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A $1\frac{1}{2}$ -dimensional version of Hopf's Theorem on the number of ends of a group

ROBERT BIERI

1. Introduction

If G is a finitely generated group then the first cohomology group with group ring coefficients $H^1(G; \mathbb{Z}G)$ is known to be free-Abelian. H. Hopf [7] has shown that its \mathbb{Z} -rank, $\text{rk } H^1(G; \mathbb{Z}G)$, attains only the values 0, 1 or ∞ , and the celebrated structure theorem of Hopf–Stallings [7], [12], classifies these three cases in terms of the group theoretic structure of G .

Of course the cohomology group $H^1(G; \mathbb{Z}G)$ carries much more information than just its Abelian group structure. As the coefficient module $\mathbb{Z}G$ is a bi-module $H^1(G; \mathbb{Z}G)$ inherits the structure of a (right) G -module; and by functoriality one can consider the restriction maps

$$\text{res: } H^1(G; \mathbb{Z}G) \rightarrow \prod^m H^1(S_i; \mathbb{Z}G) \quad (1.1)$$

where $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ is a finite family of finitely generated subgroups of G . The relative versions of Stallings's structure theorem by Swan [13] and Swarup [14] show that the kernel K of (1.1) is free-Abelian of rank 0, 1 or ∞ , and classify these three cases in terms of the structure of the pair (G, \mathcal{S}) .

In this paper we consider the cokernel $C(G, \mathcal{S})$ of the restriction map (1.1), under the assumption that G is *accessible*. (For a discussion of accessibility refer to [4], but we recall that every finitely generated torsion-free group is accessible by Grusko's Theorem and that it is unknown whether finitely generated non-accessible groups exist). We observe that Heinz Müller's result [9] on the freeness of the cokernel of the restriction map carries readily over to the case of a finite family of subgroups, so that $C(G, \mathcal{S})$ is *always free-Abelian in our situation*. Our main result asserts that *the rank m of $C(G, \mathcal{S})$ is equal to 0, 1 or ∞ except in the very special situation when G contains an infinite cyclic subgroup of finite index, in which case m can attain every value $0 \leq m < \infty$* . Then we *classify the three cases $m = 0, 1, \infty$ in terms of the structure of (G, \mathcal{S})* . The fact that, in view of the long exact cohomology sequence for the pair (G, \mathcal{S}) , the cokernel of (1.1) “lies between $H^1(G; \mathbb{Z}G)$ and $H^2(G; \mathbb{Z}G)$ ” justifies our title.

2. The results

2.1. Our main result is

THEOREM A. *Let G be a finitely generated accessible group and $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ a finite non-empty family of finitely generated infinite subgroups of G , and let $\text{rk } C(G, \mathcal{S})$ denote the rank of the (free-Abelian) cokernel of the restriction map (1.1). If G contains an infinite cyclic subgroup of finite index then*

$$\text{rk } C(G, \mathcal{S}) = \sum_{i=1}^m |G : S_i| - 1;$$

otherwise $\text{rk } C(G, \mathcal{S})$ is equal to 0 or 1, or ∞ .

Note that finite groups in the family \mathcal{S} have no influence whatsoever on the cokernel of (1.1) and so we lose no generality by assuming that all groups in \mathcal{S} are infinite.

Next we classify the three cases $\text{rk } C(G, \mathcal{S}) = 0, 1, \infty$ by exhibiting necessary and sufficient conditions for $\text{rk } C(G, \mathcal{S})$ to be 0 or 1, respectively. The case $\text{rk } C(G, \mathcal{S}) = 0$ is then, of course, given by exclusion.

2.2. $\text{rk } C(G, \mathcal{S}) = 1$. In order to state the result when $C(G, \mathcal{S})$ is infinite cyclic we introduce the following notation. Let (G, \mathcal{S}) be a pair consisting of a group G and a family $\mathcal{S} = \{S_i \mid i \in I\}$ of subgroups (possibly with repetitions!), and let $F \leq G$ be an auxiliary subgroup. For each index $i \in I$ we choose a system X_i of double coset representatives of $F \backslash G / S_i$ and consider the family

$$\mathcal{S}' = \{F \cap x_i S_i x_i^{-1} \mid x_i \in X_i, i \in I\}.$$

Up to conjugacy within F , \mathcal{S}' is independent of the choice of X_i , $i \in I$. We call (F, \mathcal{S}') the *full subpair of (G, \mathcal{S}) given by $F \leq G$* .

We define the group pair (G, \mathcal{S}) to be a *virtual Poincaré duality pair* if G contains a subgroup of finite index $F \leq G$ such that the full subpair of (G, \mathcal{S}) given by F is a Poincaré duality pair in the sense of [2]. Note that F is necessarily torsion-free and that the definition of a virtual Poincaré duality pair is independent of the particular choice of F by [2], Theorem 7.6.

THEOREM B.⁽¹⁾ *Let (G, \mathcal{S}) be as in Theorem A. Then $\text{rk } C(G, \mathcal{S}) = 1$ if and only if (G, \mathcal{S}) is a virtual Poincaré duality pair of dimension 2.*

Thus in view of [2] Theorem 9.3 we have $\text{rk } C(G, \mathcal{S}) = 1$ if and only if G

¹ Eckmann and Müller have recently obtained a different proof of Theorem B and a direct description of all virtual Poincaré duality pairs of dimension 2. See “Plane motion groups and virtual Poincaré duality of dimension 2”. Preprint, Forschungsinstitut für Mathematik 1981, ETH, Zürich.

contains a free subgroup of finite index, each S_i contains an infinite cyclic subgroup of finite index, and the relative cohomology group $H^2(G, \mathcal{S}; \mathbb{Z}G)$ is $\cong \mathbb{Z}$.

