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**Autor:** Kra, Irwin / Maskit, Bernard  
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## Bases for quadratic differentials\*

IRWIN KRA and BERNARD MASKIT

Let  $G$  be a non-elementary finitely generated Kleinian group and  $q$  be an integer bigger than one. It is well known (see, for example, Bers [7]) that every cusp form of weight  $-2q$  is the Poincaré series of a rational function with poles only on the limit set of  $G$ . There are at least three interesting problems all related, but somewhat independent, associated with the spaces of cusp forms for the group  $G$  and the Poincaré series operator.

(I) Find necessary and sufficient conditions for the Poincaré series of a rational function to vanish identically.

(II) Construct bases for the spaces of cusp forms that vary holomorphically with moduli.

(III) Construct (finite dimensional) spaces of rational functions that vary holomorphically with moduli so that the Poincaré series operator establishes an isomorphism between these spaces and the spaces of cusp forms for the Kleinian groups.

The first problem is probably the hardest. There are many formal reasons that force a Poincaré series to vanish (see, for example, [19]). A computational algorithm for determining whether or not a Poincaré series vanishes is more difficult to obtain. Deep and interesting work of Hejhal [11], [12], [13] has resulted in one solution to problem (I) for Schottky and Fuchsian groups.

Bers [5] has obtained bases for cusp forms (supported on a single component) for quasi-fuchsian groups. Earle (private communication) has observed that a set of global coordinates for Teichmüller space always leads to bases for quadratic differentials (the case  $q = 2$ ) that vary holomorphically with moduli. These are partial solutions to problem (II).

In this paper we are mainly concerned with problem (III). In the first part of the paper, we give an algorithm for determining spaces of rational functions that solve problem (III) for geometrically finite function groups and for  $q = 2$ . These spaces are constructed via the stratifications we have introduced in [18], and

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exploit techniques involving the tangent bundle to the deformation spaces of a Kleinian group.

As a by-product we exhibit large families of rational functions with non-vanishing Poincaré series, and we obtain global holomorphic trivializations of the cotangent bundle of the space of deformations of a finitely generated function group.

Similar (but by no means identical) bases for spaces of quadratic differentials were obtained previously by Bers [8] and Hejhal [11] for Schottky groups. Wolpert [27], [28] has shown that lengths of certain finite sets of geodesics are local coordinates on Teichmüller space. As a consequence he obtains bases for spaces of quadratic differentials for Fuchsian groups. The work of Earle [9] on coordinates for Teichmüller space can also be used to obtain such bases for certain Kleinian groups. Similar constructions enter the unpublished work of Bers and of Earle and Marden on coordinates for the spaces of Riemann surfaces with nodes.

The second part of our paper is an investigation of the simplest geometrically finite Kleinian groups which are not function groups; these are the groups with two components, neither invariant. Special cases of such groups appear in [9]. We show that every such group is a quasiconformal deformation of a  $\mathbf{Z}_2$ -extension of a Fuchsian group, and we stratify all such groups.

## §1. Deformation spaces and fiber spaces

1.1. Let  $G$  be a finitely generated non-elementary Kleinian group (that is,  $G$  is a discrete group of Möbius transformations which operates discontinuously at some point of the extended complex plane  $\hat{\mathbf{C}}$ ). As usual, we denote the limit set of  $G$  by  $\Lambda = \Lambda(G)$  and the region of discontinuity of  $G$  by  $\Omega = \Omega(G)$ . Let  $\Delta$  be a  $G$ -invariant union of (connected) components of  $\Omega$ .

The Banach space  $L^\infty(G, \Delta)$  is the space of  $L^\infty$  functions with support in  $\Delta$ , which satisfy

$$(\mu \circ g)\bar{g}' = \mu g' \quad \text{for all } g \in G. \quad (1)$$

In general, a bounded function on  $\hat{\mathbf{C}}$  satisfying (1) is called a *Beltrami differential* (for  $G$ ).

In the special case that  $\Delta = \Omega$ , we set  $L^\infty(G) = L^\infty(G, \Omega)$ .

A point in the open unit ball  $M(G, \Delta) \subset L^\infty(G, \Delta)$  is called a *Beltrami coefficient* (for  $G$ ).

Now let  $x_1, x_2, x_3$  be three distinct points of  $\hat{\mathbf{C}}$ , and let  $w$  be a quasiconformal

homeomorphism of  $\hat{\mathbf{C}}$ . We say that  $w$  is *normalized at*  $(x_1, x_2, x_3)$  if  $w(x_i) = x_i$ ,  $i = 1, 2, 3$ . Every quasiconformal homeomorphism  $w$  has a well defined Beltrami coefficient  $\mu = w_{\bar{z}}/w_z$  (we say that  $w$  is  $\mu$ -conformal), and conversely, given a Beltrami coefficient  $\mu$ ; there is a unique  $\mu$ -conformal homeomorphism  $w^\mu$  normalized at  $(x_1, x_2, x_3)$  (see Ahlfors–Bers [2]).

A quasiconformal homeomorphism  $w$  is said to be *compatible* with  $G$  provided that for all  $g \in G$ ,  $w \circ g \circ w^{-1}$  is a Möbius transformation. In this case,  $w$  induces an isomorphism  $\theta_w$  defined by

$$\theta_w(g) = w \circ g \circ w^{-1}, \quad g \in G,$$

of  $G$  onto another Kleinian group.

For any normalization,  $w$  is compatible with  $G$  if and only if  $w = a \circ w^\mu$  for some Beltrami coefficient  $\mu$  for the group  $G$  and some Möbius transformation  $a$ .

If  $w$  and  $w^*$  are compatible with  $G$  and induce the same isomorphism up to conjugation (that is, there is a Möbius transformation  $a$  so that  $\theta_w = \theta_{a \circ w^*}$ ) then we say that  $w$  and  $w^*$  are *equivalent*.

The *deformation space*  $T(G, \Delta)$  is defined to be the set of equivalence classes of compatible quasiconformal homeomorphisms which are conformal off  $\Delta$ . Equivalently, it is  $M(G, \Delta)$  factored by the relation:  $\mu \sim \mu^*$  if  $w^\mu$  is equivalent to  $w^{\mu^*}$ . The map

$$\Phi : M(G, \Delta) \rightarrow T(G, \Delta)$$

endows  $T(G, \Delta)$  with a topology and complex structure.

It is well known that  $T(G, \Delta)$  is a complex manifold (see [6], [15], [21]).

For the case (the primary one that concerns us here) that  $\Delta = \Omega$ , we set

$$d = d(G) = \dim T(G, \Omega).$$

A finitely generated Kleinian group is called *stratifiable* if there are  $d+3$  distinct points  $x_1, x_2, x_3, y_1, \dots, y_d \in \hat{\mathbf{C}}$  so that if we use  $(x_1, x_2, x_3)$  to normalize each  $w^\mu$ , then the mapping

$$\Psi(w^\mu) = (w^\mu(y_1), w^\mu(y_2), \dots, w^\mu(y_d))$$

defines a biholomorphic embedding of  $T(G)$  onto an open subset of  $\hat{\mathbf{C}}^d$ .

The set  $x_1, x_2, x_3, y_1, \dots, y_d$  is called a *stratification* of  $G$ ; we also say that  $x_1, \dots, y_d$  *stratify*  $G$ .

The starting point in this paper is the main result of our preceding paper.

**THEOREM (Kra-Maskit) [18]).** *Every non-elementary geometrically finite function group  $G$  is stratifiable.*

To establish the above theorem, we use Maskit's description of geometrically finite function groups to obtain generators for  $G$ . Appropriate fixed points of these generators appear in the stratification for  $G$  (see also §6). We have essentially produced an algorithm for obtaining a stratification of  $G$ , once one knows how  $G$  is built up from simpler groups.

1.2. Given the Kleinian group  $G$ , a point  $x \in \hat{\mathbb{C}}$  is called *sturdy* if the following holds. Whenever  $w, w^*$  are compatible with  $G$  and  $\theta_w = \theta_{w^*}$ , then  $w(x) = w^*(x)$ . We remark that limit points are always sturdy; points of  $\Omega$  which are not fixed points are never sturdy; fixed points of elliptic elements of order  $>2$  are always sturdy; fixed points of elliptic elements of order 2 are sometimes sturdy and sometimes not. We note that if  $G$  has exactly two components, then elliptic fixed points of order 2 are sturdy.

1.3. Let  $G$  be a non-elementary Kleinian group, and let  $x_1, x_2, x_3$  be distinct sturdy points for  $G$ . Let  $\Delta$  be an invariant union of components of  $G$ .

For each  $\mu \in M(G, \Delta)$ , we set

$$G^\mu = w^\mu G (w^\mu)^{-1},$$

$$\Delta^\mu = w^\mu(\Delta),$$

where  $w^\mu$  is normalized at  $(x_1, x_2, x_3)$ . Also, for each  $g \in G$ , we set

$$g^\mu = w^\mu \circ g \circ (w^\mu)^{-1},$$

and we denote the isomorphism  $g \mapsto g^\mu$  by  $\theta_\mu$  instead of  $\theta_{w^\mu}$ .

