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Lifting idempotents and Clifford theory

JACQUES THÉVENAZ

Let N be a normal subgroup of a finite group G and let R be a noetherian complete local commutative ring. Clifford theory deals with the relationship between RG -modules and RN -modules, using induction from N to G or restriction from G to N . Since Clifford's 1937 paper [1], the theory is well understood for irreducible representations (see also [2, §11C]). For an indecomposable RN -module W , several authors have proved a going-up theorem describing how $\text{Ind}_N^G W$ decomposes (see [2, §19C]).

One purpose of this paper is to prove (in Section 2) a going-down theorem for indecomposable modules (analogous to Clifford's theorem), based on a refinement of the lifting idempotents theorem, presented in Section 1. The going-up and going-down theorems are actually equivalent in the sense that each can be derived as a corollary to the other one. One main assumption is necessary for the going-down theorem: the RG -module we start from must be projective relative to H . The whole procedure is presented in the more general context of Clifford systems. The paper concludes in Section 3 with another application of the lifting idempotents theorem, concerning the behaviour of indecomposable modules under ground ring extensions.

1. Lifting idempotents

THEOREM 1. *Let A be a ring and J a two-sided ideal contained in $\text{Rad } A$. Assume that A is complete in the J -adic topology (that is the natural map $A \rightarrow \varprojlim A/J^n$ is an isomorphism). Let Π be a finite group acting on A by automorphisms leaving J globally invariant. Let $\{\bar{e}_1, \dots, \bar{e}_n\}$ be a set of orthogonal idempotents of $\bar{A} = A/J$ satisfying $\sum_{i=1}^n \bar{e}_i = 1$. Assume the following three conditions:*

- a) *The induced action of Π on \bar{A} permutes the idempotents \bar{e}_i transitively.*
- b) *There exists $u \in A$ such that $\text{Tr}_\Omega(u) = 1$ where Ω is the stabilizer of \bar{e}_1 and $\text{Tr}_\Omega(u) = \sum_{\omega \in \Omega} \omega u$.*
- c) *\bar{u} commutes with each \bar{e}_i .*

Then $\{\bar{e}_1, \dots, \bar{e}_n\}$ lifts to a set $\{e_1, \dots, e_n\}$ of orthogonal idempotents of A which are permuted by Π transitively and such that $\sum_{i=1}^n e_i = 1$.

Remarks. 1) If A is the ring of endomorphisms of a representation V , we shall see that the condition b) corresponds to a condition of relative projectivity for V .

2) There are two situations where c) is always satisfied: either the idempotents \bar{e}_i are central or the order $|\Omega|$ of Ω is invertible in A in which case one can choose u to be the central element $|\Omega|^{-1}$.

3) When Π acts regularly on the idempotents \bar{e}_i , that is when Ω is trivial, one can take $u = 1$ so that b) and c) are trivially satisfied. This special case appears already in [3].

Proof. It suffices to prove the theorem when J is nilpotent because, since $A \cong \varprojlim A/J^n$, the lifted idempotents are constructed as limits of idempotents of A/J^n for $n \rightarrow \infty$.

For $\sigma \in \Pi$, write $\bar{e}_\sigma = \sigma \bar{e}_1$ so that $\bar{e}_\sigma = \bar{e}_\tau$ if and only if $\sigma\Omega = \tau\Omega$. Since Π acts transitively, every idempotent \bar{e}_i can be written in that form.

We proceed by induction on the nilpotent index n of J . There is nothing to prove if $n = 1$. If $n \geq 2$, let $I = J^{n-1}$ and write \tilde{a} for the image of $a \in A$ modulo I . By induction, there exist idempotents \tilde{e}_σ of A/I such that $\sigma \tilde{e}_\tau = \tilde{e}_{\sigma\tau}$ and $\sum_{\sigma \in \Pi/\Omega} \tilde{e}_\sigma = 1$. First lift arbitrarily the idempotents \tilde{e}_σ to get orthogonal idempotents e_σ of A satisfying $\sum_{\sigma \in \Pi/\Omega} e_\sigma = 1$. This is well known to be possible (see [2, §6A]). Of course the notation implies that we keep the convention:

$$e_\sigma = e_\tau \quad \text{if and only if} \quad \sigma\Omega = \tau\Omega.$$

