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# Poincaré duality groups of dimension two, II

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## 1. Introduction

A Poincaré duality group of dimension $n$, in short a $PD^n$-group, is a group $G$ acting on $\mathbf{Z}$ such that one has natural isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbf{Z} \otimes A)$$

for all integers $k$ and all $\mathbf{Z}G$-modules $A$ (where $\mathbf{Z} \otimes A$ is the tensor product over $\mathbf{Z}$ with diagonal $G$-action). $G$ is called orientable or not according to whether or not $\mathbf{Z}$ is trivial as a $\mathbf{Z}G$-module. All "surface groups", i.e., fundamental groups of closed surfaces of genus $\geq 1$ are well-known to be $PD^2$-groups. In Eckmann–Müller [4] it was proved that a $PD^2$-group with *positive* first Betti number $\beta_1$ is isomorphic to a surface group. The purpose of the present paper is to show that the condition on $\beta_1$ is automatically fulfilled:

THEOREM 1. *The first Betti number $\beta_1$ of a $PD^2$-group is positive.*

As a consequence we thus have a complete classification of $PD^2$-groups.

THEOREM 2. *A group $G$ is a $PD^2$-group if and only if it is isomorphic to a surface group.*

For notations and properties concerning $PD^n$-groups, not explicitly mentioned here, we refer to [4] where also several (algebraic and topological) consequences are discussed.

## 2. Finitely generated projective $\mathbf{Z}G$-modules

For the proof of Theorem 1 we need the following fact, which may be of interest in connection with the conjectures of Bass (4.4 and 4.5 of [2]).

If $B$ is an abelian group, we let rank $B$ denote the dimension of the $\mathbf{Q}$-vector space $B \otimes \mathbf{Q}$.

PROPOSITION 3. *Let $G$ be a $PD^2$-group, $M \neq 0$ a finitely generated projective $\mathbf{Z}G$-module, and $\mathbf{Z}$ the trivial $\mathbf{Z}G$-module. Then* rank $(\mathbf{Z} \otimes_G M) \neq 0$.

*Proof.* Let $r_M$ denote the Hattori-Stallings trace of the identity endomorphism of $M$ as defined, e.g., in [1] and [2]. It is a finite linear combination with integral coefficients of the conjugacy classes $\tau$ in $G$,

$$r_M = \sum_{\tau} r_M(\tau)\tau.$$

For $x \in G$ let $r_M(x)$ be the coefficient of the conjugacy class of $x$. Suppose that $r_M(x) \neq 0$ for an element $x \in G$, $x \neq 1$. Then there exists, by Proposition 6.2 of [2], a prime $p$ and an integer $n > 0$ such that $x$ is conjugate to $x^{p^n}$. It follows (see the remark on p. 12 of [2]) that $x$ is contained in a subgroup $H \cong \mathbf{Z}[1/p]$ of $G$. By Strebel's theorem [5] all subgroups of infinite index in $G$ are of cohomological dimension 1 and thus free. Therefore $H$ has finite index in $G$; since $G$ is finitely generated so is $H$ and we have a contradiction. Hence $r_M(x) = 0$ for all $x \in G \setminus 1$ and it follows that $r_M(1) = \text{rank}\,(\mathbf{Z} \otimes_G M)$.

We now consider the nonzero finitely generated projective $\mathbf{C}G$-module $M \otimes \mathbf{C}$. We have $r_M(1) = r_{M \otimes \mathbf{C}}(1)$ which is positive by Kaplansky's theorem (see [1], Theorem 8.9), and the result follows.

## 3. Proof of Theorem 1. Euler characteristic

The completion of the proof is now in the same spirit as [3]. We first note that we can restrict attention to orientable $PD^2$-groups. Indeed (see [4], p. 511), if $G$ is non-orientable and $G_1$ the orientable subgroup of index 2 in $G$ then $\beta_1(G_1) > 0$ implies $\beta_1(G) > 0$.

So let $G$ be an orientable $PD^2$-group, and

$$0 \to P \to \mathbf{Z}G^d \to \mathbf{Z}G \xrightarrow{\varepsilon} \mathbf{Z} \tag{1}$$

a projective resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$. Since $PD^n$-groups are of type $(FP)$, the module $P$ is finitely generated projective. Since $H^0(G; \mathbf{Z}G) = H^1(G; \mathbf{Z}G) = 0$ and $H^2(G; \mathbf{Z}G) = \mathbf{Z}$ with trivial $G$-action for any orientable

$PD^2$-group, applying $\mathrm{Hom}_G(-, \mathbf{Z}G)$ to (1) yields an exact sequence

$$\mathbf{Z} \overset{\gamma}{\twoheadleftarrow} P^* \leftarrow \mathbf{Z}G^d \leftarrow \mathbf{Z}G \leftarrow 0 \tag{2}$$

where $P^* = \mathrm{Hom}_G(P, \mathbf{Z}G)$ is finitely generated projective. Let $IG$ be the kernel of $\varepsilon$ (the augmentation ideal) and $L$ the kernel of $\gamma$. Applying Schanuel's lemma to (1) and (2) gives

$$P^* \oplus IG \cong \mathbf{Z}G \oplus L.$$

There is a surjection $\mathbf{Z}G^d \twoheadrightarrow L$, and we obtain a surjection $\mathbf{Z}G^{d+1} \twoheadrightarrow P^* \oplus IG$ and hence a surjection $\mathbf{Z}G^{d+1} \twoheadrightarrow P^*$, with kernel $K \neq 0$. Obviously $K$ is a finitely generated projective $\mathbf{Z}G$-module, and we see from Proposition 3 that rank $(\mathbf{Z} \otimes_G K) \neq 0$. It follows that rank $(\mathbf{Z} \otimes_G P^*) \leq d$.

The Euler characteristic $\chi(G)$ of $G$ can be obtained by applying $\mathbf{Z} \otimes_G -$ to the resolution (2) and taking the alternating sum of the ranks:

$$\chi(G) = \mathrm{rank}\,(\mathbf{Z} \otimes_G P^*) - d + 1 \leq 1.$$

On the other hand $\chi(G) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1$ since the Betti numbers $\beta_0$ and $\beta_2$ of an orientable $PD^2$-group are $= 1$. Thus $2 - \beta_1 \leq 1$, i.e., $\beta_1 > 0$.

## 4. Poincaré 2-complexes

As a corollary of the above group-theoretic results the topological application mentioned in [4], Section 2 can be given an improved version.

We recall that a Poincaré $n$-complex is a $CW$-complex dominated by a finite complex and fulfilling Poincaré duality of formal dimension $n$ for arbitrary local coefficients. By results of Wall [6] a Poincaré 2-complex $X$ with finite fundamental group $\pi_1(X)$ is homotopy equivalent to the 2-sphere or to the real projective plane; if $\pi_1(X)$ is infinite, then $X$ is aspherical, i.e., an Eilenberg-MacLane complex $K(G, 1)$ for $G = \pi_1(X)$. In the latter case $G$ is a $PD^2$-group, and thus by our Theorem 2 isomorphic to $\pi_1(Y)$ where $Y$ is a closed surface of genus $\geq 1$. The isomorphism $\pi_1(X) \cong \pi_1(Y)$ yields a homotopy equivalence between $X$ and $Y$. In summary we have

THEOREM 4. *A $CW$-complex is a Poincaré 2-complex if and only if it is homotopy equivalent to a closed surface of genus* $\geq 0$.

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