1.4. We need the *Bers fiber space*

$$F(G, \Delta) = \{(\Phi(\mu), z) \mid \mu \in M(G, \Delta), z \in \Delta^\mu\}.$$

It is easy to see that  $F(G, \Delta)$ , is a complex manifold of dimension  $\dim T(G, \Delta) + 1$ .

We denote the projection on the first factor by

$$\pi : F(G, \Delta) \rightarrow T(G, \Delta).$$

The group  $G$  operates on  $F(G, \Delta)$ , so as to preserve the fiber of this projection: if  $g \in G$ ,

$$g(\Phi(\mu), z) = (\Phi(\mu), g^\mu(z)), \quad \mu \in M(G, \Delta), \quad z \in w^\mu(\Delta).$$

The quotient space  $V(G, \Delta) = F(G, \Delta)/G$  is also a complex manifold of the same dimension, but not necessarily connected ( $V(G, \Delta)$  has the same number of components as  $\Delta/G$ ).

If  $\Delta = \Omega$ , we use the notation:

$$T(G) = T(G, \Omega), \quad F(G) = F(G, \Omega), \quad V(G) = V(G, \Omega).$$

## §2. Quadratic differentials

2.1. Let  $G$  be a non-elementary stratifiable Kleinian group, and let  $y, x_1, x_2, x_3$  be four distinct sturdy points for  $G$ . We normalize each  $w^\mu$  at  $(x_1, x_2, x_3)$ .

We define the function  $\varphi_y : F(G) \rightarrow \hat{\mathbf{C}}$  as follows (here  $\mu \in M(G), z \in w^\mu(\Omega)$ ):

$$\varphi_y(\Phi(\mu), z) = \frac{w^\mu(y) - x_1}{(z - x_1)(z - x_2)(z - x_3)(z - w^\mu(y))}, \quad \text{if } x_1, x_2, x_3, w^\mu(y) \neq \infty; \tag{2}$$

$$\varphi_y(\Phi(\mu), z) = \frac{-1}{(z - x_1)(z - x_2)(z - x_3)}, \quad \text{if } w^\mu(y) = \infty;$$

$$= \frac{w^\mu(y) - x_1}{(z - x_1)(z - x_2)(z - w^\mu(y))}, \quad \text{if } x_3 = \infty;$$

$$= \frac{w^\mu(y) - x_1}{(z - x_1)(z - x_3)(z - w^\mu(y))}, \quad \text{if } x_2 = \infty;$$

$$= \frac{1}{(z - x_2)(z - x_3)(z - w^\mu(y))}, \quad \text{if } x_1 = \infty.$$

Using the well known fact [2] that  $w^\mu$  is holomorphic in  $\mu$ , one sees at once that  $\varphi_y$  is a holomorphic map  $F(G) \rightarrow \hat{\mathbf{C}}$ .

It is also quite easy to see that  $\varphi_y$  is integrable over each fiber of  $F(G)$ ; that is,

$$\iint_{\Omega^\mu} |\varphi_y(\Phi(\mu), z) dz \wedge d\bar{z}| < \infty$$

(in fact, the rational function  $\varphi_y(\Phi(\mu), \cdot)$  is integrable over all of  $\hat{\mathbf{C}}$ ).

We now define the *Poincaré series operator*  $\Theta$  by the formula

$$(\Theta\varphi_y)(\Phi(\mu), z) = \sum_{g \in G} \varphi_y(\Phi(\mu), g^\mu(z)) \left( \frac{d}{dz} g^\mu(z) \right)^2. \tag{3}$$

It is a routine computation to see that this series converges uniformly and absolutely on compact subsets of  $F(G)$ .

For each fixed  $\mu$ , the series (3) is a classical Poincaré series on  $\Omega^\mu$  and  $\Theta\varphi_y(\Phi(\mu), z)$  is an integrable quadratic differential in  $z$  on  $\Omega^\mu$ ; that is,

$$(\Theta\varphi_y)(\Phi(\mu), g^\mu(z)) \cdot \left( \frac{d}{dz} g^\mu(z) \right)^2 = (\Theta\varphi_y)(\Phi(\mu), z), \tag{4}$$

for all  $g \in G$ ; also

$$\iint_{\Omega^\mu/G^\mu} |(\Theta\varphi_y)(\Phi(\mu), z) dz \wedge d\bar{z}| \leq \iint_{\Omega^\mu} |\varphi_y(\Phi(\mu), z) dz \wedge d\bar{z}| < \infty. \tag{5}$$

We remark that  $\Theta\varphi_y(\Phi(\mu), z)$  is holomorphic even if  $y$  or some  $x_i$  is an elliptic fixed point in  $\Omega(G)$ . To see this, we fix  $\mu$  and renormalize so that  $w^\mu(y) = 0$  (or  $x_i = 0$ ), and the maximal elliptic subgroup of  $G^\mu$  with fixed point at 0 is  $\{z \mapsto e^{2\pi ip/q}z, p = 0, \dots, q-1\}$ . We let  $H$  be the corresponding subgroup of  $G$  and we write the sum in (3) as first a sum over  $H$ , then a sum over  $G/H$ . It suffices to show that the sum over  $H$  is regular at 0. We rewrite (2) in its partial fraction decomposition, and note that the coefficient of  $1/z$  for  $\varphi_y(\Phi(\mu), z)$  is  $1/x_2x_3$  (assuming  $x_2 \neq \infty \neq x_3$ ). Then the sum over  $H$  of the singular terms reduces to

$$\frac{1}{x_2x_3} \sum_{p=0}^{q-1} e^{2\pi ip/q} = 0.$$

For each  $\mu \in M(G)$ , we let  $Q(G^\mu)$  be the space of holomorphic integrable quadratic differentials for  $G^\mu$  (that is, the space of holomorphic functions on  $\Omega(G^\mu)$  satisfying (4) and (5)). We denote the dimension of this space by  $d$  (of course,  $d$  is independent of  $\mu$ ); it is well known that  $d = \dim T(G)$ .

Our main result is the following:

**THEOREM 1.** *Let  $G$  be a stratifiable Kleinian group with stratification  $x_1, x_2, x_3, y_1, \dots, y_d$ . Then for each  $\mu \in M(G)$ , the  $d$  functions*

$$\Theta\varphi_{y_1}(\Phi(\mu), z), \dots, \Theta\varphi_{y_d}(\Phi(\mu), z),$$

*defined on  $\Omega(G^\mu)$ , form a basis (over  $\mathbb{C}$ ) for  $Q(G^\mu)$ .*

As remarked before, for fixed  $\mu$  the restriction of  $\Theta$  to  $\Omega(G^\mu)$  is the classical Poincaré  $\Theta$  operator. In this situation, we will write  $\varphi_y(z)$  for  $\varphi_y(\Phi(\mu), z)$ , etc.

2.2. We now fix  $\mu$ , and consider the space  $R^\mu$  of rational functions  $f(z)$ , where the poles of  $f$  are all simple and occur only at some of the points:  $\{x_1, x_2, x_3, w^\mu(y_1), \dots, w^\mu(y_d)\}$ ; we require further that  $f(z) = O(|z|^{-4})$  near  $\infty$ , if  $\infty$  is not one of these points, and we require  $f(z) = O(|z|^{-3})$  if  $\infty$  is one of these points. One easily sees that the vector space  $R^\mu$  has dimension  $d$ .

Our theorem asserts that the Poincaré series operator establishes an isomorphism between  $R^\mu$  and  $Q(G^\mu)$ .

Since  $\{x_1, \dots, y_d\}$  is a stratification for  $G$ , the space  $R^\mu$  depends, not on  $\mu$ , but on  $\Phi(\mu) \in T(G)$ . Then  $R^\mu$  and  $Q(G^\mu)$  are the fibers of trivial  $d$ -dimensional vector bundles over  $T(G)$ .

2.3. We again return to the case that  $\Delta$  is an invariant union of components of  $G$ . We let  $Q(G, \Delta)$  be the subspace of  $Q(G)$  consisting of those quadratic differentials supported on  $\Delta$ .

For  $\varphi \in Q(G, \Delta)$  and  $\mu \in L^\infty(G, \Delta)$ , we introduce the pairing

$$\langle \varphi, \mu \rangle = \frac{1}{2} \iint_{\Delta/G} \varphi(z) \mu(z) |dz \wedge d\bar{z}|,$$

and we set

$$Q(G, \Delta)^\perp = \{ \mu \in L^\infty(G, \Delta) \mid \langle \varphi, \mu \rangle = 0 \text{ for all } \varphi \in Q(G, \Delta) \}.$$

The pairing gives us a canonical isomorphism between the dual space  $Q(G, \Delta)^*$  of  $Q(G, \Delta)$  and  $L^\infty(G, \Delta)/Q(G, \Delta)^\perp$  (see [14, Chapter III]).

