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## Linking pairings on singular spaces

MARK GORESKEY and PAUL SIEGEL

### §1. Introduction

In [GM1] [GM2] [GM3] intersection homology groups  $IH_i^{\bar{p}}(X)$  were defined for any  $n$  dimensional compact oriented pseudomanifold  $X$  and any perversity  $\bar{p}$  between  $\bar{0} = (0, 0, \dots, 0)$  and  $\bar{t} = (0, 1, 2, 3, \dots)$ . Questions concerning the torsion subgroups of the intersection homology groups have arisen in three contexts:

(A) Is the torsion in  $IH_i^{\bar{p}}(X)$  dually paired with the torsion in  $IH_j^{\bar{q}}(X)$  when  $i + j = n$  and  $\bar{p} + \bar{q} = \bar{t}$ ?

(B) Does the universal coefficient formula hold for  $IH_i^{\bar{p}}(X)$ ?

(C) For a compact  $4k$  dimensional pseudomanifold  $X$  with even dimensional strata, does the determinant of the intersection pairing on  $IH_{2k}^{\bar{m}}(X)$  equal 1?

The answer to all these questions is “yes” if  $X$  is a manifold, but “no” if  $X$  is a general singular space. However, for singular spaces which are “locally  $\bar{p}$ -torsion free” the answer is “yes” to each of these questions:

**DEFINITION.** A pseudomanifold  $X$  is locally  $\bar{p}$ -torsion free if, for each stratum of  $X$ ,

$$T_{c-2-p(c)}^{\bar{p}}(L) = 0$$

where  $L$  denotes the link of that stratum,  $c$  denotes its codimension, and  $T_i^{\bar{p}}(L)$  is the torsion subgroup of  $IH_i^{\bar{p}}(L)$ .

The answer to question (A) is:

**THEOREM 4.4.** *Suppose  $X$  is a compact oriented  $n$  dimensional pseudomanifold. Then there is a canonical torsion pairing*

$$T_i^{\bar{p}}(X) \times T_{n-i-1}^{\bar{q}}(X) \rightarrow \mathbf{Q}/\mathbf{Z} \tag{*}$$

where  $\bar{q} = \bar{t} - \bar{p}$ . If  $X$  is also locally  $\bar{p}$ -torsion free then this pairing is nondegenerate.

Similarly, the answer to question (B) is:

**THEOREM 8.1.** *Suppose  $X$  is a locally  $\bar{p}$ -torsion free pseudomanifold. Let  $G$  be an abelian group. Then there is a natural exact sequence*

$$0 \rightarrow IH_i^{\bar{p}}(X) \otimes G \rightarrow IH_i^{\bar{p}}(X; G) \rightarrow \text{Tor}(IH_{i-1}^{\bar{p}}(X), G) \rightarrow 0$$

For spaces which are not locally  $\bar{p}$ -torsion free there is a new torsion group  $R_i^{\bar{p}}(X)$  which in some sense measures the degeneracy of the torsion pairing (\*), i.e. there is a sequence

$$\cdots \rightarrow T_i^{\bar{p}}(X) \rightarrow \text{Hom}(T_{n-i-1}^{\bar{q}}(X), \mathbf{Q}/\mathbf{Z}) \rightarrow R_i^{\bar{p}}(X) \rightarrow T_{i-1}^{\bar{p}}(X) \rightarrow \cdots$$

The group  $R_i^{\bar{p}}(X)$  is a topological invariant of  $X$  and is the hypercohomology of a complex of sheaves which is supported on the singular set of  $X$ .

**THEOREM 9.3.** *For any compact oriented  $n$  dimensional pseudomanifold  $X$  there is a natural nondegenerate pairing*

$$R_i^{\bar{p}}(X) \otimes R_{n-i}^{\bar{q}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}$$

where  $\bar{q} = \bar{t} - \bar{p}$ .

This pairing gives rise to a cobordism invariant characteristic class for certain singular spaces, which was first introduced in [S]:

Suppose  $X$  is a compact oriented  $4k$  dimensional pseudomanifold with even dimensional strata (or, more generally suppose  $IH_{l/2}^{\bar{m}}(L) = 0$  whenever  $L$  is the link of a stratum with odd codimensional  $c = l + 1$ ). Then we have a nondegenerate rational pairing

$$I : IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \otimes IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \rightarrow \mathbf{Q}$$

and a nondegenerate torsion pairing

$$K : R_{2k}^{\bar{m}}(X) \otimes R_{2k}^{\bar{m}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}$$

**THEOREM 11.3.** *The Witt class (in  $W(\mathbf{Q}/\mathbf{Z})$ ) of the pairing  $K$  is equal to the torsion part of the Witt class (in  $W(\mathbf{Q})$ ) of the pairing  $I$ . This characteristic class is a cobordism invariant for cobordisms with even dimensional strata.*

This result suggests that the cobordism groups of the spaces (defined in §7.1) which satisfy Poincaré duality over the integers may coincide with Mishchenko's higher Witt groups of  $\mathbf{Z}$  (see [R]).

We are grateful to R. MacPherson for several valuable conversations and in particular for his suggestion that the “peripheral complex”  $\underline{R}_i^{\bar{p}}$  should be an interesting object to study. Many of the results in this paper have been worked out independently by P. Deligne.

