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**Autor:** Kraft, Hanspeter / Procesi, Claudio

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# On the geometry of conjugacy classes in classical groups

HANSPETER KRAFT* and CLAUDIO PROCESI

*Summary.* We study closures of conjugacy classes in the Lie algebras of the orthogonal and symplectic groups and determine which ones are normal varieties. Furthermore we give a complete classification of the minimal singularities which arise in this context, i.e. the singularities which occur in the open classes in the boundary of a given conjugacy class. In contrast to the results for the general linear group ([KP1], [KP2]) there are classes with non normal closure; they are branched in a class of codimension two and give rise to normal minimal singularities. The methods used are (classical) invariant theory and algebraic geometry.

## 0. Introduction

**0.1** The subject of this paper is the study of the singularities arising in the closure of a conjugacy class of a semisimple group. In our preceding papers [KP1], [KP2], [PK] we treated in detail the case of the *linear group*, developing a number of techniques mostly based on *classical invariant theory*. In this paper we continue the analysis for the other classical groups, obtaining various precise results that will be presently explained. The exceptional groups seem to be untreatable by the methods here developed, essentially because of the lacking of a suitable analogue of the so-called "First Fundamental Theorem of Invariant Theory" which we have for classical groups.

Before going into a detailed discussion of our results let us recall some of the main features of the theory.

**0.2** *Conjugacy classes*: If $G$ is a reductive group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, we study the *adjoint action* of $G$ on itself and on $\mathfrak{g}$. The orbits of this action are the *conjugacy classes*. If $C$ is such a conjugacy class in $G$, it is open in its closure $\bar{C}$ which is an affine algebraic variety. The dimension of each class is even and $\bar{C}$ is the union of finitely many conjugacy classes. In $\bar{C}$ there is a *unique closed class* $C'$ which is necessarily the conjugacy class of a semisimple element. By the theory of

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Luna [Lu] we can fibre $\bar{C}$ over $C'$ by a map $\varphi: \bar{C} \to C'$ which associates to each element $x$ of $\bar{C}$ its semisimple part $x_s$. The fibre of an element $s \in C'$ can be described in this way: If $y \in C \cap \varphi^{-1}(s)$ and $y = s.u$ is the Jordan decomposition then $\varphi^{-1}(s)$ is isomorphic to the closure of the conjugacy class of $u$ in the (reductive) centralizer $Z_G(s)$ of $s$.

This analysis shows that the study of singularities in closures of conjugacy classes can be reduced to the case of *unipotent classes.* Moreover, if we assume that $G$ is a *classical group* so will be the centralizer $Z_G(s)$. Thus we will restrict ourselves to unipotent elements. Finally, since we work over $\mathbb{C}$, the unipotent variety of $G$ is isomorphic in a $G$-equivariant way with the *nilpotent cone* $N$ of $\mathfrak{g}$ (under the logarithmic map). It will be more convenient to treat this case.

**0.3** *Invariant functions on* $\mathfrak{g}$ (The theory of Kostant [Ko]): Let $R$ be the ring of regular functions on $\mathfrak{g}$ invariant under $G$. $R$ is a polynomial ring in $r := \operatorname{rank} G$ *homogeneous generators* which define a map $\pi: \mathfrak{g} \to \mathbb{C}^r$ constant on conjugacy classes. By the general theory of invariants in each fibre we find a unique closed orbit, here the class of a semisimple element. Moreover, in $\pi^{-1}(x)$ there is a unique open dense class, the *regular class*, and $\pi^{-1}(\pi(0))$ is the nilpotent cone $N$ of $\mathfrak{g}$. The fibres of $\pi$ have all the *same dimension*, are *reduced* and even *normal.* In fact, the $r$ equations defining $\pi^{-1}(x)$ give us this fibre as a *normal complete intersection.*

The closure of a non regular class is not a complete intersection in general and one of our methods consists in constructing an associated variety which is a complete intersection from which the given closure can be obtained as a quotient (cf. 0.10).

There are other important features in this theory which here will not be pursued; they refer mostly to the *theory of sheets* (i.e. maximal irreducible subsets of $\mathfrak{g}$ of classes of a fixed dimension, cf. [BK]). This part of the analysis of conjugacy classes has been extensively treated by various authors and it is of course also intimately connected with ours (cf. [BK], [Kr], [P], [B], [Ke1]).

**0.4** *Rational singularities*: The result of Kostant on the global nature of the fibres $\pi^{-1}(x)$ can be usefully improved showing that this variety has *rational singularities* ([H3]). We recall:

DEFINITION. A normal variety $Z$ is said to have *rational singularities* if there is a resolution of singularities $\varphi: Y \to Z$ such that $R^i\varphi_*\mathcal{O}_Y = 0$ for all $i > 0$.

This notion is considerably stronger than the Cohen–Macaulay property and will play a role in our analysis. In fact, we have the following result of Kempf's

([KK] p. 50):

PROPOSITION. *If $Z$ is a normal variety and $\varphi: Y \to Z$ a resolution of singularities then $Z$ has rational singularities if and only if $Z$ is Cohen–Macaulay and for any differential form $\omega$ defined on the smooth part of $Z$, the pull back $\varphi^*(\omega)$ can be extended to the whole of $Y$.*

**0.5** *Kleinian singularities*: In the case of a surface, rational singularities are strongly connected with quotient singularities and with semisimple groups (cf. 0.6). This connection is established through the *simple* or *Kleinian singularities.* Let $H$ be a finite subgroup of $SL_2(\mathbb{C})$. The quotient $\mathbb{C}^2/H$ is a surface in $\mathbb{C}^3$ with an isolated singularity in zero. The finite groups $H$ and the equations defining $\mathbb{C}^2/H$ have been described by Klein; they correspond to the Dynkin diagrams $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$.

| $H$ | $\lvert H\rvert$ | equation of $\mathbb{C}^2/H \subset \mathbb{C}^3$ | Dynkin diagram |
|---|---|---|---|
| cyclic | $n+1$ | $x^{n+1}+y^2+z^2=0$ | $A_n$ |
| dihedral | $4n$ | $x^{n+1}+xy^2+z^2=0$ | $D_{n+2}$ |
| binary tetrahedral | 24 | $x^4+y^3+z^2=0$ | $E_6$ |
| binary octahedral | 48 | $x^3y+y^3+z^2=0$ | $E_7$ |
| binary icosahedral | 120 | $x^5+y^3+z^2=0$ | $E_8$ |

The connection with the Dynkin diagrams appears forming a *minimal resolution of singularities.* Then the exceptional fibre is a union of lines with selfintersection number $-2$ meeting transversally. The Dynkin diagram is constructed simply by drawing for each line a vertex and for each intersection point an edge between the two corresponding vertices (cf. for example [Br1]).

**0.6** *Subregular singularities* (Brieskorn's theory [Br2], [Sl]): We have seen that the nilpotent cone $\mathcal{N}$ in a reductive Lie algebra $\mathfrak{g}$ contains a unique dense open class, the *regular class* $C_{\text{reg}}$. If we consider $\mathcal{N}' := \mathcal{N} - C_{\text{reg}}$ we still find in $\mathcal{N}'$ a unique dense open class, the *subregular class* $C_{\text{subreg}}$ which is of codimension 2 in $\mathcal{N}$. If we slice $\mathcal{N}$ transversally to $C_{\text{subreg}}$ we find an isolated surface singularity. *It is always a Kleinian singularity and its Dynkin diagram is the one corresponding to G in the cases $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$.* In the other cases there is a simple rule to discover the corresponding Dynkin diagram which gives the following table:

| group | singularity |
|---|---|
| $B_n$ | $A_{2n-1}$ |
| $C_n$ | $D_{n+1}$ |
| $G_2$ | $D_4$ |
| $F_4$ | $E_6$ |

We will generalize these results to pairs of conjugacy classes $C$, $C'$ such that $C' \subset \bar{C}$ and $\operatorname{codim}_C C' = 2$ (cf. theorems 2 and 2′ in 0.9).

The theory of Brieskorn continues to describe a *semiuniversal deformation* of the singularity under consideration which is obtained by restricting the quotient map $\pi : \mathfrak{g} \to \mathbb{C}^r$ (0.3) to a cross section in $\mathfrak{g}$ through the subregular class $C_{\text{subreg}}$. This should have some analogue in our theory which has not yet been understood.

**0.7** *Collapsing of vector bundles and minimal singularities* (Kempf's theory [K]): Let $G$ be again a reductive group acting linearly on a vector space $V$ and $x \in V$. A way, sometimes useful, to study the orbit closure $\overline{Gx}$ is the following: Assume there is a linear subspace $U \subseteq \overline{Gx}$ with $x \in U$ such that $P := \{g \in G \mid gU \subseteq U\}$ is a parabolic subgroup. We can then construct a proper map

$$\varphi : G \times^P U \to \overline{Gx}$$

and say that $\overline{Gx}$ is obtained by *collapsing the vector bundle* $G \times^P U$ (over $G/P$). If the stabilizer $G_x$ of $x$ is contained in $P$ we have that $\varphi$ is birational, hence a resolution of singularities. Kempf's theorem implies now that, *if $P$ acts in a completely reducible way on $U$, then $\overline{Gx}$ is normal with rational singularities.* The strength of this theorem lies (also) in the fact that the normality is automatically insured, the weakness on the other hand results from the seriously restrictive hypothesis on the action of $P$ on $U$. While it is often easy to describe a resolution of singularities of the form $G \times^P U$ it seldom happens that the action of $P$ on $U$ is completely reducible.

One noteworthy example which plays also a role in our analysis is the following: Given an irreducible representation $V$ of $G$ and a highest weight vector $v \in V$, the line $\mathbb{C}v$ is fixed by a parabolic and the variety $\overline{Gv}$ has a resolution of singularities given by the line bundle $G \times^P \mathbb{C}v$ over $G/P$. In the case of the adjoint action of $G$ on $\mathfrak{g}$, $Gv$ is the unique *minimal nilpotent* (non zero) class indicated by $C_{\min}$, and $\bar{C}_{\min}$ *is normal, Cohen–Macaulay and has an isolated rational singularity in zero.*

**0.8** Having recalled these theorems we can now expose the contents of the paper. The first type of results deals with the following question: Given a nilpotent conjugacy class $C$ in the orthogonal or symplectic Lie algebra $\mathfrak{o}_n$ or $\mathfrak{sp}_n$, what can we say about its closure $\bar{C}$? E.g. is it *normal, Cohen–Macaulay or with rational singularities* like in the case of $\mathfrak{gl}_n$? W. Hesselink has shown in [H4] that $\bar{C}$ is normal if for the corresponding partition $\eta = (\eta_1, \eta_2, \ldots)$ describing the Jordan normal form of a matrix in $C$ we have $\eta_1 + \eta_2 \leqq 4$ in the orthogonal case and $\eta_1 \leqq 2$ in the symplectic case. Here we should remark that for the orthogonal group $O_{2n}$ a conjugacy class $C$ sometimes splits as the union of two $SO_{2n}$-classes $C^{(1)}$ and $C^{(2)}$. This happens precisely when all blocks in a Jordan normal form of an element of $C$ have even size. It will turn out, that for these special classes $C^{(i)}$ our information is more incomplete. For the other classes (including the non-connected orthogonal classes) we have the following result (9.2, 16.2).

