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## Poincaré duality groups of dimension two, II

BENO ECKMANN and PETER LINNELL

### 1. Introduction

A Poincaré duality group of dimension  $n$ , in short a  $PD^n$ -group, is a group  $G$  acting on  $\mathbf{Z}$  such that one has natural isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbf{Z} \otimes A)$$

for all integers  $k$  and all  $\mathbf{Z}G$ -modules  $A$  (where  $\mathbf{Z} \otimes A$  is the tensor product over  $\mathbf{Z}$  with diagonal  $G$ -action).  $G$  is called orientable or not according to whether or not  $\mathbf{Z}$  is trivial as a  $\mathbf{Z}G$ -module. All “surface groups”, i.e., fundamental groups of closed surfaces of genus  $\geq 1$  are well-known to be  $PD^2$ -groups. In Eckmann–Müller [4] it was proved that a  $PD^2$ -group with *positive* first Betti number  $\beta_1$  is isomorphic to a surface group. The purpose of the present paper is to show that the condition on  $\beta_1$  is automatically fulfilled:

**THEOREM 1.** *The first Betti number  $\beta_1$  of a  $PD^2$ -group is positive.*

As a consequence we thus have a complete classification of  $PD^2$ -groups.

**THEOREM 2.** *A group  $G$  is a  $PD^2$ -group if and only if it is isomorphic to a surface group.*

For notations and properties concerning  $PD^n$ -groups, not explicitly mentioned here, we refer to [4] where also several (algebraic and topological) consequences are discussed.

### 2. Finitely generated projective $\mathbf{Z}G$ -modules

For the proof of Theorem 1 we need the following fact, which may be of interest in connection with the conjectures of Bass (4.4 and 4.5 of [2]).

If  $B$  is an abelian group, we let  $\text{rank } B$  denote the dimension of the  $\mathbf{Q}$ -vector space  $B \otimes \mathbf{Q}$ .

**PROPOSITION 3.** *Let  $G$  be a  $PD^2$ -group,  $M \neq 0$  a finitely generated projective  $\mathbf{Z}G$ -module, and  $\mathbf{Z}$  the trivial  $\mathbf{Z}G$ -module. Then  $\text{rank}(\mathbf{Z} \otimes_G M) \neq 0$ .*

*Proof.* Let  $r_M$  denote the Hattori-Stallings trace of the identity endomorphism of  $M$  as defined, e.g., in [1] and [2]. It is a finite linear combination with integral coefficients of the conjugacy classes  $\tau$  in  $G$ ,

$$r_M = \sum_{\tau} r_M(\tau) \tau.$$

For  $x \in G$  let  $r_M(x)$  be the coefficient of the conjugacy class of  $x$ . Suppose that  $r_M(x) \neq 0$  for an element  $x \in G$ ,  $x \neq 1$ . Then there exists, by Proposition 6.2 of [2], a prime  $p$  and an integer  $n > 0$  such that  $x$  is conjugate to  $x^{p^n}$ . It follows (see the remark on p. 12 of [2]) that  $x$  is contained in a subgroup  $H \cong \mathbf{Z}[1/p]$  of  $G$ . By Strebel's theorem [5] all subgroups of infinite index in  $G$  are of cohomological dimension 1 and thus free. Therefore  $H$  has finite index in  $G$ ; since  $G$  is finitely generated so is  $H$  and we have a contradiction. Hence  $r_M(x) = 0$  for all  $x \in G \setminus 1$  and it follows that  $r_M(1) = \text{rank}(\mathbf{Z} \otimes_G M)$ .

We now consider the nonzero finitely generated projective  $\mathbf{C}G$ -module  $M \otimes \mathbf{C}$ . We have  $r_M(1) = r_{M \otimes \mathbf{C}}(1)$  which is positive by Kaplansky's theorem (see [1], Theorem 8.9), and the result follows.

### 3. Proof of Theorem 1. Euler characteristic

The completion of the proof is now in the same spirit as [3]. We first note that we can restrict attention to orientable  $PD^2$ -groups. Indeed (see [4], p. 511), if  $G$  is non-orientable and  $G_1$  the orientable subgroup of index 2 in  $G$  then  $\beta_1(G_1) > 0$  implies  $\beta_1(G) > 0$ .