It was shown by Eckmann and Müller [5] that the 2-dimensional Poincaré duality pairs are geometric, that is, given by the fundamental group and the peripheral subgroup system of a compact surface-with-boundary. This yields the

COROLLARY.⁽¹⁾ *Let (G, \mathcal{S}) be as in Theorem A and assume G is torsion-free. Then $\text{rk } C(G, \mathcal{S}) = 1$ if and only if G is a free group having a basis $\{t_1, t_2, \dots, t_{m-1}, x_1, \dots, x_n\}$, such that the subgroups $S_i \in \mathcal{S}$ are conjugate to the infinite cyclic subgroups $gp(t_1), \dots, gp(t_{m-1}), gp(t_1 \cdots t_{m-1} r)$, where*

$$r = [x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n], \quad n \text{ even} \geq 0$$

if $C(G, \mathcal{S})$ has trivial G -action, and

$$r = x_1^2 x_2^2 \cdots x_n^2, \quad n \geq 0$$

2.3. $\text{rk } C(G, \mathcal{S}) = 0$. In order to exhibit the structure of (G, \mathcal{S}) when the restriction map (1.1) is surjective we have to consider simultaneous decompositions of G and the subgroups S_i as fundamental groups of graphs of groups. In order to handle the family \mathcal{S} it is convenient to consider graphs of groups (\mathfrak{G}, X) where the underlying graph X is not necessarily connected and define its “fundamental group” $\pi_1(\mathfrak{G}, X)$ to be the family of fundamental groups of the connected components.

In more detail: Let $X(i)$, $i \in I$, denote the connected components of the (oriented) graph X , with vertices $V(X(i))$ and (positive) edges $E(X(i))$, and let $\mathfrak{G}(i)$ be the corresponding system of vertex groups G_v , $v \in V(X(i))$ and edge groups $G_e \leq G_{o(e)}$, $G_{\bar{e}} \leq G_{t(e)}$, $e \in E(X(i))$. Then $\pi_1(\mathfrak{G}, X)$ stands for the family of groups $G(i) = \pi_1(\mathfrak{G}(i), X(i))$, $i \in I$. Recall that $G(i)$ is generated by the vertex groups G_v , $v \in V(X(i))$ and stable letters p_e , $e \in E(X(i))$, subject to the following defining relations.

$$p_e^{-1} G_e p_e = G_{\bar{e}}, \quad e \in E(X(i))$$

$p_e = 1$ for all edges e in a maximal tree of $X(i)$.

So let $G = \pi_1(\mathfrak{G}, X)$, with X connected, and $\mathcal{S} = \pi_1(\mathfrak{G}, Y)$ with Y arbitrary, and let $V(X)$, $V(Y)$ be the set of vertices and $E(X)$, $E(Y)$ the set of (positive) edges of X resp. Y .

DEFINITION. We say that the decompositions of G and \mathcal{S} are compatible (via an orientation preserving graph map $f: Y \rightarrow X$) if there are elements $c_v \in G$,

$v \in V(X)$, such that the following holds

$$c_v^{-1} S_v c_v \leq G_{f(v)} \quad \text{for every vertex } v \in V(Y) \quad (2.1)$$

$$c_{o(e)} p_{f(e)} = p_e c_{t(e)} \quad \text{for every edge } e \in E(Y), \quad (2.2)$$

where p_e and $p_{f(e)}$ stand for the stable letters corresponding to the (positive) edges e resp. $f(e)$.

Note that if G and \mathcal{S} have compatible decompositions via f , $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$, then the same holds for any family $\mathcal{S}' = \{S'_i \mid i \in I\}$ with $S'_i = g_i^{-1} S_i g_i$, $g_i \in G$. Indeed, let (\mathfrak{S}_i, Y_i) , $i \in I$, be the connected components of (\mathfrak{S}, Y) . Then conjugating each $\mathcal{S}_i = \pi_1(\mathfrak{S}_i, Y_i) \leq G$ by g_i yields a decomposition $\mathcal{S}' = \pi_1(\mathfrak{S}', Y)$ satisfying (2.1), and (2.2), where for each $v \in V(Y_i)$ and $e \in E(Y_i)$ c_v is to be replaced by $g_i^{-1} c_v$ and p_e by $g_i^{-1} p_e g_i$.

Now we are in a position to state

THEOREM C. *Let (G, \mathcal{S}) be as in Theorem A. Then $C(G, \mathcal{S}) = 0$ if and only if G and \mathcal{S} have compatible decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ given by a graph map $f: Y \rightarrow X$ which is bijective on the edges, such that the following holds:*

- (i) *all edge groups of G are finite and coincide with the corresponding (conjugate) edge groups of \mathcal{S}*
- (ii) *all vertex groups of \mathcal{S} have ≤ 1 end.*

As a special case Theorem 3 contains a splitting result which is related to those of Swan [13], Lemma 7.1, and Wall [15].

COROLLARY. *Let G be a torsion-free finitely generated group and \mathcal{S} a finite family of finitely generated free subgroups of G . Then $C(G, \mathcal{S}) = 0$ if and only if G is the free product $G = S_1 * \cdots * S_m * K$ where $S'_i \leq G$ is a subgroup conjugate to S_i , $1 \leq i \leq m$, and $K \leq G$ is an auxiliary subgroup.*

Proof. If f is surjective G and \mathcal{S} have decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ satisfying the properties (i), (ii) of Proposition 7.2. Hence all edge groups are trivial and all vertex groups S_v of \mathcal{S} have ≤ 1 end. Since S_v is free this means that $S_v = 1$, and $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ is the family of fundamental groups (in the topological sense) of the connected components Y_i of Y . Since $X = f(Y_i)$ the fundamental group of X is free product of $\pi_1(f(Y_i))$ and an auxiliary group K_1 , and clearly $G \cong \pi_1(X) * K_2$ where K_2 is the tree product along a maximal tree of X . Finally $\pi_1(f(Y_i)) \cong \pi_1(Y_i) * K_{3i}$ because f identifies certain vertices; note that one has to choose base points and use conjugation to adapt the elements $c_v \in G$ so that the last isomorphism involves conjugation.

3. Two preliminary lemmas

3.1. Let G be a group and K a commutative ring with nontrivial unity. Recall that a KG -module M is said to be of type $(FP)_n$, where n is an integer ≥ 0 or $n = \infty$, if M has a projective resolution which is finitely generated in all dimensions $\leq n$. If M is of type $(FP)_\infty$ and of finite projective dimension then M is said to be of type (FP) . If the trivial G -module K is of type $(FP)_n$ (resp. of type (FP)) then we say that the group G is of type $(FP)_n$ over K (resp. of type (FP) over K).