THEOREM 1. *Let $C$ be an orthogonal or symplectic conjugacy class. Then*
(a) *$C$ is always seminormal (i.e. any homeomorphic map $Z \to \bar{C}$ is an isomorphism),*
(b) *$\bar{C}$ is normal if and only if it is normal in the classes of codimension 2.*

In particular $\bar{C}$ is normal if it does not contain a class of codimension 2 (8.3); this occurs often for low-dimensional classes (cf. tables at the end of the paper). For another consequence consider a resolution of singularities $\varphi: Y \to \bar{C}$ (cf. 10.2 and 15.1). *Then $\bar{C}$ is normal if and only if the fibres of $\varphi$ over any class of codimension 2 in $\bar{C}$ are connected.*

The first connected non-normal closures of conjugacy classes are $\bar{C}_{(3,2,2)}$ in $\mathfrak{so}_7$, $\bar{C}_{(3,3,1,1)}$ in $\mathfrak{sp}_8$ and $\bar{C}_{(5,2,2)}$ in $\mathfrak{so}_9$ (the partition always refers to the Jordan normal form of a matrix in $C$). In $\mathfrak{so}_8$ all classes have normal closure, but there is always a (connected) class with non-normal closure in $\mathfrak{sp}_n$ and $\mathfrak{so}_n$ for $n \geqslant 9$.

The more precise question about the Cohen–Macaulay property or rational singularities does not yet always have an answer; one serious difficulty which does not occur in $\mathfrak{gl}_n$ comes from the presence of *non-polarizable* classes (cf. section 10). In the third part we collect various special results and methods in this line.

**0.9** Theorem 1 shows that the normality of $\bar{C}$ is determined by the type of singularities occurring in the classes of codimension two. More precisely we ask the following question: *Given a conjugacy class $C$ and an open class $C'$ in the boundary $\partial C = \bar{C} - C$, what is the type of singularity* (up to smooth equivalence) *of $\bar{C}$ in $C'$?* We have already seen two such examples in 0.6 and 0.7: the *subregular* singularity $C_{\text{subreg}} \subseteq \bar{C}_{\text{reg}}$ and the *minimal* singularity $0 \in \bar{C}_{\min}$. It turns out that with one exception these are the only types of singularities occurring.

THEOREM 2. *Let $C$ be an orthogonal or symplectic conjugacy class and $C'$ an open class in the boundary $\partial C = \bar{C} - C$.*
(a) *If $C'$ is of codimension 2 then the singularity of $\bar{C}$ in $C'$ is smoothly equivalent to an isolated surface singularity of type $A_k$, $D_k$ or $A_k \cup A_k$, where the last one is the non-normal union of two surface singularities of type $A_k$ meeting transversally in the singular point.*

(b) *If $C'$ is of codimension $>2$ then the singularity of $\bar{C}$ in $C'$ is smoothly equivalent to a minimal singularity in $\mathfrak{so}_n$ or $\mathfrak{sp}_n$.*

In addition we give a simple combinatorial method to determine these pairs $(C, C')$ and the corresponding singularity: If $\eta$ and $\sigma$ are the Young-diagrams (partitions) of $C$ and $C'$ (describing the Jordan-decomposition of a matrix in $C$ and $C'$), we cancel all common rows and columns of $\eta$ and $\sigma$ to obtain an "irreducible" pair $\eta'$ and $\sigma'$, called the *type* of $(C, C')$.

THEOREM 2'. *In the situation of theorem 2 the singularity of $\bar{C}$ in $C'$ depends up to smooth equivalence only on the type of $(C, C')$. The types and the corresponding singularities are listed in table I* (3.4). (cf. 12.3).

E.g. the pair $(C, C')$ in $\mathfrak{sp}_{12}$ given by the partitions $(4, 3, 3, 1, 1)$ and $(4, 2, 2, 2, 2)$ has type $(2, 2), (1, 1, 1, 1)$ (cancel the first row and the first column!) with corresponding non-normal singularity $A_1 \cup A_1$.

**0.10** A main tool in the proofs of these theorems is classical invariant theory. Given a symplectic space $U$ and an orthogonal space $V$ (i.e. vector-spaces with non-degenerate skew-symmetric or symmetric form respectively) we consider $\mathrm{Hom}(U, V)$ as a representation of $Sp(U) \times O(V)$. Then the "First Fundamental Theorem" states that there are natural maps $\pi : \mathrm{Hom}(U, V) \to \mathfrak{sp}(U)$ and $\rho : \mathrm{Hom}(U, V) \to \mathfrak{so}(V)$ which are quotients with respect to $O(V)$ and $Sp(U)$ respectively (4.2). This construction allows us to proceed by induction and to associate to a conjugacy class $C$ an affine variety $Z$ and a surjective map $\vartheta : Z \to \bar{C}$ which is a quotient (0.11) under a certain product of orthogonal and symplectic groups (§5). We show that $Z$ is a *complete intersection* and compute the dimension $\vartheta^{-1}(C')$ of classes $C' \subset \bar{C}$. For this purpose we give a *classification of the orbits in* $\mathrm{Hom}(U, V)$ under $Sp(U) \times O(V)$, called "*orthosymplectic orbits*" (§6). (We thank H.-G. Quebbemann and V. Kac for explaining to us – in rather different ways – how this classification can be obtained.) Furthermore we need a *dimension formula* for these orbits $O \subset \mathrm{Hom}(U, V)$ expressing $\dim O$ in terms of the dimensions of the conjugacy classes $\pi(O) \subset \mathfrak{sp}(U)$ and $\rho(O) \subset \mathfrak{so}(V)$ (§7). With these results it is not difficult to obtain part (b) of theorem 1, whereas for the other claims we have to make a very precise and detailed analysis of the geometry of the two quotient maps $\pi$ and $\rho$. This is done in part II of the paper.

**0.11** Finally we should remark some conventions. The ground field $k$ is algebraically closed and of characteristic zero. For any variety $Z$ we denote by $\mathcal{O}(Z)$ the ring of (global) regular functions. If $G$ is a reductive group acting on an affine variety $Y$, $R = \mathcal{O}(Y)$, $R^G$ the subring of invarient functions and $W$ the maximal spectrum of $R^G$, we will often indicate $W = Y/G$ and call the map $\pi: Y \to Y/G$ the *quotient map* under $G$ (even if it has bad fibres). Sometimes $Y/G$ is contained in a larger variety $Z$ but we may still call the composed map $Y \to Y/G \hookrightarrow Z$ a quotient.

The main property of quotient maps is that *if $X \subseteq Y$ is a closed $G$-stable subvariety then $\pi(X)$ is a closed subvariety of $Y/G$ and $\pi|_X : X \to \pi(X)$ is again a quotient under $G$.* Clearly *if $Y$ is normal then $\pi(Y) = Y/G$ is also normal.* Such a permanence does not hold for the *Cohen–Macaulay property*. On the other hand one can often use the following important result of Boutot ([Bt], cf. [Ho]):

THEOREM. *If $Y$ has rational singularities then $Y/G$ has also rational singularities.*

More precisely Boutot proves that for any subring $S \subset \mathcal{O}(Y)$ which is a direct summand as $S$-module the maximal spectrum of $S$ has rational singularities.

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# Part I. The basic construction

## 1. Quadratic spaces

**1.1** Let $\varepsilon$ be $+1$ or $-1$. A finite dimensional vector space $V$ with a non-degenerate form $(\ ,\ )$ such that $(u, v) = \varepsilon(v, u)$ for all $u, v \in V$ will be called a *quadratic space of type* $\varepsilon$ (shortly an *orthogonal space* in case $\varepsilon = 1$, a *symplectic space* in case $\varepsilon = -1$). We denote by $G(V)$ the subgroup of $GL(V)$ leaving the form invariant. So we have $G(V) \cong O_n$ or $G(V) \cong Sp_n$ according to $\varepsilon = 1$ or $\varepsilon = -1$ ($n := \dim V$).

Let $?^* : \operatorname{End} V \to \operatorname{End} V$ be the canonical involution associated to the form, i.e. for any $D \in \operatorname{End} V$ the adjoint endomorphism $D^*$ is defined by

$$(Du, v) = (u, D^*v) \quad \text{for} \quad u, v \in V.$$

By definition we have

$$G(V) = \{g \in GL(V) \mid g^* = g^{-1}\},$$
$$\mathfrak{g}(V) := \operatorname{Lie} G(V) = \{D \in \operatorname{End} V \mid D^* = -D\}$$

(the space of skew endomorphisms), and

$$\dim G(V) = \dim \mathfrak{g}(V) = \frac{n^2 - \varepsilon n}{2}, \qquad n := \dim V.$$

If $V$ and $U$ are quadratic spaces of type $\varepsilon$ and $\varepsilon'$ respectively and $X: V \to U$ a linear map, the adjoint map $X^*: U \to V$ is defined by

$$(Xv, u)_U = (v, X^*u)_V \quad \text{for} \quad v \in V,\, u \in U.$$

One easily sees that $(X^*)^* = \varepsilon \cdot \varepsilon' \cdot X$.

**1.2** Let $V$ be a quadratic space of type $\varepsilon$, $U$ a quadratic space of type $-\varepsilon$. Then the compositions $XX^*$ and $X^*X$ are skew; hence we have the two maps

$$\begin{array}{ccc} \mathrm{L}(V, U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \rho \downarrow & & \\ \mathfrak{g}(V) & & \end{array}$$

defined by $\pi(X) := XX^*$, $\rho(X) := X^*X$, where $\mathrm{L}(V, U) := \mathrm{Hom}_k(V, U)$ is the space of linear maps. The group $G(U) \times G(V)$ acts on $\mathrm{L}(V, U)$ in the obvious way ($(g, h)X := gXh^{-1}$), and $\pi$ and $\rho$ are equivariant with respect to this operation and the adjoint operation of $G(U)$ and $G(V)$ on $\mathfrak{g}(U)$ and $\mathfrak{g}(V)$ respectively.

Let us assume $n := \dim V \geq m := \dim U$. The following is the first fundamental theorem of classical invariant theory ([W] II.A theorem 2.9.A and VI theorem 6.1.A, [V] §3, théorème 1 and théorème 2):

THEOREM. *$\pi$ and $\rho$ are quotient maps (under $G(V)$ and $G(U)$ respectively) and the image of $\rho$ is the determinantal variety in $\mathfrak{g}(V)$ of endomorphisms of rank $\leq m$.*

(For a characteristic free proof see [CP] theorem 5.6 (i) and theorem 6.6.)

