Z}}_p\zeta_p^*)^*/(\mathbb{Z}\zeta_p^*)^\wedge]_{(p)} \cong \hat{\mathbb{Z}}_p.$$

Proof. This is clear if $s = 0$. So fix $s > 0$, let $K \subseteq \mathbb{Q}\zeta_p^s$ be the fixed subfield of \mathbb{F}_p^* , and let $R \subseteq K$ be the ring of integers. Then $\Gamma = \text{Gal}(K/\mathbb{Q})$ is cyclic of order p^{s-1} . Set $\zeta = \zeta_p^s$, and let $\gamma \in \Gamma$ be the generator: $\gamma(\zeta) = \zeta^{p+1}$.

Let

$$\mathfrak{p} = \left\langle z = \prod_{a=1}^{p-1} (1 - \zeta^{\kappa(a)}) \right\rangle \subseteq R$$

be the prime ideal over p . Set

$$U' = \text{Ker}[N : (\hat{R}_p)^* \rightarrow (\hat{\mathbb{Z}}_p)^*],$$

where N is the norm of $\hat{K}_p/\hat{\mathbb{Q}}_p$. Note that $U' \subseteq 1 + \mathfrak{p}\hat{R}_p$.

Fix $u \in U'$. By Hilbert's Theorem 90, there is $x \in \hat{K}_p^*$ such that $u = \gamma(x)/(x)$. Write $x = z^i v$, where $v \in (\hat{R}_p)^*$ and z is the element defined above. Then

$$\gamma(z^i)/z^i = \prod_{a=1}^{p-1} \left(\frac{1 - \zeta^{(p+1)\kappa(a)}}{1 - \zeta^{\kappa(a)}} \right)^i \in R^* \cap (1 + \mathfrak{p}).$$

Furthermore, $N(v) \in \kappa(\mathbb{F}_p^*) \times (1 + p^s \hat{\mathbb{Z}}_p)$ (the norm group has index p^{s-1} by local class field theory). So there exists $w \in (\hat{\mathbb{Z}}_p)^*$ such that $N(w) = w^{p^{s-1}} = N(v)$. Then

$$N(vw^{-1}) = 1 \quad \text{and} \quad \gamma(vw^{-1})/(vw^{-1}) = \gamma(v)/v = x \cdot (\gamma(z^i)/z^i)^{-1}.$$

In other words,

$$U' = \{\gamma(v)/v : v \in U'\} \cdot (R^* \cap (1 + \mathfrak{p})). \quad (1)$$

But U' is a $\hat{\mathbb{Z}}_p[\Gamma]$ -module, and the closure of $R^* \cap (1 + \mathfrak{p})$ is a $\hat{\mathbb{Z}}_p[\Gamma]$ -submodule. Since $\hat{\mathbb{Z}}[\Gamma]$ is a local ring with maximal ideal generated by p and $\gamma - 1$, no proper submodule of U' can generate

$$U' / \langle \gamma(v)/v, v^p : v \in U' \rangle.$$

So by (1), $R^* \cap (1 + \mathfrak{p})$ is dense in $U' = \text{Ker}(N)$; and

$${}^0[(\hat{\mathbb{Z}}_p \zeta_{p^s})^* / (\mathbb{Z} \zeta_{p^s}^*)]_{(p)} = [(\hat{R}_p)^* / (R^*)]_{(p)} \cong (\text{Im}(N))_{(p)} \cong \hat{\mathbb{Z}}_p. \quad \square$$

By Proposition 6, if π is a p -group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order, then ${}^0[K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})]_{(p)}$ is a sum of one copy of $\hat{\mathbb{Z}}_p$ for each irreducible $\mathbb{Q}\pi$ -module; and is thus (abstractly, at least) isomorphic to $\hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi)$. The key remaining step is to construct a natural isomorphism between these groups; once this is done the isomorphism between ${}^0D(\mathbb{Z}\pi)$ and the Artin cokernel will follow easily.

We temporarily allow p to be an arbitrary prime (possibly $p = 2$). If A is a $\hat{\mathbb{Q}}_p$ -algebra, and V is an A -module with $\dim_{\hat{\mathbb{Q}}_p}(V) < \infty$; let $\det(u, V)$, for $u \in A$, denote the determinant over $\hat{\mathbb{Q}}_p$ of $u : V \rightarrow V$. Define

$$L : (\hat{\mathbb{Z}}_p)^* \rightarrow \hat{\mathbb{Z}}_p$$

by setting $L(u) = 1/p \log(u/\kappa(\bar{u}))$ for $u \in (\hat{\mathbb{Z}}_p)^*$ and $\bar{u} \in \mathbb{F}_p^*$ its reduction mod p (note that $u/\kappa(\bar{u}) \in 1 + p\hat{\mathbb{Z}}_p$).

