Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 58 (1983)

Artikel: Class numbers and periodic smooth maps.

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DOI: https://doi.org/10.5169/seals-44595

Comment. Math. Helvetici 58 (1983) 167–185

0010–2571/83/002167–19$01.50 + 0.20/0

Class numbers and periodic smooth maps

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§1. Introduction

For m a positive integer, let ζ_m be a primitive mth root of unity, and let $Z(\zeta_m)$ denote the ring of integers of the cyclotomic field $Q(\zeta_m)$ over the rationals Q. It is well-known that $Z(\zeta_m)$ is just the collection of polynomials in ζ_m with integral coefficients. Two ideals $\mathscr{a}$ and $\mathscr{b}$ of $Z(\zeta_m)$ are said to be equivalent if there exists $\mathscr{c} \in Q(\zeta_m)$ with $\mathscr{a} = \mathscr{b} \cdot \mathscr{c}$. The equivalence classes of ideals form the class group C_m, a finite group with order $h(m)$, the class number of m. Complex conjugation induces an involution on C_m; let C_m^- denote its (-1)-eigenspace. Then $h^-(m)$, the order of C_m^- is a factor of $h(m)$. This paper studies the significance of the parity of $h^-(m)$ for the behavior near fixed points of periodic diffeomorphisms of the sphere. From this definition, it is almost immediate that $h(m)$ and $h^-(m)$ actually have the same parity. For m a prime this relationship between $h(m)$ and $h^-(m)$, defined differently, was first noticed by Kummer [K].

Let Σ be a smooth manifold, $f : \Sigma \to \Sigma$ a diffeomorphism. Let x be a fixed point; i.e., $f(x) = x$. Then the derivative

$$(df)_x : \Sigma_x \to \Sigma_x$$

will be a linear isomorphism of the tangent space of Σ at x with itself. Then a question of P. A. Smith asks whether, for Σ a (homotopy) sphere, f periodic (i.e., $f^m = id_\Sigma$ for some m), and x and y isolated fixed points of f, $(df)_x$ and $(df)_y$ would be *linearly similar.* Linear similarity means by definition that there is a linear isomorphism $L : \Sigma_x \to \Sigma_y$ with $L(df)_x L^{-1} = (df)_y$. Results of Atiyah–Bott, Milnor (see [AB]), Bredon, and Sanchez established this in many cases. However, in [CS2], this was shown to be false even when Σ is a differentiable sphere. Petrie had previously given examples of actions of highly non-cyclic groups with inequivalent fixed point representations. His examples fail to satisfy the conclusions of classical "Smith theory"; they all have subgroups whose fixed point sets are

* This work supported by NSF Grant MCS-8002925.
† This work supported by NSF Grant MCS 80-26053.

disconnected but not isolated. The actions of [CS2] satisfy the conclusions of Smith theory in the strongest possible way, viz the fixed point set of f^j, any j, is also a differentiable sphere. Some more recent examples will appear in the Ph.D. thesis of Alan Siegel.

Given a periodic diffeomorphism f of Σ, a root of unity ξ, and a fixed point x of f, let $m_f(x, \xi)$ denote the multiplicity of ξ as an eigenvalue of $(df)_x$. The linear similarity of df_x, df_y is just the assertion that $m_f(x, \xi) = m_f(y, \xi)$, for all roots of unity ξ. By the *mod two Smith Conjecture* one means the following weaker statement: For each root of unity ξ,

$$m_f(x, \xi) \equiv m_f(y, \xi) \qquad (\mathrm{mod}\ 2).$$

Unlike the original question, the mod two version is true for smooth maps of period a power of *any* prime, by the above quoted results for odd primes and by results of Bredon for the prime two.

Recall that a periodic map g is free if, for all j, either g^j is the identity or g^j fixes no points. For smooth maps of homotopy spheres of period q or $2q$, with q odd, that are free outside of a 1-dimensional set, it follows from the arguments of Atiyah–Bott, Milnor and Sanchez that the mod two Smith conjecture is actually true.

THEOREM 1.1. *For a positive odd integer q, the following statements are equivalent:*

1. *The mod two Smith conjecture holds for periodic diffeomorphisms of homotopy spheres, of period* 4*q, that are free outside of a* 1*-dimensional set*; *and*
2. *q has at most two prime divisors and* $h^-(q) \equiv 1$ (mod 2).

Thus, the mod two Smith Conjecture holds for periodic smooth maps of period $4q$, q odd, free outside a one-dimensional set, that have period less than 116. The first counter-example is a map of period 116 on the standard sphere of dimension 17. It can actually be shown that the mod two Smith Conjecture holds, for *all* periods less than 112, for periodic diffeomorphisms that are free outside a 1-dimensional set.

When $h^{-1}(q)$ is even or q has more than two prime divisors, it can also be shown that the minimal dimension of a counter-example to the mod two Smith Conjecture is at most $2\phi(q)+1$, and that in this counter-example, the eigenvalues of $(df)_x$ and $(df)_y$ other than -1 will (necessarily) be disjoint and of multiplicity one. (Here $\phi(q)$ is the Euler ϕ-function.) Moreover, there is a counter-example to the mod two Smith Conjecture, on the standard sphere, in every odd dimensional above the minimal one. An interesting problem relating topology and number theory is to find a formula for the minimal dimension of a counter-example to the mod two Smith Conjecture, when $h^-(q)$ is odd or q has more than two prime factors.