*Remark.* Here we are reformulating the theorem for the orthogonal group. We are using the fact that, if $J$ is a non-degenerate skew $n \times n$ matrix, one can identify the space of symmetric $n \times n$ matrices with the Lie algebra of the symplectic group of $J$, by the map $Y \mapsto YJ$.

## 2. Conjugacy classes and their degenerations

**2.1** Let $V$ be a quadratic space of type $\varepsilon$. The following is the crucial result for the classification of conjugacy classes and their closures in classical Lie algebras (Freudenthal, Gerstenhaber, Hesselink; cf. [SS] IV.2.19, [H1] theorem 3.10). For any subgroup $G \subseteq GL(V)$ and any element $D \in \operatorname{End} V$ we denote by $G \cdot D$ the conjugacy class of $D$ under $G$.

PROPOSITION. *If* $D \in \mathfrak{g}(V)$ *is an endomorphism, then*
(a) $GL(V) \cdot D \cap \mathfrak{g}(V) = G(V) \cdot D$,
(b) $\overline{GL(V) \cdot D} \cap \mathfrak{g}(V) = \overline{G(V) \cdot D}$.

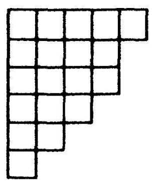
This implies that the conjugacy class $C_D := G(V) \cdot D$ of a nilpotent $D \in \mathfrak{g}(V)$ is determined by its associated *partition* $\eta = (\eta_1, \eta_2, \ldots, \eta_t)$,

$$\eta_1 \geq \eta_2 \geq \cdots \geq \eta_t, \quad |\eta| := \sum_{i=1}^{t} \eta_i = \dim V,$$

given by the *sizes of the blocks of the Jordan normal form of D* (in End $V$). If we denote by $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_s)$ the dual partition (i.e. $\hat{\eta}_i := \#\{j \mid \eta_j \geq i\}$) we have for all $j$

$$\dim \operatorname{Ker} D^j = \sum_{i=1}^{j} \hat{\eta}_i.$$

It is convenient to represent the partitions geometrically as *Young-diagrams* with rows consisting on $\eta_1, \eta_2, \ldots, \eta_t$ boxes respectively. Then the dual partition $\hat{\eta}$ is defined by setting $\hat{\eta}_i$ equal to the length of the $i$th column of the diagram $\eta$. E.g. the partition $(5, 4, 4, 3, 2, 1)$ is represented by



**2.2** The diagram $\eta$ associated to a nilpotent $D \in \mathfrak{g}(V)$ satisfies the following condition $Y_\varepsilon$ ([SS] IV.2.15):

($Y_\varepsilon$) *The number* $\#\{j \mid \eta_j = i\}$ *is even for* $i \equiv \frac{1-\varepsilon}{2} \bmod 2$.

This means that for *orthogonal* $V$ (*symplectic* $V$) the rows of *even* length (of *odd* length) occur *pairwise*. Furthermore *any diagram of this type comes from a nilpotent conjugacy class in* $\mathfrak{g}(V)$.

DEFINITION. A Young-diagram $\eta$ satisfying condition $Y_\varepsilon$ is called an *$\varepsilon$-diagram*. We denote by $C_{\varepsilon,\eta}$ the associated nilpotent conjugacy class in $\mathfrak{g}(V)$, $V$ a quadratic space of type $\varepsilon$ of dimension $|\eta|$.

*Remark.* There are Young-diagrams $\eta$ satisfying both conditions $Y_1$ and $Y_{-1}$. Such a diagram determines two different conjugacy classes $C_{1,\eta}$ and $C_{-1,\eta}$, an orthogonal and a symplectic one.

Let us summarize these results.

THEOREM. *Let $V$ be a quadratic space of type $\varepsilon$ and $\eta$ a Young-diagram of size $|\eta| = \dim V$. Denote by $C_\eta$ the corresponding nilpotent conjugacy class in* $\mathfrak{gl}(V)$.

(i) $C_\eta \cap \mathfrak{g}(V) \neq \emptyset$ *if and only if $\eta$ is an $\varepsilon$-diagram.*

(ii) *If $\eta$ is an $\varepsilon$-diagram, then*

(a) $C_\eta \cap \mathfrak{g}(V) = C_{\varepsilon,\eta}$ *is a single conjugacy class in* $\mathfrak{g}(V)$,

(b) $\overline{C_\eta} \cap \mathfrak{g}(V) = \bar{C}_{\varepsilon,\eta}$.

**2.3** Let us remark that a conjugacy class $C$ under the *orthogonal group* is connected if and only if $C$ is also a conjugacy class under the *special orthogonal group*. To determine these classes we need the following analysis ([SS] IV.2.27).

DEFINITION. A Young-diagram $\eta$ is called *very even*, if all rows are of even length and occur an even number of times.

PROPOSITION. *For an $\varepsilon$-diagram $\eta$ the conjugacy class $C_{\varepsilon,\eta}$ is disconnected if and only if* $\varepsilon = 1$ (hence $V$ orthogonal) *and $\eta$ is very even. In this case* $|\eta| \equiv 0$ mod 4 *and $C_{\varepsilon,\eta}$ splits into two conjugacy classes with respect to $SO(V)$.*

*Remark.* If $\eta$ is very even and $C^{(1)}$, $C^{(2)}$ are the two components of $C_{1,\eta}$, then we have $C \subseteq \overline{C^{(1)}} \cap \overline{C^{(2)}}$ for any conjugacy classes $C \subseteq \bar{C}_{1,\eta}$, $C \neq C_{1,\eta}$.

**2.4** We recall the formula giving the dimension of a conjugacy class $C_{\varepsilon,\eta}$ in terms of the diagram $\eta$ (cf. [H1] corollary 3.8(a)).

PROPOSITION. *Let $V$ be a quadratic space of type $\varepsilon$ and $D \in \mathfrak{g}(V)$ a nilpotent element with associated Young-diagram $\eta$. Then*

$$\dim \operatorname{Cent}_{G(V)} D = \frac{1}{2}\left(\sum_i \hat{\eta}_i^2 - \varepsilon \#\{j \mid \eta_j \text{ odd}\}\right)$$

$$\dim C_{\varepsilon,\eta} = \frac{1}{2}\left(|\eta|^2 - \varepsilon\, |\eta| - \sum_i \hat{\eta}_i^2 + \varepsilon \#\{j \mid \eta_j \text{ odd}\}\right).$$

*Remark.* One has $\dim \operatorname{Cent}_{GL(V)} D = \sum_i \hat{\eta}_i^2$ (cf. [H1] corollary 3.8(a)), hence

$$\dim \operatorname{Cent}_{GL(V)} D = 2 \dim \operatorname{Cent}_{G(V)} D + \varepsilon \#\{j \mid \eta_j \text{ odd}\}$$

and

$$\dim C_\eta = 2 \dim C_{\varepsilon,\eta} + \varepsilon(|\eta| - \#\{j \mid \eta_j \text{ odd}\}),$$

where $C_\eta$ is the conjugacy class in $\mathfrak{gl}(V)$ generated by $C_{\varepsilon,\eta}$.

**2.5** DEFINITION. Let $\eta$ be an $\varepsilon$-diagram. An *$\varepsilon$-degeneration* of $\eta$ is an $\varepsilon$-diagram $\sigma$ such that $|\sigma| = |\eta|$ and $C_{\varepsilon,\sigma} \subseteq \bar{C}_{\varepsilon,\eta}$. We describe this ordering by $\sigma \leq \eta$.

The following is the basic result on degenerations of conjugacy classes ([H1] theorem 3.10; cf. also theorem 2.2(ii)(b) and [KP1] proposition 1.3(a)).

PROPOSITION. *Given two $\varepsilon$-diagrams $\sigma$ and $\eta$ with $|\sigma| = |\eta|$ we have $C_{\varepsilon,\sigma} \subseteq \bar{C}_{\varepsilon,\eta}$ if and only if $\sum_{i=1}^{j} \sigma_i \leq \sum_{i=1}^{j} \eta_i$ for all $j$. This is equivalent to $\sum_{k>j} \hat{\sigma}_k \leq \sum_{k>j} \hat{\eta}_k$ for all $j$.*

*Remark.* One can show that any $\varepsilon$-degeneration of $\eta$ is obtained by "moving down some boxes", taking care of the fact that the result has to be again an $\varepsilon$-diagram. E.g. ($\varepsilon = 1$)

$\eta = (4, 4.2, 2, 1) > \sigma = (3, 3, 3, 3, 1)$

## 3. Minimal degenerations (combinatorial description)

**3.1** In order to state the main result of this section we need the following definition.

DEFINITION. An $\varepsilon$-degeneration $\alpha \leq \eta$ is called *minimal*, if $\sigma \neq \eta$ and there is no $\varepsilon$-diagram $\nu$ such that $\sigma < \nu < \eta$ (i.e. $\sigma < \eta$ are adjacent in the ordering of $\varepsilon$-diagrams).

In geometrical terms this means that the conjugacy class $C_{\varepsilon,\sigma}$ is *open in the complement of* $C_{\varepsilon,\eta}$ *in* $\bar{C}_{\varepsilon,\eta}$.

**3.2** To explain the classification of minimal degenerations we introduce a combinatorial equivalence on $\varepsilon$-degenerations suggested by the following result.

PROPOSITION. *Let $\sigma \leq \eta$ be an $\varepsilon$-degeneration. Assume that for two integers $r$ and $s$ the first $r$ rows and the first $s$ columns of $\eta$ and $\sigma$ coincide and that $(\eta_1, \eta_2, \ldots, \eta_r)$ is an $\varepsilon$-diagram. Denote by $\eta'$ and $\sigma'$ the diagrams obtained by erasing these rows and columns of $\eta$ and $\sigma$ respectively and put $\varepsilon' := (-1)^s\varepsilon$. Then $\sigma' \leq \eta'$ is an $\varepsilon'$-degeneration and*

$$\operatorname{codim}_{\bar{C}_{\varepsilon',\eta'}} C_{\varepsilon',\sigma'} = \operatorname{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma}.$$

*Proof.* By induction it is enough to consider the two cases $s=1$, $r=0$ and $s=0$, $r>0$.