Since $\sigma \tilde{e}_\tau = \tilde{e}_{\sigma\tau}$, we have:

$$\sigma e_\tau = e_{\sigma\tau} + r_{\sigma,\tau} \quad \text{for some} \quad r_{\sigma,\tau} \in I.$$

We list several properties of the elements $r_{\sigma,\tau}$:

$$(1) \quad \text{If } \omega \in \Omega, r_{\sigma,\tau\omega} = r_{\sigma,\tau}.$$

This follows from $e_{\eta\omega} = e_\eta$ for all $\eta \in \Pi$.

$$(2) \quad \sum_{\tau \in \Pi/\Omega} r_{\sigma,\tau} = 0.$$

This follows when σ is applied to $1 = \sum_{\tau \in \Pi/\Omega} e_\tau$.

$$(3) \quad \eta r_{\sigma,\tau} = r_{\eta\sigma,\tau} - r_{\eta,\sigma\tau}.$$

This is a consequence of $(\eta\sigma)e_\tau = \eta(\sigma e_\tau)$.

$$(4) \quad r_{\sigma,\tau} = e_{\sigma\tau} r_{\sigma,\tau} + r_{\sigma,\tau} e_{\sigma\tau}.$$

This follows from the equality $\sigma e_\tau = (\sigma e_\tau)^2$ using also $I^2 = 0$. Multiplying (4) by e_η on the right or e_λ on the left (or both in the first case below), we get:

$$(5) \quad e_\lambda r_{\sigma,\tau} e_\eta = \begin{cases} 0 & \text{if } \lambda\Omega \neq \sigma\tau\Omega \neq \eta\Omega \\ r_{\sigma,\tau} e_\eta & \text{if } \lambda\Omega = \sigma\tau\Omega \neq \eta\Omega \\ e_\lambda r_{\sigma,\tau} & \text{if } \lambda\Omega \neq \sigma\tau\Omega = \eta\Omega \\ 0 & \text{if } \lambda\Omega = \sigma\tau\Omega = \eta\Omega \end{cases}$$

(6) If $\lambda\Omega \neq \eta\Omega$, $e_{\mu\lambda} r_{\mu,\eta} + r_{\mu,\lambda} e_{\mu\eta} = 0$.

This is a consequence of $(\mu e_\lambda) \cdot (\mu e_\eta) = 0$ using again $I^2 = 0$.

Now define: $f_\sigma = e_\sigma + \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} \cdot e_\lambda \cdot \lambda u$ where $u \in A$ satisfies hypotheses b) and c). By (1), we have:

$$(7) \quad f_{\sigma\omega} = f_\sigma \quad \text{if } \omega \in \Omega.$$

$$(8) \quad \sum_{\sigma \in \Pi/\Omega} f_\sigma = 1.$$

For

$$\begin{aligned} \sum_{\sigma \in \Pi/\Omega} f_\sigma &= \sum_{\sigma \in \Pi/\Omega} e_\sigma + \sum_{\sigma \in \Pi/\Omega} \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} \cdot e_\lambda \cdot \lambda u \\ &= 1 + \sum_{\lambda \in \Pi} \left(\sum_{\sigma \in \Pi/\Omega} r_{\lambda,\lambda^{-1}\sigma} \right) e_\lambda \cdot \lambda u = 1 \quad \text{by (2).} \end{aligned}$$

$$(9) \quad f_\sigma f_\tau = 0 \quad \text{if } \sigma\Omega \neq \tau\Omega.$$

$$f_\sigma f_\tau = \sum_{\lambda \in \Pi} e_\sigma r_{\lambda,\lambda^{-1}\tau} e_\lambda \cdot \lambda u + \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} e_\lambda \cdot \lambda u \cdot e_\tau.$$

By hypothesis c), $\lambda \bar{u} \cdot \bar{e}_\tau = \lambda(\bar{u} \cdot \bar{e}_{\lambda^{-1}\tau}) = \lambda(\bar{e}_{\lambda^{-1}\tau} \cdot \bar{u}) = \bar{e}_\tau \cdot \lambda \bar{u}$. Hence λu commutes with e_τ modulo J . Since $I \cdot J = J^{n-1} \cdot J = 0$, we have $r \cdot \lambda u \cdot e_\tau = r \cdot e_\tau \cdot \lambda u$ for all $r \in I$ and so we can permute λu and e_τ in the second sum. Therefore, the only non-zero terms appear for $\lambda \in \tau\Omega$. By (5), the same holds for the first sum. Consequently:

$$f_\sigma f_\tau = \sum_{\omega \in \Omega} (e_\sigma r_{\tau\omega,\omega^{-1}} + r_{\tau\omega,\omega^{-1}\tau^{-1}\sigma} e_{\tau\omega}) e_{\tau\omega} \cdot \tau\omega u.$$