## §2. Notation

Our notation follows [GM2] and [GM3].  $X$  will denote an  $n$ -dimensional compact oriented piecewise linear pseudomanifold with a P.L. stratification

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-2} = \Sigma = X_{n-1} \subset X_n = X$$

such that each point  $x \in X_i - X_{i-1}$  has a neighborhood of the form  $U = (i\text{-simplex}) \times \text{cone}(L)$  where  $L$  is the *link* of the stratum containing  $x$ .

The symbol  $IH_i^{\bar{p}}(X)$  denotes the  $i$ th intersection homology group of  $X$ , with perversity  $\bar{p} = (p_2, p_3, p_4, \dots)$  where  $p_c \leq p_{c+1} \leq p_c + 1$  and  $p_2 = 0$ . This group is canonically isomorphic to the hypercohomology group  $\mathcal{H}^{-i}(\underline{IC}_{\bar{p}})$  of the complex of sheaves  $\underline{IC}_{\bar{p}}$  which was constructed by Deligne [GM3]. It does not depend on the choice of P.L. structure or on the choice of stratification of  $X$ .

## §3. Linking products in intersection homology

3.1. Let  $X$  be a compact  $n$  dimensional piecewise linear stratified pseudomanifold and suppose  $\bar{p} + \bar{q} = \bar{r}$  are perversities as in [GM2] §1. We shall define a product

$$L : T_i^{\bar{p}}(X) \times T_j^{\bar{q}}(X) \rightarrow IH_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z}) \quad (1)$$

where  $T_i^{\bar{p}}(X)$  denotes the torsion subgroup of  $IH_i^{\bar{p}}(X)$ . Let  $\xi \in IC_i^{\bar{p}}(X)$  and  $\eta \in IC_j^{\bar{q}}(X)$  be cycles which represent torsion classes  $[\xi] \in T_i^{\bar{p}}(X)$  and  $[\eta] \in T_j^{\bar{q}}(X)$ . Then there are integers  $m_1$  and  $m_2$  and chains  $\tilde{\xi} \in IC_{i+1}^{\bar{p}}(X)$ ,  $\tilde{\eta} \in IC_{j+1}^{\bar{q}}(X)$  such that  $\partial\tilde{\xi} = m_1\xi$  and  $\partial\tilde{\eta} = m_2\eta$ . We may choose  $\xi$  and  $\tilde{\xi}$  so as to be dimensionally transverse to  $\eta$  and  $\tilde{\eta}$  by [Mc].

Define  $L([\xi], [\eta])$  to be the homology class of the intersection cycle  $(1/m_1)\tilde{\xi} \cap \eta \in IC_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z})$  (see [GM2] §2.1). It is easy to check that  $L([\xi], [\eta])$  is well defined and is equal to the homology class of the cycle  $(-1)^i(1/m_2)\xi \cap \tilde{\eta}$ . Furthermore  $L([\xi][\eta]) = (-1)^{(n-i)(n-j)}L([\eta], [\xi])$ .

3.2. The torsion product for complementary dimensions ( $i+j=n-1$ ) and perversities ( $\bar{p} + \bar{q} = \bar{r} = (0, 1, 2, 3, \dots)$ ) may also be constructed by the sheaf

theoretic techniques of [GM3] from the intersection product, as follows: If  $\underline{\underline{\mathbf{D}}}_X$  denotes the dualizing complex on  $X$ , we have the product morphism

$$\underline{\underline{IC}}_{\bar{p}} \otimes \underline{\underline{IC}}_{\bar{q}} \rightarrow \underline{\underline{\mathbf{D}}}_X[n]$$

and its adjoint

$$\underline{\underline{IC}}_{\bar{p}} \rightarrow R \underline{\underline{\text{Hom}}}(\underline{\underline{IC}}_{\bar{q}}, \underline{\underline{\mathbf{D}}}_X)[n] \quad (2)$$

Applying the hypercohomology functor  $\mathcal{H}^{-i}$  and the universal coefficient theorem [B], we obtain a commuting diagram with exact columns:

$$\begin{array}{ccc} \begin{array}{c} 0 \\ \downarrow \\ T_i^{\bar{p}} \\ \downarrow \\ IH_i^{\bar{p}}(X) \\ \downarrow \\ IH_i^{\bar{p}}/T_i^{\bar{p}} \\ \downarrow \\ 0 \end{array} & \longrightarrow & \begin{array}{c} 0 \\ \downarrow \\ \text{Ext}(IH_{n-i-1}^{\bar{t}-\bar{p}}(X), \mathbf{Z}) \cong \text{Hom}(T_{n-i-1}^{\bar{t}-\bar{p}}, \mathbf{Q}/\mathbf{Z}) \\ \downarrow \\ \mathcal{H}^{-i}(X; R \underline{\underline{\text{Hom}}}(\underline{\underline{IC}}_{\bar{t}-\bar{p}}, \underline{\underline{\mathbf{D}}}_X)[n]) \\ \downarrow \\ \text{Hom}(IH_{n-i}^{\bar{t}-\bar{p}}(X), \mathbf{Z}) \\ \downarrow \\ 0 \end{array} \end{array}$$

The adjoint of the homomorphism on the top line is the desired product

$$L: T_i^{\bar{p}} \times T_{n-i-1}^{\bar{t}-\bar{p}} \rightarrow \mathbf{Q}/\mathbf{Z} \quad (3)$$

**3.3. PROPOSITION.** *The linking product (3) coincides with the augmented product (1):*

$$T_i^{\bar{p}} \times T_j^{\bar{q}} \rightarrow IH_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z}) \rightarrow H_0(\text{point}, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z} \quad (*)$$

when  $j = n - i - 1$  and  $\bar{q} = \bar{t} - \bar{p}$ .