(a) $s=1$, $r=0$: Then $\varepsilon' = -\varepsilon$, $\hat{\sigma}_1 = \hat{\eta}_1$, $\eta_i' = \eta_i - 1$ and $\sigma_i' = \sigma_i - 1$. In addition $\#\{j \mid \eta_j' \text{ odd}\} = \#\{j \mid \eta_j \text{ even}\} = \hat{\eta}_1 - \#\{j \mid \eta_j \text{ odd}\}$ and similarly for $\sigma$. Moreover it is clear from the second description of the ordering that $\sigma' \leq \eta'$. Using the dimension formula 2.4 one gets

$$\begin{aligned} 2\operatorname{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma} &= \sum \hat{\sigma}_i^2 - \sum \hat{\eta}_i^2 - \varepsilon(\#\{j \mid \sigma_j \text{ odd}\} - \#\{j \mid \eta_j \text{ odd}\}) \\ &= \sum \hat{\sigma}_i'^2 - \sum \hat{\eta}_i'^2 - \varepsilon'(\#\{j \mid \sigma_j' \text{ odd}\} - \#\{j \mid \eta_j' \text{ odd}\}) \\ &= 2\operatorname{codim}_{\bar{C}_{\varepsilon',\eta'}} C_{\varepsilon'\sigma'}. \end{aligned}$$

(b) $s=0$, $r>0$: Then $\varepsilon' = \varepsilon$, $\eta_i = \sigma_i$ for $1 \leq i \leq r$, $\hat{\eta}_i = \hat{\eta}_i' + r$, $\hat{\sigma}_i = \hat{\sigma}_i' + r$ for $1 \leq i \leq t := \eta_r = \sigma_r$ and $\hat{\eta}_j = \hat{\sigma}_j$ for $j > t$. Hence

$$\sum_i \hat{\sigma}_i^2 - \sum_i \hat{\eta}_i^2 = \sum_i \hat{\sigma}_i'^2 - \sum_i \hat{\eta}_i'^2$$

since

$$\sum_i \hat{\sigma}'_i - \sum_i \hat{\eta}'_i = |\sigma'| - |\eta'| = 0.$$

Furthermore $\#\{j \mid \sigma_j \text{ odd}\} - \#\{j \mid \eta_j \text{ odd}\} = \#\{j \mid \sigma'_j \text{ odd} - \#\{j \mid \eta'_j \text{ odd}\}$. Again by the dimension formula 2.4 we get the required result, since clearly $\sigma' \leq \eta'$ from the first description of the ordering (2.5). qed.

*Remark.* A similar statement holds in the linear case ([KP2], proposition 3.1).

**3.3** DEFINITIONS. (a) In the setting of the proposition above we say that the *$\varepsilon$-degeneration $\sigma \leq \eta$ is obtained from the $\varepsilon'$-degeneration $\sigma' \leq \eta'$ by adding rows and columns.*

(b) An $\varepsilon$-degeneration $\sigma \leq \eta$ is called *irreducible* if it cannot be obtained by adding rows and columns in a non trivial way.

*Remarks.* (i) In the previous setting we have $\operatorname{codim}_{C_{\varepsilon',\mu'}} C_{\varepsilon',\sigma'} = \operatorname{codim}_{C_{\varepsilon,\eta}} C_{\varepsilon,\sigma}$ (3.2) and $\sigma' \leq \eta'$ is *minimal* if and only if $\sigma \leq \eta$ is *minimal.*

(ii) Any $\varepsilon$-degeneration is obtained in a unique way from an irreducible $\varepsilon'$-degeneration by adding rows and columns.

**3.4** The previous analysis suggests that, for the classification of the minimal $\varepsilon$-degenerations, one should first describe the *minimal irreducible $\varepsilon$-degenerations.* They are given in the following table. (The meaning of the last line of the table is

Table I
Irreducible minimal $\varepsilon$-degenerations

| Type | *a* | *b* | *c* | *d* | *e* | *f* | *g* | *h* |
|---|---|---|---|---|---|---|---|---|
| Lie algebra | $\mathfrak{sp}_2$ | $\mathfrak{sp}_{2n}$ $n>1$ | $\mathfrak{so}_{2n+1}$ $n>0$ | $\mathfrak{sp}_{4n+2}$ $n>0$ | $\mathfrak{so}_{4n}$ $n>0$ | $\mathfrak{so}_{2n+1}$ $n>1$ | $\mathfrak{sp}_{2n}$ $n>1$ | $\mathfrak{so}_{2n}$ $n>2$ |
| $\varepsilon$ | $-1$ | $-1$ | $1$ | $-1$ | $1$ | $1$ | $-1$ | $1$ |
| $\eta$ | $(2)$ | $(2n)$ | $(2n+1)$ | $(2n+1, 2n+1)$ | $(2n, 2n)$ | $(2, 2, 1^{2n-3})$ | $(2, 1^{2n-2})$ | $(2, 2, 1^{2n-4})$ |
| $\sigma$ | $(1, 1)$ | $(2n-2, 2)$ | $(2n-1, 1, 1)$ | $(2n, 2n, 2)$ | $(2n-1, 2n-1, 1, 1)$ | $(1^{2n+1})$ | $(1^{2n})$ | $(1^{2n})$ |
| $\operatorname{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma}$ | 2 | 2 | 2 | 2 | 2 | $4n-4$ | $2n$ | $4n-6$ |
| $\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$ | $A_1$ | $D_{n+1}$ | $A_{2n-1}$ | $A_{2n-1}$ | $A_{2n-1} \cup A_{2n-1}$ | $b_n$ | $c_n$ | $d_n$ |

explained in section 14, cf. 14.2 and 14.3.) It is clear that these $\varepsilon$-degenerations are irreducible and one easily sees that they are minimal. Furthermore it is not hard to deduce from [H1] (proposition 3.1) that the list is complete.

*Remark.* For the types $a, b, c, f$ and $g$ of table I the $\varepsilon$ is determined by $\eta$ and $\sigma$ (because of condition $Y_\varepsilon$). The pair $(\eta, \sigma)$ in case $e$ and $h$ is an $\varepsilon$-degeneration also for $\varepsilon = -1$, but *not a minimal* one.

DEFINITION. Let $t \in \{a, b, c, d, e, f, g, h\}$. An $\varepsilon$-degeneration $\sigma \leq \eta$ is said to be *of type t*, if it is obtained from the corresponding minimal irreducible degeneration by adding rows and columns.

The previous analysis implies that each minimal degeneration $\sigma \leq \eta$ has a *uniquely determined type* $t \in \{a, b, c, d, e, f, g, h\}$ and that $\operatorname{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma}$ equals the codimension of the type $t$ (3.2). In particular $\operatorname{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma} = 2$ if and only if $t \in \{a, b, c, d, e\}$.

## 4. The induction lemma

**4.1** Let $V$ be a quadratic space of type $\varepsilon$ and $D \in \mathfrak{g}(V)$ a nilpotent element with conjugacy class $C_{\varepsilon,\eta}$. Consider the new form on $V$ given by $|u, v| := (u, Dv)$. Clearly it is of type $-\varepsilon$ and its kernel is exactly Ker $D$. Thus we have canonically defined a *non-degenerate form* on $U := \operatorname{Im} D$ of type $-\varepsilon$, and one sees that the two maps

$$V \underset{I}{\overset{X}{\rightleftarrows}} U,$$

given by the canonical decomposition $D = I \circ X : V \twoheadrightarrow U = \operatorname{Im} D \hookrightarrow V$, are *adjoint*, in the sense that $X^* = I$ (cf. 1.1). We have

$$D = IX = X^*X$$

and

$$D' := D\,|_U = XI = XX^*.$$

In particular $D' \in \mathfrak{g}(U)$ (1.2) and it follows from the construction that $D' \in C_{-\varepsilon,\eta'}$ where $\eta'$ is obtained from $\eta$ erasing the first column (cf. [KP1] 2.2 and 2.3).

**4.2** Consider the two maps as in 1.2:

$$\begin{array}{ccc} \mathrm{L}(V, U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \downarrow{\scriptstyle \rho} & & \\ \mathfrak{g}(V) & & \end{array}$$

and define $\mathrm{L}'(V, U) := \{Y \in \mathrm{L}(V, U) \mid Y \text{ surjective}\}$.

LEMMA. *For any $Y \in \mathrm{L}'(V, U)$ the stabilizer of $Y$ in $G(U)$ is trivial and $\rho^{-1}(\rho(Y))$ is an orbit under $G(U)$.*

*Proof.* The first statement is clear since $Y: V \to U$ is surjective. Let $Z \in \rho^{-1}(\rho(Y))$, i.e. $Z^*Z = Y^*Y$. Since $Y^*Y$ has rank $m := \dim U$ the map $Z$ is necessarily surjective (and $Z^*$ injective) and $\operatorname{Ker} Z = \operatorname{Ker} Z^*Z = \operatorname{Ker} Y^*Y = \operatorname{Ker} Y$. Hence we can find a $g \in GL(U)$ such that $gZ = Y$, and so

$$Z^*Z = Y^*Y = Z^*g^*gZ.$$

Since $Z^*$ is injective and $Z$ surjective this implies $g^*g = 1$, i.e. $g \in G(U)$. qed.

**4.3** Now we are ready to prove our main induction lemma. Using the notations introduced in 4.1 we put $N_{\varepsilon,\eta} := \pi^{-1}(\bar{C}_{-\varepsilon,\eta'})$.

LEMMA. (i) $\rho(N_{\varepsilon,\eta}) = \bar{C}_{\varepsilon,\eta}$,
(ii) *$\rho^{-1}(C_{\varepsilon,\eta})$ is a single orbit under $G(U) \times G(V)$ contained in $N_{\varepsilon,\eta} \cap \mathrm{L}'(V, U)$,*
(iii) $\pi(\rho^{-1}(C_{\varepsilon,\eta})) = C_{-\varepsilon,\eta'}$.

$$\begin{array}{ccc} N_{\varepsilon,\eta} & \xrightarrow{\pi} & \bar{C}_{-\varepsilon,\eta'} \\ \downarrow{\scriptstyle \rho} & & \\ \bar{C}_{\varepsilon,\eta} & & \end{array}$$

*Proof.* Clearly the closed set $N_{\varepsilon,\eta}$ is stable under $G(U) \times G(V)$. The construction in 4.1 shows that $\rho(X) = D \in C_{\varepsilon,\eta}$ and $\pi(X) = D|_U \in C_{-\varepsilon,\eta'}$, hence $C_{\varepsilon,\eta} \subseteq \rho(N_{\varepsilon,\eta})$. Since $\rho$ is a quotient map (1.2) the image $\rho(N_{\varepsilon,\eta})$ is closed (0.11) and so $\bar{C}_{\varepsilon,\eta} \subseteq \rho(N_{\varepsilon,\eta})$. On the other hand we have for each $Y \in N_{\varepsilon,\eta}$

$$\operatorname{rk}(YY^*)^{h-1} \leq \sum_{j \geq h} \hat{\eta}'_j = \sum_{j > h} \hat{\eta}_j, \qquad h = 1, 2, \ldots$$

(cf. 2.5 and 1.2; $\hat{\eta}'_i = \hat{\eta}_{i+1}$ by construction). Hence

$$\mathrm{rk}(Y^*Y)^h = \mathrm{rk}\, Y^*(YY^*)^{h-1}Y \leq \mathrm{rk}(YY^*)^{h-1} \leq \sum_{j>h} \hat{\eta}_j.$$

This, again by 2.5, implies that $\rho(Y) = Y^*Y \in \bar{C}_{\varepsilon,\eta}$, proving assertion (i). By construction we have $X \in L'(V, U)$, hence $\rho^{-1}(\rho(X))$ is the orbit of $X$ under $G(U)$ (lemma 4.2). It follows that $\rho^{-1}(C_{\varepsilon,\eta})$ is the orbit of $X$ under $G(U) \times G(V)$, which implies the assertions (ii) and (iii). qed.