Now assume A is a finite dimensional semisimple $\hat{\mathbb{Q}}_p$ -algebra, and let $\mathfrak{A} \subseteq A$ be any order. Let V_1, \dots, V_k be the distinct irreducible A -modules, and set

$$n_i = [\text{End}_A(V_i) : \hat{\mathbb{Q}}_p].$$

Define a homomorphism

$$\delta = \delta_{\mathfrak{A}} : K_1(\mathfrak{A}) \rightarrow \hat{\mathbb{Q}}_p \otimes_{\mathbb{Z}} K_0(A)$$

by setting, for any matrix $u \in GL_r(\mathfrak{A})$,

$$\delta([u]) = \sum_{i=1}^k \frac{1}{n_i} L(\det(u, V_i)) \cdot [V_i].$$

PROPOSITION 7. *For any prime p , the maps $\delta_{\mathfrak{A}}$ are natural with respect to homomorphisms between orders in semisimple $\hat{\mathbb{Q}}_p$ -algebras.*

Proof. We must show, for any homomorphism $\alpha: A \rightarrow B$, orders $\mathfrak{A} \subseteq A$ and $\mathfrak{B} \subseteq B$ such that $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$, and $u \in \mathfrak{A}^*$, that

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \delta_{\mathfrak{B}}(\alpha(u)) \in \hat{\mathbb{Q}}_p \otimes K_0(B).$$

Let V_1, \dots, V_s be the irreducible A -modules, and W_1, \dots, W_t the irreducible B -modules. Define $a_{ij}, b_{ij} \in \mathbb{Z}$ by setting

$$\alpha_*(V_i) = \sum_{j=1}^t a_{ij} W_j, \quad \alpha^*(W_j) = \sum_{i=1}^s b_{ij} V_i$$

(where $\alpha_*(V_i) = B \otimes_A V_i$, and $\alpha^*(W_j)$ is W_j regarded as an A -module). We also set

$$m_i = [\text{End}_A(V_i) : \hat{\mathbb{Q}}_p], \quad n_j = [\text{End}_B(W_j) : \hat{\mathbb{Q}}_p],$$

and write $L_i = L(\det(u, V_i))$ for short. Then

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \alpha_*\left(\sum_{i=1}^s (L_i/m_i)[V_i]\right) = \sum_{i,j} (a_{ij}L_i/m_i)[W_j]$$

and

$$\delta_{\mathfrak{B}}(\alpha(u)) = \sum_{j=1}^t n_j^{-1} L(\det(u, \alpha^*(W_j)))[W_j] = \sum_{i,j} (b_{ij}L_i/n_j)[W_j].$$

It remains to check that $(b_{ij}/n_j) = (a_{ij}/m_i)$ for all i, j . But

$$\dim \text{Hom}_A(V_i, W_j) = m_i b_{ij}, \quad \dim \text{Hom}_B(\alpha_* V_i, W_j) = n_j a_{ij};$$

and these two dimensions are equal by [1, Theorem 2.19]. \square

We now again restrict to the case where p is odd.

PROPOSITION 8. *Let π be a p -group, let $\mathfrak{M} \subseteq \mathbb{Q}\pi$ be any maximal order, and set $\delta_\pi = \delta_{\hat{\mathbb{Q}}_p\pi}$. Then*

$$\text{Im} [\delta_\pi : K_1(\mathfrak{M}_p) \rightarrow \hat{\mathbb{Q}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi)] = \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi);$$

and δ_π induces an isomorphism

$$\delta' = \delta'_\pi : {}^0[K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi) \cong \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi).$$

Proof. Using Proposition 1, it will suffice to show that whenever $A \cong M_r(\mathbb{Q}\zeta_p)$ and $\mathfrak{M} \subseteq A$ is a maximal order, then $\delta = \delta_{\hat{A}_p}$ induces an isomorphism

$$\delta' : {}^0[K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(A).$$

By [9, Theorem 21.6], we may assume that $\mathfrak{M} = M_r(\mathbb{Z}\zeta_p)$.

Let $V \cong (\hat{\mathbb{Q}}_p\zeta_p)^r$ be the irreducible \hat{A}_p -representation. For any $u \in 1 + J(\mathfrak{M}_p)$ (where $J(\mathfrak{M}_p)$ is the Jacobson radical),

$$\begin{aligned} \delta(u) &= \frac{1}{\varphi(p^s)} L(\det(u, V)) \cdot [V] \\ &= \frac{1}{p\varphi(p^s)} \log(N_{\hat{\mathbb{Q}}_p\zeta_p/\mathbb{Q}_p}(\det_{\hat{\mathbb{Q}}_p\zeta_p}(u))) \cdot [V]. \end{aligned}$$

Furthermore, by local class field theory,

$$N \circ \det(1 + J(\mathfrak{M}_p)) = 1 + p^s \hat{\mathbb{Z}}_p$$

(or $1 + p\hat{\mathbb{Z}}_p$ if $s = 0$). Since $\log(1 + p^s \hat{\mathbb{Z}}_p) = p^s \hat{\mathbb{Z}}_p$ for $s \geq 1$, we have

$$\delta(K_1(\mathfrak{M}_p)_{(p)}) = \delta(1 + J(\mathfrak{M}_p)) = \hat{\mathbb{Z}}_p \cdot [V] = \hat{\mathbb{Z}}_p \otimes K_0(A).$$

If u is a global unit, then $N(\det(u)) = \pm 1$, and so $\delta(u) = 0$. Furthermore, δ is \mathbb{F}_p^* -linear when $K_0(A)$ is given the trivial action; and so δ induces a surjection

$$\delta' : {}^0[K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})]_{(p)} \rightarrow \hat{\mathbb{Z}}_p \otimes K_0(A) \cong \hat{\mathbb{Z}}_p.$$

But the two groups are isomorphic by Proposition 6, and so δ' is an isomorphism. \square

We can now prove the main result. Recall that the Artin cokernel $A_{\mathbf{Q}}(\pi)$ is defined by

$$A_{\mathbf{Q}}(\pi) = \text{Coker} [\text{Ind} : \sum \{R_{\mathbf{Q}}(\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \rightarrow R_{\mathbf{Q}}(\pi)].$$