LEMMA 3.1 (Stallings [12]). *Let K be a field and assume that G has no K -torsion. Let V be a non-trivial KG -module of finite K -dimension. then we have*

- (a) *The KG -module V is of type $(FP)_n$ if and only if the group G is of type $(FP)_n$ over K .*
- (b) *The projective dimension of the KG -module V is equal to the homology dimension $cd_K G$ of G over K .*

Proof. Let $\mathbf{P} \rightarrow K$ be a projective resolution of the KG -module K . Then $\mathbf{P} \otimes_K V$ is a projective resolution of V . And if \mathbf{P} is finitely generated (resp. of finite length) so is $\mathbf{P} \otimes_K V$.

Conversely: Assume first that V is of type $(FP)_n$. By induction one may assume that P_0, P_1, \dots, P_{n-1} are finitely generated, hence so are $P_i \otimes_K V$, $i = 1, 2, \dots, n-1$.

Let $R = \ker(P_{n-1} \rightarrow P_{n-2})$. Since V is of type $(FP)_n$, $R \otimes_K V$ is finitely generated over KG ; hence so is R , and therefore G is of type $(FP)_n$ over K .

Now assume V is of projective dimension $\leq n$. Then $R \otimes_K V$ is a projective KG -module. Let F be a free KG -module and $f: F \rightarrow R$ an epimorphism. There is a KG -homomorphism $g: R \otimes_K V \rightarrow F \otimes_K V$ which splits $f \otimes 1$. Stallings defines to such a map g the “transfer trace” $g_V^*: R \rightarrow V$ as follows: for every $r \in R$ and a fixed basis $\{v_1, v_2, \dots, v_n\}$ of V one has

$$g(r \otimes v_i) = \sum_{i=1}^n g_{ij}(r) \otimes v_j$$

and we can put

$$g_V^*(r) = \sum_{i=1}^n g_{ii}(r)$$

It is easy to check that $g_V^*: R \rightarrow F$ is a KG -homomorphism which does not

depend upon the choice of the basis $\{v_1, v_2, \dots, v_n\}$, and that the composite map $f \cdot g_V^*: F \rightarrow R$ is multiplication by $n = \dim_K V$. Since G has no K -torsion $\frac{1}{n} g_V^*: R \rightarrow F$ splits f , and R is projective.

3.2. There is an immediate Corollary which improves Lemma 3.2(b) provided the cohomology dimension $\text{cd}_K G$ is known to be finite.

COROLLARY 3.2. *Let K be a field, G a group of finite cohomology dimension over K , and M a KG -module containing a non-trivial submodule $V \leq M$ of finite K -dimension. Then $\text{cd}_K G$ is equal to the projective dimension of M .*

Proof. Let A be a KG -module such that $\text{Ext}_{KG}^m(V, A) \neq 0$, where $m = \text{cd}_K G$. Since the projective dimension of any KG -module is $\leq m$ we obtain from the long exact Ext-sequence

$$\text{Ext}_{KG}^m(M, A) \rightarrow \text{Ext}_{KG}^m(V, A) \rightarrow \underbrace{\text{Ext}_{KG}^{m+1}(M/V, A)}_{=0}$$

that $\text{Ext}_{KG}^m(M, A) \neq 0$. Hence the projective dimension of M is $\geq m$ and hence $= m = \text{cd}_K G$.

4. Resolutions of end groups by permutation modules

4.1. Let G be an infinite finitely generated accessible group and $\mathcal{S} = \{S_i \mid i \in I\}$ a finite family of finitely generated subgroups of G . In this section we deduce a finite resolution of the relative cohomology group $H^1(G, \mathcal{S}; \mathbb{Z}G)$ regarded as a right G -module. For definitions and notation concerning the cohomology of a pair (G, \mathcal{S}) we refer to [2]. Thus we consider the short exact sequence

$$\Delta_{G/\mathcal{S}} \rightarrow \mathbb{Z}G/\mathcal{S} \xrightarrow{\epsilon} \mathbb{Z} \tag{4.1}$$

where $\mathbb{Z}(G/\mathcal{S})$ is an abbreviation for the direct sum of all permutation modules $\mathbb{Z}G/S_i$, $i \in I$, and ϵ is the obvious augmentation. Then

$$H^k(G, \mathcal{S}; \mathbb{Z}G) = \begin{cases} H^k(G; \mathbb{Z}G), & \text{if } \mathcal{S} = \emptyset \\ \text{Ext}_G^{k-1}(\Delta_{G/\mathcal{S}}, \mathbb{Z}G) & \text{if } \mathcal{S} \neq \emptyset \end{cases}$$

Note that $H^0(G, \mathcal{S}; A) = 0$ for $\mathcal{S} \neq \emptyset$; and replacing the subgroups $S_i \in \mathcal{S}$ by

conjugates leads to an isomorphic relative group. Finally, we shall use the abbreviation $H^n(\mathcal{S}; \mathbb{Z}G)$ for the direct product of the groups $H^n(S_i; \mathbb{Z}G)$, $i \in I$.

4.2. Let I_{fin} (resp. I_{inf}) denote the set of all $i \in I$ with S_i finite (resp. infinite), and put

$$\mathcal{S}_{\text{fin}} = \{S_i \mid i \in I_{\text{fin}}\}, \quad \mathcal{S}_{\text{inf}} = \{S_i \mid i \in I_{\text{inf}}\}.$$

From $\mathbb{Z}G/\mathcal{S} = \mathbb{Z}G/\mathcal{S}_{\text{fin}} \oplus \mathbb{Z}G/\mathcal{S}_{\text{inf}}$ one easily obtains a short exact sequence of left G -modules.

$$\mathbb{Z}G/\mathcal{S}_{\text{fin}} \rightarrowtail \Delta_{G/\mathcal{S}} \twoheadrightarrow \Delta_{G/\mathcal{S}_{\text{inf}}}$$

and the corresponding Ext-Sequence yields the short exact sequence of right G -modules.

$$0 \rightarrow H^0(\mathcal{S}_{\text{fin}}; \mathbb{Z}G) \rightarrow H^1(G, \mathcal{S}; \mathbb{Z}G) \rightarrow H^1(G, \mathcal{S}_{\text{inf}}; \mathbb{Z}G) \rightarrow 0. \quad (4.2)$$

Now, $H^0(\mathcal{S}_{\text{fin}}; \mathbb{Z}G)$ is the direct product of the (right) permutation modules $\mathbb{Z}(S_i \setminus G)$, $i \in I_{\text{fin}}$.