We will use this lemma, in the spirit of [KP1], to present the variety $\bar{C}_{\varepsilon,\eta}$ as a quotient of a suitable variety $Z$ which is a complete intersection (cf. the following section 5).

*Remark.* The construction and the lemma above depend only on the first column of $\eta$. In the future we will freely apply it to a partition $\eta$ and all its degenerations which have the same first column.

## 5. The variety $Z$

**5.1** Let us start with a nilpotent endomorphism $D \in \mathfrak{g}(V)$ with conjugacy class $C_D = C_{\varepsilon,\eta}$. In the previous section we have canonically defined a non degenerate form on $D(V)$ such that the two maps

$$V \underset{I}{\overset{X}{\rightleftarrows}} D(V), \qquad D = I \cdot X \text{ the canonical decomposition,}$$

are adjoint (i.e. $X^* = I$) and that $D|_{D(V)} = X \cdot I$ is skew symmetric. Thus proceeding by induction we construct spaces

$$V_0 := V,\ V_1 := D(V), \ldots, V_i := D^i(V), \ldots$$

endowed with non-degenerate forms of type $\varepsilon, -\varepsilon, \ldots, (-1)^i\varepsilon, \ldots$. Since $D$ is nilpotent, for some minimal $t \geq 0$ we have $V_{t+1} = 0$.

The analysis in 4.1 shows that the skew endomorphism $D|_{V_i}$ belongs to the conjugacy class $C_{\varepsilon^i,\eta^i}$, where $\varepsilon^i = (-1)^i\varepsilon$ and $\eta^i$ is obtained from $\eta$ deleting the first $i$ columns.

**5.2** We construct now from these quadratic spaces a variety

$$Z \subseteq M := L(V_0, V_1) \times L(V_1, V_2) \times \cdots \times L(V_{t-1}, V_t)$$

defined by the following equations:

$$\begin{aligned} X_1X_1^* &= X_2^*X_2 \\ X_2X_2^* &= X_3^*X_3 \\ &\vdots \\ X_{t-1}X_{t-1}^* &= X_t^*X_t \\ X_tX_t^* &= 0 \end{aligned} \qquad (*)$$

The variety $Z$ is clearly stable under the obvious action of $G(V_0)\times G(V_1)\times\cdots\times G(V_t)$ on $M$. The given equation imply for a point $(X_1,\ldots,X_t)\in Z$ that

$$\text{rk}\,(X_1^*X_1)^h = \text{rk}\,X_1^*(X_1X_1^*)^{h-1}X_1 \le \text{rk}\,(X_1X_1^*)^{h-1} = \text{rk}\,(X_2^*X_2)^{h-1} \text{ etc.,}$$

i.e.

$$\text{rk}\,(X_1^*X_1)^h \le \dim V_h = \text{rk}\,D^h.$$

Thus (2.5) $X_1^*X_1\in\overline{C_D}$. On the other hand the string $(X_1^0, X_2^0, \ldots, X_t^0)$ defined by $X_i^0 := D\,|_{V_{i-1}}: V_{i-1}\to V_i$ is clearly in $Z$ and $X_1^{0*}X_1^0 = D$.

Thus we have defined a map

$$\vartheta: Z\to\overline{C_D}$$

by $(X_1,\ldots,X_t)\mapsto X_1^*X_1$, which is $G(V_0)$-equivariant by construction, and so $\vartheta(Z)\supseteq C_d$.

**5.3** THEOREM. (i) *The variety $Z$ is a reduced complete intersection in $M$ with respect to the equations* $(*)$.

(ii) *The map $\vartheta: Z\to\overline{C_D}$ is surjective and a quotient map under* $G(V_1)\times G(V_2)\times\cdots\times G(V_t)$.

The proof of this theorem is rather similar to the one in [KP1] for the linear group; it will be given in 5.5. The new feature is that $Z$ is *in general singular in codimension* 1. The consequences of this phenomena will be extensively analysed in part II.

**5.4** We need a crucial lemma whose proof will be given in 8.2 as a consequence of the theory of *orthosymplectic orbits* (section 6 and 7).

LEMMA. *For every conjugacy class* $C \subseteq \overline{C_D}$ *we have*

$$\operatorname{codim}_Z \vartheta^{-1}(C) \geq \tfrac{1}{2} \operatorname{codim}_{\overline{C_D}} C.$$

**5.5** *Proof of theorem* 5.3: Consider the map

$$\zeta : M \to \prod_{i=1}^{t} \mathfrak{g}(V_i) =: N$$

given by $(X_1, \ldots, X_t) \mapsto (X_1X_1^* - X_2^*X_2, X_2X_2^* - X_3^*X_3, \ldots, X_tX_t^*)$. Then $Z$, as a scheme, is the fibre $\zeta^{-1}(0)$. We first claim that $\zeta$ is smooth in

$$M' := \{(X_1, \ldots, X_t) \mid \text{all } X_i \text{ surjective}\} = \prod_{i=1}^{t} L'(V_{i-1}, V_i).$$

For this we compute the differential $d\zeta$ at a point $\alpha = (X_1, \ldots, X_t) \in M'$. Taking a tangent vector $(P_1, \ldots, P_t) \in M$ we get:

$$(d\zeta)_\alpha(P_1, \ldots, P_t) = (P_1X_1^* + X_1P_1^* - P_2^*X_2 - X_2^*P_2, \ldots, P_{t-1}X_{t-1} + X_{t-1}P_{t-1} - P_t^*X_t^*P_t, P_tX_t^* + X_tP_t^*).$$

Since each $X_i$ is surjective we can solve the equation $(d\zeta)_\alpha(P_1, \ldots, P_t) = (T_1, \ldots, T_t)$ inductively: If $P_t, P_{t-1}, \ldots, P_{j+1}$ have been determined, one has to solve an equation

$$P_jX_j^* + X_jP_j^* = S_j$$

for some $S_j$ satisfying $S_j^* = -S_j$. This can be done setting $S_j = R_j - R_j^*$ and then solving $X_jP_j^* = R_j$ using the fact that $X_j$ is surjective. Thus $(d\zeta)_\alpha$ is surjective for $\alpha \in M'$, proving the claim.

In particular $Z$, as a scheme, is smooth in $Z' := Z \cap M'$. Furthermore by an easy induction using lemma 4.3 (ii) and (iii) we see that $\zeta^{-1}(C_D) \subseteq Z'$, hence $Z' \neq \emptyset$ and $\operatorname{codim}_M Z' = \dim N$.

Since $\overline{C_D} - C_D$ consists of finitely many conjugacy classes $C_i$ and for each we have $\operatorname{codim}_Z \vartheta^{-1}(C_i) \geq \frac{1}{2} \operatorname{codim}_{\overline{C_D}} C_i \geq 1$ by lemma 5.4, we deduce that $Z$ is a complete intersection smooth in codimension 0. Thus $Z$ is a reduced Cohen–Macaulay variety ([EGA] IV, proposition 5.8.5) and $Z = \overline{Z'}$. This proves (i). For (ii) we proceed by inverse induction. By theorem 1.2 the quotient of $M$ under $G(V_t)$ is given by the map $(X_1, \ldots, X_t) \mapsto (X_1, \ldots, X_{t-1}, X_t^*X_t)$. But, on $Z$, we have $X_t^*X_t = X_{t-1}X_{t-1}$, and so the quotient map restricted to $Z$ is just the projection $Z \ni (X_1, \ldots, X_t) \mapsto (X_1, \ldots, X_{t-1})$.

Proceeding in this way we see that the quotient of $Z$ under $G(V_2) \times G(V_3) \times \cdots \times G(V_t)$ is given by the projection $(X_1, \ldots, X_t) \mapsto X_1$, and finally that $\vartheta$ is the quotient under $G(V_1)\times\cdots\times G(V_t)$, as desired. Since we have already remarked in 5.2 that $C_D\subseteq\vartheta(Z)$, we must have $\vartheta(Z)=\overline{C_D}$ (0.11). qed.

## 6. Orthosymplectic orbits

**6.1** Let $U$ be an orthogonal and $V$ a symplectic space. In this section we want to recall the classification theory of the orbits in $L(V, U)$ under the group $G:=O(U)\times Sp(V)$, shortly "*orthosymplectic orbits*". For simplicity we will restrict ourselves to *unstable orbits* (in the sense of geometric invariant theory). It is easily seen that the representation of $G$ on $L(V, U)$ is a $\Theta$-group in the sense of Vinberg–Kac, and that the ring of invariants is the polynominal ring in the elements $Tr((X^*X)^i)$, $i=1,2,\ldots,\min(\dim V, \dim U)$. If we associate to $X\in L(V, U)$ the pair

$$(X^*, X)\in L(U, V)\times L(V, U)$$

we have that $X$ is *unstable* if and only if $(X^*, X)$ is a "*nilpotent pair*" ([KP1] 4.1), i.e. if $(X^*, X)$ as an endomorphism of $U\oplus V$ is nilpotent.

**6.2** We will always consider

$$L(V, U)\subseteq L(U, V)\times L(V, U)$$

by the previous map $X\mapsto(X^*, X)$. The classification follows the same pattern as the one relative to $\mathfrak{g}(V)\subset\mathfrak{gl}(V)$ (cf. 2.1, 2.2); it has been explained to us independently by H. Quebbemann (cf. [Q]) and V. Kac (using the method developed in [GV]).

If $X\in L(V, U)$ we denote by $O_X$ its $G$-orbit and by $P_X$ the $GL(U)\times GL(V)$-orbit of the corresponding pair $(X^*, X)$. The first step in the classification is given by

$$P_X\cap L(V, U)=O_X.$$

Thus the orbit $O_X$ is determined by the *ab-diagram* of the pair $(A, B)=(X^*, X)$; we refer the reader to [KP1] 4.2 and 4.3 for a discussion of nilpotent pairs and their *ab*-diagrams.

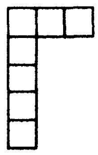
**6.3** To complete the classification we need to describe the *ab*-diagrams which occur in this way; these diagrams will be called *orthosymplectic*. As in the theory of Jordan blocks for classical Lie algebras (cf. [SS]) one can form *direct sums* and speak of *indecomposables*. (Of course the *ab*-diagram of a direct sum is just the union of the two *ab*-diagrams.) Thus one is reduced to the classification of indecomposable orthosymplectic $ab$-diagrams. This is given in the following table. (For the number of $a$'s and $b$'s in the $ab$-diagram of $X \in L(V, U)$ one has $\#a = \dim U$ and $\#b = \dim V$.)