It is well known (see for example [16] and the literature quoted there) that the tangent space to  $T(G, \Delta)$  at  $\Phi(\mu)$  is canonically isomorphic to  $Q(G^\mu, \Delta^\mu)^*$ , and (using the pairing) the cotangent space is (canonically isomorphic to)  $Q(G^\mu, \Delta^\mu)$ .

2.4. We now proceed to the proof of our theorem. For ease of computation we assume first that  $x_3 = \infty$ . It was shown by Ahlfors and Bers [2] that for fixed  $\mu \in M(G, \Delta)$ , for  $t^{-1} > \|\mu\|_\infty$ , and for fixed  $z \neq x_1, x_2, x_3$ , the function  $t \mapsto w^{t\mu}(z)$  is a holomorphic function of  $t$ . They further showed that for  $x_3 = \infty$ ,

$$\frac{d}{dt} w^{t\mu}(z) \Big|_{t=0} = \frac{1}{2\pi i} (z - x_1)(z - x_2) \iint_{\Omega} \frac{\mu(\zeta) d\zeta \wedge d\bar{\zeta}}{(\zeta - z)(\zeta - x_1)(\zeta - x_2)}. \tag{6}$$

We fix  $\mu \in M(G)$  and we choose  $\nu_1, \dots, \nu_d \in L^\infty(G^\mu)$  so that  $\nu_1, \dots, \nu_d$  forms a basis for  $L^\infty(G^\mu)/Q(G^\mu)^\perp$ .

We return to our map  $\Psi: T(G) \rightarrow \hat{\mathbf{C}}^d$ , since  $x_3 = \infty$ ,  $\Psi: T(G) \rightarrow \mathbf{C}^d$ , and we need to study the differential  $D\Psi$  at the point  $\Phi(\mu)$ . We consider a point  $\Psi(\Phi(\mu)) \in \mathbf{C}^d$ , and we write  $\Psi_1, \dots, \Psi_d$  as natural parameters on  $\mathbf{C}^d$ . Regarding  $L^\infty(G)/Q(G^\mu)^\perp$  as the tangent space to  $T(G)$ , an arbitrary tangent vector to  $T(G)$  at  $\Phi(\mu)$  can be written as  $\sum_{i=1}^d z_i \nu_i$  with  $z_i \in \mathbf{C}$ . Then formula (6) yields

$$\begin{aligned} & \frac{\partial}{\partial z_k} (\Psi_j(\Phi(\mu))) \left( \sum z_i \nu_i \right) \Big|_{z=0} \\ &= \frac{\partial}{\partial z} w^{z\nu_k}(w^\mu(y_j)) \Big|_{z=0} \\ &= \frac{1}{2\pi i} (w^\mu(y_j) - x_1)(w^\mu(y_j) - x_2) \iint_{\Omega^\mu} \frac{\nu_k(\zeta) d\zeta \wedge d\bar{\zeta}}{(\zeta - x_1)(\zeta - x_2)(\zeta - w^\mu(y_j))} \\ &= \frac{1}{2\pi i} (w^\mu(y_j) - x_2) \iint_{\Omega^\mu} \nu_k(\zeta) \varphi_{y_j}(\Phi(\mu), \zeta) d\zeta \wedge d\bar{\zeta} \\ &= \frac{1}{2\pi i} (w^\mu(y_j) - x_2) \langle \nu_k(\cdot), \Theta_{\varphi_{y_j}}(\Phi(\mu), \cdot) \rangle. \end{aligned}$$

We showed in [18] that  $\Psi$  is a holomorphic injection (onto an open set); hence  $D\Psi$  is a linear isomorphism. Since the  $\nu_k$  span the tangent space to  $T(G)$  at  $\Phi(\mu)$ , the functions  $\Theta_{\varphi_{y_j}}(\Phi(\mu), \cdot)$  must span the cotangent space. We have shown that for every  $\mu$ , the functions  $\Theta_{\varphi_{y_j}}(\Phi(\mu), \cdot)$ ,  $j = 1, \dots, d$ , are linearly independent.

In order to complete the proof of our theorem, it remains only to remove the restriction  $x_3 = \infty$ . We do this by choosing a Möbius transformation  $a$  with  $a(x_1) = x_1$ ,  $a(x_2) = x_2$ , and  $a(\infty) = x_3$ . An easy computation shows that there is a constant  $c$ , depending only on  $a$ , so that the quadratic differentials  $\Theta_{\varphi_{y_1}}, \dots, \Theta_{\varphi_{y_d}}$  for  $G$ , and  $\Theta_{\varphi_{a(y_1)}}, \dots, \Theta_{\varphi_{a(y_d)}}$  for  $aGa^{-1}$  are related by

$$\Theta_{\varphi_{a(y_i)}}(\Phi(\nu), a(z))(a'(z))^2 = c \Theta_{\varphi_{y_i}}(\Phi(\mu), z),$$

where  $\mu(z)a'(z) = \nu(az)\overline{a'(z)}$ ,  $z \in \Omega^\mu$ .

2.5. For any finitely generated Kleinian group,  $T(G)$  is a domain in  $\mathbf{C}^n$  (see [18]), hence the cotangent space is trivial. We have shown the following.

**COROLLARY 1.** *Let  $G$  be a stratifiable Kleinian group, with  $\dim T(G) = d$ . Then there are  $d$  holomorphic functions  $\varphi_1, \dots, \varphi_d$  on  $F(G)$ , so that  $\Theta\varphi_1, \dots, \Theta\varphi_d$  exhibits a global holomorphic trivialization of the cotangent bundle of  $T(G)$ .*

2.6. We now turn to the case where we have selected an invariant union of components  $\Delta$ . Since  $T(G, \Delta)$  is a submanifold of  $T(G)$ , the cotangent space of  $T(G, \Delta)$  is a subspace of the cotangent space of  $T(G)$ . This yields the following.

**COROLLARY 2.** *Let  $G$  be a stratifiable Kleinian group and let  $\Delta$  be an invariant union of components. Let  $r = \dim T(G, \Delta)$ . Then there are  $r$  functions  $\psi_1, \dots, \psi_r$  meromorphic on  $F(G, \Delta)$ , so that for any  $\mu \in M(G, \Delta)$ , the functions  $\Theta\psi_1, \dots, \Theta\psi_r$  form a basis for  $Q(G^\mu, \Delta^\mu)$  and vanish identically on  $\Omega^\mu \setminus \Delta^\mu$ .*

**§3. Fuchsian groups**

3.1. Let  $\Gamma$  be a finitely generated Fuchsian group of the first kind operating on the upper half plane  $U$  (and on the lower half plane  $U^*$ ). We assume that  $d = \dim T(\Gamma, U) > 0$ . We showed in [18] that  $\Gamma$  can be stratified by real points,  $x_1, x_2, x_3, y_1, \dots, y_{2d}$ , and that the Fuchsian groups in  $T(\Gamma)$  are precisely those points for which the  $2d$  coordinates  $w^\mu(y_i)$  are all real.

One easily sees that if  $\Gamma^\mu$  is Fuchsian, and  $w^\mu(y_j)$  are all real, then for each  $j$ ,

$$\Theta\varphi_{y_j}(\Phi(\mu), z) = \overline{\Theta\varphi_{y_j}(\Phi(\mu), \bar{z})}.$$

We restate our main result in this case as follows.

**THEOREM 2.** *Let  $\Gamma$  be a finitely generated Fuchsian group of the first kind, with  $\dim T(\Gamma, U) = d$ . Let  $x_1, x_2, x_3, y_1, \dots, y_{2d}$  be a real stratification for  $\Gamma$ . Let  $\mu \in M(\Gamma)$  be such that  $\Gamma^\mu$  is Fuchsian. Then the functions  $\Theta\varphi_{y_1}, \dots, \Theta\varphi_{y_{2d}}$  (a) commute with complex conjugation, (b) form a basis over  $\mathbf{C}$  for  $Q(\Gamma^\mu)$ , and (c) form a basis over  $\mathbf{R}$  for  $Q(\Gamma^\mu, U)$ .*

We remark that conclusion (a) is false if  $\Gamma^\mu$  is not Fuchsian; conclusion (b) is true even if  $\Gamma^\mu$  is quasifuchsian. Nothing is known about conclusion (c) if  $\Gamma^\mu$  is not Fuchsian.

3.2. Our main theorem asserts that certain sets of Poincaré series form a basis for quadratic differentials; in particular, these Poincaré series do not identically vanish.

**COROLLORY 3.** *Let  $\Gamma$  be a finitely generated Fuchsian group of the first kind, and let  $x_1, x_2, x_3, y$  be part of a stratification for  $\Gamma$ . Then the Poincaré series  $\Theta_{\varphi_y}(0, \cdot)$  does not vanish identically in  $U$  or in  $U^*$ .*

We showed in [18], that every elliptic fixed point can be made part of a stratification of a Fuchsian group. Hence we obtain:

**COROLLORY 4.** *Let  $y$  be an elliptic fixed point of the Fuchsian group of the first kind  $\Gamma$ . Then there are real points  $x_1, x_2, x_3$  so that the Poincaré series  $\Theta_{\varphi_y}(0, \cdot)$  does not vanish identically in both  $U$  and  $U^*$ .*

One often can choose both fixed points of an elliptic element as part of a stratification set.