4.3. It remains to consider the cohomology group $H^1(G, \mathcal{S}_{\text{inf}}; \mathbb{Z}G)$, which – by the long exact sequence for the pair $(G, \mathcal{S}_{\text{inf}})$ – is isomorphic to the kernel of the restriction map $H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}_{\text{inf}}; \mathbb{Z}G)$. If the kernel is $=0$ then, by Swarup's relative version of Stallings's Structure Theorem [14] one can replace the groups in \mathcal{S}_{inf} by suitable conjugates in such a way that G can be written as the fundamental group of a graph of groups (\mathcal{G}, X) with finite edge groups and with every group of \mathcal{S}_{inf} contained in one of the vertex groups. Let V be the set of vertices and E the set of positive edges of X . \mathcal{S}_{inf} can be written as a disjoint union of families \mathcal{S}_v of subgroups of the edge groups, G_v , $v \in V$. If $H^1(G_v, \mathcal{S}_v; \mathbb{Z}G) \neq 0$ for some $v \in V$ one can repeat the decomposition procedure. But as G is accessible the decomposition stops after a finite number of steps. Hence we can assume that $H^1(G_v, \mathcal{S}_v; \mathbb{Z}G) = 0$ for all $v \in V$.

The relative Mayer–Vietoris sequence (cf. [2], Theorems 3.2 and 3.3, which can be generalized to arbitrary graphs of groups) now yields a short exact sequence of right G -modules.

$$0 \rightarrow \prod_v H^0(G_v, \mathcal{S}_v; \mathbb{Z}G) \rightarrow \prod_E H^0(G_e; \mathbb{Z}G) \rightarrow H^1(G, \mathcal{S}_{\text{inf}}; \mathbb{Z}G) \rightarrow 0. \quad (4.3)$$

Of course $H^0(G_v, \mathcal{S}_v; \mathbb{Z}G) = 0$ if either $\mathcal{S}_v \neq \emptyset$ or G_v is infinite. If G_v is finite then $\mathcal{S}_v = \emptyset$ and $H^0(G_v, \mathcal{S}_v; \mathbb{Z}G) \cong \mathbb{Z}(G_v \setminus G)$, and similarly for $H^0(G_e; \mathbb{Z}G)$. Thus

(4.3) can be written as

$$0 \rightarrow \prod_{V_{\text{fin}}} \mathbb{Z}(G_v \setminus G) \rightarrow \prod_E \mathbb{Z}(G_e \setminus G) \rightarrow H^1(G, \mathcal{S}_{\text{inf}}; \mathbb{Z}G) \rightarrow 0, \quad (4.4)$$

where $V_{\text{fin}} \subseteq V$ is the set of all vertices v with G_v finite.

4.4. From the short exact sequence (4.2) and (4.4) we deduce three things. Firstly, the permutation modules $\mathbb{Z}(U \setminus G)$ for a finite subgroup $U \leq G$ are of type $(FP)_{\infty}$. Since the index sets V_{fin} , E and I_{fin} are finite it follows that the G -module $H^1(G, \mathcal{S}; \mathbb{Z}G)$ is of type $(FP)_{\infty}$. Secondly, when tensored with \mathbb{Q} , a permutation module $\mathbb{Z}(U \setminus G)$, U finite, becomes a projective $\mathbb{Q}G$ -module. Hence using (4.4) and (4.2) one can construct a finite projective resolution of $H^1(G, \mathcal{S}; \mathbb{Q}G)$. This yields a bound for the projective dimension and the Euler characteristic of this $\mathbb{Q}G$ -module. Using the notation of [3] (in fact extending it slightly) we write $\chi(M)$ for the Hattori–Stallings-rank of a $\mathbb{Q}G$ -module of type (FP) – recall that $\chi(M)$ is a finite \mathbb{Q} -linear combination of conjugacy classes in G – and $\mu(M) \in \mathbb{Q}$ for its coefficient of $1 \in G$.

We summarize:

THEOREM 4.1. *Let G be a finitely generated infinite accessible group and $\mathcal{S} = \{S_i \mid i \in I\}$ a finite family of finitely generated subgroups of G . Then the right G -module $H^1(G, \mathcal{S}; \mathbb{Z}G)$ is of type $(FP)_{\infty}$. The $\mathbb{Q}G$ -module $H^1(G, \mathcal{S}; \mathbb{Q}G)$ is of type (FP) and of projective dimension ≤ 2 ; and its Euler characteristic is given by*

$$\mu(H^1(G, \mathcal{S}; \mathbb{Q}G)) = \sum_E \frac{1}{|G_e|} - \sum_{V_{\text{fin}}} \frac{1}{|G_v|} + \sum_{I_{\text{fin}}} \frac{1}{|S_i|}. \quad (4.5)$$

Proof. If K is a finite group then the trivial $\mathbb{Q}K$ -module \mathbb{Q} is projective and has Euler characteristic $\mu(\mathbb{Q}) = 1/|U|$. If U is a subgroup of G then $\mathbb{Q}G$ is free as a $\mathbb{Q}U$ -module, hence $\mathbb{Q} \otimes_{\mathbb{Q}U} \mathbb{Z}G \cong \mathbb{Q}(U \setminus G)$ is $\mathbb{Q}G$ -projective; and by the covariance property of χ (and μ) we get $\mu(\mathbb{Q}(U \setminus G)) = \mu(\mathbb{Q})$. Using the behaviour of χ (and μ) with respect to exact sequence yields formula (4.5).