Now apply (6) with $\eta = 1$, $\mu = \tau\omega$ and $\lambda = \omega^{-1}\tau^{-1}\sigma$, using also (1). The condition $\lambda\Omega \neq \eta\Omega$ is equivalent to $\sigma\Omega \neq \tau\Omega$. We get $f_\sigma f_\tau = 0$, as required.

Clearly (8) and (9) imply that f_σ is idempotent. There remains to prove the additional property we are looking for:

$$(10) \quad \tau f_\sigma = f_{\tau\sigma}.$$

By (3), we have:

$$\tau f_\sigma = e_{\tau\sigma} + r_{\tau,\sigma} + \sum_{\lambda \in \Pi} (r_{\tau\lambda, \lambda^{-1}\sigma} - r_{\tau,\sigma}) \cdot (e_{\tau\lambda} + r_{\tau,\lambda}) \cdot \tau\lambda u.$$

Since $I^2 = 0$, we get:

$$\begin{aligned} \tau f_\sigma &= e_{\tau\sigma} + \sum_{\lambda \in \Pi} r_{\tau\lambda, \lambda^{-1}\sigma} \cdot e_{\tau\lambda} \cdot \tau\lambda u + r_{\tau,\sigma} \left(1 - \sum_{\lambda \in \Pi} e_{\tau\lambda} \cdot \tau\lambda u \right) \\ &= e_{\tau\sigma} + \sum_{\mu \in \Pi} r_{\mu, \mu^{-1}\tau\sigma} \cdot e_\mu \cdot \mu u + r_{\tau,\sigma} \left(1 - \sum_{\mu \in \Pi/\Omega} e_\mu \cdot \sum_{\omega \in \Omega} \mu\omega u \right) \\ &= f_{\tau\sigma} + r_{\tau,\sigma} \left(1 - \sum_{\mu \in \Pi/\Omega} e_\mu \cdot \mu Tr_\Omega(u) \right) = f_{\tau\sigma}, \end{aligned}$$

using $Tr_\Omega(u) = 1$ and $\sum_{\mu \in \Pi/\Omega} e_\mu = 1$. ■

2. Clifford theory

Let N be a normal subgroup of a finite group G and $S = G/N$. Throughout this section, R denotes a noetherian local commutative ring which is complete in its natural topology of local ring. These assumptions are made in order to have the following properties:

(i) Every finitely generated RG -module is a direct sum of indecomposable submodules.

(ii) If M is an indecomposable RG -module, then $\text{End}_{RG} M$ is a local ring. Hence Krull-Schmidt theorem holds for RG -modules.

In order to study the restriction to N of an indecomposable RG -module, we consider the more general case of an S -graded Clifford system $A = \bigoplus_{s \in S} A_s$ over R , in the sense of [2, §11C]. The case of group algebras corresponds to $A = RG$ and $A_1 = RN$. Recall that there exist units $a_s \in A_s$ such that $A_s = a_s A_1 = A_1 a_s$. Also $a_s a_t a_{st}^{-1} \in A_1$ because $A_s A_t = A_{st}$.

For the rest of this paper, all modules will be finitely generated left modules. For an A_1 -module W , denote by W^A the induced module $\text{Ind}_{A_1}^A W = A \otimes_{A_1} W$, while for an A -module V , we denote by V_{A_1} the restriction $\text{Res}_{A_1}^A V$. If V is an A -module, then S acts on $\text{End}_{A_1} V$ by $sf = a_s f a_s^{-1}$ and the set of fixed points is exactly $\text{End}_A V$.

DEFINITIONS. 1) An A -module V is said to be *projective relative* to A_1 if V is a direct summand of a module induced from A_1 which actually can be chosen to

be $(V_{A_1})^A$. This is equivalent to the existence of an endomorphism $u \in \text{End}_{A_1} V$ such that $\text{Tr}_S(u) = 1$ where $\text{Tr}_S(u) = \sum_{s \in S} su$. The equivalence of these definitions is well known in the case of group algebras [2, §19A], but the proof can be carried over without change to the case of Clifford systems.