**COROLLORY 5.** *Let  $x_1, x_2, x_3, y$  be part of a stratification set for the Fuchsian group of the first kind  $\Gamma$ , where  $\{x_1, x_2, x_3, y\}$  is invariant under complex conjugation. Then the Poincaré series  $\Theta_{\varphi_y}(0, \cdot)$  does not vanish identically in either  $U$  or  $U^*$ .*

*Remarks.* (1) Let  $\Gamma$  be a torsion free Fuchsian group operating on  $U$  such that  $U/\Gamma$  is a compact surface of genus  $g \geq 2$ . In [27], Wolpert constructs  $6g - 6$  Poincaré series that form a basis for  $Q(\Gamma, U)$  over  $\mathbf{R}$ . Further, by using the Petersson scalar product and a geometric interpretation of the Poincaré series he constructs, he is able to select  $3g - 3$  series that form a basis over  $\mathbf{C}$ .

(2) In [13], Hejhal studies Poincaré series of rational functions that are invariant under conjugation. He produces finite spanning sets for  $Q(\Gamma)$  (and hence also  $Q(\Gamma, U)$ ) that are not necessarily linearly independent. His methods are not limited to quadratic differentials, but also work for  $q$ -differentials,  $q > 2$ .

## §4. Other applications

4.1. If  $G$  is a stratifiable function group with invariant component  $\Delta$ , where  $T(G) = T(G, \Delta)$  (for example  $G$  might be a Schottky group, or  $G$  might be such that all the components other than  $\Delta$  are thrice punctured spheres), then a stratification gives us global coordinates for  $T(G)$ , and Theorem 1 gives us a global holomorphic trivialization of the cotangent space. In the special case that  $\Delta$  is simply-connected, we get global coordinates for the Teichmüller space (every Teichmüller space can be so realized; see Maskit [24]).

4.2. We also remark that if  $G$  is as in the preceding section, we can achieve the same result with functions which are not necessarily rational. To this end, we

let  $\Gamma$  be the Fuchsian model of  $G$ ; that is, there is a holomorphic projection  $h:U \rightarrow \Delta$ , where for each  $\gamma \in \Gamma$ , there is a  $g \in G$  so that  $h \circ \gamma = g \circ h$ . This projection map induces a holomorphic covering  $T(\Gamma, U) \rightarrow T(G, \Delta)$ , (see [6], [15], [21]) and this map can be further extended to a fiber preserving holomorphic covering  $H:F(\Gamma, U) \rightarrow F(G, \Delta)$  (see, for example, [10]).

The map  $H$  induces a map  $H_*$  from functions on  $w^\mu(\Delta)$  to functions on  $w^\sigma(U)$  (where  $\mu$  and  $\sigma$  are appropriately related). If we transform functions as quadratic differentials (that is,  $H_*\varphi(z) = \varphi(h(z))h'(z)^2$ ), then, denoting the Poincaré series operator for  $G$  by  $\Theta_G$ , an easy computation shows that  $\Theta_G \circ H_* = H_* \circ \Theta_\Gamma$ . We choose  $\varphi_1, \dots, \varphi_d$  so that  $\Theta_\Gamma \varphi_1, \dots, \Theta_\Gamma \varphi_d$  form a basis for  $Q(\Gamma, U)$ . Then  $\Theta_G H_* \varphi_1, \dots, \Theta_G H_* \varphi_d$  form a basis for  $Q(G)$ , and conversely a basis for  $Q(\Gamma, U)$  can be obtained from a basis for  $Q(G)$ .

**§5. Extensions of Fuchsian groups of the second kind**

5.1. In this and the next section we give stratifications for all  $\mathbf{Z}_2$ -extensions of quasifuchsian groups of the first kind. In this section we focus on certain groups which are  $\mathbf{Z}_2$ -extensions of Fuchsian groups of the second kind.

Let  $\Gamma$  be a finitely generated, non-elementary Fuchsian group of the second kind. There are standard generators for  $\Gamma$  of the form  $A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_n, F_1, \dots, F_m$ , where the  $A_i, B_i$  and  $F_j$  are hyperbolic and the  $E_j$  are elliptic or parabolic. The signature of  $\Gamma$  is  $(g, n, m; \nu_1, \dots, \nu_n)$  where  $\nu_i$  is the order of  $E_i$  if  $E_i$  is elliptic, and  $\nu_i = \infty$  if  $E_i$  is parabolic. Since  $\Gamma$  is of the second kind,  $m > 0$ . The defining relations for  $\Gamma$  are

$$\prod_{j=1}^g [A_j, B_j] \circ \prod_{k=1}^n E_k \circ \prod_{l=1}^m F_l = 1,$$

$$E_k^{\nu_k} = 1, \quad \text{if } \nu_k < \infty, \quad k = 1, \dots, n,$$

where as usual

$$[A_j, B_j] = A_j \circ B_j \circ A_j^{-1} \circ B_j^{-1}.$$

One easily sees that viewing  $\Gamma$  as a Kleinian group,  $\Omega(\Gamma)/\Gamma$  is a surface of genus  $2g + m - 1$  with  $2n$  distinguished points on it. Hence  $\dim T(\Gamma) = 3(2g + m) - 6 + 2n$ .

5.2. We want to adjoin square roots  $f_p$  of the elements  $F_p$ , where  $f_p$  interchanges upper and lower half planes. We do this as follows.

We choose some  $F_p (1 \leq p \leq m)$  and we let  $I_p$  denote the axis of  $F_p$  in  $U$ . We denote reflection in the real axis by  $j$ , and set

$$C_p = I_p \cup j(I_p) \cup \{\text{fixed points of } F_p\}.$$

The circle  $C_p$  bounds two discs; one of them, call it  $D_p$  is precisely invariant under  $F_p$  in  $\Gamma$  (that is, if  $A \in \Gamma$  is not a power of  $F_p$  then  $A(D_p) \cap D_p = \emptyset$ ).

Since  $F_p$  is hyperbolic, there are two distinct Möbius transformations whose square is  $F_p$ . One square root is hyperbolic and it preserves  $D_p$ ; call the other  $f_p$ .

We let  $\Gamma_0 = \Gamma$ , and for  $p = 1, \dots, m$ , we define  $\Gamma_p$  to be generated by  $\Gamma_{p-1}$  and  $f_p$ . The construction of  $\Gamma_p$  from  $\Gamma_{p-1}$  and  $f_p$  requires a version of Combination Theorem II (see, for example, [22]) which is unfortunately not in print. We briefly outline the proof here.

We let  $\omega_p$  be a fundamental domain for  $\Gamma_p$  (acting on  $\Omega(\Gamma_p)$ ), where  $\omega_p \cap D_{p+1}$  is a fundamental domain for the action of  $\langle F_{p+1} \rangle$  on  $D_{p+1}$ ,  $p = 0, 1, \dots, m-1$ . We denote the complement of  $D_p$  by  $\hat{D}_p$ .

**LEMMA.** *For  $p = 0, \dots, m-1$ , the group  $\Gamma_{p+1}$  is discrete;  $\hat{D}_{p+1} \cap \omega_p$  is a fundamental domain for  $\Gamma_{p+1}$ ; the relations in  $\Gamma_{p+1}$  are the relations of  $\Gamma_p$  together with  $f_{p+1}^2 = F_{p+1}$ .*

*Proof.* Any element of  $\Gamma_{p+1}$  can be written as  $A = A_{k+1} \circ f_{p+1}^{\pm 1} \circ A_k \circ \dots \circ f_{p+1}^{\pm 1} \circ A_1$ , where  $A_1, A_{k+1}$  might be trivial, but otherwise no  $A_i$  is a power of  $F_{p+1}$  (there is also the easy case  $A = f_{p+1}^{\pm 1} \circ F_{p+1}$ ). Let  $z$  be a point of  $\hat{D}_{p+1} \cap \omega_p$ . Then  $A_1(z) \notin \omega_p$  (if  $A_1 \neq 1$ ) and  $A_i(z) \in \hat{D}_{p+1}$ ;  $f_{p+1}^{\pm 1} \circ A_1(z) \notin \hat{D}_{p+1}$ ,  $A_2 \circ f_{p+1}^{\pm 1} \circ A_1(z) \in \hat{D}_{p+1} \setminus \omega_p$ , etc.

We have shown that no two distinct points of  $\omega_p \cap \hat{D}_{p+1}$  are equivalent under any word of the above form; hence  $\Gamma_{p+1}$  is discrete, and the relations are as stated. The remainder of the proof that  $\hat{D}_{p+1} \cap \omega_p$  is a fundamental domain is standard (see for example [22]).

We will later need a similar version of Combination Theorem II, where instead of  $f_p^2 = F_p \in G_p$ , we have  $f_p^2 = 1$ . The proof is almost identical and is left to the reader.