4.5. **Remark.** For the proof of the main result we shall actually only need the case $\mathcal{S} = \emptyset$ of Theorem 4.1. In this case (4.2) is irrelevant and hence the projective dimension of $H^1(G; \mathbb{Q}G)$ is even ≤ 1 .

5. The cokernel $C(G, S)$ of res is free-Abelian

5.1. Next we observe that H. Müller’s result [9] on the cokernel of the restriction map extends to the case of a family of subgroups:

THEOREM 5.1 (H. Müller). *Let G be a finitely generated accessible group and $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ a finite family of finitely generated subgroups. Then the cokernel $C(G, \mathcal{S})$ of the restriction map $H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G)$ is free-Abelian.*

Proof. Following the proof of [9], Corollary 1.9 one can embed S_1 into a certain accessible group \bar{S}_1 with $C(\bar{S}_1, S_1)$ free-Abelian and such that there is a short exact sequence

$$C(\bar{G}, \bar{\mathcal{S}}) \rightarrow C(G, \mathcal{S}) \otimes_G \mathbb{Z}\bar{G} \rightarrow C(\bar{S}_1, S_1) \otimes_{S_1} \mathbb{Z}\bar{G},$$

where \bar{G} stands for the amalgamated free product $\bar{G} = G *_{S_1} \bar{S}_1$ and $\bar{\mathcal{S}}$ for the family $\bar{\mathcal{S}} = \{\bar{S}_1, S_2, \dots, S_m\}$ of subgroups of \bar{G} . Hence it suffices to prove that $C(\bar{G}, \bar{\mathcal{S}})$ is free-Abelian. Repeating the argument shows that we may assume that all subgroups S_1, \dots, S_m are accessible. The proof of [9], Corollary 1.4 now carries over.

6. The case when $0 < \text{rk } C(G, \mathcal{S}) < \infty$

6.1. Throughout this section we assume G to be a finitely generated accessible group and $\mathcal{S} = \{S_1, \dots, S_m\}$ a finite non-empty family of finitely generated infinite subgroups such that the cokernel $C(G, \mathcal{S})$ of (1.1) is of finite \mathbb{Z} -rank > 0 .

LEMMA 6.1. *Under these assumptions the restriction map (1.1) is injective, so that one has the short exact sequence of G -modules.*

$$H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G) \rightarrow C(G, \mathcal{S}). \quad (6.1)$$

Proof. If not, then by Swarup's relative version of Stalling's structure theorem [14], after replacing the groups S_i by suitable conjugates, the pair (G, \mathcal{S}) decomposes non-trivially as an amalgamated product of two pairs (G_i, \mathcal{S}_i) , $i = 1, 2$ or as an HNN-extension over a pair (G_1, \mathcal{S}_1) , where in either case the amalgamated (associated) subgroup is finite. Writing C_i for the cokernel $C(G_i, \mathcal{S}_i)$ we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} H^1(G; \mathbb{Z}G) & \rightarrow & H^1(\mathcal{S}; \mathbb{Z}G) & \rightarrow & C(G, \mathcal{S}) & \rightarrow 0 \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\ \prod H^1(G_i; \mathbb{Z}G) & \rightarrow & \prod H^1(\mathcal{S}_i; \mathbb{Z}G) & \rightarrow & \prod C_i \otimes_{G_i} \mathbb{Z}G & \rightarrow 0 & \end{array}$$

α is the restriction which occurs in the Mayer–Vietoris sequence for G ; hence, as the amalgamated subgroup is finite, α is epimorphic. \mathcal{S} is the disjoint union of \mathcal{S}_1

and \mathcal{S}_2 ; hence β is the identity. It follows by the 5-Lemma that γ is an isomorphism. Therefore one of the G -modules $C_i \otimes_{G_i} \mathbb{Z}G$ is of finite \mathbb{Z} -rank >0 . But this implies that G_i is of finite index in G which is impossible.

6.2. Dunwoody's accessibility criterion [4] asserts that a group G is accessible if and only if the cohomology group $H^1(G; \mathbb{Z}G)$ is finitely generated as a right G -module. From our assumption that G is accessible and $C(G, \mathcal{S})$ free-Abelian of finite rank it thus follows that $H^1(\mathcal{S}; \mathbb{Z}G)$ and hence each $H^1(S_i; \mathbb{Z}G) \cong H^1(S_i; \mathbb{Z}S_i) \otimes_{S_i} \mathbb{Z}G$ is finitely generated over $\mathbb{Z}G$. As $\mathbb{Z}G$ is a free $\mathbb{Z}S_i$ -module we can infer that $H^1(S_i; \mathbb{Z}S_i)$ is finitely generated over $\mathbb{Z}S_i$. Hence all groups S_i , $1 \leq i \leq m$, are accessible by the criterion again.

Thus the absolute version of Theorem 4.1 applies for both G and S_i , $1 \leq i \leq m$. Hence the G -modules $H^1(G; \mathbb{Z}G)$ and $H^1(\mathcal{S}; \mathbb{Z}G)$ are of type $(FP)_{\infty}$, and in view of the short exact sequence (6.1) so is $C(G, \mathcal{S})$. Moreover the $\mathbb{Q}G$ -modules $H^1(G; \mathbb{Q}G)$ and $H^1(\mathcal{S}; \mathbb{Q}G)$ are of type (FP) and of projective dimension ≤ 1 . Hence the short exact sequence (6.1), when tensored with \mathbb{Q} , shows that $C_{\mathbb{Q}}(G, \mathcal{S}) = C(G, \mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}G$ -module of type (FP) and of projective dimension ≤ 2 .

By Lemma 3.1 we can now infer that the group G is of type $(FP)_{\infty}$ over \mathbb{Z} and of type (FP) with $\text{cd}_{\mathbb{Q}} G \leq 2$ over \mathbb{Q} .