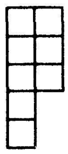
Table II
Indecomposable orthosymplectic $ab$-diagrams

| Type | $\alpha_n$ | $\beta_n$ | $\gamma_n$ | $\delta_n$ | $\varepsilon_n$ |
|---|---|---|---|---|---|
| $ab$-diagram | $aba \cdots ba$ | $bab \cdots ab$ | $aba \cdots ba$<br>$aba \cdots ba$ | $bab \cdots ab$<br>$bab \cdots ab$ | $aba \cdots ab$<br>$bab \cdots ba$ |
| $n$ | — | — | odd | even | — |
| $\#a$<br>$\#b$ | $2n \times 1$<br>$2n$ | $2n-1$<br>$2n$ | $2(n+1)$<br>$2n$ | $2n$<br>$2(n+1)$ | $2n$<br>$2n$ |

**6.4** *Remark.* If $\tau$ is the $ab$-diagram of an (unstable) element $X \in L(V, U)$ the Young-diagram of $\pi(X) = XX^*$ and $\rho(X) = X^*X$ are obtained from $\tau$ erasing the $b$'s (respectively the $a$'s). The reader may observe that from the ortho-symplectic diagram $\tau$ we obtain, of course, a $(+1)$-diagram and a $(-1)$-diagram in this way. We will write also $\pi(\tau)$ and $\rho(\tau)$ for these two Young-diagrams. E.g.

$$\begin{array}{lll} & ab & & a & & & b \\ & ba & & a & & & b \\ \tau = & ababa & \text{gives}\quad \pi(\sigma) = & aaa = & \text{and} & \rho(\tau) = & bb = \\ & bab & & a & & & bb \\ & bab & & a & & & bb \end{array}$$





We remark also that $X$ is *injective* (respectively *surjective*) if and only if the corresponding $ab$-diagram $\tau$ is formed by indecomposables of types $\alpha_n$ and $\gamma_n$ (respectively $\beta_n$ and $\delta_n$) (cf. [KP1] 4.4, remark 2).

**6.5** If $\tau$ is an $ab$-diagram we denote by $P_\tau$ the corresponding orbit under $GL(U) \times GL(V)$ (in $L(U, V) \times L(V, U)$). If $\tau$ is orthosymplectic $O_\tau$ denotes the corresponding orbit under $G = O(U) \times Sp(V)$ (in $L(V, U) \subseteq L(U, V) \times L(V, U)$). To summarize we have the following result (cf. theorem 2.2).

THEOREM. *Let $\tau$ be an ab-diagram.*

(i) *$\tau$ is orthosymplectic if and only if it is a union of diagrams of types* $\alpha, \beta, \gamma, \delta, \varepsilon$ (table II).

(ii) *$P_\tau \cap L(V, U) \neq \varnothing$ if and only if $\tau$ is orthosymplectic*

(iii) *If $\tau$ is orthosymplectic then $P_\tau \cap L(V, U) = O_\tau$.*

## 7. Dimension formula for orthosymplectic orbits

**7.1** To any $ab$-diagram $\tau$ we associate the number

$$\Delta_\tau := \sum_{i \text{ odd}} a_i \cdot b_i$$

where $a_i$ (resp. $b_i$) is the number of rows of $\tau$ of length $i$ starting with $a$ (resp. with $b$) (cf. [KP1] 5.3). If $\tau$ is orthosymplectic we have $\Delta_\tau = 0$ if the corresponding map $X: V \to U$ is injective or surjective. More precisely one easily finds

$$\Delta_\tau = 2\sum_k (\#\beta_k \cdot \#\gamma_{2k-1} + \#\alpha_k \cdot \#\delta_{2k})$$

where $\#\alpha_k, \#\beta_k, \ldots$ denotes the number of indecomposable factors of $\tau$ of type $\alpha_k, \beta_k, \ldots$ (cf. 6.3).

PROPOSITION. *Let $O \subseteq \mathrm{L}(V, U)$ be an orthosymplectic orbit with associated $ab$-diagram $\tau$. Then*

$$\dim O = \tfrac{1}{2}(\dim \pi(O) + \dim \rho(O) + \dim U \cdot \dim V - \Delta_\tau).$$

For the proof we need some preparation.

**7.2** We first describe $\mathrm{L}(V, U)$ as a $\Theta$-group in the sense of Vinberg–Kac (cf. [Vi]). As in section 6 we will always denote by $U$ an orthogonal space and by $V$ a symplectic space. Consider the group $\tilde{G} := GL(U \oplus V)$ and the automorphism $\Theta: \tilde{G} \to \tilde{G}$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}^{-1}.$$

We have $\Theta^4 = Id$ and $\Theta^2 = \mathrm{Int}\, J$, the conjugation with $J := \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}$. Furthermore one easily determines the fixed point groups and finds

$$G := \tilde{G}^\Theta = O(U) \times Sp(V)$$
$$G' := \tilde{G}^{\Theta^2} = GL(U) \times GL(V).$$

$\Theta$ determines an automorphism of order 4 of the Lie algebra $\tilde{\mathfrak{g}} := \mathrm{Lie}\, \tilde{G} = \mathrm{End}\,(U \oplus V)$, also denoted by $\Theta$, given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto -\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

Fixing a 4th root of unity $\zeta$ we obtain a $\mathbb{Z}/4\mathbb{Z}$-graduation of $\tilde{\mathfrak{g}}$

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}^{(0)}\oplus\tilde{\mathfrak{g}}^{(1)}\oplus\tilde{\mathfrak{g}}^{(2)}\oplus\tilde{\mathfrak{g}}^{(3)},\ \tilde{\mathfrak{g}}^{(i)}:=\{x\in\tilde{\mathfrak{g}}\mid \Theta X=\zeta^i X\}$$

which is clearly $G$-stable. By definition

$$\tilde{\mathfrak{g}}^{(0)}=\mathfrak{g}:=\operatorname{Lie} G \quad \text{and} \quad \tilde{\mathfrak{g}}^{(0)}\oplus\tilde{\mathfrak{g}}^{(2)}=\mathfrak{g}':=\operatorname{Lie} G'.$$

Furthermore

$$\tilde{\mathfrak{g}}^{(1)}=\left\{\begin{pmatrix}0 & B\\ \zeta B^* & 0\end{pmatrix}\Big|\, B\in \mathrm{L}(V,U)\right\},$$

hence we can identify $\tilde{\mathfrak{g}}^{(1)}$ and $\mathrm{L}(V, U)$ as $G$-modules (cf. 6.2).

**7.3** We recall that a triple $(X, H, Y)$ of elements of a Lie algebra $\mathfrak{g}$ is called an $\mathfrak{sl}_2$-*triple*, if they satisfy the following relations:

$$[H,X]=2X, \qquad [H,Y]=-2Y, \qquad [X,Y]=H, \tag{$*$}$$

i.e. if the linear map $\mathfrak{sl}_2\to\mathfrak{g}$ defined by $\begin{pmatrix}0&1\\0&0\end{pmatrix}\mapsto X$, $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\mapsto H$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}\mapsto Y$ is a Lie algebra homomorphism.

LEMMA. *Let $X\in\tilde{\mathfrak{g}}^{(1)}$ be a nilpotent element. Then there is an $\mathfrak{sl}_2$-triple $(X, H, Y)$ in $\tilde{\mathfrak{g}}$ with $H\in\tilde{\mathfrak{g}}^{(0)}$ and $Y\in\tilde{\mathfrak{g}}^{(3)}$.*

*Proof.* By the Jacobson–Morozov theorem there exists an $\mathfrak{sl}_2$-triple $(X, H', Y')$ in $\tilde{\mathfrak{g}}$. In particular $[H', X]=2X$ and $H'\in[X,\tilde{\mathfrak{g}}]$. Denoting by $H$ the component of $H'$ in $\tilde{\mathfrak{g}}^{(0)}$ we get $[H, X]=2X$ and $H\in[X,\tilde{\mathfrak{g}}]$, since $X\in\tilde{\mathfrak{g}}^{(1)}$. Hence there is a $Y''\in\tilde{\mathfrak{g}}$ such that $(X, H, Y'')$ is an $\mathfrak{sl}_2$-triple ([Bo] chap. VIII, §11, lemme 6). Denoting by $Y$ the component of $Y''$ in $\tilde{\mathfrak{g}}^{(3)}$ the relations $(*)$ for $(X, H, Y'')$ immediatly imply that $(X, H, Y)$ is an $\mathfrak{sl}_2$-triple too. qed.

*Remark.* It is clear from the proof above that the lemma holds for any $\Theta$-group.

**7.4** Let $(X, H, Y)$ be an $\mathfrak{sl}_2$-triple in $\tilde{\mathfrak{g}}$. The semisimple element $H$ defines a $\mathbb{Z}$-graduation of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}}=\bigoplus_i \tilde{\mathfrak{g}}_i, \qquad \tilde{\mathfrak{g}}_i:=\{X'\in\tilde{\mathfrak{g}}\mid [H,X']=iX'\}.$$

It is easy to see that $\tilde{\mathfrak{p}}:=\bigoplus_{i\geq 0}\tilde{\mathfrak{g}}_i$ is a parabolic subalgebra with nilradical

$\tilde{\mathfrak{n}} := \bigoplus_{i>0} \tilde{\mathfrak{g}}_i$ and Levi-decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{n}}$. We denote by $\tilde{P}$ the parabolic subgroup of $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{p}}$.

The following proposition is proved in [SS] (III 4.16, 4.11 and 4.19 (i)).

PROPOSITION. (a) *The stabilizer* $\tilde{G}_X$ *of* $X$ *is contained in* $\tilde{P}$.

(b) *All* $\mathfrak{sl}_2$*-triples of the form* $(X, H', Y')$ *are conjugate under* $\tilde{G}_X$. *In particular the parabolic* $\tilde{\mathfrak{p}}$ *depends only on* $X$.

(c) $X \in \tilde{\mathfrak{n}}_2 := \bigoplus_{i\geq 2} \tilde{\mathfrak{g}}_i$ *and the map* ad $X: \tilde{\mathfrak{p}} \to \tilde{\mathfrak{n}}_2$ *is surjective.*

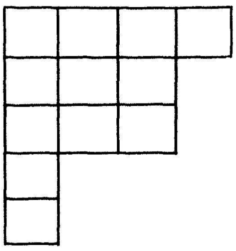
We remark that the assertions (a) and (c) imply that the canonical map

$$\tilde{G} \times^{\tilde{P}} \tilde{\mathfrak{n}}_2 \to \overline{C_X}$$

is a desingularisation, where $C_X$ is the conjugacy class of $X$ in $\tilde{\mathfrak{g}}$. In particular we have

$$\dim \overline{C_X} = \dim \tilde{\mathfrak{n}} + \dim \tilde{\mathfrak{n}}_2.$$

**7.5** *Remark.* It is easy to calculate the dimensions of the weight spaces of $H$ in terms of the Young-diagram $\lambda$ of the nilpotent endomorphism $X$ of $U \oplus V$. These dimensions depend only on the conjugacy class $C_X$ and not on the choice of an $\mathfrak{sl}_2$-triple $(X, H, Y)$ (cf. proposition 7.4(b)). The boxes of $\lambda$ correspond to a Jordan basis of $X$. Choosing $H$ diagonal with respect to this basis with entries $(2n, 2n-2, \ldots, 2, 0, -2, \ldots, -2n)$ in a row of $\lambda$ of length $2n+1$ and with entries $(2n-1, 2n-3, \ldots, 1, -1, \ldots, -2n+1)$ in a row of length $2n$, it is well known (and easy to check) that there exists a $Y \in \tilde{\mathfrak{g}}$ such that $(X, H, Y)$ is an $\mathfrak{sl}_2$-triple. In particular the zero weight space of $H$ is spanned by the base vectors corresponding to the middle boxes of the rows of odd length.