**COROLLARY.** *If the morphism (2) is a quasi-isomorphism, then the linking pairing (\*) is nondegenerate.*

The proof of Prop. 3.3 is similar to the proof of Corollary 3.6 in [GM3] and will be omitted.

#### §4. Spaces for which the linking pairing is nondegenerate

4.1. DEFINITION. *A stratified pseudomanifold  $X$  is locally  $\bar{p}$ -torsion free if for each stratum of  $X$  we have*

$$T_{q(c)}^{\bar{p}}(L) = 0 \quad (4)$$

where  $L$  is the link of the stratum,  $c$  is the codimension and  $q(c) = c - 2 - p(c)$ .

4.2. Remark. If  $X$  is a locally  $\bar{p}$ -torsion free space and  $L$  is the link of any stratum of  $X$ , then  $L$  is also a locally  $\bar{p}$ -torsion free space.

4.3. PROPOSITION.  *$X$  is locally  $\bar{p}$ -torsion free with respect to one stratification iff the same is true with respect to any refinement of that stratification.*

*Proof.* The link  $L'$  of a stratum in the refinement has the form of a join,  $L' = S^k * L$  where  $L$  is the link of a stratum in the original stratification. We must verify that

$$T_r^{\bar{p}}(L') = 0 \quad (4')$$

where  $r = l + k - 1 - p(l + k + 1)$  and  $l = \dim(L)$ . For  $k = 0$ ,  $L'$  is the suspension of  $L$  and

$$IH_i^{\bar{p}}(\Sigma L) = \begin{cases} IH_{i-1}^{\bar{p}}(L) & \text{if } i > l - p(l) - 1 \\ 0 & \text{if } i = l - p(l) - 1 \\ IH_i^{\bar{p}}(L) & \text{if } i < l - p(l) - 1 \end{cases}$$

There are three possibilities:  $p(l+2) = p(l)$ ,  $p(l+2) = p(l) + 1$ , or  $p(l+2) = p(l) + 2$ . In each case one calculates  $T_r^{\bar{p}}(L) = 0$  assuming (4) holds.

For  $k > 0$ , equation (4') may be verified by repeated application of the case  $k = 0$ . Q.E.D.

4.4. THEOREM. *Suppose  $X$  is a compact  $n$  dimensional piecewise linear stratified pseudomanifold which is locally  $\bar{p}$ -torsion free. Then the morphism (2) is a quasi-isomorphism, so the linking pairing (\*) is non-singular.*

Theorem 1 depends on a result from homological algebra which we now describe.

## §5. Truncation of complexes

5.1. If  $C^\cdot$  is a (cochain) complex of free abelian groups,

$$\dots \xrightarrow{d} C^a \xrightarrow{d} C^{a+1} \xrightarrow{d} C^{a+2} \xrightarrow{d} \dots$$

we denote by  $\text{Hom}^\cdot(C^\cdot, \mathbf{Z})$  the dual complex,

$$\text{Hom}^b(C^\cdot, \mathbf{Z}) = \text{Hom}(C^{-b}, \mathbf{Z}).$$

Deligne has defined truncation functors ([D1] [D2] [GM3])

$$(\tau_{\leq a} C^\cdot)^n = \begin{cases} 0 & \text{if } n > a \\ \ker d & \text{if } n = a \\ C^n & \text{if } n < a \end{cases}$$

$$(\tau^{\geq a} C^\cdot)^n = \begin{cases} C^n & \text{if } n > a \\ \text{coker } d^{a-1} & \text{if } n = a \\ 0 & \text{if } n < a \end{cases}$$

It is easy to verify the following facts from homological algebra:

5.2. PROPOSITION. *Let  $C^\cdot$  be a complex of free abelian groups. Then the following natural sequence is split exact*

$$0 \rightarrow \text{Ext}(H^{-m+1}(C^\cdot), \mathbf{Z}) \rightarrow H^m(\text{Hom}^\cdot(C^\cdot, \mathbf{Z})) \rightarrow \text{Hom}(H^{-m}(C^\cdot), \mathbf{Z}) \rightarrow 0$$

Consequently,

$$H^a \text{Hom}(\tau^{\geq b} C^\cdot, \mathbf{Z}) = \begin{cases} 0 & \text{if } a \geq -b + 2 \\ \text{Ext}(H^b(C^\cdot), \mathbf{Z}) & \text{if } a = -b + 1 \\ H^a \text{Hom}^\cdot(C^\cdot, \mathbf{Z}) & \text{if } a \leq -b \end{cases}$$

$$H^a \text{Hom}(\tau_{\leq b} C^\cdot, \mathbf{Z}) = \begin{cases} H^a(\text{Hom}^\cdot(C^\cdot, \mathbf{Z})) & \text{if } a \geq -b + 1 \\ \text{Hom}(H^b(C^\cdot), \mathbf{Z}) & \text{if } a = -b \\ 0 & \text{if } a \leq -b - 1 \end{cases}$$

(The same result holds if we drop the hypothesis that  $C^\cdot$  is free, and replace  $\mathbf{Z}$  by its injective resolution  $\mathbf{Q} \rightarrow \mathbf{Q}/\mathbf{Z}$ ).