The class number actually plays a role even when q has more than two prime divisors. In fact, it will actually be shown below that 1. of Theorem 1.1 is equivalent to the assertion that the index of the Stickelberger ideal S^- in $Z[G(q)]^-$ is odd, $G(q)$ the Galois group of $Q(\zeta_q)$ over Q. Theorem 1.1 then follows from the results of Iwasawa that the index of this ideal is $h^-(q)$ for q divisible by at most two primes and of Sinnott that this index is $2^{g-2}h^-(q)$ for g the number of primes dividing q, $g>1$. (See [SI].) Thus, the failure of the mod two Smith Conjecture for more than three prime divisors is due to the extraneous powers of two in Sinnott's theorem. It is possible to relate with more refinement the type of behavior that can occur at the fixed points with the structure of $(Z[G(q)]^-/S^-)$. Even when $g>2$, it will still be possible to distinguish phenomena related to the class number $h^-(q)$ and to formulate, as above, a purely topological necessary and sufficient condition that $h^-(q)$ be odd. These matters will be the subject of a future paper.

The results of [CS2] [CS3] also give much, and in some cases complete, information on the possible pairs of linear maps $(df)_x$, $(df)_y$ that can arise as derivatives at fixed points of the same periodic smooth f map of a homotopy sphere Σ. A periodic smooth map f of Σ is said to be of Smith type if the fixed points set of f^j, all j, is either discrete or connected. The results of [CS2] [CS3] provide evidence for the revised conjecture that if f is of Smith type and x and y are fixed points, then $(df)_x$ and $(df)_y$ are *topologically similar*, i.e., there is a homeomorphism $\phi:\Sigma_x\to\Sigma_y$ with $\phi(df)_x\phi^{-1}=(df)_y$.

[*Historical Note:* It should be pointed out that the conjecture that for rotations topological and linear similarity would be equivalent was first proposed at the 1935 International Topology Conference in Moscow by deRham. He reduced it to the periodic case and made a number of other fundamental contributions to this problem as well. In [KR], this conjecture was extended to include all linear endomorphisms with all eigenvalues of modules one, and their extended conjecture also reduced to the periodic case. They and other authors (see [CS1] for details) produced further evidence. In [CS1], however, this conjecture was settled in the negative.]

The revised Smith Conjecture is established in [CS2] for periodic smooth maps that are free outside a 1-dimensional set. For period $4q$, q odd, the converse is also established (compare Theorem 4.1 below). That is, given A and B, linear isomorphisms of a m-dimensional vector space, that are periodic of period $4q$ and free outside of 1-dimensional sets, there exists a periodic smooth map f of S^{m+1}, with isolated fixed points x and y, with $(df)_x$ linearly similar to A and $(df)_y$ to B, if and only if A and B are typologically similar. Thus, to Theorem 1.1 one can add the following statement as equivalent to 1 or 2:

3. *If A and B are periodic topologically similar linear maps of real vector spaces, with A free outside a 1-dimensional set, then every root of unity appears as eigenvalues of A and of B with multiplicities congruent mod two.*

For $2 \mid q$, both the algebraic and geometric situations are more involved and will be taken up in later papers.

The first two sections of this paper will be purely algebraic and number-theoretic. Using results of Iwasawa–Sinnott, we relate the notion of tempered numbers to class-numbers of cyclotomic fields. The notion of tempered numbers is a measure of how multiples of integers distribute when reduced mod a given integer. In the final section, geometric results of [CS1] and [CS2] are compared with the algebra to obtain the theorem.

§2. Tempered numbers

Let n be a positive integer and a any integer. Let $R_n(x)$ denote the least non-negative number congruent to x mod n. Then, for x any *integer*, the function[(1)] whose value is 1 if $0 < R(ax) \leq n/2$ and is zero otherwise depends only upon the congruence class of x mod n and so defines a map

$$f_a^{(n)} : Z/nZ \to \{0, 1\} = Z/2Z$$

which also depnds only upon the congruence class of a mod n.

Now let $n = 4q$, with q odd, $q > 1$, and consider only the functions $f_a^{(4q)}$ with the least common divisor $(a, 4q) = 1$, a defined mod n (i.e., $a \in (Z/nZ)^*$), and $a \equiv 1 \pmod 4$. These functions satisfy the obvious linear relations

$$(\dagger)_n : f_a + f_{2q-a} = f_1 + f_{2q-1}.$$

As in [CS1], we say that the number $n = 4q$ is *tempered* if all linear relations among the functions $\{f_a^{(n)} \mid a \in (Z)^*,\ a \equiv 1 \pmod 4\}$ are consequences of the relations $(\dagger)_n$.

In this section we wish to reduce the notion that $4q$ is tempered to a statement about modules over $Z_2[G(q)]$. The functions $f_a^{(q)}$, $(a, q) = 1$, satisfy the obvious linear relations

$$(*)_q : f_a^{(q)} + f_{-a}^{(q)} = f_1^{(q)} + f_{-1}^{(q)}.$$

* It would be equivalent to consider the function that is zero for $R(ax) \leq n/2$ and 1 otherwise.

We will say that the *odd* number q is *tempered* if all linear relations among the functions $\{f_a^{(q)} \mid a \in (Z/qZ)^*\}$ are consequences of the relations $(*)_q$.

THEOREM 2.1. *Let q be odd. Then $4q$ is tempered if and only if q is tempered.*

Proof. The proof is based on a series of simple identities. For example, suppose that $y = 4k+1$. Then

$$(q-1)y = 4kq + (q-y)$$

and

$$(2q-2)y = 8kq + 2(q-y).$$

Hence, if $0 \le y < 4q-1$, the sum

$$f_1^{(4q)}((q-1)y) + f_1^{(4q)}((2q-2)y)$$

is zero for $0 \le y \le 2q$ and 1 for $y > 2q$. Thus, if we set $\delta : Z/nZ \to \{0, 1\}$ to be function that is 1 except for $\delta(0) = 0$, we obtain that for $y \equiv 1 \pmod 4$

$$f_1^{(4q)}(y) = \delta + f_1^{(4q)}((q-1)y) + f_1^{(4q)}((2q-2)y).$$

Since $f_a^{(4q)}(x) = f_1^{(4q)}(ax)$, it follows that

$$\text{(2.1.1)} \quad f_a^{(4q)}(x) = \delta + f_a^{(4q)}((q-1)x) + f_a^{(4q)}((2q-2)x)), \qquad x \equiv 1 \pmod 4$$

Similarly one can obtain the formula

$$\text{(2.1.2)} \quad f_a^{(4q)}(x) = \delta_{2q} + f_a^{(4q)}((1-q)x) + f_a^{(4q)}((2-2q)x)), \qquad x \equiv 1 \pmod 4$$

Here

$$\delta_i(x) = \begin{cases} 0 & x \ne i \\ 1 & x = i \end{cases}.$$

This can actually be obtained from 2.1.1 by noting that $f_a(x) = f_{-a}(x)$ and $f_a + f_{-a} = \delta + \delta_{2q}$.