So let  $G$  be an orientable  $PD^2$ -group, and

$$0 \rightarrow P \rightarrow \mathbf{Z}G^d \rightarrow \mathbf{Z}G \xrightarrow{\epsilon} \mathbf{Z} \tag{1}$$

a projective resolution of the trivial  $\mathbf{Z}G$ -module  $\mathbf{Z}$ . Since  $PD^n$ -groups are of type  $(FP)$ , the module  $P$  is finitely generated projective. Since  $H^0(G; \mathbf{Z}G) = H^1(G; \mathbf{Z}G) = 0$  and  $H^2(G; \mathbf{Z}G) = \mathbf{Z}$  with trivial  $G$ -action for any orientable

$PD^2$ -group, applying  $\text{Hom}_G(-, \mathbf{Z}G)$  to (1) yields an exact sequence

$$\mathbf{Z} \xleftarrow{\gamma} P^* \leftarrow \mathbf{Z}G^d \leftarrow \mathbf{Z}G \leftarrow 0 \quad (2)$$

where  $P^* = \text{Hom}_G(P, \mathbf{Z}G)$  is finitely generated projective. Let  $IG$  be the kernel of  $\varepsilon$  (the augmentation ideal) and  $L$  the kernel of  $\gamma$ . Applying Schanuel's lemma to (1) and (2) gives

$$P^* \oplus IG \cong \mathbf{Z}G \oplus L.$$

There is a surjection  $\mathbf{Z}G^d \twoheadrightarrow L$ , and we obtain a surjection  $\mathbf{Z}G^{d+1} \twoheadrightarrow P^* \oplus IG$  and hence a surjection  $\mathbf{Z}G^{d+1} \twoheadrightarrow P^*$ , with kernel  $K \neq 0$ . Obviously  $K$  is a finitely generated projective  $\mathbf{Z}G$ -module, and we see from Proposition 3 that  $\text{rank}(\mathbf{Z} \otimes_G K) \neq 0$ . It follows that  $\text{rank}(\mathbf{Z} \otimes_G P^*) \leq d$ .

The Euler characteristic  $\chi(G)$  of  $G$  can be obtained by applying  $\mathbf{Z} \otimes_G -$  to the resolution (2) and taking the alternating sum of the ranks:

$$\chi(G) = \text{rank}(\mathbf{Z} \otimes_G P^*) - d + 1 \leq 1.$$

On the other hand  $\chi(G) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1$  since the Betti numbers  $\beta_0$  and  $\beta_2$  of an orientable  $PD^2$ -group are  $= 1$ . Thus  $2 - \beta_1 \leq 1$ , i.e.,  $\beta_1 \geq 1$ .

#### 4. Poincaré 2-complexes

As a corollary of the above group-theoretic results the topological application mentioned in [4], Section 2 can be given an improved version.

We recall that a Poincaré  $n$ -complex is a  $CW$ -complex dominated by a finite complex and fulfilling Poincaré duality of formal dimension  $n$  for arbitrary local coefficients. By results of Wall [6] a Poincaré 2-complex  $X$  with finite fundamental group  $\pi_1(X)$  is homotopy equivalent to the 2-sphere or to the real projective plane; if  $\pi_1(X)$  is infinite, then  $X$  is aspherical, i.e., an Eilenberg-MacLane complex  $K(G, 1)$  for  $G = \pi_1(X)$ . In the latter case  $G$  is a  $PD^2$ -group, and thus by our Theorem 2 isomorphic to  $\pi_1(Y)$  where  $Y$  is a closed surface of genus  $\geq 1$ . The isomorphism  $\pi_1(X) \cong \pi_1(Y)$  yields a homotopy equivalence between  $X$  and  $Y$ . In summary we have

**THEOREM 4.** *A  $CW$ -complex is a Poincaré 2-complex if and only if it is homotopy equivalent to a closed surface of genus  $\geq 0$ .*

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