### §6. Proof of Theorem 4.4

Let us assume by induction that the theorem has been proven for pseudomanifolds of dimension  $\leq n-1$ . Let  $X$  be a pseudomanifold of dimension  $n$ . The morphism (2) is clearly a quasi-isomorphism over the nonsingular part of  $X$ , so it suffices to verify that the complex of sheaves

$$\underline{\underline{S}}^\bullet = R \underline{\underline{\text{Hom}}}^\bullet(\underline{\underline{IC}}_{\bar{q}}^\bullet, \underline{\underline{D}}_X^\bullet)[n]$$

satisfies the axioms [AX1] of [GM3] §3.3, since these axioms uniquely determine  $\underline{\underline{IC}}_p^\bullet$ . Specifically, we will verify the support axioms

$$\begin{aligned} \text{[AX1](c)} \quad \underline{\underline{H}}^m(\underline{\underline{S}}^\bullet | U_{k+1}) &= 0 && \text{for all } m > p(k) - n \\ \text{[AX1](d')} \quad \underline{\underline{H}}^m(j_k^! \underline{\underline{S}}^\bullet | U_{k+1}) &= 0 && \text{for all } m \leq p(k) - n + 1 \end{aligned}$$

where  $j_k: Y = X_{n-k} - X_{n-k-1} \rightarrow U_{k+1} = X - X_{n-k-1}$  is the closed inclusion of the stratum with codimension  $k$  into  $U_{k+1}$ .

*Verification of [AX1](d').* Let  $\underline{\underline{D}}_Y^\bullet$  and  $\underline{\underline{D}}_{k+1}^\bullet$  denote the dualizing complexes of  $Y$  and  $U_{k+1}$  respectively. Then

$$\begin{aligned} j_k^! \underline{\underline{S}}^\bullet &\cong \text{dual } j_k^* \text{ dual } \underline{\underline{S}}^\bullet \\ &\cong R \underline{\underline{\text{Hom}}}^\bullet(j_k^* \underline{\underline{IC}}_{\bar{q}}^\bullet, \underline{\underline{D}}_Y^\bullet)[n] \\ &\cong R \underline{\underline{\text{Hom}}}^\bullet(j_k^* \underline{\underline{IC}}_{\bar{q}}^\bullet[k-2n], \underline{\underline{Z}}_Y) \end{aligned}$$

These complexes are cohomologically locally constant on  $Y$ , so the stalk of the sheaf  $R \underline{\underline{\text{Hom}}}$  is the  $R \text{ Hom}$  of the stalks. Let  $j_y$  denote the inclusion of a point  $y \in Y$ . Then the stalk cohomology at  $y$  is

$$\begin{aligned} \underline{\underline{H}}^m(j_k^! \underline{\underline{S}}^\bullet)_y &= \text{Hom}(j_y^* \underline{\underline{IC}}_{\bar{q}}^\bullet[k-2n], \mathbf{Z}) \\ &\cong \text{Ext}(H^{-m+1}(j_y^* \underline{\underline{IC}}_{\bar{q}}^\bullet[k-2n], \mathbf{Z})) \oplus \text{Hom}(H^{-m}(j_y^* \underline{\underline{IC}}_{\bar{q}}^\bullet[k-2n], \mathbf{Z})) \\ &= 0 \quad \text{whenever } -m+k-2n > q(k)-n \text{ by [AX1](c) for } \underline{\underline{IC}}_{\bar{q}}^\bullet. \end{aligned}$$

This holds if  $m \leq p(k) - n + 1$ . (This verification did *not* use the assumption  $T_{q(k)}^p(L) = 0$ ).

*Verification of [AX1](c).* We shall show the stalk cohomology over points  $y \in Y$  satisfies the required vanishing condition

$$\begin{aligned} j_k^* \underline{\underline{S}}^\bullet | U_{k+1} &\cong R \underline{\underline{\text{Hom}}}^\bullet(j_k^! \underline{\underline{IC}}_{\bar{q}}^\bullet, \underline{\underline{D}}_Y^\bullet)[n] \\ &\cong R \underline{\underline{\text{Hom}}}^\bullet(j_k^! \underline{\underline{IC}}_{\bar{q}}^\bullet, \underline{\underline{Z}}_Y)[2n-c] \\ &\cong R \underline{\underline{\text{Hom}}}^\bullet(\tau^{\geq q(k-n)+1} j_k^* Ri_* i^* \underline{\underline{IC}}_{\bar{q}}^\bullet[-1], \mathbf{Z})[2n-c] \end{aligned}$$

because of the distinguished triangle

$$\begin{array}{ccc} Rj_k * j_k^! \underline{\underline{IC}}_{\bar{q}} | U_{k+1} & \longrightarrow & \underline{\underline{IC}}_{\bar{q}} | U_{k+1} = \tau_{\leq q(k)-n} Ri_* i^* \underline{\underline{IC}}_{\bar{q}} \\ & \begin{array}{c} \uparrow [1] \searrow \\ \downarrow \end{array} & \\ & Ri_* i^* \underline{\underline{IC}}_{\bar{q}} | U_{k+1} & \end{array}$$

(Here,  $i: U_k \rightarrow U_{k+1}$  is the inclusion.)