6.3. Our next aim is to show that the kernel $\Delta = \Delta_{G/\mathcal{S}}$ of the augmentation map $\varepsilon: \mathbb{Z}G/\mathcal{S} \rightarrow \mathbb{Z}$ (4.1) is a G -module of type $(FP)_1$. To that end take an arbitrary direct power $\prod \mathbb{Z}G$ of copies of $\mathbb{Z}G$, and apply $\text{Tor}_1^{\mathbb{Z}G}(\prod \mathbb{Z}G, -)$ to the short exact sequence (4.1). This yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} \text{Tor}_1^{\mathbb{Z}G}(\prod \mathbb{Z}G, \mathbb{Z}) & \rightarrow & (\prod \mathbb{Z}G) \otimes_G \Delta & \rightarrow & (\prod \mathbb{Z}G) \otimes_G \mathbb{Z}(G/\mathcal{S}) & \rightarrow & (\prod \mathbb{Z}G) \otimes_G \mathbb{Z} \rightarrow 0 \\ \downarrow \mu_1 & & \downarrow \mu_2 & & \downarrow \mu_3 & & \\ 0 & \rightarrow & \prod \Delta & \rightarrow & \prod \mathbb{Z}G/\mathcal{S} & \rightarrow & \prod \mathbb{Z} \rightarrow 0 \end{array} .$$

where the vertical arrows stand for the limiting homomorphism (e.g., $\mu_1(\prod \lambda_i \otimes d) = \prod \lambda_i d$, $\lambda_i \in \mathbb{Z}G$, $d \in \Delta$). Since \mathbb{Z} is of type $(FP)_{\infty}$ as a G -module $\text{Tor}_1^{\mathbb{Z}G}(\prod \mathbb{Z}G, \mathbb{Z}) = 0$ and μ_3 is an isomorphism. \mathcal{S} is a finite family of finitely generated subgroups of G , hence $\mathbb{Z}G/\mathcal{S}$ is of type $(FP)_1$ and μ_2 is an isomorphism. It follows that μ_1 is an isomorphism, whence Δ is of type $(FP)_1$ (see e.g. [1], chapter I).

6.4. From Section 6.3. we infer that the $\mathbb{Q}G$ -module $\Delta_{\mathbb{Q}} = \Delta \otimes \mathbb{Q}$ is of type $(FP)_1$. So let us choose a $\mathbb{Q}G$ -projective resolution

$$P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \longrightarrow \Delta_{\mathbb{Q}} \quad (6.2)$$

which is finitely generated in dimensions 0 and 1, and which we can use to compute the relative cohomology groups $H^n(G, \mathcal{S}; \mathbb{Q}G)$ for $n = 1$ and 2. Also, we have the long exact sequence for the pair (G, \mathcal{S})

$$\cdots \rightarrow H^0(\mathcal{S}; \mathbb{Q}G) \rightarrow H^1(G, \mathcal{S}; \mathbb{Q}G) \rightarrow H^1(G; \mathbb{Q}G) \xrightarrow{\text{res}} H^1(\mathcal{S}; \mathbb{Q}G) \rightarrow H^2(G, \mathcal{S}; \mathbb{Q}G) \rightarrow$$

where res is injective by Lemma 6.1. Since all groups in \mathcal{S} are infinite $H^0(\mathcal{S}; \mathbb{Q}G) = 0$ and hence $H^1(G, \mathcal{S}; \mathbb{Q}G) = 0$. This shows that

$$0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \text{coker } (\partial_1^*) \rightarrow 0,$$

with $P_i^* = \text{Hom}_{\mathbb{Q}G}(P_i, \mathbb{Q}G)$ is a short exact sequence. But P_0^* and P_1^* are finitely generated projective right $\mathbb{Q}G$ -modules, hence $\text{coker } (\partial_1^*)$ is a $\mathbb{Q}G$ -module of projective dimension ≤ 1 . Clearly $\text{coker } (\partial_1^*)$ contains $\ker \partial_2^*/\text{im } \partial_1^* = H^2(G, \mathcal{S}; \mathbb{Q}G)$ which, in turn, contains the submodule $C_{\mathbb{Q}}(G, \mathcal{S})$ of finite \mathbb{Q} -dimension. By Corollary 3.2 this implies that the cohomology dimension of G over \mathbb{Q} is in fact ≤ 1 .⁽²⁾ Hence by Dunwoody's generalization of Stallings' theorem [4] G contains a free subgroup of finite index.

We summarize

THEOREM 6.2. *Let G be a finitely generated accessible group and \mathcal{S} a finite family of finitely generated subgroups of G . If the cokernel $C(G, \mathcal{S})$ of the restriction map*

$$H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G)$$

is of finite \mathbb{Z} -rank > 0 then G contains a free subgroup of finite index.

Remark. It follows, in particular, that in the situation of Theorem 6.2 one has $H^2(G; \mathbb{Z}G) = 0$. Hence the long exact sequence for (G, \mathcal{S}) shows that $H^2(G, \mathcal{S}; \mathbb{Z}G) \cong C(G, \mathcal{S})$.

6.5. It remains to examine the situation when G is a finitely generated infinite free-by-finite group and \mathcal{S} a finite family of m infinitely generated, infinite subgroups. Then G can be thought of as the fundamental group of a finite graph

² This type of argument was used by Farrell [6]

(\mathcal{G}, X) of finite groups. Let V denote the set of vertices and E the set of positive edges of X . Then Theorem 4.1 yields the formula

$$\mu(H^1(G; \mathbb{Q}G)) = \sum_E \frac{1}{|G_e|} - \sum_V \frac{1}{|G_v|}.$$

But this is precisely the negative of the formula for the Euler characteristic $\mu(G) = \mu(\mathbb{Q})$ (e.g. [3], Theorem 2). Hence we have

$$\mu(H^1(G; \mathbb{Q}G)) = -\mu(G),$$

and similar for S_i ,

$$\begin{aligned} \mu(H^1(S_i; \mathbb{Q}G)) &= \mu(H^1(S_i; \mathbb{Q}S_i) \otimes_{S_i} \mathbb{Z}G) \\ &= \mu(H^1(S_i; \mathbb{Q}S_i)) = -\mu(S_i). \end{aligned}$$

From the short exact sequence (6.1) we now obtain the formula

$$\mu(C_{\mathbb{Q}}(G, \mathcal{S})) = \mu(G) - \sum_{i=1}^m \mu(S_i) \tag{6.3}$$