For $y = 4k+2$, $\quad 0 \le y \le 4q-1$,

$$(q-1)y = 4kq + (2q-y).$$

Hence,

$$f_1(y)=\delta_0+\delta_{2q}+f_1((q-1)y).$$

Thus, we obtain, as above

(2.1.3) For $x\equiv 2 \pmod 4$,

$$f_a^{(4q)}(x)=\delta_0+\delta_{2q}+f_a^{(4q)}((2q-2)x).$$

Finally, since $f_a(x)+f_a(-x)=\delta+\delta_{2q}$, from (2.1.1) and (2.1.2) one also obtains:

(2.1.4) $f_a^{(4q)}(x)=\delta_{2q}+f_a^{(4q)}((q-1)x)+f_a^{(4q)}((2q-2)x), \qquad x\equiv 3 \pmod 4;$

(2.1.5) $f_a^{(4q)}(x)=\delta+f_a^{(4q)}((1-q)x)+f_a^{(4q)}((2-2q)x), \qquad x\equiv 3 \pmod 4.$

Now let $H_q\subset(Z/4qZ)^*$ be the set of elements congruent to 1 mod 4. Then the above identities will be used to obtain;

(2.1.6) *Let* $\lambda_a\in\{0, 1\}$ *for each* $a\in H_q$. *Then*

$$\sum_{a\in H_q}\lambda_a f_a=0 \qquad (\text{in } Z/2Z)$$

if and only if

$$\sum_{a\in H_q}\lambda_a f_a(x)=o \qquad \forall x\equiv 0 \pmod 4.$$

One implication follows by restriction. Conversely, suppose

$$\sum_{a\in H_q}\lambda_a f_a(x)=0 \qquad \forall x\equiv 0 \pmod 4.$$

Then since

$$f_a+f_{-a}=\delta+\delta_{2q}$$

and since $f_a(-x)=f_{-a}(x)$, it follows that

$$\sum_{a\in H_q}\lambda_a(\delta(x)+\delta_{2q}(x))=0 \qquad \forall x\equiv 0 \pmod 4.$$

Let $x = 4$, then $\delta(x) = 1$, $\delta_{2q}(x) = 0$. Hence

$$\sum_{a \in H_q} \lambda_a \equiv 0 \pmod 2.$$

Suppose now that $q \equiv 1 \pmod 4$. Then for $x \equiv 1 \pmod 4$,

$$\sum_{a \in H_q} \lambda_a f_a(x) = \sum_{a \in H_q} \lambda_a(\delta + f_a^{(4q)}((q-1)x) + f_a^{(4q)}((2q-2)x)).$$

Since $q - 1 \equiv 2q - 2 \equiv 0 \pmod 4$ and since $\sum_a \lambda_a \equiv 0 \pmod 2$, the right side vanishes. Hence

$$\sum_{a \in H_q} \lambda_a f_a(x) = 0 \qquad \text{for } x \equiv 1 \pmod 4.$$

The same argument also proves that

$$\sum_{a \in H_q} \lambda_a f_a(x) = 0,$$

for $x \equiv 2 \pmod 4$, using (2.1.2), and $x \equiv 3 \pmod 4$, using (2.1.4). For $q \equiv -1 \pmod 4$, we use (2.1.2) and (2.1.5) instead. Thus

$$\sum_{a \in H_q} \lambda_a f_a(x) = 0 \qquad \forall x \in Z/4qZ,$$

so (2.1.6) is proven.

Let $\pi : H_q \to (Z/qZ)^*$ be reduction mod q. Then π is an isomorphism, and the following identity is immediately verified:

(2.1.7) $\quad f_a^{(4q)}(4x) = f_{\pi(a)}^{(q)}(x), \qquad x = 0, 1, \ldots, q-1.$

Combining this with (2.1.6) yields

(2.1.8) $\quad \sum_{a \in H_q} \lambda_a f_a^{(4q)} = 0$ *if and only if*

$$\sum_{b \in (Z/qZ)^*} \omega_b f_b^{(q)} = 0, \quad \textit{where} \quad \omega_b = \lambda_\pi - 1_{(b)}.$$

Since $2q \equiv 2 \pmod 4$, it follows that $\pi(2q - 1) = -\pi(a)$, $a \in H_q$, or, equivalently, that $\pi^{-1}(-b) = 2q - \pi^{-1}(b)$. With (2.1.8) and this observation, Theorem 2.1 is proven.

From now on, we identify $(Z/qZ)^*$ with the Galois group $G(q)$ of $Q(\zeta_q)$ over Q, ζ_q a primitive qth root of unity. Under this identification, a number $a \bmod q$, $(a, q)=1$, corresponds to the unique element σ_a of $G(q)$ with $\sigma_a(\zeta_q)=\zeta_q^a$.