Thus the stalk at a point  $y \in Y$  of  $H^m(S')$  is

$$\underline{\underline{H}}^m(\underline{\underline{S}}_y) = \begin{cases} H^{m+2n-c-1}(\text{Hom}(j_y^* i_* i^* \underline{\underline{IC}}_{\bar{q}}, \mathbf{Z})) & \text{if } m \leq p(k) - n \\ \text{Ext}(H^{q-n+1}(j_y^* i_* i^* \underline{\underline{IC}}_{\bar{q}}), \mathbf{Z}) & \text{if } m = p(k) - n + 1 \\ 0 & \text{if } m > p(k) - n + 1 \end{cases}$$

by Proposition 5.2.

Therefore axiom (c) will be satisfied iff the following group vanishes:

$$\text{Ext}(H^{q-n+1}(j_y^* i_* i^* \underline{\underline{IC}}_{\bar{q}}), \mathbf{Z}).$$

This is isomorphic to  $\text{Ext}(IH_{k-2-q(k)}^{\bar{q}}(L), \mathbf{Z})$  by [GM3] §2.2. However the link  $L$  of the stratum  $Y$  is a  $k-1$  dimensional pseudomanifold which is locally  $\bar{q}$ -torsion free (see Remark 4.2) so the theorem applies to  $L$  by induction and

$$T_{c-2-q(c)}^{\bar{q}}(L) \text{ is } \mathbf{Q}/\mathbf{Z}\text{-dual to } T_{q(c)}^{\bar{p}}(L)$$

which is 0 by assumption.

## §7. Spaces which satisfy Poincaré duality

In this section we describe a class of spaces such that the intersection homology group with middle perversity  $\bar{m} = (0, 0, 1, 1, 2, 2, \dots)$  satisfies Poincaré duality over the integers.

7.1. THEOREM. *Suppose the compact oriented  $n$  dimensional pseudo-manifold  $X$  satisfies the following two conditions:*

(a) *For each stratum of odd codimension  $c$ ,*

$$IH_{(c-1)/2}^{\bar{m}}(L; \mathbf{Z}) = 0$$

(b) For each stratum of even codimension  $c$ ,

$$T_{c/2-1}^{\bar{m}}(L) = 0$$

where  $L$  is the link of the stratum in question. Then there is a canonical split exact sequence

$$0 \rightarrow \text{Hom}(T_{i-1}^{\bar{m}}(X), \mathbf{Q}/\mathbf{Z}) \rightarrow IH_{n-i}^{\bar{m}}(X; \mathbf{Z}) \rightarrow \text{Hom}(IH_i^{\bar{m}}(X; \mathbf{Z}), \mathbf{Z}) \rightarrow 0$$

which is compatible with the intersection pairing and the linking pairing.

*Proof.* Conditions (a) and (b) guarantee (by Theorem 4.4) that the morphism induced by the product

$$\underline{IC}_{\bar{m}} \rightarrow R \underline{\text{Hom}}(\underline{IC}_{\bar{n}}, \underline{D}_{\bar{x}})[n]$$

is a quasi-isomorphism, and condition (a) guarantees that the natural map  $\underline{IC}_{\bar{m}} \rightarrow \underline{IC}_{\bar{n}}$  is a quasi-isomorphism (see the obstruction sequence argument in [S] or [GM3] §5.6). Together they imply that  $\underline{IC}_{\bar{m}}$  is self dual. The exact sequence is the universal coefficient theorem of [8].

**7.2. Remarks.** If  $X$  satisfies properties (a) or (b) above, with respect to one stratification then it satisfies the same properties with respect to any refinement of the stratification.

## §8. Change of coefficients in intersection homology

In this section we will assume  $G$  is an abelian group. Recall that  $IH_i^{\bar{p}}(X; G)$  is defined to be the  $i$ th homology group of the complex of chains  $IC_i^{\bar{p}}(X; C)$  which consists of those  $\xi \in C_i(X) \otimes G$  such that  $|\xi|$  is  $(\bar{p}, i)$ -allowable and  $|\partial\xi|$  is  $(\bar{p}, i-1)$ -allowable ([GM2] §6.3).

**8.1. THEOREM.** *Suppose  $X$  is a P.L. stratified pseudomanifold and for each stratum of  $X$ ,*

$$\text{Tor}(IH_{q(c)}^{\bar{p}}(L), G) = 0$$

where  $L$  is the link of the stratum,  $c$  is its codimension, and  $q(c) = c - 2 - p(c)$ . Then

there is a canonical exact sequence

$$0 \rightarrow IH_i^{\bar{p}}(X) \otimes G \rightarrow IH_i^{\bar{p}}(X; G) \rightarrow \text{Tor}(IH_{i-1}^{\bar{p}}(X), G) \rightarrow 0$$

which is split.

*Remark.* If  $X$  is locally  $\bar{p}$ -torsion free then the hypothesis holds for any abelian group  $G$ .