Looking at the identifications of the sides of the fundamental domains, we see that for  $p < m$ ,  $\Omega(\Gamma_p)$  is connected and the surfaces  $\Omega(\Gamma_p)/\Gamma_p$  are all surfaces of the same conformal type  $(2g + m - 1, 2n)$ .

However  $\Omega(\Gamma_m)$  has two components  $U$  and  $U^*$ , but  $\Omega(\Gamma_m)/\Gamma_m$  is still just one surface of the same conformal type  $(2g + m - 1, 2n)$ . In particular the spaces  $T(\Gamma_p)$  all have the same dimension.

**5.3. THEOREM 3.** *The Kleinian groups  $\Gamma_p$  are all stratifiable.*

*Proof.* For  $p < m$ , these are geometrically finite function groups and so there is nothing to prove [18].

We now assume that  $p = m$ , and notice that the two components of  $\Gamma_m$  are simply connected. Hence  $T(\Gamma_m)$  is simply connected and so for every element  $\gamma$  of  $\Gamma_m$ ,  $\text{tr } w \circ \gamma \circ w^{-1}$ , the trace of  $w \circ \gamma \circ w^{-1}$  is well defined on  $T(\Gamma_m)$  once we have chosen  $\text{tr } \gamma$ .

There are several cases to consider.

*Case I.*  $g > 0$ .

For  $i = 1, \dots, m$ , let  $z_{i,1}$  and  $z_{i,2}$  be the attractive and repulsive fixed points, respectively of  $F_i$ .

For  $i = 1, \dots, g - 1$ , we let  $x_{i,1}, \dots, x_{i,6}$ , be respectively, the attractive fixed point of  $A_i$ , the repulsive fixed point of  $A_i$ , the attractive fixed point of  $B_i$ , the repulsive fixed point of  $B_i$ ,  $A_i(z_{1,1})$ , and  $B_i(z_{1,1})$ .

We let  $C$  be the commutator  $[A_g, B_g]$ ; we set  $u_1 = A_g$  (attractive fixed point of  $C$ ),  $u_2 = A_g$  (repulsive fixed point of  $C$ ),  $u_3 = B_g$  (attractive fixed point of  $C$ ).

For  $i = 1, \dots, n$ , we let  $y_{i,1}, y_{i,2}$  be the fixed points of  $E_i$  if  $E_i$  is elliptic; if  $E_i$  is parabolic, we let  $y_{i,1}$  be the fixed point of  $E_i$ , and  $y_{i,2} = E_i(z_{1,1})$ .

Finally for  $i = 1, \dots, m$ , we set  $z_{1,3} = f_i(u_1)$ .

We note that we have defined  $6g - 6 + 3 + 2n + 3m = \dim T(\Gamma_m) + 3$  complex parameters.

Let  $w$  be some deformation in  $T(\Gamma_m)$ . We need to show that the parameters  $w(x_{1,1}), \dots, w(z_{m,3})$  determine the generators  $w \circ A_1 \circ w^{-1}, \dots, w \circ f_m \circ w^{-1}$  of  $w\Gamma_m w^{-1}$ . By changing the origin of the deformation space, this and all similar arguments in subsequent cases, are reduced to showing that the parameters  $x_{1,1}, \dots, z_{m,3}$  determine the generators  $A_1, \dots, f_m$  of  $\Gamma_m$ .

Obviously  $A_1, B_1, \dots, A_{g-1}, B_{g-1}, E_1, \dots, E_n, f_1, \dots, f_m$  are all determined. Hence  $C$  is determined; the choice of  $\text{tr } C^{-1}$  together with  $u_1, u_2, u_3$ , determine  $A_g$  and  $B_g$  [18].

*Case II.*  $g = 0, n \geq 2$ .

For  $i = 1, \dots, m$ , we define  $z_{i,1}$  and  $z_{i,2}$  as above. Then for  $i = 1, \dots, n - 2$ , we define  $y_{i,1}$  and  $y_{i,2}$  as above, and we let  $y$  be a fixed point of  $E_{n-1}$ . Finally, for  $i = 1, \dots, m$ , we define  $z_{i,3} = f_i(y)$ .

The elements  $E_1, \dots, E_{n-2}, f_1, \dots, f_m$  are all determined. Hence  $E_{n-1} \circ E_n$  is determined. It was shown in [18] that  $E_{n-1} \circ E_n$ , together with  $y$  determine  $E_{n-1}$  and  $E_n$ .

*Case III.*  $g = 0, n = 1$ .

We remark that in this case  $m \geq 2$ , and we choose our parameters as follows. Let  $y$  be a fixed point of  $E_1$ . For  $i = 1, \dots, m-1$ , let  $z_{i,1}, z_{i,2}, z_{i,3}$  be, respectively, the attractive fixed point of  $F_i$ , the repulsive fixed point of  $F_i$ , and  $f_i(y)$ . Let  $z_m$  be the attractive fixed point of  $F_m$ .

We have defined  $3m-1 = \dim T(\Gamma_m) + 3$  parameters. We see at once that  $f_1, \dots, f_{m-1}$  are determined. We normalize  $\Gamma_m$  so that  $z_m = \infty, y = 0$ ; we write  $F_1 \circ \dots \circ F_{m-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ ,  $E_1 = \begin{pmatrix} K & 0 \\ p & K^{-1} \end{pmatrix}$ ,  $f_m = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ ,  $|\alpha| > 1$ . Since  $T(\Gamma_m)$  is simply connected, we can choose  $K, a, b, c, d$  so that in  $SL(2; \mathbf{C})$

$$\begin{pmatrix} K & 0 \\ p & K^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & -\beta \\ 0 & \alpha \end{pmatrix}.$$

This yields

- (1)  $Ka\alpha = \alpha^{-1}$ ,
- (2)  $Ka\beta + Kb\alpha^{-1} = -\beta$ ,
- (3)  $\alpha(p\alpha + K^{-1}c) = 0$ .

Since  $a\alpha \neq 0$ , we can solve (3) for  $p$ . Equation (1) yields  $\alpha$  up to sign. We can solve (2) to obtain  $\beta = -(Kb/1 + aK)\alpha^{-1}$ . Hence  $f_m$  is determined in  $PSL(2; \mathbf{C})$  (note that  $1 + aK \neq 0$ ; since otherwise  $b = 0$  which is impossible).

*Case IV.*  $g = 0, n = 0$ .

In this case  $m \geq 3$ . Let  $y_1$  be the repulsive fixed point of  $F_1$ . For  $i = 2, \dots, m-1$ , we let  $z_{i,1}, z_{i,2}, z_{i,3}$  be, respectively, the attractive fixed point of  $F_i$ , the repulsive fixed point of  $F_i$ ,  $f_i(y_1)$ . Let  $y_2$  be the repulsive fixed point of  $F_m$ , and let  $y_3 = f_1(y_2)$ .

We look at a deformation of  $\Gamma_m$  and we note that  $f_2, \dots, f_{m-1}$  are determined by the  $z_{i,j}$ . We normalize  $\Gamma_m$  so that  $y_1 = 0, y_2 = \infty, y_3 = 1$ . Then we can write  $F_2 \circ \dots \circ F_{m-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ ,  $f_1 = \begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix}$ ,  $|K| > 1$ ,  $f_m = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$ ,  $|\alpha| < 1$ , where  $a, b, c, d$  are known, and the choice is made so that

$$\begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix} \begin{pmatrix} K & 0 \\ K & K^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & -\beta \\ 0 & \alpha \end{pmatrix}$$

(that is, we choose  $a, b, c, d, K, \alpha, \beta$  for the original  $\Gamma_m$  so that this relation holds;

then we can regard  $a, b, c, d, K, \alpha, \beta$  as functions on  $T(\Gamma_m)$ , where this relation continues to hold; since  $T(\Gamma_m)$  is simply connected, the functions are well defined).

From routine calculation we obtain the following four equations:

$$(1) K^2 a \alpha = \alpha^{-1},$$

$$(2) K^2 a \beta + K^2 b \alpha^{-1} = -\beta,$$

$$(3) (K^2 + 1) a \alpha + K^{-2} c \alpha = 0,$$

$$(4) (K^2 + 1)(a \beta + b \alpha^{-1}) + K^{-2}(c \beta + d \alpha^{-1}) = \alpha.$$

Since  $\alpha \neq 0$ , we can solve (3) for  $K^2$  and obtain

$$K^2 = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{c}{a}}.$$

Since  $|K| > 1$ , we can never have  $c/a = \frac{1}{4}$ , hence on  $T(\Gamma_m)$  there is a unique solution for  $K^2$ , and equally well for  $K$ .

We then solve (1) for  $\alpha^2$  and (2) for  $\beta$  in terms of  $\alpha^{-1}$ ; hence as above, we can solve for  $\alpha$  and  $\beta$ .