Let Λ_q be the vector space over $Z/2Z=\{0, 1\}$ of functions from Z/qZ to $Z/2Z$. Then Λ_q is a module over the ring $R_q=Z_2[G(q)]$, via the action of $G(q)$ defined by

$$(\sigma_a f)(x)=f(ax).$$

Let $A_q \subset R_q$ be the *augmentation ideal*; i.e., the kernel of the map to $Z/2Z$ that sends $\sum_a \lambda_a \sigma_a$ to $\sum_a \lambda_a$. If M is any module over a ring R and $m \in M$, let

$$(m)=\{\lambda m \mid \lambda \in R\} \text{ and let } \operatorname{Ann}(m)=\{\lambda \in R \mid \lambda m=0\}.$$

PROPOSITION 2.2. *The odd number q is tempered if and only if*

$$\operatorname{Ann}(f_1^{(q)})=A_q(\sigma_1+\sigma_{-1}).$$

Proof. Clearly $\sigma_a f_1^{(q)}=f_a^{(q)}$, $(a, 4q)=1$. Hence the relation $(*)_q$ can be rewritten as

$$(\sigma_1+\sigma_a)(\sigma_1+\sigma_{-1})f_1^{(q)}=0.$$

The ideal A_q is easily seen to be generated over $Z/2Z$ by the elements $\sigma_1+\sigma_a$, $(a, q)=1$. Thus the ideal $A_q(\sigma_1+\sigma_{-1})$ consists precisely of linear combinations over $Z/2Z$ of the elements of $(\sigma_1+\sigma_a)(\sigma_1+\sigma_{-1})$ for $(a, q)=1$. The Proposition follows easily.

In view of 1.1, the equation of 2.2 is also necessary and sufficient for $4q$ to be tempered. In a future paper the general problem of when is $2^B q$ tempered, q odd, will be solved.

§3. Stickelberger ideals

Throughout this section let q be a fixed odd integer, $q>1$, and let $G=G(q)$, the Galois group of $Q(\zeta_q)$ over Q as above. Recall from 2.2 that q is tempered iff $\operatorname{Ann}(f_1^{(q)})=(\sigma_1+\sigma_{-1})A_q$.

Let $S \subset Z[G]$ be the Stickelberger ideal, as defined in [SI]. For any real number x, let $\langle x\rangle$ denote the least non-negative residue of $x \bmod Z$. For $c \in Z$, let

$$\theta(c)=\sum_{a\in(Z/qZ)^*}\left\langle\frac{-ca}{q}\right\rangle\sigma_a^{-1},$$

where $\sigma_a \in G$ is the element with $\sigma_a(\zeta_q) = \zeta_q^a$. Then the subgroup (i.e., Z-module) S' of $Q[G]$ generated by the elements $\theta(c)$ is actually a $Z[G]$-module, as $\sigma_b\theta(c) = \theta(bc)$. Then, by definition, $S = S' \cap Z[G]$.

For any $Z[G]$-module, M, let $M^- = \{x \in M \mid \sigma_{-1}x = -x\}$.

LEMMA 3.1* ([L] Chap. 2, §1, Lemma 1). $Z[G]^- = (\sigma_1 - \sigma_{-1})$.

In view of this lemma, S^- is contained in the $Z[G]$ ideal $(\sigma_1 - \sigma_{-1})$ generated by $\sigma_1 - \sigma_{-1}$. Let $\gamma: Z[G] \to Z_2[G] = R_q$ be mod 2 reduction. Then $\gamma S^- \subset \gamma(\sigma_1 - \sigma_{-1}) = (\sigma_1 + \sigma_{-1})$. Note that Ann $(\sigma_1 + \sigma_{-1}) \supset (\sigma_1 + \sigma_{-1})$; actually it will be seen that there are equal.

THEOREM 3.2. *The* R_q*-modules* Ann $(f_1^{(q')})/A_q(\sigma_1 + \sigma_{-1})$ *and* Ann $(\gamma S^-)/(\sigma_1 + \sigma_{-1})$ *are isomorphic.*

THEOREM 3.3. *The odd number* q *is tempered if and only if* $\gamma S^- = (\sigma_1 + \sigma_{-1})$.

COROLLARY 3.4. *The odd number* q *is tempered if and only if* q *has at most two prime factors and* $h^-(q)$ *is odd.*

Corollary 3.4 follows from Theorem 3.3 and the main result of [SI].

LEMMA 3.4* Ann $(\sigma_1 + \sigma_{-1}) = (\sigma_1 + \sigma_{-1})$. (Hence $R_q/(\sigma_1 + \sigma_{-1})$ and $(\sigma_1 + \sigma_{-1})$ are isomorphic.)

Proof. $(\sigma_1 + \sigma_{-1})^2 = \sigma_1 + \sigma_1 = 0$, so one inclusion is obvious. Suppose

$$(\sigma_1 + \sigma_{-1})\left(\sum_{g \in G} z(g)g\right) = 0.$$

Then for all g,

$$z(g) + z(\sigma_{-1}g) = 0,$$

so that, for $\Gamma \subset G$ any set of representatives of $G/\{\sigma_1, \sigma_{-1}\}$,

$$\sum_{g \in G} z(g)g = \sum_{g \in \Gamma} z(g)g + \sum_{g \in \Gamma} z(g)\sigma_{-1}g = (\sigma_1 + \sigma_{-1}) \sum_{g \in \Gamma} z(g)g.$$

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* Results valid for all q.

LEMMA 3.5* $\mathrm{Ann}(\gamma S^-)=(\sigma_1+\sigma_{-1})$ *if and only if* $\gamma S^-=(\sigma_1+\sigma_{-1})$.