8.2. *Proof.*  $IH_i^{\bar{p}}(X; G)$  is the hypercohomology group  $\mathcal{H}^{-i}(\underline{IC}_{\bar{p}}^{\cdot}(G))$  of the complex of sheaves which is obtained by applying Deligne's construction to the constant sheaf  $G$  on  $X - \Sigma$ . We shall show that  $\underline{IC}_{\bar{p}}^{\cdot}(G)$  and  $\underline{IC}_{\bar{p}}^{\cdot} \otimes G$  are quasi-isomorphic under the hypotheses of the theorem. (The short exact sequence is then the statement of the universal coefficient theorem for the complex  $\underline{IC}_{\bar{p}}^{\cdot} \otimes G$ ).

The quasi-isomorphism is obtained by verifying the axioms [AX1] for the complex  $\underline{IC}_{\bar{p}}^{\cdot} \otimes G$ . Since the verification is analogous to that in §6, we omit it here but remark that the relevant lemma from homological algebra is the following:

LEMMA. Let  $C^{\cdot}$  be a chain complex of free abelian groups. Then

$$H^n_{\tau_{\leq a}}(C^{\cdot} \otimes G) = \begin{cases} 0 & \text{for } n > a \\ H^n[(\tau_{\leq a} C^{\cdot}) \otimes G] \oplus \text{Tor}(H^{n+1}(C^{\cdot}), G) & \text{for } n = a \\ H^n[(\tau_{\leq a} C^{\cdot}) \otimes G] & \text{for } n < a \end{cases}$$

### §9. The peripheral complex $\underline{R}_{\bar{p}}^{\cdot}$

9.1. Let  $X$  be an  $n$  dimensional compact oriented pseudomanifold.

DEFINITION.  $\underline{R}_{\bar{p}}^{\cdot}$  is the (algebraic) mapping cone of the morphism (2).  $R_i^{\bar{p}}(X)$  is the hypercohomology group  $\mathcal{H}^{-i}(\underline{R}_{\bar{p}}^{\cdot})$ .

9.2. *Remarks.* (1) We have a distinguished triangle in  $D^b(X)$ ,

$$\begin{array}{ccc} \underline{IC}_{\bar{p}}^{\cdot} & \xrightarrow{\quad} & R \underline{\text{Hom}}^{\cdot}(\underline{IC}_{\bar{q}}^{\cdot}, \underline{D}_X^{\cdot})[n] \\ & \swarrow \scriptstyle [1] & \searrow \\ & \underline{R}_{\bar{p}}^{\cdot} & \end{array}$$

Thus,  $X$  is locally  $\bar{p}$ -torsion free if and only if  $\underline{R}_{\bar{p}}^{\cdot} \cong 0$ .

(2) The cohomology sheaves associated to  $\underline{R}_{\bar{p}}^{\dot{}}$  are supported on the singular set of  $X$  since the morphism (2) is a quasi-isomorphism over the nonsingular part of  $X$ .

(3) The hypercohomology groups  $R_i^{\bar{p}}(X) = \mathcal{H}^{-i}(\underline{R}_{\bar{p}}^{\dot{}})$  are torsion groups since the morphism (2) becomes a quasi-isomorphism when both sides are tensored with the rationals (see [GM3] §5.3). Thus, the torsion sub-groups of the hypercohomology groups of the complexes in the above triangle can be identified as follows:

$$\cdots \longrightarrow T_i^{\bar{p}} \longrightarrow \text{Hom}(T_{n-i-1}^{\bar{q}}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\alpha_i} R_i^{\bar{p}}(X) \xrightarrow{\beta_i} T_{i-1}^{\bar{p}} \longrightarrow \cdots$$

This sequence is exact except at  $R_i^{\bar{p}}(X)$  (see diag. 11.3).

9.3. PROPOSITION. *There is a canonical nondegenerate pairing*

$$K: R_i^{\bar{p}}(X) \times R_{n-i}^{\bar{q}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}$$

such that  $K(\alpha_i(a), b) = a \cdot \beta_{n-i}(b)$  for all  $a \in \text{Hom}(T_{n-i-1}^{\bar{q}}, \mathbf{Q}/\mathbf{Z})$  and all  $b \in R_{n-i}^{\bar{q}}$ .

*Proof.* First we define  $K$ . The intersection product ([GM3] §5.2)

$$\underline{IC}_{\bar{p}}^{\dot{}} \otimes \underline{IC}_{\bar{q}}^{\dot{}} \rightarrow \underline{D}_{\dot{X}}[n]$$

induces a pair of adjoint morphisms

$$\phi_1: \underline{IC}_{\bar{p}}^{\dot{}} \rightarrow R \underline{\text{Hom}}^*(\underline{IC}_{\bar{p}}^{\dot{}}, \underline{D}_{\dot{X}})[n]$$

$$\phi_2: \underline{IC}_{\bar{q}}^{\dot{}} \rightarrow R \underline{\text{Hom}}^*(\underline{IC}_{\bar{p}}^{\dot{}}, \underline{D}_{\dot{X}})[n]$$

and  $\underline{R}_{\bar{p}}^{\dot{}} = \text{mapping cone}(\phi_1)$ ;  $\underline{R}_{\bar{q}}^{\dot{}} = \text{mapping cone}(\phi_2)$ . Dualizing  $\phi_2$  gives rise to a pair of distinguished triangles