## §6. Global coordinates of Teichmüller spaces II: Earle slices

6.1. Let  $G$  be a finitely generated Kleinian group with exactly two components, neither of them invariant. We shall see (Lemma 6.3) that such a group must be a  $\mathbf{Z}_2$ -extension of a finitely generated quasifuchsian group of the first kind. It was remarked by Earle [9] that  $T(G)$ , the deformation space of  $G$ , is in this case the Teichmüller space of  $\Delta_0/G_0$ , where

$\Delta$  = one of the components of  $\Omega(G)$ ,

$G_0$  = stabilizer of  $\Delta$  in  $G$ , and

$\Delta_0 = \{z \in \Delta; z \text{ is not a fixed point of an elliptic element of } G_0\}$ .

In this section we shall prove the following.

**THEOREM 4.** *Let  $G$  be a finitely generated Kleinian group with two components and  $\Omega(G)/G$  connected. Then the group  $G$  is stratifiable:*

6.2. The results of the next two sections, while apparently obvious, have, to the best of our knowledge, never appeared in print. The proofs seem to require deep results. For the convenience of the reader, we include complete details.

**LEMMA.** *Let  $G_0$  be a finitely generated quasifuchsian group with components  $\Delta$  and  $\Delta^*$ . Then there exists a unique extremal quasireflection  $J: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  which commutes with every element of  $G_0$  and which maps  $\Delta$  onto  $\Delta^*$ .*

*Proof.* As usual let  $U$  be the upper half-plane and  $f: \Delta \rightarrow U$ , a Riemann map. Define the finitely generated Fuchsian group  $F_0$  of the first kind by  $F_0 = fGf^{-1}$ . Let  $U^*$  be the lower half-plane and let  $j(z) = \bar{z}$ . We let  $f^*: U^* \rightarrow \Delta^*$  be the unique Teichmüller (extremal) mapping that induces the isomorphism

$$F_0 \ni \gamma \mapsto f \circ \gamma \circ f^{-1} \in G_0.$$

Define

$$J(z) = \begin{cases} f^* \circ j \circ f(z), & z \in \Delta \cup \Lambda(G), \\ f^{-1} \circ j \circ (f^*)^{-1}(z), & z \in \Delta^* \cup \Lambda(G). \end{cases}$$

It is easy to see that  $J$  commutes with every element of  $G_0$ . Since  $J$  preserves the fixed points of elements of  $G_0$ ,  $J$  is the identity on  $\Lambda(G)$ . Hence  $J$  is a global quasireflection. Further  $J$  has minimal maximal dilatation among all quasireflections commuting with  $G_0$ , and  $J$  is the unique quasireflection with these properties.

We shall call  $J$  the *extremal quasireflection* for  $G_0$ .

6.3. **LEMMA.** *Let  $G$  be an arbitrary finitely generated Kleinian group with two components, neither of them invariant. Then there exists a  $\mathbf{Z}_2$ -extension  $F$  of a finitely generated Fuchsian group of the first kind so that  $G$  is a quasiconformal deformation of  $F$ .*

*Proof.* Let  $\Delta$  and  $\Delta^*$  be the two components of  $G$ , and let  $G_0$  be the stabilizer of  $\Delta$  (therefore also of  $\Delta^*$ ). We conclude that  $G_0$  is a finitely generated quasifuchsian group of the first kind (see, for example, [20] or [17]) and that  $G$  is a  $\mathbf{Z}_2$ -extension of  $G_0$ .

Let  $g$  be some element of  $G \setminus G_0$ . Let  $J$  be the extremal quasireflection for  $G_0$ . Since  $J$  is unique,  $g \circ J \circ g^{-1} = J$ ; that is,  $g$  commutes with  $J$ . Hence  $(g \circ J)^2 = g^2 \in G_0$ ; that is,  $g \circ J$  acts as an orientation reversing (quasiconformal) involution on  $\Delta/G_0$ . It is classical (see, for example, [4]) that there is a  $\mathbf{Z}_2$ -extension  $F'$  of a finitely generated Fuchsian group of the first kind  $F_0$  by an orientation reversing

conformal self-map of  $U$  which topologically uniformizes  $\Delta/G'$ , where  $G'$  is the group generated by  $G_0$  and  $g \circ J$ . Thus there exists a quasiconformal map  $f: \Delta \rightarrow U$  that conjugates  $G'$  to  $F'$ . For  $z \in \Delta^*$ , we define  $f(z) = j \circ f \circ J(z)$ , and observe that  $f$  extends to a global quasiconformal homeomorphism which conjugates  $G'$  onto  $F'$  [1]. Obviously  $f$  conjugates  $J$  to  $j$  and  $G_0$  to  $F_0$ , and so  $f \circ g \circ f^{-1} = (f \circ g \circ J \circ f^{-1}) \circ (f \circ J \circ f^{-1})$  is a fractional linear transformation interchanging  $U$  and  $U^*$ , where  $(f \circ g \circ f^{-1})^2 \in F_0$ . Finally, we let  $F$  be the group generated by  $F_0$  and  $f \circ g \circ f^{-1}$ .

6.4. Using Lemma 6.3, we conclude that to prove Theorem 4, it suffices to assume that  $G$  is a  $\mathbf{Z}_2$ -extension of a finitely generated Fuchsian group  $G_0$  of the first kind acting on  $U$ , and that the extra generator of  $G$  interchanges  $U$  and  $U^*$ . We shall assume that we are in a slightly more general situation. We are studying extensions  $G$  of non-elementary finitely generated Fuchsian groups of the first or second kind  $G_0$  acting on  $U$  by an element  $g_0$  that maps  $U$  onto  $U^*$ . We let  $g \in G \setminus G_0$ , and we form the group  $G'$  generated by  $G_0$  and  $j \circ g$ . Then  $G'$  acts as a group of conformal and anti-conformal automorphisms of  $U$ ,  $G'$  is isomorphic to  $G$ , and  $G'$  is independent of our choice of the element  $g \in G \setminus G_0$ . Furthermore  $j \circ g$  induces an anti-conformal involution  $J$  on  $S = U/G_0$ .

It is classical that  $S/J$  is a surface, perhaps non-orientable, of some genus, with some number of boundary curves and some number of cross-caps. We give this a precise statement, and for the convenience of the reader, we include a proof.

There is a unique closed orientable surface  $\bar{S}$  which conformally contains  $S$ ; the difference  $\bar{S} \setminus S$  is a finite set of parabolic punctures.

**LEMMA.** *There is a finite set of simple disjoint loops  $w_1, \dots, w_s$  on  $\bar{S}$  with the following properties.*

- (1) *The loops  $w_1, \dots, w_s$  divide  $\bar{S}$  into two subsurfaces;  $J$  interchanges these two subsurfaces and keeps each  $w_i$  invariant.*
- (2) *For each  $i$ ,  $J$  either fixes every point of  $w_i$ , or has no fixed point on  $w_i$ .*
- (3) *If  $J$  fixes every point of  $w_i$ , then  $w_i$  may pass through some elliptic ramification points or parabolic punctures. Off these punctures,  $w_i$  is a geodesic on  $S_0 = S \setminus \{\text{ramification points}\}$ .*
- (4) *If  $J$  has no fixed point on  $w_i$ , then  $w_i$  is a smooth geodesic on  $S_0$ , (and doesn't pass through any elliptic or parabolic punctures).*
- (5) *Every fixed point of  $J$  is a point of some  $w_i$ .*

*Proof.* We look at the set of fixed points in  $U$  of elements of  $G'$ ; these consist of fixed points of elliptic elements of  $G_0$  and fixed axes of orientation reversing elements of  $G'$ . A point of intersection of two or more of these axes is necessarily

an elliptic fixed point. Looking at all these axes near an elliptic fixed point; one sees that they project onto a simple path on  $S$ . We conclude that the projection of the fixed axes is a set of simple disjoint paths.

We next show that if  $A$  is the line of fixed points of the element  $g \in G'$ ; then either  $A$  is the axis of a hyperbolic element of  $G_0$ , or both endpoints of  $A$  are parabolic fixed points. Let  $x$  and  $y$  be the endpoints of  $A$ , and assume  $x$  is not a parabolic fixed point. Then [3]  $x$  is a point of approximation for  $G_0$  and so there is a sequence  $g_n$  of distinct elements of  $G_0$ , with  $g_n(x) \rightarrow x'$ ,  $g_n(y) \rightarrow y' \neq x'$ . Since  $G'$  is discrete, we must have  $g_n(x) = x'$ ,  $g_n(y) = y'$  for almost all  $n$ ; that is, for fixed  $m$  and  $n$  sufficiently large  $g_n \circ g_m^{-1}(x) = x$ ,  $g_n \circ g_m^{-1}(y) = y$ .

We have shown that the projection to  $\bar{S}$  of the set of fixed points of reflections in  $G'$  is a set  $w_1, \dots, w_q$  of simple disjoint loops.

By looking at paths connecting these loops one easily sees that  $w_1, \dots, w_q$  divides  $S$  into at most two surfaces.