Proof. The if part follows from 3.4. To see the converse, note that multiplication induces a monomorphism.

$$\alpha: R_q/\mathrm{Ann}\,\gamma S^- \to \mathrm{Hom}_{R_q}(\gamma S^-, R_q),$$

$\alpha(x)(y)=xy$. Given $\beta\in\mathrm{Hom}_{R_q}(\gamma S^-, R_q)$, set

$$\beta(x)=\sum_{g\in G}\beta_g(x)g.$$

Then the assignment of β_{σ_1} to β is easily seen to induce an isomorphism of $\mathrm{Hom}_{R_q}(\gamma S^-, R_q)$ and $\mathrm{Hom}_{Z/2Z}(\gamma S^-; Z/2Z)$. Thus $\dim_{Z/2Z}(R_q/\mathrm{Ann}\,\gamma S^-)\le \dim_{Z/2Z}\gamma S^-$. So if $\mathrm{Ann}\,\gamma S^-=(\sigma_1+\sigma_{-1})$, then by Lemma 3.4

$$R_q/\mathrm{Ann}\,\gamma S^-=R_q/(\sigma_1+\sigma_{-1})\cong(\sigma_1+\sigma_{-1}),$$

and so $\dim_{Z/2Z}(\sigma_1+\sigma_{-1})\le\dim_{Z/2Z}\gamma S^-$. Hence, as $\gamma S^-\subset(\sigma_1+\sigma_{-1})$, they must be equal.

Clearly Theorem 3.3 follows from Lemma 3.5 and Theorem 3.2. The remainder of this section will therefore be devoted to proving 3.2. To do this we first define a map

$$\psi:\Lambda_q\to Z_2[G]=R_q$$

as follows. Let $\delta_d\in\Lambda_q$, $0\le d<q$, be given by $\delta_d(x)=1$ iff $x=d$. Then, if $d=sr_d$, $s\mid q$ and $(r_d,q)=1$, let $\Gamma_d=\{a\mid 1\le a<q,\ (a,q)=1,\ a\equiv r_d \pmod{q/s}\}$, and define

$$\psi(\delta_d)=\sum_{a\in\Gamma_d}\sigma_a^{-1}\left(=\sigma_{r_d}^{-1}\sum_{a\in\Gamma_s}\sigma_a^{-1}\right)$$

Then, for any $f\in\Lambda_q$, write $f=\sum_{d=0}^{q-1}\beta(d)\delta_d$ and let $\psi(f)=\sum_{d=0}^{q-1}\beta(d)\psi(\delta_d)$. Clearly ψ is a map of $Z/2Z$-vector spaces.

Let $(\sum_{g\in G}\alpha_g g)^-=\sum_g\alpha_g g^{-1}$, as usual.

LEMMA 3.6* *For* $\lambda\in R_q$, $\psi(\lambda f)=\bar\lambda\psi(f)$, *(i.e.,* ψ *is an* R_q*-module map, where* R_q *has the module structure* $\lambda\cdot\omega=\bar\lambda\omega$.)

Proof. If $(b,q)=1$, $\Gamma_{db}=\sigma_b\cdot\Gamma_d$ (where r_b acts on $\{0,\ldots,q-1\}$ by $\sigma_b(x)=$

$R_q(xb)$). Hence

$$\psi(\sigma_b\delta_b)=\sum_{a\in\Gamma_d}\sigma_{ba}^{-1}=\sigma_b^{-1}\Big(\sum_{a\in\Gamma_d}\sigma_a^{-1}\Big),$$

which clearly implies the result.

For $s \mid m$, let $M_s \subset \Lambda_q$ be generated over $Z/2Z$ by the functions δ_d with $d=sr$, $(r,q)=1$. Clearly $\sigma_a M_s = M_s$, $(a,q)=1$ and so M_s is an R_q-submodule. For example, M_0 is the submodule generated by δ_0

LEMMA 3.7* $\Lambda_q=\bigoplus_{s\mid m} M_s$.

The proof is quite simple and so is omitted. If follows from this lemma that if we let h_s be the component of $f_1^{(q)}$ in M_s, then $\operatorname{Ann}(f_1^{(q)})=\bigcap_{s\mid m}\operatorname{Ann}(h_s)$.

LEMMA 3.8* $\psi \mid M_s : M_s \to Z_2[G]$ is a monomorphism.

Proof. For $\delta_d\in M_s$, $0\le d<q$, write $d=sr_d$, $(q,r_d)=1$. Suppose $\psi(\sum_{\Delta_s}\beta(d)\delta_d)=0$. where Δ_s consists of those d with $\delta_d\in M_s$, $0\le d<q$. Then

$$\sum_{\Delta_s}\beta(d)\Big(\sigma_{r_d}^{-1}\sum_{a\in\Gamma_s}\sigma_a^{-1}\Big)=0;$$

i.e.,

$$\Big(\sum_{\Delta_s}\beta(d)\sigma_{r_d}^{-1}\Big)\Big(\sum_{a\in\Gamma_s}\sigma_a^{-1}\Big)=0;$$

i.e.,

$$\text{(3.8.1)}\quad \sum_{\Delta_s}\sum_{\Gamma_s}\beta(d)\sigma_{r_d a}^{-1}=0.$$

Suppose $\sigma_{r_d a}=\sigma_{r_c}$ where $d,c\in\Delta_s$, $a\in\Gamma_s$. Then $r_d a=r_c \pmod q$. Since $a\in\Gamma_s$, $a\equiv 1 \pmod{q/s}$. Hence $r_d a\equiv r_d \bmod (q/s)$; hence

$$da\equiv d \pmod q$$

as $d=sr_d$. Thus $d\equiv c \pmod q$, so $d=c$, as $0\le d$, $c<q$, and then $r_d=r_c$, and $a\equiv 1 \pmod q$. Hence the left side of (3.8.1) has the form

$$\sum_{\Delta_s}\beta(d)\sigma_{r_d}^{-1}+\varepsilon,$$

where ε is a linear combination of elements of the set $\{\sigma_a^{-1} \mid a \not\equiv r_d \pmod q$ for all $d \in \Delta_s\}$. Clearly this implies that $\beta(d)=0$ for all $d \in \Delta_s$, i.e., $\sum_{\Delta_s} \beta(d)\delta_d = 0$. Thus $\psi \mid M_s$ is $1-1$.