2) If W is an A_1 -module, then $a_s \otimes W$ has a natural structure of A_1 -module and is called a *conjugate* of W .

3) Let $M = \bigoplus_{i,j} M_{ij}$ be a decomposition of a module M into indecomposable summands such that $M_{ij} \cong M_{ik}$ for all i, j, k and $M_{ij} \not\cong M_{km}$ if $i \neq k$. Then $M_i = \bigoplus_j M_{ij}$ is called a *homogeneous component* of M . Contrary to the case of semi-simple modules, note that in general M_i is not uniquely determined by M .

Now we can state the going-down theorem analogous to Clifford's theorem:

THEOREM 2. *Let A be an S -graded Clifford system over R and V an indecomposable A -module. Assume that V is projective relative to A_1 , that is there exists an indecomposable summand W of V_{A_1} such that V is a direct summand of W^A . Let $T = \{t \in S \mid a_t \otimes W \cong W\}$ be the inertial subgroup of W and let $\{s_1, \dots, s_n\}$ be a set of coset representatives of T in S . Finally let $B = \bigoplus_{t \in T} A_t$ be the T -graded subalgebra of A . Then:*

- (i) V_{A_1} is isomorphic to a direct sum of conjugates of W .
- (ii) $\{a_{s_i} \otimes W \mid i = 1, \dots, n\}$ is a complete set of non-isomorphic conjugates of W and each appears with the same multiplicity in a decomposition of V_{A_1} .
- (iii) There exists a decomposition $V_{A_1} = \bigoplus_{i=1}^n U_i$ into homogeneous components which are permuted transitively by $\{a_s \mid s \in S\}$ and such that $\{a_t \mid t \in T\}$ stabilizes U_1 .
- (iv) U_1 is an indecomposable B -module and V is isomorphic to U_1^A .

Beside Theorem 1, the main ingredient for the proof of Theorem 2 is the following:

PROPOSITION 3. *Let A be an R -algebra, finitely generated as R -module, and M an A -module. Denote by a bar the reduction modulo the radical of $\text{End}_A M$. Let $M = \bigoplus_{i=1}^n M_i$ (respectively $M = \bigoplus_{i=1}^n M'_i$) be any decomposition of M corresponding to idempotents $e_1, \dots, e_n \in \text{End}_A M$ (respectively e'_1, \dots, e'_n).*

(i) *The modules M_i are homogeneous components of M if and only if $\bar{e}_1, \dots, \bar{e}_n$ are the primitive central idempotents of $\overline{\text{End}_A M}$.*

(ii) *Assume the modules M_i and M'_i are homogeneous components of M , labelled in order to have $M_i \cong M'_i$ for all i . Then there exists $f \in \text{Aut}_A M$ such that $f(M_i) = M'_i$ for all i and $\bar{f} = 1$.*

(iii) *Assume the modules M_i and M'_i are homogeneous components of M . Then $M_1 \cong M'_1$ if and only if $\bar{e}_1 = \bar{e}'_1$.*

Proof. (i) If the modules M_i are homogeneous components of M , write $M_i \cong m_i N_i$ with N_i indecomposable. Let $E_i = \text{End}_A N_i$ and $D_i = \overline{\text{End}_A N_i}$. By Fitting's theorem [2, §19C, lemma], there is a commutative diagram

$$\begin{array}{ccc} & E = \text{End}_A M & \\ & \uparrow & \searrow \\ \prod_{i=1}^n M_{m_i}(E_i) \cong \prod_{i=1}^n \text{End}_A M_i & & \overline{\text{End}_A M} \cong \prod_{i=1}^n M_{m_i}(D_i) \end{array}$$

Since e_i is the unit matrix of $M_{m_i}(E_i)$ (with zeros in all other components), \bar{e}_i is the unit matrix of $M_{m_i}(D_i)$, i.e. \bar{e}_i is a primitive central idempotent of $\overline{\text{End}_A M}$.