If  $w_1, \dots, w_q$  does not divide  $S$  then there is a homotopically non-trivial loop  $v$  on  $S$ , where  $v$  is disjoint from all  $w_i$  and the element of  $G_0$  corresponding to  $v$  is hyperbolic (this follows easily from the fact that  $G_0$  is non-elementary). Let  $h$  be some hyperbolic element of  $G_0$  whose axis is disjoint from all reflection axes in  $G'$ , and let  $r$  be some reflection in  $G'$ . Normalize  $G'$  so that  $r(z) = -\bar{z}$ , and  $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,  $ad - bc = 1$ ,  $a, b, c, d$  real; then  $-bc > 0$ , and  $|a + d| > 2$ . Observe that

$$(h \circ r)^2 = \begin{pmatrix} a^2 - bc & b(d - a) \\ c(a - d) & d^2 - bc \end{pmatrix}.$$

If this were the identity, then we would have  $a = d$ ,  $-bc > 0$  and  $a^2 - bc = 1$ , so that  $|a + d| < 2$ . We have shown that if  $w_1, \dots, w_q$  does not divide  $S$ , then  $S/J$  is non-orientable, or equivalently, that  $G'$  contains freely acting orientation reversing elements.

Let  $u$  be the shortest orientation-reversing loop on  $(S \setminus (w_1 \cup \dots \cup w_q))/J$ . Then  $u^2 = w_{q+1}$ , is a simple loop on  $S \setminus (w_1 \cup \dots \cup w_q)$  which is invariant under  $J$ .

If  $w_1, \dots, w_{q+1}$ , does not divide  $S$ , then we repeat the above argument; after a finite number of steps we arrive at the required  $w_1, \dots, w_s$ .

6.5. In proving Theorem 4 under the simplifying assumption of §6.4, we first take up the case that none of the loops  $w_1, \dots, w_s$  is pointwise fixed by  $J$ .

This means that  $j \circ g$  has no fixed points in  $U$ , for any choice of  $g \in G \setminus G_0$ .

Every element  $g \in G \setminus G_0$  is either loxodromic or elliptic of order 2; one easily sees that  $g$  is elliptic if and only if  $j \circ g$  has fixed points in  $U$ . Hence  $J$  has no fixed points if and only if every element of  $G \setminus G_0$  is loxodromic.

We look at all lifts of all  $w_i$ ; these divide  $U$  into regions; we choose one of these regions, call it  $R$ , and let  $\Gamma$  be the stability subgroup of  $R$  in  $G_0$ . One sees at once that  $R/\Gamma$  is one of the halves of  $S$  cut along  $w_1, \dots, w_s$ . Hence  $\Gamma$  is a finitely generated Fuchsian group of the second kind representing a surface of some genus, with some number of elliptic or parabolic punctures, and  $s$  holes. We observe that for each primitive  $F \in \Gamma$  representing one of these holes, there is an element  $f \in G$ , with  $f^2 = F$ , and  $f$  interchanges upper and lower half planes (that is,  $j \circ f$  preserves the axis of  $F$  but interchanges the two non-euclidean half planes bounded by it).

We choose  $F_1, \dots, F_s$  to be non-conjugate primitive such elements and let  $f_1, \dots, f_s$  be their square roots. Then the group  $\tilde{G}$  generated by  $\Gamma$  and  $f_1, \dots, f_s$  is a subgroup of  $G$ , has two components, neither invariant, and, as we observed in §5,  $\Omega(\tilde{G})/\tilde{G}$  is the two halves of  $S$  with their boundaries glued together; that is,  $\Omega(\tilde{G})/\tilde{G} = \Omega(G)/G$ . We conclude that  $\tilde{G} = G$  and hence, using §5.3,  $G$  is stratifiable.

6.6. We turn now to the case that  $J$  has fixed points on  $S$ . As we remarked earlier, this is equivalent to there being an involution  $g \in G$ , which interchanges the upper and lower half planes. We conjugate  $G$  so that  $g(z) = -z$ .

Then the involution  $J$  is induced by  $j \circ g : z \mapsto -\bar{z}$ ; it has the positive imaginary axis as fixed point set. Let  $w$  be the fixed loop of  $J$  on  $S$  containing the projection of the positive imaginary axis, and let  $n$  be the number of elliptic and parabolic punctures on  $w$ . There are several cases to consider.

*Case V.  $n = 0$ .*

In this case  $w$  is a simple closed curve. Let  $\mathcal{A}$  be the set of all translates in both  $U$  and  $U^*$  of  $\{\operatorname{Re} z = 0, \operatorname{Im} z > 0\}$  under  $G$ . Let  $R$  be the subset of  $\hat{C}$  cut out by  $\mathcal{A}$ , where  $R$  is bounded by the imaginary axis and lies in the right half plane. Let  $G_1$  be the stabilizer of  $R$  in  $G$  and let  $G_{01} = G_1 \cap G_0$ .

Exactly as in §5.2 (except that  $g^2 = 1$ ), we can form the group  $\tilde{G}$  generated by  $G_1$  and  $g$ . We know that  $\tilde{G} \subset G$ ,  $\tilde{G}$  has two components, neither invariant, and  $\Omega(\tilde{G})/\tilde{G}$  is homeomorphic to  $\Omega(G_1)/G_1$ . Since  $G_1$  has one component and contains no degenerate subgroups, it is stratifiable. The fixed points of  $g$  are hyperbolic fixed points of  $G_1$ ; hence  $\tilde{G}$  is stratifiable. It remains to show that  $\tilde{G} = G$ .

If  $w$  does not divide  $S = U/G_0$ , then  $R \cap U/G_{01}$ , and  $R \cap U^*/G_{01}$ , are both equal to  $S$  cut along  $w$ . Then there is some loop  $w_1$ , disjoint from  $w$ , which is also invariant under  $J$ . Lifting  $J$  so that it keeps a lift  $C \subset R \cap U$  of  $w_1$  invariant, we get an orientation reversing element  $g_1 \in G'$ , which keeps  $R \cap U$  invariant. Then  $j \circ g_1$

maps  $R \cap U$  onto  $R \cap U^*$ ; that is,  $j \circ g_1 \in G_1$ . We conclude that  $R/G_1 = R \cap U/G_{01}$  is  $S$  cut along  $w$ . Hence  $\Omega(\tilde{G})/\tilde{G} = \Omega(G)/G$ , so  $\tilde{G} = G$ .

If  $w$  divides  $S$ , then  $R \cap U/G_{01}$  and  $R \cap U^*/G_{01}$  are either equal or they are the two halves of  $s$  cut along  $w$ . Since  $j(R \cap U) = R \cap U^*$ , it must be the latter. The result now follows as above.

*Case VI.  $n = 1$ .*

We let  $P$  be the point of ramification or the puncture on  $w$ . Deforming  $w$  to lie on “either side” of  $P$ , we get two non-homotopic simple loops on  $S$ ; we let  $w'$  and  $w''$  be the geodesics on  $S$  in the corresponding homotopy classes (such geodesics exist except when  $S$  is a sphere with three elliptic or parabolic punctures in which case  $\dim T(G) = 0$ ).

We note that  $J(w') = w''$ . The loops  $w'$  and  $w''$  bound a subsurface  $S_2 \subset S$ , where  $P \in S_2$ . We let  $S_1 = S \setminus \bar{S}_2$ .

We let  $C'$  be a geodesic in  $U$  lying over  $w'$ . For the sake of definiteness we assume that  $C'$  is in the first quadrant. We extend  $C'$  to be a complete circle in  $\hat{C}$ , and let  $\mathcal{A} = \bigcup_{\gamma \in G} \gamma(C')$ . As before  $\mathcal{A}$  is a  $G'$ -invariant union of disjoint circles accumulating at all points of  $\hat{R}$ . Let  $R_1$  be the region in the first quadrant cut out by  $\mathcal{A}$ , bounded in part by  $C'$ , where the projection of  $R_1$  to  $U/G_0$  does not contain the curve  $w$ .

As before we let  $G_1$  be the stabilizer of  $R_1 \cup jR_1$  in  $G$ , and  $G_{01}$  be the stabilizer of  $R_1$  in  $G_0$ .

The surface  $S_1$  may or may not be connected. If  $S_1$  is not connected, then  $R_1/G_{01}$  is half of  $S_1$  and  $(R_1 \cup jR_1)/G_1 = S_1$ . In this case  $G_{01} = G_1$ . If  $S_1$  is connected then  $G_1$  is a  $\mathbf{Z}_2$ -extension of  $G_{01}$  and  $R_1/G_{01} = S_1 = (R_1 \cup jR_1)/G_1$ . In either case  $\Omega(G_1)/G_1$  is  $S_1$  with a tube attaching the two boundary components; that is,  $\Omega(G_1)/G_1$  is the surface  $S$  with the point  $P$  no longer a puncture or ramification point. We conclude that

$$d = d(G) = d(G_1) + 1.$$

We also remark that  $G_1$  is a function group – it is either a finitely generated non-elementary Fuchsian group of the second kind or a  $\mathbf{Z}_2$ -extension of such a group. In particular, every structure subgroup of  $G_1$  is elementary. Hence  $G_1$  is geometrically finite [23], and so it is stratifiable [18]. We let  $x_1, \dots, x_{d+2}$  be a stratification of  $G_1$ .