In view of 3.7 and 3.8

$$\mathrm{Ann}\,(f_1^{(q)}) = \bigcap_{s\mid q} \mathrm{Ann}\,(\psi(h_s)).$$

If we let $\mathfrak{a}$ be the ideal generated by the $\psi(h_s)$, $s \mid q$, then the right side is just $\mathrm{Ann}\,\mathfrak{a} = \{\lambda \in R_q \mid \lambda x = 0$ for all $x \in \mathfrak{a}\}$. So

$$\mathrm{Ann}\,(f_1^{(q)}) = \mathrm{Ann}\,(\mathfrak{a}).$$

Let $\Delta_s = \{d \mid 0 \leq d < g-1,\ s \mid d,$ and $(d/s, q) = 1\}$, as above. Let $\Delta_s' = \{d \in \Delta_s \mid 0 < d \leq [q/2]\}$. Then

$$h_s = \sum_{d \in \Delta_s'} \delta_d.$$

Hence, writing $d = sr_d$ as above for $d \in \Delta_s$,

$$\psi(h_s) = \sum_{d \in \Delta_s'} \sum_{a \in \Gamma_s} \sigma_{r_d}^{-1} \sigma_a^{-1}$$

So clearly

$$\psi(h_s) = \sum_{\substack{b=1 \\ (b,q)=1}}^{[q/2s]} \sum_{a \in \Gamma_s} \sigma_{ab}^{-1}. \qquad \text{(Thus, } \psi(h_s) = 0.)$$

Let $\mathcal{B} \subset \mathfrak{a}$ be the ideal consisting of linear combinations

$$\sum_{s\mid q} \lambda_s \psi(h_s), \quad \text{where} \quad \sum_{s\mid q} \lambda_s \in A_q, \quad \text{and} \quad \lambda_q = 0.$$

PROPOSITION 3.9. $\mathcal{B} = \gamma(S^-)$.

Proof. Recall

$$\theta(s) = \sum_{a \in (Z/qZ)^*} \left\langle \frac{-as}{q} \right\rangle \sigma_a^{-1}.$$

Hence

$$q\theta(s) = \sum_a R_q(-as)\sigma_a^{-1} \in S \subset Z[G].$$

Clearly $R_q(-cs)$ depends only on the class of $c \bmod q/s$, and for $1 \le c < q/s$, $R_q(-cs) \equiv 1 \pmod 2$ iff c is even. Hence

$$\gamma(q\theta(s)) = \sum_{\substack{b=1 \\ (b,q)=1}}^{q/s} \sigma_b^{-1}\Big(\sum_{a \in \Gamma_s} \sigma_a^{-1}\Big).$$

Thus one easily sees that

$$\psi(q\theta(s)) = \sigma_2^{-1}\psi(h_s).$$

Hence, for $d \in \Delta_s$, $s \mid q$,

$$\gamma(q\theta(d)) = \sigma_{2r_d}^{-1}\psi(h_s).$$

According to [SI], S^- can be described as the intersection of $Z[G]$ with the subgroup of $Q[G]$ generated by the elements

$$e^-\theta(c) = \theta(c) - (1/2)s(G),$$

$s(G)$ the sum of the group elements. Let

$$\rho : Z[G] \to Z$$

be the augmentation. Given for each $c \ge 0$, $\omega_c \in Z[G]$, all but a finite number being 0,

$$\sum_c \omega_c\theta(c) = \sum_c \omega_c e^-\theta(c) + \tfrac{1}{2}\rho\Big(\sum_c \omega_c\Big)s(G).$$

Hence, since $q\theta(c) \in Z[G]\ \forall c$,

$$\sum_c \omega_c q\theta(c) \in S^- \quad \text{if} \quad \rho\Big(\sum_c \omega_c\Big) = 0.$$

Given $\lambda_s \in Z_2[G]$ with

$$\sum_{s \mid q} \lambda_s \in A_q \quad \text{and} \quad \lambda_q = 0,$$

as in the definition of $\mathcal{B}$, it is easy to see that there exist $\omega_s \in Z[G]$ with $\gamma(\omega_s) = \lambda_s$

and $\rho(\sum_{s|q}\omega_s)=0$. Hence

$$\sum_{s|q}\lambda_s\psi(h_s)=\sum_{s|q}\lambda_s\gamma(q\sigma_2\theta(s))=\sum_{s|q}\lambda_s\gamma(q\theta(2s))$$
$$=\gamma\sum_{s|q}\omega_s q\theta(2s))\in\gamma S^-.$$

Hence $\mathscr{B}\subset\gamma S^-$.

Now suppose that $\xi\in S^-$. Then we can write

$$\xi=\sum_{c>0}\lambda_c(\theta(c)-\tfrac{1}{2}s(G)),$$

where λ_c, $c>0$, $c\in Z$, are integers and all but a finite number are zero. Since q is odd and $q\theta(c)\in Z[G]$, it follows easily that $\omega=\frac{1}{2}(\sum_c\lambda_c)$ is an integer. Again, since q is odd,

$$\gamma(\xi)=\gamma(q\xi)=\sum_{c>0}\lambda_c\gamma(q\theta(c))+\omega s(G)=\sum_{c>0}\lambda_c\sigma_{2r_c}^{-1}\psi(h_{s_c})+\omega s(G),$$

where we now write $c\equiv s_c r_c \pmod q$ with $s_c\mid q$, $(r_c,q)=1$, $0\le s$, $r_c<q$. Since $\sum_{c>0}\lambda_c\equiv 0(2)$, $\sum_{c>0}\lambda_c\sigma_{2r_c}^{-1}\in A_q$ and so

$$\sum_{c>0}\lambda_c\sigma_{2r_c}^{-1}\psi(h_{s_c})\in\mathscr{B}.$$

Since $\Gamma_1=\{\sigma_1\}$ and $\Delta_1=\{a\mid(a,q)=1\}$,

$$\psi(h_1)=\sum_{\substack{b=1\\(b,q)=1}}^{[q/2]}\sigma_b^{-1}.$$

Hence one sees easily that

(3.9.1) $(\sigma_1+\sigma_{-1})\psi(h_1)=s(G)$,

and so $\omega s(G)\in\mathscr{B}$ also. Thus

$$\gamma S^-\subset\mathscr{B},$$

proving the proposition.