THEOREM 9. *For any p -group π (p -odd), δ' induces an isomorphism*

$$\delta''_{\pi} : {}^0D(\mathbb{Z}\pi) \xrightarrow{\cong} A_{\mathbf{Q}}(\pi).$$

Proof. Let C be the set of cyclic subgroups of π . Propositions 2, 7, and 8 combine to give the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \sum_{\sigma \in C} {}^0K'_1(\hat{\mathbb{Z}}_p\sigma) & \xrightarrow{\Sigma\eta_{\sigma}} & \sum_{\sigma \in C} \hat{\mathbb{Z}}_p \otimes R_{\mathbf{Q}}(\sigma) & \xrightarrow{\Sigma\theta_{\sigma}} & \sum_{\sigma \in C} {}^0D(\mathbb{Z}\sigma) \longrightarrow 0 \\ \downarrow I_1 & & \downarrow I_2 & & \downarrow I_3 \\ {}^0K'_1(\hat{\mathbb{Z}}_p\pi) & \xrightarrow{\eta_{\pi}} & \hat{\mathbb{Z}}_p \otimes R_{\mathbf{Q}}(\pi) & \xrightarrow{\theta_{\pi}} & {}^0D(\mathbb{Z}\pi) \longrightarrow 0 \end{array}$$

Here I_1 , I_2 , and I_3 are the induction maps; and θ_{π} and η_{π} are the composites ($\mathfrak{M} \subseteq \theta\pi$ a maximal order):

$$\theta_{\pi} : \hat{\mathbb{Z}}_p \otimes R_{\mathbf{Q}}(\pi) \xrightarrow[\cong]{(\delta')^{-1}} {}^0[K_1(\mathfrak{M}_p)/K'_1(\mathfrak{M})^{\wedge}]_{(p)} \longrightarrow {}^0D(\mathbb{Z}\pi)$$

(the second map being the map of Proposition 2), and

$$\eta_{\pi} : {}^0K'_1(\hat{\mathbb{Z}}_p\pi) \rightarrow {}^0K_1(\mathfrak{M}_p) \xrightarrow{\delta_{\pi}} \hat{\mathbb{Z}}_p \otimes R_{\mathbf{Q}}(\pi).$$

By Proposition 3(ii), I_1 is onto. Assume that ${}^0D(\mathbb{Z}\sigma) = 0$ for any cyclic p -group σ . It then follows by diagram chasing that

$${}^0D(\mathbb{Z}\pi) \cong \text{Coker}(I_2) \cong A_{\mathbf{Q}}(\pi).$$

($A_{\mathbf{Q}}(\pi)$ is a p -group by the Artin induction theorem: see, for example, [1, Theorem 15.4]).

It remains to check that ${}^0D(\mathbb{Z}\sigma) = 0$ for cyclic σ : This is implicit in [6], [5], and [15]; but doesn't seem to be stated explicitly. If $|\sigma| \leq p$, then $D(\mathbb{Z}\sigma) = 0$ by [10, Theorem 6.24].

So assume $|\sigma| = p^n$ for $n \geq 2$. Let $\rho \subseteq \sigma$ be the order p subgroup, and assume inductively that ${}^0D(\mathbb{Z}[\sigma/\rho]) = 0$. There is a commutative diagram

$$\begin{array}{ccccc} \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\rho) & \xrightarrow{i_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma) & \xrightarrow{j_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho) \\ \downarrow \theta_p & & \downarrow \theta_\sigma & & \downarrow \theta_{\sigma/\rho} \\ 0 = {}^0D(\mathbb{Z}\rho) & \longrightarrow & {}^0D(\mathbb{Z}\sigma) & \longrightarrow & {}^0D(\mathbb{Z}[\sigma/\rho]) = 0, \end{array}$$

where i_* and j_* are induced by inclusion and projection. Since $K_1(\hat{\mathbb{Z}}_p\sigma)$ maps onto $K_1(\hat{\mathbb{Z}}_p[\sigma/\rho])$,

$$j_*(\text{Ker}(\theta_\sigma)) = \text{Ker}(\theta_{\sigma/\rho}) = \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho).$$

In other words, $\theta_\sigma \mid \text{Ker}(j_*)$ is onto. Furthermore, $\text{Ker}(j_*) \subseteq \text{Im}(i_*)$: if $V \cong \mathbb{Q}\zeta_p^n$ and $W \cong \mathbb{Q}\zeta_p$ are the faithful irreducible $\mathbb{Q}\sigma$ - and $\mathbb{Q}\rho$ -representations, then $[V]$ generates $\text{Ker}(j_*)$, and $V = \text{Ind}_\rho^\sigma(W)$. We thus get that $\theta_\sigma \mid \text{Im}(i_*)$ is onto, but $\theta_\sigma \circ i_* = 0$, and so ${}^0D(\mathbb{Z}\sigma) = 0$.