If conversely \bar{e}_i is primitive central, decompose it into primitive idempotents $\bar{e}_i = \bar{e}_{i1} + \cdots + \bar{e}_{im_i}$ and lift them to get $e_i = e_{i1} + \cdots + e_{im_i}$. Now $e_{ij}E \cong e_{ik}E$ because $\bar{e}_{ij}\bar{E} \cong \bar{e}_{ik}\bar{E}$. Therefore:

$$e_{ij}M \cong e_{ij}E \otimes_E M \cong e_{ik}E \otimes_E M \cong e_{ik}M.$$

So $e_i M = \bigoplus_{j=1}^{m_i} e_{ij}M$ is a homogeneous decomposition of $e_i M$ into indecomposable summands. If some indecomposable summand of $e_i M$ was isomorphic to a summand of $e_k M$ for $k \neq i$, there would be less than n homogeneous components in M and so, by the first part of the proof, less than n primitive central idempotents in \bar{E} .

(ii) Consider again the commutative diagram

$$\begin{array}{ccc} & \text{End}_A M & \\ & \uparrow j & \searrow q \\ \prod_{i=1}^n \text{End}_A M_i & & \overline{\text{End}_A M} \end{array}$$

$\nearrow p$

We emphasize that not only q but also p is surjective. Choose an isomorphism $g_i : M_i \rightarrow M'_i$ for each i and define an automorphism g of M by $g|_{M_i} = g_i$. Since g is invertible, so is $q(g)$ and since p is onto, there exists $h \in \prod_{i=1}^n \text{End}_A M_i$ such that $p(h) = q(g)^{-1}$. Clearly $f = g \cdot j(h)$ satisfies $f(M_i) = M'_i$ and $\bar{f} = 1$.

(iii) By (ii), if $M_1 \cong M'_1$, there exists $f \in \text{Aut}_A M$ such that $f(M_1) = M'_1$, $f(\bigoplus_{i=2}^n M_i) = \bigoplus_{i=2}^n M'_i$ and $\bar{f} = 1$. It follows easily that $e'_1 = f e_1 f^{-1}$ and therefore $\bar{e}'_1 = \bar{e}_1$.

Conversely suppose $\bar{e}'_1 = \bar{e}_1$. By Krull–Schmidt theorem, $M'_1 \cong M_i$ for some i . By the first part of this proof, $\bar{e}'_1 = \bar{e}_i$. Hence $\bar{e}_i = \bar{e}_1$ and so $i = 1$. ■

Proof of Theorem 2. (i) Write $V_{A_1} = \bigoplus_{i=1}^r W_i$ with the W_i indecomposable. Since V is a direct summand of W^A , V_{A_1} is a summand of $(W^A)_{A_1} \cong \bigoplus_{s \in S} a_s \otimes W$. By Krull–Schmidt theorem, each W_i is isomorphic to some $a_s \otimes W$.

(ii) Changing notations write $V_{A_1} = \bigoplus_{i=1}^n m_i W_i$ where $m_i W_i$ denotes the direct sum of m_i copies of W_i and $W_i \not\cong W_j$ if $i \neq j$. By (i), $W_i \cong a_s \otimes W$ for some s . Applying a_s to V , we get:

$$\bigoplus_{i=1}^n m_i W_i \cong V_{A_1} = (a_s V)_{A_1} \cong \bigoplus_{i=1}^n m_i (a_s \otimes W_i).$$

Comparing the multiplicities of W_i in both decompositions, we get $m_i = m_1$. The same argument applied with an arbitrary a_s shows that $a_s \otimes W$ must be isomorphic to some W_i . Therefore, by definition of T , $\{a_s \otimes W \mid i = 1, \dots, n\}$ is a complete set of non-isomorphic conjugates of W .

(iii) Let $E = \text{End}_{A_1} V$ and $\bar{E} = E/\text{rad}(E)$. The group S acts on E via $sf = a_s f a_s^{-1}$ and induces an action on \bar{E} which necessarily permutes the primitive central idempotents of \bar{E} .