Since $(\sigma_1+\sigma_{-1})=\operatorname{Ann}(\sigma_1+\sigma_{-1})\subset\operatorname{Ann}\gamma S^-$, it follows from 3.9 that $(\sigma_1+\sigma_{-1})\subset\operatorname{Ann}\mathcal{B}$ and that

$$\operatorname{Ann}\mathcal{B}/(\sigma_1+\sigma_{-1})=\operatorname{Ann}\gamma S^-/(\sigma_1+\sigma_{-1}).$$

Since $a=\operatorname{Ann}(f_1^{(q)})$, the following lemma will complete the proof of 3.2

LEMMA 3.10. *The modules* $\operatorname{Ann}\mathcal{B}/(\sigma_1+\sigma_{-1})$ *and* $\operatorname{Ann}a/A_q(\sigma_1+\sigma_{-1})$ *are isomorphic.*

Proof. Since $a\subset\mathcal{B}$, $\operatorname{Ann}a\subset\operatorname{Ann}\mathcal{B}$, and this inclusion induces a mapping

$$\tau:\frac{\operatorname{Ann}a}{A_q\cdot(\sigma_1+\sigma_{-1})}\to\frac{\operatorname{Ann}\mathcal{B}}{(\sigma_1+\sigma_{-1})}$$

Lemma 3.10 will be proven by showing this map to be an isomorphism.

Suppose that $\lambda\in\operatorname{Ann}a$ and that λ represents an element in the kernel of τ. Then $\lambda=\omega\cdot(\sigma_1+\sigma_{-1})$ for some ω. Hence, since $(\sigma_1+\sigma_{-1})\psi(h_1)=s(G)$ (see (3.9.1) above), $o=\lambda\psi(h_1)=\omega s(G)$. Since $\sigma_a s(G)=s(G)$ for all $\sigma_a\in G(q)$,

$$\Big(\sum_{g\in G}\alpha_g g\Big)s(G)=\Big(\sum_{g\in G}\alpha_g\Big)s(G).$$

Hence $\omega s(G)=0$ if and only if $\omega\in A_q$. Therefore, $\lambda\in A_q\cdot(\sigma_1+\sigma_{-1})$, which shows that τ is one-to-one.

Suppose that $\lambda\in\operatorname{Ann}\mathcal{B}$. Then, for $s\mid q$, $s\neq q$, $\lambda(\psi(h_s)+\psi(h_1)=0$, i.e.,

(3.10.1) $\quad\lambda\psi(h_1)=\lambda\psi(h_s).$

Now, if $\omega\in A_q$, $\lambda\omega\psi(h_1)=0$. Hence, $\lambda\psi(h_1)\in\operatorname{Ann}A_q$. It is easy to see that $\operatorname{Ann}A_q=(s(G))$; hence

$$\lambda\psi(h_1)=\varepsilon s(G),\qquad \varepsilon=0\text{ or }1,$$

as $(s(G))=\{0,s(G)\}$. But then

$$(\lambda+\varepsilon(\sigma_1+\sigma_{-1}))\psi(h_1)=0,$$

using 3.9.1. By (3.10.1), applied to $\lambda+\varepsilon(\sigma_1+\sigma_{-1})$,

$$\lambda+\varepsilon(\sigma_1+\sigma_{-1})\in\operatorname{Ann}\alpha.$$

Thus τ is also surjective.

§4. Class numbers

In this section we prove the equivalence of statements 1. and 2. of Theorem 1.1. Recall that a linear periodic automorphism ϕ of a real vector space V is said to be pseudo-free if it is *free* outside of a 1-dimensional, invariant set L; i.e., $\phi(L)=L$ and ϕ^j is either the identity or fixes no points of $V-L$, for all j.

THEOREM 4.1. *Let ϕ_i be automorphisms of vector spaces V_i, $i=1,2$, with ϕ_1 of period m and pseudo-free. Then* (i) *implies* (ii) *and, for $m\neq 0 \pmod 8$ or* $\dim V_1<10$, (i) *is equivalent to* (ii) *where*:

(i) *There is a periodic smooth map $f:\Sigma\to\Sigma$ of a homotopy sphere Σ, free outside of a 1-dimensional set, with isolated fixed points, x_1, x_2 and with linear isomorphisms ψ_i: $\Sigma_{x_i}\to V_i$, $i=1,2$, so that $\psi_i^{-1}(df)_{x_i}\psi_i=\phi_i$ (i.e., $(df)_{x_i}$ and ϕ_i are linearly similar)*; *and*

(ii) *ϕ_1 and ϕ_2 are topologically similar, i.e., there is a homeomorphism $\psi: V_1\to V_2$ with $\psi^{-1}\phi_1\psi=\phi_2$.*

(Note that ϕ_1 and ϕ_2 topologically similar and ϕ_1 pseudo-free implies ϕ_2 psuedo-free also.)

This is just part of Theorem 2 of [CS2]. Now consider pseudo-free automorphisms ϕ_i, of period $4q$, q odd, on vector spaces V_i. Let $p_i(t)=\det(\phi_i-tI)$ be the characteristic polynomial of ϕ_i.