One easy consequence of Theorem 9 is an alternate proof, for odd p -groups, of Martin Taylor's theorem [13] involving the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism. $T(\mathbb{Z}\pi)$ is the group of all elements

$$[\Sigma, n] - [\mathbb{Z}\pi] \in D(\mathbb{Z}\pi),$$

for $(n, |\pi|) = 1$, where $[\Sigma, n]$ is the projective module

$$[\Sigma, n] = n\mathbb{Z}\pi + \mathbb{Z} \cdot \left(\sum_{g \in \pi} g \right) \subseteq \mathbb{Z}\pi.$$

So if π is a p -group and $\mathfrak{M} \subseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then $[\Sigma, n] - [\mathbb{Z}\pi]$ corresponds, under the identification in Proposition 2, to the element of $K_1(\mathfrak{M}_p)$ which is $n \in (\hat{\mathbb{Z}}_p)^*$ at the identity component and 1 at all other components (in particular, $T(\mathbb{Z}\pi) \subseteq {}^0D(\mathbb{Z}\pi)$). The isomorphism of Theorem 9 thus sends $T(\mathbb{Z}\pi)$ to the group of multiples of the identity in

$$R_{\mathbb{Q}}(\pi) / \sum \{ \text{Ind}_\sigma^\pi(R_{\mathbb{Q}}(\sigma)) : \sigma \subseteq \pi \text{ cyclic} \}.$$

In other words:

THEOREM 10. (*M. Taylor [13]*) *For any p -group π , $T(\mathbb{Z}\pi)$ is cyclic of order equal to the Artin exponent of π . \square*

The computation of $|{}^0D(\mathbb{Z}\pi)|$ can now be carried out, using the same idea as for the calculation in [3]: that of comparing discriminants. We first consider the Artin cokernel of an arbitrary finite group.

THEOREM 11. *Let π be any finite group, and write*

$$\mathbb{Q}\pi \cong \prod_{i=1}^k M_{r_i}(D_i),$$

where the D_i are division algebras. Let X be a set of conjugacy class representatives for cyclic subgroups $\sigma \subseteq \pi$. Then

$$|A_{\mathbb{Q}}(\pi)| = \left[\left(\prod_{\sigma \in X} \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \right) / \left(\prod_{i=1}^k [D_i : \mathbb{Q}] \right) \right]^{1/2}$$

Proof. For convenience, set

$$G = \sum_{\sigma \in X} \text{Ind}_{\sigma}^{\pi} (R_{\mathbb{Q}}(\sigma)) \subseteq R_{\mathbb{Q}}(\pi).$$

Then

$$|R_{\mathbb{Q}}(\pi)/G| = [d(G)/d(R_{\mathbb{Q}}(\pi))]^{1/2}; \quad (1)$$

where $d(-)$ denotes discriminant with respect to the usual inner product

$$\langle [V], [W] \rangle = \frac{1}{|\pi|} \sum_{g \in \pi} \chi_V(g) \chi_W(g).$$

For each i , let V_i denote the irreducible representation of $M_{r_i}(D_i)$; then

$$\begin{aligned} \langle [V_i], [V_j] \rangle &= \dim_{\mathbb{Q}} (\text{Hom}_{\mathbb{Q}\pi} (V_i, V_j)) = 0 & \text{if } i \neq j \\ &= [D_i : \mathbb{Q}] & \text{if } i = j. \end{aligned}$$

So

$$d(R_{\mathbb{Q}}(\pi)) = \prod_{i=1}^k [D_i : \mathbb{Q}]. \quad (2)$$

To compute $d(G)$, consider first the set

$$S = \{[\mathbb{Q}(\pi/\sigma)] : \sigma \in X\} \subseteq R_{\mathbb{Q}}(\pi);$$

where $\mathbb{Q}(\pi/\sigma)$ denotes the permutation representation with \mathbb{Q} -basis π/σ . These elements generate G : since

$$\mathbb{Q}(\pi/\sigma) = \text{Ind}_\sigma^\pi(\mathbb{Q}) \in G,$$

and $R_{\mathbb{Q}}(\sigma)$ ($\sigma \in X$) is generated by the elements

$$\{[\mathbb{Q}(\sigma/\tau)] = \text{Ind}_\tau^\sigma([\mathbb{Q}]) : \tau \subseteq \sigma\}.$$

Also, $\text{rk}(R_{\mathbb{Q}}(\pi)) = |X|$ (see [1, Theorem 21.5]); and so S is a basis for G .

It follows that

$$d(G) = \det(M) \tag{3}$$

where $M = (M_{\sigma\tau})_{\sigma, \tau \in X}$ is the matrix defined by

$$M_{\sigma\tau} = \langle [\mathbb{Q}(\pi/\sigma)], [\mathbb{Q}(\pi/\tau)] \rangle.$$

For $\sigma \in X$, let χ_σ denote the character of $\mathbb{Q}(\pi/\sigma)$. For any $x \in \pi$,

$$\chi_\sigma(x) = \frac{1}{|\sigma|} \cdot \#\{g \in \pi : xg\sigma = g\sigma\} = \frac{1}{|\sigma|} \cdot \#\{g \in \pi : x \in g\sigma g^{-1}\}.$$

Hence, for $\sigma, \tau \in X$,

$$\begin{aligned} M_{\sigma\tau} &= \frac{1}{|\pi|} \sum_{x \in \pi} \chi_\sigma(x) \chi_\tau(x) = \frac{1}{|\sigma| \cdot |\tau|} \cdot \frac{1}{|\pi|} \sum_{g, h \in \pi} |g\sigma g^{-1} \cap h\tau h^{-1}| \\ &= \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} |\sigma \cap g\tau g^{-1}|. \end{aligned} \tag{4}$$

To simplify what follows, define, for $n \geq 1$ and $m \geq 1$,

$$\varphi_m(n) = n - \sum_{\substack{d|n \\ d < m}} \varphi(d).$$

Note in particular that $\varphi_1(n) = n$, $\varphi_n(n) = \varphi(n)$, and $\varphi_m(n) = 0$ for $m > n$. Let $N = \max\{|\sigma| : \sigma \in X\}$; and define, for $1 \leq m \leq N$:

$$X_m = \{\sigma \in X : |\sigma| = m\} \quad Y_m = \{\sigma \in X : |\sigma| \geq m\} = \bigcup_{i \geq m} X_i.$$

For all $0 \leq m \leq N$, define a matrix $M^{(m)} = (M_{\sigma\tau}^{(m)})_{\sigma, \tau \in Y_m}$, by setting

$$M_{\sigma\tau}^{(m)} = \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|).$$

In particular, $M^{(1)} = M$.