On the other hand $C_{\mathbb{Q}}(G, \mathcal{S})$ is a $\mathbb{Q}G$ -module of finite \mathbb{Q} -dimension, whence $\mu(C_{\mathbb{Q}}(G, \mathcal{S})) = \dim C_{\mathbb{Q}}(G, \mathcal{S}) \cdot \mu(G)$ (see e.g. [3], Lemma 8). Together with (6.3) this yields the equation

$$\mu(G)(\text{rk } C(G, \mathcal{S}) - 1) + \sum_{i=1}^m \mu(S_i) = 0. \tag{6.4}$$

Let F be a free subgroup of finite index in G and n the rank of F . Then $\mu(F) = 1 - n = |G : F| \cdot \mu(G)$. This shows that $\mu(G)$ is ≤ 0 and $\mu(G) = 0$ if and only if G is infinite cyclic-by-finite. Of course the same holds for $\mu(S_i)$; hence we can deduce from (6.4) that $\mu(S_i) = 0$ for $1 \leq i \leq m$ and either $\mu(G) = 0$ or $\text{rk } C(G, \mathcal{S}) = 1$. In other words: all groups S_i , $1 \leq i \leq m$, contain an infinite cyclic subgroup of finite index and either the same holds for G itself or one has $C(G, \mathcal{S}) \cong \mathbb{Z}$.

Remark. Instead of using Euler characteristics A. Freudenberger [Diplomarbeit 1982, University of Freiburg im Breisgau, Germany] obtains formula (6.4) by computing the \mathbb{Q} -dimensions in the long exact homology sequence of G with coefficients in (6.1) tensored with \mathbb{Q} .

6.6. The proof of Theorems A and B is now easily completed: If G is infinite cyclic-by-finite then the index $|G : S_i|$ is finite for all $1 \leq i \leq m$, $H^1(G; \mathbb{Z}G) \cong \mathbb{Z}$, and $H^1(\mathcal{S}; \mathbb{Z}G) = \prod H^1(S_i; \mathbb{Z}S_i) \otimes_{S_i} \mathbb{Z}G = \mathbb{Z}(\mathcal{S} \setminus G)$ is free-Abelian of rank $\sum |G : S_i|$. By the short exact sequence (6.1) we thus have

$$\text{rk } C(G, \mathcal{S}) = \sum_{i=1}^m |G : S_i| - 1.$$

On the other hand, if $\mu(G) \neq 0$ and hence $C(G, \mathcal{S}) \cong \mathbb{Z}$ we consider a free subgroup F of finite index in G and the full subpair (F, \mathcal{S}') of (G, \mathcal{S}) given by F (c.f. Section 2.2). By [2], Proposition 7.5, we have

$$H^2(F, \mathcal{S}'; \mathbb{Z}F) \cong H^2(G, \mathcal{S}; \mathbb{Z}G) \cong C(G, \mathcal{S}) \cong \mathbb{Z}.$$

Hence (F, \mathcal{S}') is a 2-dimensional Poincaré duality pair by the PD^2 -criterion [2] Theorem 9.3.

7. The case when $C(G, \mathcal{S}) = 0$.

7.1. Here we have to consider compatible decompositions of the pair (G, \mathcal{S}) as defined in Section 2.3. That is, both G and \mathcal{S} are “fundamental groups of graphs of groups” $G = \pi_1(\mathbb{G}, X)$ $\mathcal{S} = \pi_1(\mathbb{S}, Y)$ – where the graph Y is not necessarily connected – and there is given an orientation preserving graph map $f: Y \rightarrow X$ and for each vertex v of Y a group element $c_v \in G$ such that the equations (2.1) and (2.2) are satisfied.

One feature of compatible decompositions is a natural homomorphism between the Mayer–Vietoris sequences of G and \mathcal{S} . Indeed one has the commutative diagram of G -modules.

$$\begin{array}{ccccccc}
 \bigoplus \mathbb{Z}G/S_e & \longrightarrow & \bigoplus \mathbb{Z}G/S_v & \longrightarrow & \mathbb{Z}G/S & & \\
 \downarrow f_1 & & \downarrow f_0 & & \downarrow \varepsilon & & \\
 \bigoplus \mathbb{Z}G/G_e & \longrightarrow & \bigoplus \mathbb{Z}G/G_v & \longrightarrow & \mathbb{Z} & & \\
 \downarrow E(X) & & \downarrow V(Y) & & & & \\
 \end{array} \tag{7.1}$$

where $V(X)$, $V(Y)$ stand for the set of vertices and $E(X)$, $E(Y)$ for the set of positive edges of X resp. Y . The rows are the short exact sequences [9], p. 168

and the vertical maps are given by

$$\left. \begin{aligned} f_1(gS_e) &= gc_{o(e)}G_{f(e)} \\ f_0(gS_v) &= gc_vG_{f(v)} \end{aligned} \right\} g \in G, \quad v \in V(Y), \quad e \in E(Y)$$

Commutativity of the diagram is guaranteed by (2.2). Applying the functor $\text{Ext}_{\mathbb{Z}G}^*(-, A)$, A an arbitrary G -module, and using the Shapiro Lemma thus yields the commutative ladder

$$\begin{array}{ccccccc} \cdots & H^k(G; A) & \rightarrow & \prod_{V(X)} H^k(G_v; A) & \rightarrow & \prod_{E(X)} H^k(G_e; A) & \rightarrow H^{k+1}(G; A) \rightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots & H^k(\mathcal{S}; A) & \rightarrow & \prod_{V(Y)} H^k(S_v; A) & \rightarrow & \prod_{E(Y)} H^k(S_e; A) & \rightarrow H^{k+1}(\mathcal{S}; A) \rightarrow \cdots \end{array} \quad (7.2)$$

7.2. We are now in a position to prove

PROPOSITION 7.1. *Let G be a group and \mathcal{S} a family of subgroups. Assume that G and \mathcal{S} have compatible decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$, via a graph map $f: Y \rightarrow X$ which is injective on the edges, and such that the following two conditions hold*

- (i) $S_e^{c_{o(e)}} = G_{f(e)}$ for every edge $e \in E(Y)$
- (ii) all vertex groups S_v of \mathcal{S} have ≤ 1 end.