$$\begin{array}{ccccc}
 & & \underline{R}_{\bar{p}}^{\dot{}} & & \\
 & \swarrow [1] & \vdots & \searrow & \\
 \underline{IC}_{\bar{p}}^{\dot{}} & \xrightarrow{\phi_1} & R \underline{\text{Hom}}^*(\underline{IC}_{\bar{q}}^{\dot{}}, \underline{D}_{\dot{X}})[n] & & \\
 \downarrow \text{biduality} \cong & & \downarrow & & \downarrow \cong \text{identity} \\
 \text{isomorphism} & & R \underline{\text{Hom}}^*(\underline{R}_{\bar{q}}^{\dot{}}, \underline{D}_{\dot{X}})[n] & & \\
 & \swarrow & \downarrow [1] & \searrow & \\
 R \underline{\text{Hom}}^*(R \underline{\text{Hom}}^*(\underline{IC}_{\bar{p}}^{\dot{}}, \underline{D}_{\dot{X}}), \underline{D}_{\dot{X}}) & \xrightarrow{\phi_2^*} & R \underline{\text{Hom}}^*(\underline{IC}_{\bar{q}}^{\dot{}}, \underline{D}_{\dot{X}})[n] & & 
 \end{array}$$

From this diagram we obtain a quasi-isomorphism,

$$\underline{R}_{\bar{p}} \rightarrow R \underline{\text{Hom}}'(\underline{R}_{\bar{q}}, \underline{D}_X)[n+1]$$

Applying hypercohomology and the universal coefficient theorem we obtain an isomorphism

$$\tilde{K} : R_i^{\bar{p}}(X) \rightarrow \text{Hom}(R_{n-i}^{\bar{q}}(X), \mathbf{Q}/\mathbf{Z})$$

whose adjoint is the desired  $K : R_i^{\bar{p}}(X) \otimes R_{n-i}^{\bar{q}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}$ .

The compatibility of  $K$  with  $\alpha$  and  $\beta$  is equivalent to the statement that the following diagram commutes:

$$\begin{array}{ccc} R_i^{\bar{p}}(X) & \xleftarrow{\alpha} & \text{Hom}(T_{n-i-1}^{\bar{q}}, \mathbf{Q}/\mathbf{Z}) \\ \tilde{K} \downarrow & & \downarrow \text{identity} \\ \text{Hom}(R_{n-i}^{\bar{q}}(X), \mathbf{Q}/\mathbf{Z}) & \xleftarrow{\beta^*} & \text{Hom}(T_{n-i-1}^{\bar{q}}, \mathbf{Q}/\mathbf{Z}) \end{array}$$

However this diagram is simply the torsion in the hypercohomology of the right hand face of the preceding diagram.

9.4. EXAMPLE. If the singular set of  $X$  consists of a single stratum  $\Sigma$  of codimension  $c$ , then the stalk homology of  $\underline{R}_{\bar{p}}$  is, for any  $x \in \Sigma$ ,

$$\mathcal{H}^{-i}(\underline{R}_{\bar{p}})_x = \begin{cases} T_{c-2-p(c)}(L) & \text{if } i = n - p(c) - 1 \\ 0 & \text{if } i \neq n - p(c) - 1 \end{cases}$$

where  $L$  is the link of the stratum.

If  $X$  is obtained from an  $n$ -dimensional manifold  $M$  by attaching the cone on its boundary  $\partial M$ , then

$$R_i^{\bar{m}}(X) = \begin{cases} T_{[(n-1)/2]}(\partial M) & \text{if } i = [(n+1)/2] \\ 0 & \text{if } i \neq [(n+1)/2] \end{cases}$$

where  $[ \ ]$  denotes the integer part. If  $\dim(M) = 4k$  then the equivalence class in the Witt ring  $W(\mathbf{Q}/\mathbf{Z})$  of the torsion pairing

$$T_{2k-1}^{\bar{m}}(\partial M) \times T_{2k-1}^{\bar{m}}(\partial M) \rightarrow \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* in [AHV].

**§10. Spaces for which the peripheral complex is self dual**

The canonical morphism  $\underline{IC}_{\bar{p}} \rightarrow \underline{IC}_{\bar{q}}$  (where  $\bar{p} \leq \bar{q}$ ) induces canonical morphisms on the peripheral complexes,  $\underline{R}_{\bar{p}} \rightarrow \underline{R}_{\bar{q}}$ . The spaces for which  $\underline{R}_{\bar{m}}$  is self dual over  $\mathbf{Q}/\mathbf{Z}$  are the spaces such that  $\underline{R}_{\bar{m}} \rightarrow \underline{R}_{\bar{n}}$  is a quasi-isomorphism.

10.1. THEOREM. Suppose  $X$  is a compact oriented  $n$  dimensional pseudomanifold such that, for each stratum with odd codimension  $c$ ,

$$IH_{(c-1)/2}^{\bar{m}}(L; \mathbf{Z}) = 0$$

where  $L$  is the link of the stratum in question. Then  $\underline{R}_{\bar{m}} \rightarrow \underline{R}_{\bar{n}}$  is a quasi-isomorphism so  $K$  induces a nondegenerate product

$$K : R_{\bar{i}}^{\bar{m}}(X) \times R_{\bar{n}-i}^{\bar{m}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}.$$

*Proof.* The assumption implies  $\underline{IC}_{\bar{m}} \rightarrow \underline{IC}_{\bar{n}}$  and  $R \underline{Hom}(\underline{IC}_{\bar{n}}, \underline{D}_X) \rightarrow R \underline{Hom}(\underline{IC}_{\bar{m}}, \underline{D}_X)$  are quasi-isomorphisms (see [S] or [GM3] §5.6). Therefore the induced map  $\underline{R}_{\bar{m}} \rightarrow \underline{R}_{\bar{n}}$  is also a quasi-isomorphism. Q.E.D.