Fix $1 \leq m \leq N$. For $\sigma, \tau \in X_m$ (i.e., $|\sigma| = |\tau| = m$),

$$\begin{aligned} M_{\sigma\tau}^{(m)} &= m^{-2} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) \\ &= (\varphi(m)/m^2) \cdot \#\{g \in \pi : \sigma = g\tau g^{-1}\} = 0 & \text{if } \sigma \neq \tau \\ &= \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| & \text{if } \sigma = \tau \end{aligned}$$

In particular,

$$\det(M^{(N)}) = \prod_{\sigma \in X_N} \left[\frac{\varphi(|\sigma|)}{|\sigma|} |N(\sigma)/\sigma| \right]. \quad (5)$$

If $1 \leq m < N$ and $\sigma, \tau \in Y_{m+1}$ (i.e., $|\sigma|, |\tau| > m$), consider the entries $M_{\sigma\rho}^{(m)}$ for $\rho \in X_m$ ($|\rho| = m$). By definition, $M_{\sigma\rho}^{(m)} = 0$ unless $|\sigma \cap g\rho g^{-1}| \geq m$ for some g ; i.e., unless $g\rho g^{-1} \subseteq \sigma$. If $m \nmid |\sigma|$, then these $M_{\sigma\rho}^{(m)}$ all vanish; and also $M_{\sigma\tau}^{(m)} = M_{\sigma\tau}^{(m+1)}$ ($\varphi_m(n) = \varphi_{m+1}(n)$ if $m \nmid n$). If $m \mid |\sigma|$, let $\rho \in X_m$ be the unique element conjugate to a subgroup of σ ; then

$$\begin{aligned} M_{\sigma\tau}^{(m)} - (M_{\sigma\rho}^{(m)}/M_{\rho\rho}^{(m)}) \cdot (M_{\rho\tau}^{(m)}) &= \frac{1}{|\sigma| \cdot |\tau|} \left[\sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) - \sum_{g \in \pi} \varphi_m(|\rho \cap g\tau g^{-1}|) \right] \\ &= \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_{m+1}(|\sigma \cap g\tau g^{-1}|) = M_{\sigma\tau}^{(m+1)}. \end{aligned}$$

In other words, for all $\sigma, \tau \in Y_{m+1}$,

$$M_{\sigma\tau}^{(m+1)} = M_{\sigma\tau}^{(m)} - \sum_{\rho \in X_m} (M_{\sigma\rho}^{(m)}/M_{\rho\rho}^{(m)}) \cdot M_{\rho\tau}^{(m+1)};$$

and $M^{(m+1)}$ is obtained from $M^{(m)}$ by elementary operations which eliminate all entries $M_{\rho\tau}^{(m)}$ for $\rho \in X_m, \tau \in Y_{m+1}$. It follows that

$$\begin{aligned} \det(M^{(m)}) &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} M_{\sigma\sigma}^{(m)} \\ &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} \left[\frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \right]. \end{aligned}$$

Combining this with (5) gives

$$d(G) = \det(M) = \det(M^{(1)}) = \prod_{\sigma \in X} \left[\frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \right].$$

Finally, combined with (1) and (2), this gives the desired formula for $|R_{\mathbb{Q}}(\pi)/G|$. \square

When π is a p -group (recall that p is always odd), the above formula can be reformulated solely in terms of cyclic subgroups:

THEOREM 12. *Let π be a p -group, and let X be a set of conjugacy class representatives for cyclic subgroups $\sigma \subseteq \pi$. Then*

$$|{}^0D(\mathbb{Z}\pi)| = \left[\prod_{\sigma \in X} \frac{|N(\sigma)/\sigma|^2}{|Z(\sigma)|} \right]^{1/2}.$$

Proof. By Proposition 1, $\mathbb{Q}\pi \cong \prod_{i=1}^k M_{r_i}(D_i)$, when the D_i are fields, and

$$\begin{aligned} \prod_{i=1}^k [D_i : \mathbb{Q}] &= \prod_{\sigma \in X} \varphi(|\sigma| \cdot |Z(\sigma)|/|N(\sigma)|) \\ &= \prod_{\sigma \in X} [\varphi(|\sigma|) \cdot |Z(\sigma)|/|N(\sigma)|]. \end{aligned}$$

The result now follows by substitution into the formula of Theorem 11. \square

Finally, for the sake of completeness, we extend Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ in [3] to arbitrary (not necessarily abelian) p -groups π . For any such π , $\hat{\mathbb{Q}}_p\pi$ will denote the group ring modulo conjugation:

$$\overline{\hat{\mathbb{Q}}_p\pi} = \hat{\mathbb{Q}}_p\pi / \langle x - gxg^{-1} : x \in \hat{\mathbb{Q}}_p\pi, g \in \pi \rangle = \hat{\mathbb{Q}}_p\pi / \langle xy - yx : x, y \in \hat{\mathbb{Q}}_p\pi \rangle.$$

This can be regarded as the $\hat{\mathbb{Q}}_p$ -vector space with basis the set of conjugacy classes in π . Let $\overline{\hat{\mathbb{Z}}_p\pi} \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$ be the image of $\hat{\mathbb{Z}}_p\pi$; and let $\overline{\mathfrak{M}} \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$ denote the image of any maximal order $\mathfrak{M} \subseteq \hat{\mathbb{Q}}_p\pi$.