Let $V_{A_1} = \bigoplus_{i=1}^n U_i$ be a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents e_1, \dots, e_n . Assume W is a summand of U_1 . For $s \in S$, $V_{A_1} = \bigoplus_{i=1}^n a_s U_i$ is also a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents $a_s e_i a_s^{-1} = s e_i$. By Proposition 3(i), $\{\bar{e}_1, \dots, \bar{e}_n\}$ are the primitive central idempotents of \bar{E} . Since $a_s U_1 \cong a_s \otimes U_1 \cong U_i$ for some i , we have $s \bar{e}_1 = \bar{e}_i$ by Proposition 3(iii). Moreover each U_i is isomorphic to some $a_s U_1$ by part (i) and (ii). This implies that S acts transitively on the set $\{\bar{e}_1, \dots, \bar{e}_n\}$. Since W is a summand of U_1 , T is the stabilizer of \bar{e}_1 (again by Proposition 3(iii)).

Now since V is projective relative to A_1 , there exists $v \in \text{End}_{A_1} V$ such that $\text{Tr}_S(v) = 1$. Let $u = \sum_{i=1}^n r_i v$ where r_1, \dots, r_n are representatives of the cosets Tr . Then $\text{Tr}_T(u) = \sum_{t \in T} t u = \text{Tr}_S(v) = 1$. Moreover \bar{u} commutes with \bar{e}_i for \bar{e}_i is central. Therefore the hypotheses of Theorem 1 are satisfied. It follows that there exist orthogonal idempotents f_1, \dots, f_n of E (lifting $\bar{e}_1, \dots, \bar{e}_n$) which are permuted transitively by S and such that T stabilizes f_1 .

By Proposition 3(i), the modules $f_i V_{A_1}$ are homogeneous components of V_{A_1} . The equation $f_i = s f_1 = a_s f_1 a_s^{-1}$ means exactly that $a_s (f_1 V_{A_1}) = f_i V_{A_1}$. This completes the proof of part (iii).

(iv) Since $\{a_t \mid t \in T\}$ stabilizes $U_1 = f_1 V_{A_1}$, U_1 is a B -module. Now $V = \bigoplus_{i=1}^n a_s U_1$ which is the definition of an induced module. Finally U_1 is indecomposable otherwise V would be decomposable. ■

Counter-example. Without the assumption of relative projectivity for V , Theorem 2 does not hold any more. Take K a field of characteristic 2, $G = C_4$, $N = C_2$ and $V = K[X]/(X-1)^3$ (the generator of C_4 acting by multiplication by X). Then: $\text{Res}_N V = S_1 \oplus S_2$ where $S_i = K[Y]/(Y-1)^i$ (the generator of C_2

acting by multiplication by Y). Since S_1 and S_2 do not have the same dimension, they cannot be conjugate. In fact, the two primitive central idempotents of $\overline{\text{End}_{KN} V}$ are fixed under the action of $S = G/N$, and each of them can be lifted in four ways in $\text{End}_{KN} V$. But no idempotent of $\text{End}_{KN} V$ is fixed by S .

Now we can recall the going-up theorem, which we shall prove to be equivalent to Theorem 2.

THEOREM 4 (Conlon, Tucker, Ward [2, §19C]). *Let A be an S -graded Clifford system over R , W an indecomposable A_1 -module, T the inertial subgroup of W and $B = \bigoplus_{i \in T} A_i$. If $W^B = \bigoplus_{i=1}^m Z_i$ is a decomposition of W^B into indecomposable B -modules, then each Z_i^A is an indecomposable A -module, that is $W^A = \bigoplus_{i=1}^m Z_i^A$ gives a decomposition of W^A into indecomposable A -modules.*

Proof. The notation $X \mid Y$ will mean: X is a direct summand of Y . Let Z be an indecomposable summand of W^B . Since T is the inertial subgroup of W , $(W^B)_{A_1} = |T| \cdot W$ and so Z_{A_1} is a multiple of W . Since $Z \mid (Z^A)_B$, there exists an indecomposable summand V of Z^A such that $Z \mid V_B$. Then $V \mid W^A$ and $W \mid V_{A_1}$. By Theorem 2, there exists an indecomposable B -module U such that $V \cong U^A$ and U_{A_1} is a multiple of W . Now $U \mid (Z^A)_B$ because $V \mid Z^A$ and $U \mid (U^A)_B = V_B$. But Z is the only indecomposable summand of $(Z^A)_B$ whose restriction to A_1 is a multiple of W , for $(Z^A)_{A_1} = \bigoplus_{i=1}^n a_{s_i} \otimes Z_{A_1}$ (where $\{s_1, \dots, s_n\}$ is a set of coset representatives of T in S) and $a_{s_i} \otimes Z_{A_1}$ is a proper conjugate of Z_{A_1} (a multiple of a proper conjugate of W). It follows that $U \cong Z$ and so $Z^A \cong U^A \cong V$ is indecomposable. ■