E.g. $\lambda =$  has weights $\begin{array}{cccc} 3 & 1 & -1 & 3 \\ 2 & 0 & -2 & \\ 2 & 0 & -2 & \\ 0 & & & \\ 0 & & & \end{array}$ ;

hence the dimension of the weight space $W_i$ of weight $i$ are given by $\dim W_0 = 4$, $\dim W_1 = \dim W_{-1} = 1$, $\dim W_2 = \dim W_{-2} = 2$, $\dim W_3 = \dim W_{-3} = 1$.

If in addition $X \in \tilde{\mathfrak{g}}^{(1)} \oplus \tilde{\mathfrak{g}}^{(3)}$ has associated $ab$-diagram $\tau$, the definition of $H$ above implies that $H \in \mathfrak{g}' = \tilde{\mathfrak{g}}^{(0)} \oplus \tilde{\mathfrak{g}}^{(2)}$. Hence the weight spaces of $H$ are of the form $U_i \oplus V_i$, and it is clear how to calculate the dimensions of $U_i$ and $V_i$ in terms of the $ab$-diagram $\tau$. In particular $\dim U_0$ is given by the number of rows of $\tau$ consisting of an odd number of $a$'s and an even number of $b$'s; similarly for dim $V_0$.

$$\tau = \begin{matrix} ababab a \\ bababab \\ ababa \\ ab \\ ba \\ a \end{matrix}$$

E.g. $\tau = ababa$ ; then dim $U_0 = 3$, dim $V_0 = 1$.

**7.6** Now let $X \in \tilde{\mathfrak{g}}^{(1)}$ be a nilpotent element with associated orthosymplectic $ab$-diagram $\tau$. We choose an $\mathfrak{sl}_2$-triple $(X, H, Y)$ with $H \in \tilde{\mathfrak{g}}^{(0)} = \mathfrak{g}$ (lemma 7.3). Then $H$ defines a $\mathbb{Z}$-graduation of $\mathfrak{g}'$ and $\mathfrak{g}$, both induced by the $\mathbb{Z}$-graduation of $\tilde{\mathfrak{g}}$ (7.4). Hence $\mathfrak{p}' := \tilde{\mathfrak{p}} \cap \mathfrak{g}'$ and $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$ are parabolic subalgebras of $\mathfrak{g}'$ and $\mathfrak{g}$ with Levi decompositions

$$\mathfrak{p}' = \mathfrak{g}_0' \oplus \tilde{\mathfrak{n}}', \qquad \mathfrak{g}_0' := \tilde{\mathfrak{g}}_0 \cap \mathfrak{g}', \qquad \tilde{\mathfrak{n}}' := \tilde{\mathfrak{n}} \cap \mathfrak{g}'$$

and

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}, \qquad \mathfrak{g}_0 := \tilde{\mathfrak{g}}_0 \cap \mathfrak{g}, \qquad \mathfrak{n} := \tilde{\mathfrak{n}} \cap \mathfrak{g}.$$

Denoting by $P'$ and $P$ the corresponding parabolic subgroups of $G' = GL(U) \times GL(V)$ and $G = O(U) \times Sp(V)$ it follows from proposition 7.4 (a) that $G_X' \subset P'$ and $G_X \subset P$. Defining

$$\mathfrak{n}_2' := \tilde{\mathfrak{n}}_2 \cap (\tilde{\mathfrak{g}}^{(1)} \oplus \tilde{\mathfrak{g}}^{(3)}) \quad \text{and} \quad \mathfrak{n}_2 := \tilde{\mathfrak{n}}_2 \cap \tilde{\mathfrak{g}}^{(1)}$$

proposition 7.4 (c) implies that the maps

$$\operatorname{ad} X : \mathfrak{p}' \to \mathfrak{n}_2', \qquad \operatorname{ad} X : \mathfrak{p} \to \mathfrak{n}_2$$

are surjective. From this we easily deduce assertion (a) and (b) of the following lemma.

LEMMA. *Let $O_X'$, $O_X$ and $O_X^0$ denote the orbits of $X$ under $G'$, $G$ and $G^0 = SO(U) \times Sp(V)$ respectively.*

(a) *The canonical maps*

$$G' \times^{P'} \mathfrak{n}_2' \to \overline{O_X'} \quad \textit{and} \quad G^0 \times^{P} \mathfrak{n}_2 \to \overline{O_X^0}$$

*are desingularisations,*

(b) $\dim O_X' = \dim \mathfrak{n}' + \dim \mathfrak{n}_2'$, $\dim O_X = \dim O_X^0 = \dim \mathfrak{n} + \dim \mathfrak{n}_2$

(c) $\dim \mathfrak{n}_2' = 2 \dim \mathfrak{n}_2$.

*Proof of* (c). We have seen in 7.2 that the automorphism $\Theta^2 : \tilde{G} \to \tilde{G}$ is the conjugation with $J=\begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix}\in G$. Since $H\in\tilde{\mathfrak{g}}^{(0)}=\tilde{\mathfrak{g}}^{\Theta}$ we have $\Theta H=H$, hence $JH=HJ$. Furthermore $J\tilde{\mathfrak{g}}^{(1)}=\tilde{\mathfrak{g}}^{(3)}$ and so $J(\tilde{\mathfrak{g}}^{(1)}\cap\tilde{\mathfrak{g}}_i)=\tilde{\mathfrak{g}}^{(3)}\cap\tilde{\mathfrak{g}}_i$ for all $i$. In particular $J(\tilde{\mathfrak{g}}^{(1)}\cap\tilde{\mathfrak{n}}_2)=\tilde{\mathfrak{g}}^{(3)}\cap\tilde{\mathfrak{n}}_2$. Since $\mathfrak{n}_2'=(\tilde{\mathfrak{g}}^{(1)}\cap\tilde{\mathfrak{n}}_2)\oplus(\tilde{\mathfrak{g}}^{(3)}\cap\tilde{\mathfrak{n}}_2)$ and $\mathfrak{n}_2=\tilde{\mathfrak{g}}^{(1)}\cap\tilde{\mathfrak{n}}_2$ the claim follows. qed.

**7.7** *Proof of the dimension formula* 7.1: We first compare the dimension of the orbit $O_X$ of a nilpotent element $X\in\tilde{\mathfrak{g}}^{(1)}\cong L(V,U)$ under $G=O(U)\times Sp(V)$ with the dimension of the orbit $O_X'$ of $X$ under $G'=GL(U)\times GL(V)$. We choose an $\mathfrak{sl}_2$-triple $(X,H,Y)$ with $H\in\mathfrak{g}=\tilde{\mathfrak{g}}^{(0)}$ (lemma 7.3) and consider the associated parabolic subalgebras

$$\mathfrak{p}'=\mathfrak{g}_0'\oplus\mathfrak{n}'\subseteq\mathfrak{g}' \quad\text{and}\quad \mathfrak{p}=\mathfrak{g}_0\oplus\mathfrak{n}\subseteq\mathfrak{g}$$

(cf. 7.6). By definition the Levi factors $\mathfrak{g}_0'$ and $\mathfrak{g}_0$ are the stabilizers of $H$ in $\mathfrak{g}'$ and $\mathfrak{g}$. If $U=\bigoplus_i U_i$ and $V=\bigoplus_j V_j$ are the weight space decompositions of $U$ and $V$ with respect to $H$ (i.e. $U_i:=\{u\in U\mid Hu=i\cdot u\}$ and similarly for $V$), we find

$$\mathfrak{g}_0'=\bigoplus_i \mathfrak{gl}(U_i)\oplus\Big(\bigoplus_j \mathfrak{gl}(V_j)\Big).$$

Furthermore the subspaces $U_i+U_{-i}$ of $U$ are non degenerate orthogonal spaces and $V_j+V_{-j}$ are non degenerate symplectic subspaces of $V$. Hence

$$\mathfrak{g}_0\cong\Big(\bigoplus_{i>0}\mathfrak{gl}(U_i)\Big)\oplus\mathfrak{o}(U_0)\oplus\Big(\bigoplus_{j>0}\mathfrak{gl}(V_j)\Big)\oplus\mathfrak{sp}(V_0).$$

Putting $d_a:=\dim U_0$ and $d_b:=\dim V_0$ we get (cf. 1.1)

$$\begin{aligned}2\dim\mathfrak{g}_0-\dim\mathfrak{g}_0'&=(2\dim\mathfrak{o}(U_0)-d_a^2)+(2\dim\mathfrak{sp}(V_0)-d_b^2)\\&=(d_a(d_a-1)-d_a^2)+(d_b(d_b+1)-d_b^2)\\&=d_b-d_a.\end{aligned}$$

Using lemma 7.6 and putting $m:=\dim U$, $n:=\dim V$ we obtain (cf. 1.1)

$$\begin{aligned}4\dim O_X-2\dim O_X'=4\dim\mathfrak{n}-2\dim\mathfrak{n}'=&\\&=2(\dim\mathfrak{g}-\dim\mathfrak{g}_0)-(\dim\mathfrak{g}'-\dim\mathfrak{g}_0')\\&=(2\dim\mathfrak{g}-\dim\mathfrak{g}')-(d_b-d_a)\\&=(m(m-1)+n(n+1)-m^2-n^2)-(d_b-d_a),\end{aligned}$$

hence

$$4\dim O_X=2\dim O_X'+(n-m)(d_b-d_a). \tag{1}$$

Now consider the conjugacy classes $C_a := \pi(O_X) \subseteq \mathfrak{o}(U)$ and $C_b := \rho(O_X) \subseteq \mathfrak{sp}(V)$ and denote by $C'_a$ and $C'_b$ the conjugacy classes in $\mathfrak{gl}(U)$ and $\mathfrak{gl}(V)$ generated by $C_a$ and $C_b$. The dimension formula in the linear case ([KP1]; proposition 5.3) gives

$$\dim O'_X = \tfrac{1}{2}(\dim C'_a + \dim C'_b) + nm - \Delta_\tau. \tag{2}$$

Moreover we have (remark 2.4)

$$\dim C'_a = 2 \cdot \dim C_a + m - r_a, \qquad \dim C'_b = 2 \cdot \dim C_b - n + r_b, \tag{3}$$