If F is any field and $r \geq 1$, it is easy to check that

$$\langle xy - yx : x, y \in M_r(F) \rangle = \text{Ker} [\text{tr} : M_r(F) \rightarrow F].$$

Thus, if $\hat{\mathbb{Q}}_p\pi \cong \prod M_{r_i}(F_i)$, then $\overline{\hat{\mathbb{Q}}_p\pi} \cong \prod F_i$, and the projection $\hat{\mathbb{Q}}_p\pi \rightarrow \overline{\hat{\mathbb{Q}}_p\pi}$ is the product of the trace maps. If $R_i \subseteq F_i$ is the ring of integers, then any maximal

order $\mathfrak{M}_i \subseteq M_{r_i}(F_i)$ is conjugate to $M_{r_i}(R_i)$ [9, Theorem 21.6]; and so $\text{tr}(\mathfrak{M}_i) = R_i$. In particular, $\overline{\mathfrak{M}} = \prod R_i$ under the above identification (and is thus independent of the choice of maximal order).

PROPOSITION 13. *Let π be a p -group, and let $\overline{\hat{\mathbb{Z}}_p \pi}$, $\overline{\mathfrak{M}} \subseteq \overline{\hat{\mathbb{Q}}_p \pi}$ be as above. Then, for any odd $1 \leq t \leq p-2$,*

$$\begin{aligned} |D(\mathbb{Z}\pi)| &= |{}^t(\overline{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p \pi})| && \text{if } t \neq 1 \\ &= |{}^1(\overline{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p \pi})| \cdot \frac{|\pi^{ab}|}{|\text{tors}_p(\mathbb{Q}\pi)^*|} && \text{if } t = 1. \end{aligned}$$

Proof. Let $\mathfrak{M} \supseteq \mathbb{Z}\pi$ be a maximal order, and write

$$\hat{\mathbb{Q}}_p \pi = \prod_{i=1}^k A_i; \quad A_i \cong M_{r_i}(F_i); \quad \mathfrak{M}_p \cong \prod_{i=1}^k \mathfrak{M}_i;$$

where F_i are fields and $\mathfrak{M}_i \subseteq A_i$ is a maximal order for all i . Given any $x \in \mathfrak{M}_i$ which is topologically nilpotent (i.e., $p \mid x^n$ for some n), the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

converges in A_i . We claim that for such x ,

$$\text{tr}(\log(1+x)) = \log(\det(1+x)) \in F_i. \quad (1)$$

To see (1), choose n such that $p \mid x^{p^n}$. Then for any $m \geq 0$, $(1+x)^{p^{n+m}} = 1 + p^{m+1}y$ for some $y \in \mathfrak{M}_i$, and

$$\begin{aligned} \log(\det(1+x)^{p^{n+m}}) &= \log(\det(1+p^{m+1}y)) \equiv p^{m+1} \cdot \text{tr}(y) \\ &\equiv \text{tr}(\log(1+p^{m+1}y)) = \text{tr}(\log(1+x)^{p^{n+m}}) \pmod{p^{2m+2}}. \end{aligned}$$

So for all $m \geq 0$,

$$\log(\det(1+x)) \equiv \text{tr}(\log(1+x)) \pmod{p^{m-n+2}};$$

and (1) holds.

In particular, $\log(1+x)$ for $x \in J(\hat{\mathbb{Z}}_p \pi)$ or $x \in J(\mathfrak{M}_p)$ induces homomorphisms

$$L_1: K'_1(\hat{\mathbb{Z}}_p \pi)_{(p)} \rightarrow \overline{\hat{\mathbb{Q}}_p \pi}; \quad L_2: K_1(\mathfrak{M}_p)_{(p)} \rightarrow \overline{\hat{\mathbb{Q}}_p \pi}$$

such that $L_1 = L_2 \mid K'_1(\hat{\mathbb{Z}}_p\pi)_{(p)}$. (Here J means Jacobson radical; note that $J(\hat{\mathbb{Z}}_p\pi) \not\subseteq J(\mathfrak{M}_p)$ in general.) Furthermore,

$$\text{Ker}(L_2) = \text{tors}_p(\mathfrak{M}_p^*) = \text{tors}_p(\mathbb{Q}\pi)^*; \quad \text{Ker}(L_1) \cong \pi^{ab};$$

and so by Proposition 2, for all odd t :

$$|{}^tD(\mathbb{Z}\pi)| = [{}^t\text{Im}(L_2) : {}^t\text{Im}(L_1)]. \quad (2)$$