Then the restriction map $H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G)$ is surjective.

Proof. The condition (i) implies that the vertical map f_1 in the diagram (7.1) is the injection of a direct summand. Hence $\text{Hom}_G(f_1, \mathbb{Z}G) = f_1^*$ is surjective and (7.2) with $A = \mathbb{Z}G$ yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} \prod H^0(G_e; \mathbb{Z}G) & \longrightarrow & H^1(G; \mathbb{Z}G) & \longrightarrow & \prod H^1(G_v; \mathbb{Z}G) & & \\ f_1^* \downarrow & & \text{res} \downarrow & & \downarrow & & \\ \prod H^0(S_e; \mathbb{Z}G) & \xrightarrow{\delta} & H^1(\mathcal{S}; \mathbb{Z}G) & \longrightarrow & \prod H^1(S_v; \mathbb{Z}G) & & \end{array}$$

Now, condition (ii) asserts that $H^1(S_v; \mathbb{Z}G) = 0$ for all $v \in V(Y)$; hence δ is an epimorphism and so is res .

7.3. It remains to prove the following converse of Proposition 7.1.

PROPOSITION 7.2. *Let G be a finitely generated group and $\mathcal{S} = \{S_i \mid i \in I\}$ a finite family of finitely generated accessible subgroups. If the restriction map $\text{res} : H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G)$ is surjective then G and \mathcal{S} have compatible decompositions $G = \pi_1(\mathfrak{G}, X)$, $\mathcal{S} = \pi_1(\mathfrak{S}, Y)$ via an orientation preserving graph map $f : Y \rightarrow X$ which is bijective on the edges and such that*

- (i) *for every edge $e \in E(Y)$ the edge group G_e is finite and coincides with $S_e^{c_{\alpha(e)}}$,*
- (ii) *all vertex groups S_v of \mathcal{S} have ≤ 1 end.*

Proof. Since \mathcal{S} is a finite family of finitely generated accessible groups \mathcal{S} can be written as the “fundamental group” of some finite graph of groups (\mathfrak{S}, Y) with all edge groups S_e finite and all vertex groups S_v having ≤ 1 end. If we arrange (\mathfrak{S}, Y) such that all embeddings $S_e < G_{o(e)}$ are proper, then the number of edge pairs of Y is an invariant of \mathcal{S} which we call the *complexity*.

We shall prove Proposition 7.2 by induction on the complexity of \mathcal{S} . If the complexity is $= 0$ then every S_i has ≤ 1 end and the proposition holds with X consisting of one vertex and no edges and Y consisting of an isolated vertex for every $i \in I$. If \mathcal{S} has complexity > 0 then $H^1(\mathcal{S}; \mathbb{Z}G) \neq 0$. So assume $H^1(S_1; \mathbb{Z}G) \neq 0$, and put $\mathcal{T} = \mathcal{S} - \{S_1\}$. Since res is surjective H. Müller's first decomposition Theorem applies ([7], Corollary 3.1.). Thus after replacing the groups in \mathcal{T} by suitable conjugates the pair (G, \mathcal{T}) and the subgroup S_1 have a proper simultaneous decomposition in the following sense. Either $G = G_1 *_K G_2$ and $S_1 = S_{11} *_K S_{12}$ where K is finite, $S_{1i} \leq G_i$ ($i = 1, 2$), and \mathcal{T} is the disjoint of families $\mathcal{T}_1, \mathcal{T}_2$ of subgroups of G_1 , resp. G_2 ; or $G = G_1 *_K p$ is an HNN-group with stable letter p and finite associated subgroups K, pKp^{-1} , \mathcal{T} consists of subgroups of G_1 , and S_1 is either $= S_{11} *_K p$ or $= S_{11} *_K p S_{12} p^{-1}$, with $S_{11}, S_{12} \leq G_1$. Note that all these decompositions of G and S_1 are compatible in the sense of Section 2.3.

We restrict the discussion to the first case, the other cases being similar. By (7.2) we have a map between the Mayer–Vietoris sequences for G and S_1 , and adding $H^1(\mathcal{T}, \mathbb{Z}G)$ to the latter yields the commutative diagram with exact rows

$$\begin{array}{ccccccc}
 H^0(K; \mathbb{Z}G) & \rightarrow & H^1(G; \mathbb{Z}G) & \rightarrow & \bigoplus_{i=1}^2 H^1(G_i; \mathbb{Z}G) & \rightarrow & 0 \\
 \downarrow = & \downarrow \text{res} & & & \downarrow \oplus \text{res}_i & & \\
 H^0(K; \mathbb{Z}G) & \rightarrow & H^1(\mathcal{S}; \mathbb{Z}G) & \rightarrow & \bigoplus_{i=1}^2 H^1(S_{1i}; \mathbb{Z}G) \oplus H^1(\mathcal{T}; \mathbb{Z}G) & \rightarrow & 0
 \end{array}$$

from which we deduce that the restriction map

$$\text{res}_i : H^1(G_i; \mathbb{Z}G_i) \rightarrow H^1(\mathcal{T} \cup \{S_{1i}\}; \mathbb{Z}G_i)$$

is surjective for $i = 1, 2$. Now the complexity of $\mathcal{T}_i \cup \{S_{1i}\}$ is less than that of \mathcal{S} . Hence, by induction, G_i and $\mathcal{T}_i \cup \{S_{1i}\}$ have a compatible decomposition satisfying the assertion of Proposition 7.2. Putting these together yields a compatible decomposition of G and \mathcal{S} with the required properties.

Remark. Instead of assuming that the subgroups in \mathcal{S} are accessible in Proposition 7.2 one could also assume that the group G is accessible. Indeed, by Dunwoody's criterion [4] this would mean that $H^1(G; \mathbb{Z}G)$ is finitely generated as a right G -module. Since $\text{res}: H^1(G; \mathbb{Z}G) \rightarrow H^1(\mathcal{S}; \mathbb{Z}G)$ is assumed to be surjective the same holds for \mathcal{S} , implying that every $S_i \in \mathcal{S}$ is accessible.

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