10.2 DEFINITION. If  $X$  is a  $4k$  dimensional space which satisfies the hypotheses of Theorem 9.1 then the equivalence class in  $W(\mathbf{Q}/\mathbf{Z})$  of the pairing

$$K : R_{2k}^{\bar{m}}(X) \times R_{2k}^{\bar{m}}(X) \rightarrow \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* of  $X$ . (Here  $W(\mathbf{Q}/\mathbf{Z})$  is the Witt ring of  $\mathbf{Q}/\mathbf{Z}$  and it consists of certain equivalence classes of symmetric  $\mathbf{Q}/\mathbf{Z}$ -valued pairings on finite abelian groups [MH]).

**§11. Relation between the Witt class and the peripheral invariant**

11.1. DEFINITION. An oriented pseudomanifold  $X$  is a rational Witt space if, for each stratum of  $X$  with odd codimension  $c$ ,

$$IH_{(c-1)/2}^{\bar{m}}(L; \mathbf{Q}) = 0$$

where  $L$  is the link of that stratum. If  $\dim(X) = 4k$  define  $w(X) \in W(\mathbf{Q})$  to be the equivalence class (in the Witt ring of  $\mathbf{Q}$ ) of the nondegenerate symmetric

intersection pairing

$$IH_{2k}^{\overline{m}}(X; \mathbf{Q}) \times IH_{2k}^{\overline{m}}(X; \mathbf{Q}) \rightarrow \mathbf{Q}.$$

We recall the following fact from [S]:

**THEOREM.** *If  $X$  is a rational Witt space then  $w(X)$  is a cobordism invariant (for cobordisms which are also rational Witt spaces). The association  $X \mapsto w(x)$  determines an isomorphism.*

$$\Omega_{4k}^{\text{Witt}} \cong W(\mathbf{Q})$$

$$\Omega_j^{\text{Witt}} = 0 \quad \text{if } j \not\equiv 0 \pmod{4}.$$

11.2. The structure of  $W(\mathbf{Q})$  is given by the following split exact sequence [MH]:

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(\mathbf{Z}) & \longrightarrow & W(\mathbf{Q}) & \xrightarrow{\delta} & W(\mathbf{Q}/\mathbf{Z}) \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathbf{Z} & & \bigoplus_{p \text{ prime}} W(\mathbf{Z}/p\mathbf{Z}) & & \end{array}$$

11.3. **THEOREM.** *Suppose  $X$  is a  $4k$  dimensional oriented pseudomanifold which satisfies the hypothesis of §9.1, i.e.,*

$$IH_{(c-1)/2}^{\overline{m}}(L; \mathbf{Z}) = 0$$

*whenever  $L$  is the link of a stratum with odd codimension,  $c$ . Then  $X$  is a rational Witt space, and  $\delta w(X)$  is equal to the peripheral invariant of  $X$ .*

*Proof.* Consider the exact sequence on hypercohomology which is associated to the distinguished triangle of §9.2:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & T_{2k}^{\overline{m}}(X) & \longrightarrow & \text{Hom}(T_{2k-1}^{\overline{n}}, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\alpha} & R_{2k}^{\overline{m}} & \xrightarrow{\beta} & T_{2k-1}^{\overline{m}}(X) & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & \\ \cdots & \longrightarrow & IH_{2k}^{\overline{m}}(X) & \longrightarrow & IH_{2k}^{2k}(X) & \longrightarrow & R_{2k}^{\overline{m}} & \longrightarrow & IH_{2k-1}^{\overline{m}}(X) & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & IH_{2k}^{\overline{m}}/T_{2k}^{\overline{m}} & \xrightarrow{\theta} & \text{Hom}(IH_{2k}^{\overline{n}}, \mathbf{Z}) & \longrightarrow & 0 & \longrightarrow & IH_{2k-1}^{\overline{m}}/T_{2k-1}^{\overline{m}} & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & & 0 & & 0 & & 0 & & 0 & \end{array}$$

By [AHV] (Lemma 1.4),  $\delta w(X)$  coincides with the Witt class of the induced pairing on  $K = \text{coker}(\theta)$ . However,  $K \cong \ker \beta / \text{Im } \alpha$  since we may view the diagram above as a short exact sequence of chain complexes with the middle complex acyclic. But we have already shown (§9.4) that  $(\ker \beta) = (\text{Im } \alpha)^\perp$  so by [AHV] (Lemma 1.3), the Witt class of the pairing on  $R_{2k}^{\overline{m}}(X)$  also coincides with the Witt class of the pairing on  $\ker \beta / \text{Im } \alpha$  Q.E.D.

*Remark.* The diagram and preceding argument may be found in [BM] in the case that  $X$  has isolated singularities.

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