For any $\hat{\mathbb{Z}}_p$ -lattices $M_1, M_2 \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$, we write for short

$$[M_1 : M_2] = [M_1 : M_1 \cap M_2] / [M_2 : M_1 \cap M_2].$$

By Theorem 2 in [8], for any $1 \leq t \leq p-2$,

$$\begin{aligned} \left(1 - \frac{1}{p}\Phi\right)({}^t\text{Im}(L_1)) &= \overline{{}^t\hat{\mathbb{Z}}_p\pi} & \text{if } t \neq 1 \\ &= \text{Ker}[{}^1\overline{\hat{\mathbb{Z}}_p\pi} \rightarrow \pi^{ab}] & \text{if } t = 1 \end{aligned}$$

Here, $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$; and Φ is nilpotent ($\overline{{}^t\hat{\mathbb{Q}}_p\pi}$ lies in the augmentation ideal, since $t \neq 0$). So

$$\det\left(1 - \frac{1}{p}\Phi\right) = 1,$$

and hence

$$\begin{aligned} [{}^t\overline{\hat{\mathbb{Z}}_p\pi} : {}^t\text{Im}(\mathbb{L}_1)] &= 1 & \text{if } t \neq 1 \\ &= |\pi^{ab}| & \text{if } t = 1. \end{aligned} \quad (3)$$

Finally, note that for $s \geq 0$,

$$\begin{aligned} [{}^t\hat{\mathbb{Z}}_p\zeta_{p^s} : \log({}^t(\hat{\mathbb{Z}}_p\zeta_{p^s})^*)] &= 1 & \text{if } t \neq 1 \\ &= p^s & \text{if } t = 1: \end{aligned} \quad (4)$$

this follows by noting that $\log(1 + p\hat{\mathbb{Z}}_p\zeta_{p^s}) = p\hat{\mathbb{Z}}_p\zeta_{p^s}$, and then counting orders of the quotients. Since by (1),

$$\text{Im}(L_2) = \prod_{i=1}^k \log(R_i^*)_{(p)} \subseteq \prod_{i=1}^k F_i \cong \overline{\hat{\mathbb{Q}}_p\pi}$$

($R_i \subseteq F_i$ the ring of integers), (4) implies that

$$\begin{aligned} [{}^t\bar{\mathcal{M}} : {}^t\text{Im}(L_2)] &= 1 & \text{if } t \neq 1 \\ &= \prod |\text{tors}_p(R_i^*)| = |\text{tors}_p(\mathbb{Q}\pi)^*| & \text{if } t = 1. \end{aligned} \quad (5)$$

So (2), (3), and (5) combine to prove the proposition. \square

Generalizing Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ is now straightforward:

THEOREM 14. *Let π be a p -group (p odd). Let $S \subseteq \pi$ be a set of conjugacy class representatives for all $1 \neq g \in \pi$. Set*

$$p^n = |\pi^{ab}| \quad \text{and} \quad p^k = \prod_{g \in S} |Z(g)|.$$

For $s \geq 1$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$. Then $|D(\mathbb{Z}\pi)^-| = p^N$, where

$$N = \frac{1}{4} \left[k + 4n - \sum_{s \geq 1} w_s (sp^s - (s+1)p^{s-1} + 4s + 1) \right].$$

Proof. Let $\hat{\mathbb{Z}}_p\pi \subseteq \bar{\mathcal{M}} \subseteq \hat{\mathbb{Q}}_p\pi$ be as above. By Proposition 13,

$$|D(\mathbb{Z}\pi)^-| = |(\bar{\mathcal{M}}/\hat{\mathbb{Z}}_p\pi)^-| \cdot p^n \cdot \left[\prod_{s \geq 1} p^{sw_s} \right]^{-1}. \quad (1)$$

Write $\hat{\mathbb{Q}}_p\pi = \prod_{i=1}^k A_i$, where $A_i \cong M_{r_i}(F_i)$ and the F_i are fields. As before, the trace maps $\text{tr}_i : A_i \rightarrow F_i$ induce an identification of $\hat{\mathbb{Q}}_p\pi$ with $\prod F_i$. Let $\text{pr}_i : \hat{\mathbb{Q}}_p\pi \rightarrow A_i$ be the projection; and define an inner product on $\hat{\mathbb{Q}}_p\pi$ by setting

$$\langle x, y \rangle = \sum_{i=1}^k \text{tr}_{F_i/\mathbb{Q}_p} (\text{tr}_i \circ \text{pr}_i(x) \cdot \text{tr}_i \circ \text{pr}_i(y)) \quad (x, y \in \hat{\mathbb{Q}}_p\pi).$$

Since $\bar{\mathcal{M}} \subseteq \prod F_i$ is the product of the rings of integers, we have by definition discriminants

$$d(\bar{\mathcal{M}}) = \prod_i \Delta(F_i) \quad \text{and} \quad d(\bar{\mathcal{M}}^+) = 2^{rk(\bar{\mathcal{M}})-1} \cdot \prod_i \Delta(F_i \cap \mathbb{R}).$$

Here $\Delta(F_i)$, $\Delta(F_i \cap \mathbb{R})$ denote the discriminants over \mathbb{Q} ; and the power of 2 arises due to using the trace over F_i instead of $F_i \cap \mathbb{R}$.