Equivalence of Theorems 2 and 4. If Theorem 4 is proved independently (e.g. by the proof of [2, §19C]), then Theorem 2 can be derived as corollary in the following way: Let V be an indecomposable A -module which is a summand of W^A for some indecomposable summand W of V_{A_1} . Let T be the inertial subgroup of W . By Theorem 4, there exists an indecomposable summand U of W^B such that $V = U^A$. Now $U_{A_1} \cong mW$ for some m because $(W^B)_{A_1} \cong |T| W$. Then clearly $V \cong \bigoplus_{i=1}^n a_{s_i} \otimes U$ and $V_{A_1} \cong \bigoplus_{i=1}^n m(a_{s_i} \otimes W)$ where s_1, \dots, s_n are coset representatives of T in S . This completes the proof of Theorem 2. ■

3. Ground field extensions

Let K be a field and A a finite dimensional K -algebra. Let F be a finite Galois extension of K , with Galois group Π , and consider the F -algebra $F \otimes A$ (note that throughout this section \otimes will always mean \otimes_K). Every element $\sigma \in \Pi$ induces a semi-linear automorphism $\sigma: F \otimes A \rightarrow F \otimes A$. If W is an $F \otimes A$ -module, one can define a new $F \otimes A$ -module structure on W by scalar extension via σ (or

equivalently restriction via σ^{-1}). Explicitly the new structure is given by $a \cdot w = \sigma^{-1}(a)w$, $a \in F \otimes A$, $w \in W$. This module is called a Galois conjugate of W .

Now if V is a finitely generated indecomposable A -module, then $F \otimes V$ has a natural structure of $F \otimes A$ -module. Moreover, Π acts on $F \otimes V$ via $\sigma(f \otimes v) = \sigma f \otimes v$, $\sigma \in \Pi$, $f \in F$, $v \in V$. This action is semi-linear with respect to $F \otimes A$, i.e. $\sigma(aw) = \sigma(a)\sigma(w)$, $\sigma \in \Pi$, $a \in F \otimes A$, $w \in F \otimes V$. If $F \otimes V = \bigoplus_{i=1}^n W_i$ is a decomposition of $F \otimes V$ into homogeneous components, then so is $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$. One can readily check that σW_i is a Galois conjugate of W_i . By Krull–Schmidt theorem, $\sigma W_i \cong W_j$ for some j . Moreover, it is easy to see that for given i and j , there exists $\sigma \in \Pi$ such that $\sigma W_i \cong W_j$. The purpose of this section is to derive from Theorem 1 a stronger result, namely that for a suitable choice of the submodules W_i , one can replace this isomorphism by an equality:

PROPOSITION 5. *In the above notations, there exists a decomposition $F \otimes V = \bigoplus_{i=1}^n W_i$ of $F \otimes V$ into homogeneous components such that the modules W_i are permuted transitively under the natural action of Π on $F \otimes V$.*

Proof. Let $E = \text{End}_A V$ and $\bar{E} = E/\text{Rad } E$. Since V is indecomposable, \bar{E} is a division algebra containing K in its center. Now $F \otimes E = \text{End}_{F \otimes A}(F \otimes V)$ and let $\overline{F \otimes E} = F \otimes E/\text{Rad}(F \otimes E)$. Since F/K is separable, $\overline{F \otimes E} \cong F \otimes \bar{E}$. Let $F \otimes V = \bigoplus_{i=1}^n W_i$ be a decomposition of $F \otimes V$ into homogeneous components corresponding to idempotents $e_1, \dots, e_n \in F \otimes E$. The decomposition $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$ corresponds to the idempotents $\sigma e_1 \sigma^{-1}, \dots, \sigma e_n \sigma^{-1}$ (where σ is viewed as a semi-linear automorphism of $F \otimes V$).