We now let  $R_2$  be the region cut out by  $\mathcal{A}$  on the other side of  $C'$ . Observe that the projection of  $R_2$  to  $S$  contains the curve  $w$ . As before we let  $G_2$  be the stabilizer of  $R_2 \cup jR_2$  in  $G$  and  $G_{02}$  be the stabilizer of  $R_2$  in  $G_0$ . Note that  $R_2$

contains the positive imaginary axis, and that hence  $G_2$  is  $G_{02}$  extended by  $g: z \mapsto -z$ . Thus  $S_2 = R_2/G_{02} = (R_2 \cup jR_2)/G_2$ . Observe that  $R_2$  is invariant under the map  $z \mapsto -\bar{z}$ . As a matter of fact  $R_2/G_{02}$  is a sphere with two holes and one point of ramification order  $\nu (2 \leq \nu \leq \infty)$ ,  $JS_2 = S_2$ ,  $J$  has the reflection line  $w$  on  $S_2$ , this line passes through the ramification point  $P$ , and  $J$  interchanges the two holes. We conclude that  $G_{02}$  is a Fuchsian group of signature  $(0, 1, 1; \nu)$ . We also see that  $\Omega(G_2)/G_2$  is a torus with one ramification point of order  $\nu$ .

We let  $H$  be the stabilizer of  $C'$  in  $G$ . Then  $H$  is a hyperbolic cyclic group with generator  $h$ ; also  $H = G_1 \cap G_2$ . It is quite easy to see that  $G_2$  is generated by  $h$  and  $g$ , and that these satisfy the relations

$$g^2 = 1, (g \circ h^{-1} \circ g \circ h)^\nu = 1 \tag{7}$$

( $\gamma^\infty = 1$  means that  $\gamma$  is parabolic). (One can see from this that there is a loop  $v$  on  $\Omega(G_2)/G_2$ , where  $v$  cuts  $w, w', w''$  each exactly once, and  $v^2$  lifts to a loop on  $\Omega(G_2)$ . This in fact proves that there can be no other relations in  $G_2$  (see [23], [25]).)

We show finally how to extend the stratification of  $G_1$  by adding one parameter to obtain a stratification of  $G$ . Our last parameter is

$$y = g(\text{attractive fixed point of } h).$$

Note that  $h \in G_1$ ; hence  $h$  is determined by the stratification of  $G_1$ . We must show that the extra parameter determines the extra generator of  $G$ . We normalize so that

$$h = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad 0 < |\tau| < 1,$$

$$y = g(0) = 1.$$

(We can assume that  $\tau$  is determined since  $T(G)$  is simply connected.) Hence (because  $g^2 = 1$ )

$$g = \begin{pmatrix} \alpha & -\alpha \\ \beta & -\alpha \end{pmatrix}, \quad \alpha(\beta - \alpha) = 1.$$

The second of the relations (7) implies that

$$\text{tr}(g \circ h^{-1} \circ g \circ h) = 2\alpha^2 - \alpha\beta(\tau^2 + \tau^{-2})$$

is constant on  $T(G)$ . Since  $h^2$  is loxodromic, the last two equations determine  $\alpha\beta$  and  $\alpha^2$  uniquely. Hence these two equations have solutions  $(\alpha, \beta)$  and  $(-\alpha, -\beta)$ , and so we have determined  $g$  from our parameters. This completes Case VI.

*Case VII.  $n > 1$ .*

We let  $P_1$  and  $P_2$  be two adjacent ramification points or parabolic punctures on some reflection arc  $w$  of the anti-conformal involution  $J$  on  $S$ . We denote the orders of these ramification points by  $\nu_i$ ,  $2 \leq \nu_i \leq \infty$ , and we find a simple loop  $v$  on  $S$  with the following properties. The loop  $v$  divides  $S$  into two subsurfaces  $S_1$  and  $S_2$ ; both invariant under  $J$ . The subsurface  $S_2$  has genus 0, contains the two points  $P_1$  and  $P_2$  and no other ramification points or punctures.

We may assume that  $d(G) > 0$  (as otherwise there is nothing to prove). Then there is a shortest geodesic in the homotopy class of  $v$ . We now replace  $v$  by the geodesic in its equivalence class and note that the statement that  $P_1$  and  $P_2$  are adjacent means that if  $\nu_1 < \infty$ ,  $\nu_2 < \infty$ , then  $S_2$  contains exactly two fixed points of  $J$ . We remark that if  $\nu_1 = 2 = \nu_2$ , then the geodesic is no longer a loop, but a segment between the ramified points, and all our arguments require minor modifications, which we will ignore.

Exactly as before, we let  $C'$  be a lift of  $v$ , where  $C'$  intersects the positive imaginary axis; we complete it to a circle and let  $\mathcal{A} = \bigcup_{\gamma \in G} \gamma(C')$ . We let  $R_1$  and  $R_2$  be the regions in  $U$  cut out by  $\mathcal{A}$  with the boundaries of  $R_1$  and  $R_2$  containing  $C'$  so that the projections of these regions are  $S_1$  and  $S_2$  respectively. Both  $R_1$  and  $R_2$  are  $j \circ g$ -invariant. As in the preceding cases, for  $i = 1, 2$  we let  $G_i$  be the stabilizer of  $R_i \cup jR_i$  in  $G$ , and let  $G_{0i}$  be the stabilizer of  $R_i$  in  $G_0$ . We know that  $R_i$  is invariant under the reflection  $j \circ g$ , and thus  $(R_i \cup jR_i)/G_i = R_i/G_{0i}$ . Further  $R_i/G_{0i}$  are two parts of  $S$  cut along  $v$ . We also know that  $j \circ g$  is a reflection that conjugates  $G_{01}$  into itself, and that the fixed line of  $j \circ g$  cuts  $C'$ . If we let  $H_0$  be the hyperbolic cyclic group stabilizing  $C'$ , then we see that  $gH_0g^{-1} = H_0$ , and we conclude that the stabilizer  $H$  of  $C'$  in  $G_1$  is a non-abelian  $\mathbf{Z}_2$ -extension of  $H_0$ ; that is, as a Fuchsian group acting on the inside of  $C'$ ,  $H$  represents a disc with two ramification points each of order 2. We conclude that  $\Omega(G_1)/G_1$  is homeomorphic to  $\Omega(G)/G$  as a surface with ramification points, except that the points  $P_i$  ( $i = 1, 2$ ) no longer have ramification index  $\nu_i$ ; now they both have ramification index 2. Hence

$$d = d(G) = d(G_1).$$

Of course  $G_1$  is a geometrically finite function group; hence stratifiable. We let  $x_1, \dots, x_{d+3}$  be a stratification for  $G_1$ .

We repeat the above analysis for  $G_2$  and we conclude that  $\Omega(G_2)/G_2$  is a sphere with four ramification points of indices  $\nu_1, \nu_2, 2, 2$ . We choose generators  $g_1, g_2$  for  $G_{02}$ , where  $g_i^{\nu_i} = 1$  ( $i = 1, 2$ ),  $g_2 \circ g_1 = h$ , a generator for  $H_0$ , and  $(g \circ j) \circ g_1 \circ (g \circ j) = g_1^{-1}$ . Then  $G_2$  is generated by  $g_1, g_2, g$ . These satisfy the relations  $g_1^{\nu_1} = g_2^{\nu_2} = g^2 = (g \circ g_2 \circ g_1)^2 = (g \circ g_1)^2 = 1$ . (We remark that from the theory of signatures of Kleinian groups [26], we know  $\Omega(G_2)/G_2$ , and so  $G_2$  has a presentation of three elliptic or parabolic generators where a product of two of them is elliptic, parabolic or the identity, and a product of all three is elliptic or parabolic.)

We must show that the stratification of  $G_1$  already stratifies  $G$ . The stratification of  $G_1$  determines  $h$  and  $g$ . Again we change normalization so that ( $G_0$  is no longer Fuchsian)

$$h = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We write

$$g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Now  $\text{tr } g_1$  and  $\text{tr } g_2$  are known constants. But

$$\text{tr } g_2 = \text{tr } h \circ g_1^{-1} = \tau\delta + \tau^{-1}\alpha,$$

$$\text{tr } g_1 = \alpha + \delta.$$

Hence we can solve for  $\alpha$  and  $\delta$ . Since  $(g \circ g_1)^2 = 1$ , we also have  $\gamma = \beta$ . Finally to solve for  $\gamma$  we use  $\gamma^2 = \alpha\delta - 1$ . In general we have two solutions (note that  $\alpha\delta \neq 1$ ). The connectivity and simply connectivity of  $T(G)$  force the selection of square root.

This completes the proof of Theorem 4.

*Remark.* The cuts we made in the surface  $S$  had to be chosen with care. For example, had we chosen in the last case to cut  $S$  along a simple non-dividing loop  $v$  where  $J$  has exactly two fixed points on  $v$ , then we would have obtained a group  $G_1$  with  $d(G_1) > d(G)$ .

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Department of Mathematics  
 State University of New York at  
 Stony Brook, NY 11794

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