THEOREM 4.2. *The following are equivalent:*

(i) *ϕ_1 and ϕ_2 are topologically similar*; *and*

(ii) *There are factorizations (over the reals) $p_1(t)=k(t)h(t)$ and $p_2(t)=k(t)h(-t)$, where $k(-1)=0$ if $h(t)\neq 1$, so that $h(t)=\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j)$, with $\xi_j=$* $\exp((2\pi b_j)/4q)$, $(b_j,4q)=1$, $b_j\equiv 1 \pmod 4$, *with l even, and with*

$$(4.2.1)\quad (l/2)\Bigg(\sum_{\substack{c=1\\ c\neq q}}^{2q-1}\delta_{2c}^{(4q)}\Bigg)+\sum_{j=1}^{l} f_{b_j}^{(4q)}=0.$$

Theorem 4.2 is just a minor reformulation to suit the present notation of some of the results of [CS1]. Here $\delta_d^{(4q)}$ is just the function from $Z/4qZ$ to $\{0,1\}$ that vanishes on x iff $x\neq d$, $0\leq x<4q$. (See Theorems I, II, and compare 7.14 along with paragraph preceding 7.12, all in [CS1].)

It is easy to see that

$$\sum_{c=1,c\neq q}^{2q-1}\delta_{2c}=f_1^{(4q)}+f_{2q-1}^{(4q)}.$$

Hence (4.2.1) can be rewritten as

$$(4.3)\quad (l/2)(f_1^{(4q)}+f_{2q-1}^{(4q)})+\sum_{j=1}^{l} f_{b_j}^{(4q)}=0.$$

Suppose now that q has at most two prime factors and that $h^-(q)\equiv 1 \pmod 2$. Then, by Corollary 3.4 and Theorem 2.2, $4q$ is tempered. Let f be a periodic smooth map of the homotopy sphere Σ, with isolated fixed points x_1 and x_2, and suppose that f is free outside of a 1-dimensional set. Let $V_i=\Sigma_{x_i}$ and $(df)_{x_i}=\phi_i$, $i=1,2$. Then by 4.1 and 4.2, the characteristic polynomials of ϕ_i must factor as indicated in 4.2(ii). Hence, with b_j as in 4.2, (4.2) also holds. From the definition that $4q$ be *tempered*, it then follows easily that, after reordering of indices,

$$(b_i,\ldots,b_l)=(c_1,\ldots,c_s,2q-c_1,\ldots,2q-c_s,d_1,d_1,d_2,d_2,\ldots,d_t,d_t).$$

However, if $\xi=\exp(2\pi ib/4q)$, then $-\bar{\xi}=\exp(2\pi i(2q-b)/4q)$. It follows that multiplicity of a root of unity as a root of $h(t)=\prod_1^l(t-\xi_j)(t-\bar{\xi}_j)$ is congruent to its multiplicity as a root of $h(-t)$. From this and the fact that ϕ_1 and ϕ_2 have $k(t)h(t)$ and $k(t)h(-t)$ as characteristic polynomials, respectively, it is immediate the $\phi_1=(df)_{x_1}$ and $\phi_2=(df)_{x_2}$ satisfy the conclusion of the mod two Smith conjecture. Thus, in Theorem 1.1, it is proven that 2. implies 1.

To prove the converse, suppose that $h^-(q)\equiv 0 \pmod 2$ or that q has more than two prime factors. Then $4q$ is *not* tempered. Hence, there is a linear relation

$$\sum_{j=1}^{l} f_{b_j}=0,$$

with $1\le b_j<4q$, $b_j\equiv 1 \pmod 4$, $(b_j,4q)=1$, $j=1,2,\ldots,l$, that is *not* a consequence of the linear relations $(\ddagger)_q$ (See §2.). Without loss of generality, it may, of course, be assumed that $b_i\neq b_j$ if $i\neq j$.

Since $f_{b_i}(2q)=1$, it is immediate by evaluation at $2q$ that l is even. Let

$$\xi_j=\exp((2\pi ib_j)/4q),\qquad \tau=\exp((2\pi i)/4q),\qquad j=1,\ldots,l.$$

If $l/2$ is even, let

$$h(t)=\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j),$$

and if $l/2$ is odd, let

$$h(t)=(t-\tau)(t-\bar{\tau})(t+\tau)(t+\bar{\tau})\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j).$$

Let ϕ_1 and ϕ_2 be periodic automorphism of real vectors space V_i with characteristic polynomials $(t+1)h(t)$ and $(t+1)h(-t)$, respectively; these are elementary to construct. Then (4.3) and hence (4.2.1) is satisfied, so that, by 4.1 and 4.2, there is a periodic smooth map $f:\Sigma^{2l+1}\to\Sigma^{2l+1}$ (and actually even of S^{2l+1}) or Σ^{2l+5} (S^{2l+5}), free outside a 1-dimensional set, with fixed points x_1, x_2 and with $(df)_{x_i}$ linearly similar to ϕ_i, $i=1,2$.

We claim that f is a counter-example to the mod 2 Smith Conjecture. Since the b_j are pairwise distinct, ϕ_1 and ϕ_2 have all their eigenvalues of multiplicity one. Hence, in this case the mod 2 Smith Conjecture would actually imply the linear simplicity of ϕ_1 and ϕ_2; in particular, it would follows that $h(t)=h(-t)$. From this it follows easily that we must have

$$\{b_1,\ldots,b_l\}=\{c_1,\ldots,c_{l/2},2q-c_1,\ldots,2q-c_{l/2}\}.$$

Hence our original equation has the form

$$\sum_{j=1}^{l/2}(f_{c_j}+f_{2q-c_j})=0.$$

Since $(f_a+f_{2q-a})(0)=f_a(0)+f_a(2q)=1$, it follows by evaluation at zero that $l/2$ is also even; thus, our original equation $\sum_1^l f_{b_j}=0$ is a linear combination of relations of the form $(\ddagger)_q$, namely it is the sum of the relations

$$f_{c_j}+f_{2q-c_j}=f_1+f_{2q-1},\qquad 1\le j\le l/2,$$

a contradiction.

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Received June 10, 1982