Now Π acts on $F \otimes E$ via $\sigma \cdot (f \otimes e) = \sigma f \otimes e$, $\sigma \in \Pi$, $f \in F$, $e \in E$. We claim that $\sigma z \sigma^{-1} = \sigma \cdot z$ for all $z \in F \otimes E$. Indeed, if $z = f \otimes e$, $f \in F$, $e \in E$, and if $g \otimes v \in F \otimes V$, then:

$$\begin{aligned} (\sigma z \sigma^{-1})(g \otimes v) &= \sigma(f \otimes e)(\sigma^{-1} g \otimes v) = \sigma(f \cdot \sigma^{-1} g) \otimes ev = \sigma f \cdot g \otimes ev \\ &= (\sigma f \otimes e)(g \otimes v) = (\sigma \cdot z)(g \otimes v). \end{aligned}$$

It follows that $\{\sigma \cdot e_1, \dots, \sigma \cdot e_n\}$ are the idempotents corresponding to the decomposition $F \otimes V = \bigoplus_{i=1}^n \sigma W_i$. By Proposition 3(i), $\{\bar{e}_1, \dots, \bar{e}_n\} = \overline{\{\sigma \cdot e_1, \dots, \sigma \cdot e_n\}}$ is the set of primitive central idempotents of $F \otimes \bar{E}$. Now Π acts transitively on $\{\bar{e}_1, \dots, \bar{e}_n\}$ for if $\{\bar{e}_1, \dots, \bar{e}_k\}$ is a Π -orbit, then $\bar{e} = \sum_{i=1}^k \bar{e}_i$ is an idempotent, invariant under Π , hence lies in $K \otimes \bar{E} = \bar{E}$. Since 1 is the only idempotent of \bar{E} , we get $\bar{e} = 1$ and so $k = n$.

Since F/K is separable, $\text{Tr}_{F/K}$ is surjective. Therefore there exists $x \in F$ such that $\text{Tr}_{F/K}(x) = \sum_{\sigma \in \Pi} \sigma x = 1$. In particular, if Ω denotes the stabilizer of \bar{e}_1 and $\sigma_1, \dots, \sigma_n$ are coset representatives of Ω in Π , then $u = \sum_{i=1}^n \sigma_i x$ satisfies $\sum_{\omega \in \Omega} \omega u = 1$. Also $u \otimes \bar{1} \in F \otimes \bar{E}$ commutes with every \bar{e}_i . Therefore $u \otimes 1 \in F \otimes E$

satisfies the hypotheses of Theorem 1. Consequently $\{\bar{e}_1, \dots, \bar{e}_n\}$ lifts to a set of orthogonal idempotents f_1, \dots, f_n of $F \otimes E$ which are permuted transitively by Π and such that $\sum_{i=1}^n f_i = 1$. By Proposition 3(i), the modules $W'_i = f_i(F \otimes V)$ are homogeneous components of $F \otimes V$. Finally, since $\sigma f_i = f_j$ for some j , we have:

$$\sigma W'_i = \sigma(f_i(F \otimes V)) = (\sigma f_i \sigma^{-1})(F \otimes V) = (\sigma \cdot f_i)(F \otimes V) = f_j(F \otimes V) = W'_j. \quad \blacksquare$$

Remarks. 1) If one replace homogeneous components of $F \otimes V$ by indecomposable summands, then one must consider sets of primitive idempotents $\{\bar{e}_1, \dots, \bar{e}_n\}$ of \bar{E} instead of primitive central idempotents of \bar{E} . If one can show that there exists such a set which is stable under the action of Π (this happens quite often), then the whole proof works without change, so that there exists a decomposition $F \otimes V = \bigoplus_{i=1}^n W_i$ into *indecomposable submodules* such that the modules W_i are permuted transitively under the natural action of Π on $F \otimes V$.

2) Proposition 5 holds more generally if one replaces the field K by a complete discrete valuation ring R and the extension F by an unramified Galois extension S (so that the Galois group of S/R is isomorphic to the Galois group of the residue field extension). Moreover, A must be an R -algebra which is finitely generated as R -module.

3) The similarity between restriction to a normal subgroup (Theorem 2) and ground field Galois extension (Proposition 5) extends a little further. If Ω denotes the stabilizer of the homogeneous component W_1 of $F \otimes V$ and if L is the fixed field of Ω , then W_1 is realizable over L , that is there exists an $L \otimes_K A$ -module U such that $F \otimes_L U = W_1$. Moreover, by analogy with part (iv) of Theorem 2 (replacing group induction by scalar restriction), one can easily show that $V \cong \text{Res}_K U$.

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