By [16, Proposition 7-5-7], for $s \geq 1$,

$$\Delta(\mathbb{Q}\zeta_{p^s}) = p^{p^s(ps-s-1)}.$$

By the same proof, or by the composition formula applied to the fields $\mathbb{Q}\zeta_{p^s}/\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}$ [16, Corollary 3-7-20]:

$$\Delta(\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}) = p^{\frac{1}{2}[sp^s-(s+1)p^{s-1}-1]}.$$

Hence, $d(\overline{\mathcal{M}}^-) = p^{N_0}$, where

$$N_0 = \frac{1}{2} \sum_{s \geq 1} (sp^s - (s+1)p^{s-1} + 1). \quad (2)$$

Now fix $g, h \in \pi$. For any given $1 \leq i \leq k$,

$$\text{tr}_{F_i/\hat{\mathbb{Q}}_p}(\text{tr}_i \circ \text{pr}_i(g) \cdot \text{tr}_i \circ \text{pr}_i(h)) = \sum_{j=1}^t \chi_j(g)\chi_j(h),$$

where χ_1, \dots, χ_t are the distinct irreducible (complex) characters contained in the summand A_i . Let π^* denote the set of all irreducible complex characters. Then

$$\begin{aligned} \langle g, h \rangle &= \sum_{\chi \in \pi^*} \chi(g)\chi(h) = 0 && \text{if } g \text{ not conjugate to } h^{-1} \\ &= |Z(g)| && \text{if } g \text{ is conjugate to } h^{-1} \end{aligned}$$

by the second orthogonality relation [1, Proposition 9.26]. Hence, eliminating factors prime to p ,

$$d(\overline{\hat{\mathbb{Z}}_p\pi}) = \left[\prod_{g \in S} |Z(g)| \right]^{1/2} = p^{k/2}. \quad (3)$$

By (2) and (3), $|(\overline{\mathcal{M}'\hat{\mathbb{Z}}_p\pi})^-| = p^{N_1}$, where

$$N_1 = \frac{1}{4} \left[k - \sum_{s \geq 1} (sp^s - (s+1)p^{s-1} + 1) \right].$$

Together with (1), this proves the theorem. \square

This can also be reformulated solely in terms of cyclic subgroups of π :

THEOREM 15. *Let π be any p -group, and let X_0 be a set of conjugacy class representatives for cyclic subgroups $1 \neq \sigma \subseteq \pi$. For each $\sigma \in X_0$, set*

$$a_\sigma = \text{ord}_p |N(\sigma)/\sigma|; \quad b_\sigma = \text{ord}_p (|\sigma| \cdot |Z(\sigma)|/|N(\sigma)|).$$

Then

$$\text{ord}_p |D(\mathbb{Z}\pi)^-| = \text{ord}_p |\pi^{ab}| + \frac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1].$$

Proof. Let $w_s (s \geq 1)$ and k be as in Theorem 14. Then each $\sigma \in X_0$ has $\varphi(p^{b_\sigma})$ conjugacy classes of generators, and so

$$k = \sum_{\sigma \in X_0} \varphi(p^{b_\sigma}) \cdot (a_\sigma + b_\sigma).$$

By Proposition 1, w_s is the number of $\sigma \in X_0$ such that $b_\sigma = s$. So Theorem 14 takes the form

$$\begin{aligned} \text{ord}_p |D(\mathbb{Z}\pi)^-| &= \text{ord}_p |\pi^{ab}| \\ &\quad + \frac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma + b_\sigma)\varphi(p^{b_\sigma}) - b_\sigma p^{b_\sigma} + (b_\sigma + 1)p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \text{ord}_p |\pi^{ab}| + \frac{1}{4} \sum_{\sigma \in X_0} [a_\sigma \varphi(p^{b_\sigma}) + p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \text{ord}_p |\pi^{ab}| + \frac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1]. \quad \square \end{aligned}$$

REFERENCES

- [1] C. CURTIS and I. REINER, *Methods of representation theory with applications to finite groups and orders*, vol. 1, Wiley Interscience (1981).
- [2] A. FRÖLICH, *On the classgroup of integral groupings of finite Abelian groups*, Mathematika 16 (1969), 143–152.
- [3] A. FRÖLICH, *On the classgroup of integral groupings of finite Abelian groups II*, Mathematika 19 (1972), 51–56.
- [4] A. FRÖLICH, *Locally free module over arithmetic orders*, J. reine ang. Mathematik 274/275 (1975), 112–124.
- [5] S. GALOVICH, *The class group of a cyclic p -group*, J. Algebra 30 (1974), 368–387.
- [6] M. KERVAIRE and M. P. MURTHY, *On the projective class group of cyclic groups of prime power order*, Comment. Math. Helv. 52 (1977), 415–452.
- [7] S. LANG, *Cyclotomic Fields*, Springer-Verlag (1978).

- [8] R. OLIVER, SK_1 for finite groupings: II, Math. Scand. 47 (1980), 195–231.
- [9] I. REINER, *Maximal Orders*, Academic Press (1975).
- [10] D. S. RIM, *Modules over finite groups*, Annals of Math. 69 (1969), 700–712.
- [11] P. ROQUETTE, *Realisierung von Darstellungen endlicher nilpotente Gruppen*, Arkiv der Math. 9 (1958), 241–250.
- [12] R. SWAN and F. EVANS, *K-theory of finite groups and orders*, Lecture notes in mathematics no. 149, Springer-Verlag (1970).
- [13] M. TAYLOR, *Locally free classgroups of groups of prime power order*, J. Algebra 50 (1978), 463–487.
- [14] S. ULLOM, *Fine structure of class groups of cyclic p-groups*, J. Algebra 49 (1977), 112–124.
- [15] S. ULLOM, *Class groups of cyclotomic fields and group rings*, J. London Math. Soc. 17 (1978), 231–239.
- [16] E. WEISS, *Algebraic Number Theory*, McGraw Hill (1963).

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