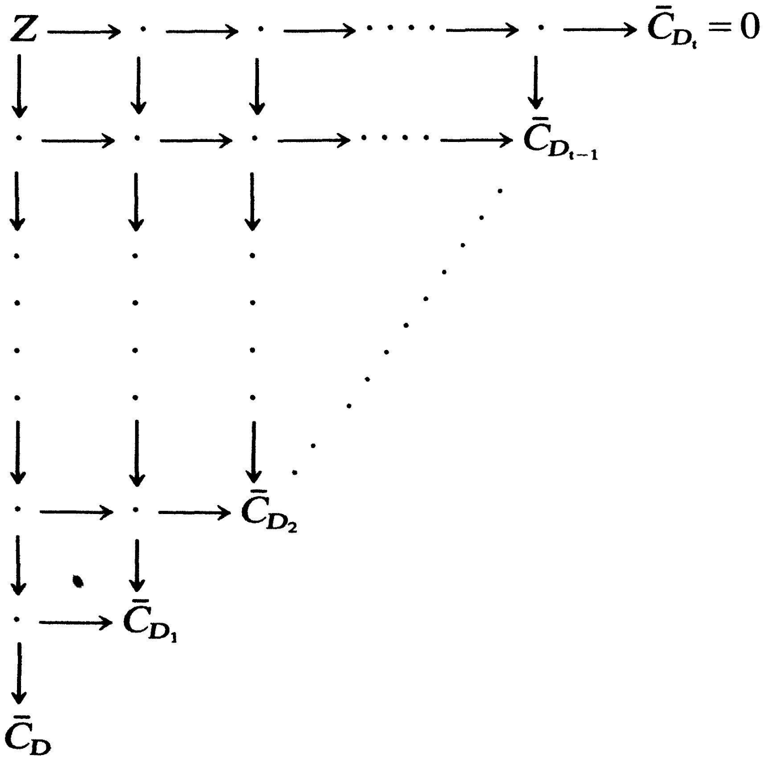
where $r_a$, $r_b$ are the number of odd rows in the Young-diagram of $C_a$ and $C_b$ respectively. From (1), (2) and (3) we obtain

$$\begin{aligned} 4 \dim O_X &= \dim C'_a + \dim C'_b + 2nm - 2\Delta_\tau + (n-m) - (d_b - d_a) \\ &= 2 \dim C_a + 2 \dim C_b + 2nm - 2\Delta_\tau + (r_b - r_a) - (d_b - d_a). \end{aligned}$$

It remains to show that $r_b - r_a = d_b - d_a$. Denoting by $a_\nu$, the number of $a$'s in the $\nu$th row of $\tau$ and by $b_\nu$ the number of $b$'s, we have (cf. remark 7.5) $d_a = \#\{\nu \mid a_\nu$ odd and $b_\nu$ even$\}$ and $d_b = \#\{\nu \mid b_\nu$ odd and $a_\nu$ even$\}$. Hence $d_b - d_a = \#\{\nu \mid b_\nu \text{ odd}\} - \#\{\mu \mid a_\mu \text{ odd}\} = r_b - r_a$. qed.

## 8. Stratification and singularities of $Z$

**8.1** We now go back to the variety $Z$ constructed out of a given endomorphism $D \in \mathfrak{g}(V)$, $V$ a quadratic space of type $\varepsilon$ (5.2). We recall that $Z$ is essentially an iterated fibre product:



(*)

with $D_i := D|_{V_i}$, $V_i := D^i(V)$. Consider the *finite set* $\Lambda$ *of strings* $\lambda = (\tau_1, \tau_2, \ldots, \tau_t)$ *of orthosymplectic ab-diagrams* $\tau_i$ *corresponding to orthosymplectic orbits* $O_{\tau_i} \subseteq \mathrm{L}(V_{i-1}, V_i)$ *satisfying*

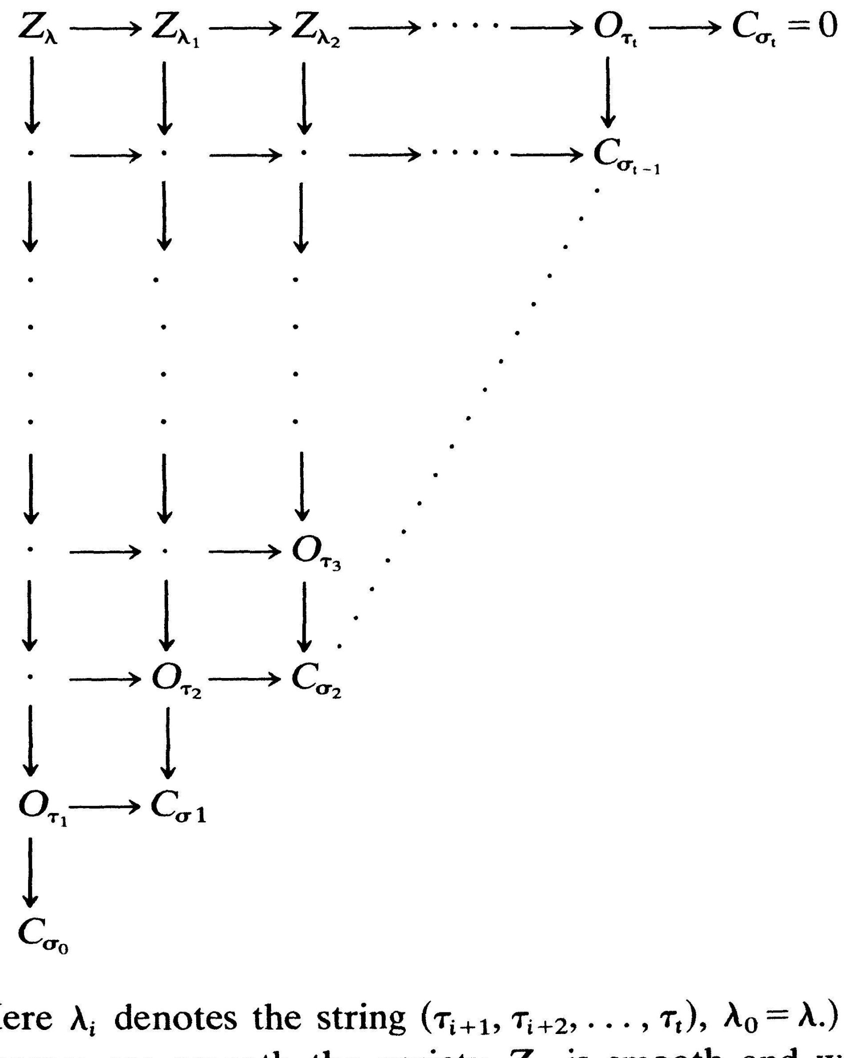
(a) $\pi(\tau_i) = \rho(\tau_{i+1}) := \sigma_i$ for $i = 1, 2, \ldots, t-1$ (i.e. $\pi(O_{\tau_i}) = C_{\sigma_i} = \rho(O_{\tau_{i+1}})$, cf. 6.4),

(b) $\sigma_t = 0$ (i.e. $C_{\sigma_t} = 0$).

It follows from the construction of $Z$ that $C_{\sigma_i} \subseteq \bar{C}_{D_i}$ for $i = 0, 1, \ldots, t$, $\sigma_0 := \rho(\tau_1)$. For $\lambda \in \Lambda$ we define a locally closed subset $Z_\lambda \subseteq Z$ by

$$Z_\lambda := \{(X_1, X_2, \ldots, X_t) \in Z \mid X_i \in O_{\tau_i}\}.$$

The definition of $\Lambda$ implies that we have a fibre product diagram subordinate to the basic diagram (*) constructing $Z$:




(Here $\lambda_i$ denotes the string $(\tau_{i+1}, \tau_{i+2}, \ldots, \tau_t)$, $\lambda_0 = \lambda$.) Since all the maps in this diagram are smooth the variety $Z_\lambda$ is smooth and we get from the dimension formula 7.1 (putting $n_i := \dim V_i$)

$$\begin{aligned}\dim Z_\lambda &= \dim O_{\tau_1} - \dim C_{\sigma_1} + \dim Z_{\lambda_1}\\ &= \tfrac{1}{2}(\dim C_{\sigma_0} + \dim C_{\sigma_1} + n_0 n_1 - \Delta_{\tau_1}) + \dim C_{\sigma_1} + \dim Z_{\lambda_1},\end{aligned}$$

hence $\dim Z_\lambda - \frac{1}{2}\dim C_{\sigma_0} = \frac{1}{2}(n_0 n_1 - \Delta_{\tau_1}) + \dim Z_{\lambda_1} - \frac{1}{2}\dim C_{\sigma_1}$. By induction this implies the following result.

PROPOSITION. *For any $\lambda = (\tau_1, \ldots, \tau_t) \in \Lambda$ we have*

$$\dim Z_\lambda = \frac{1}{2}\dim C_\sigma + \frac{1}{2}\sum_{i=0}^{t-1} n_i n_{i+1} - \frac{1}{2}\Delta_\lambda$$

*where $\sigma := \rho(\tau_1)$, $n_i := \dim V_i$ and $\Delta_\lambda := \sum_{i=1}^{t} \Delta_{\tau_i}$.*

**8.2** We are now ready to prove lemma 5.4, i.e. to show that *for each conjugacy class $C \subseteq \bar{C}_D$ we have*

$$\operatorname{codim}_Z \vartheta^{-1}(C) \geq \frac{1}{2}\operatorname{codim}_{\bar{C}_D} C.$$

We first remark that there is a *unique* (open) *stratum* $Z_{\lambda^0}$ on top of the open orbit $C_D$, $\lambda^0 = (\tau_1^0, \ldots, \tau_t^0)$, where $\tau_i^0$ is the $ab$-diagram of $X_i^0 := D|_{V_{i-1}} : V_{i-1} \to V_i$. This is an easy consequence of lemma 4.3 (ii) and (iii) (cf. also 5.2). For this stratum we have $Z_{\lambda^0} \subseteq Z' = Z \cap M'$ (5.5) and one easily sees that $\Delta_{\lambda^0} = 0$ (6.4), hence

$$\dim Z_{\lambda^0} = \frac{1}{2}\dim C_D + \frac{1}{2}\sum_{i=0}^{t-1} n_i n_{i+1}$$

and $\dim Z_\lambda \leq \dim Z_{\lambda^0} - 1$ for all other $\lambda \in \Lambda$.

This implies $\dim Z = \dim Z_{\lambda^0} = \dim Z'$ and also the claim, since $\vartheta^{-1}(C)$ is a finite union of strata $Z_\lambda$ satisfying

$$\operatorname{codim}_Z Z_\lambda = \dim Z_{\lambda^0} - \dim Z_\lambda = \frac{1}{2}(\operatorname{codim}_{C_D} C + \Delta_\lambda). \qquad (*)$$

In particular we see that the strata $Z_\lambda$ of codimension 1 lie on top of a conjugacy class $C \subseteq \overline{C_D}$ of codimension 2 and satisfy $\Delta_\lambda = 0$. This already has the following implication.

**8.3** PROPOSITION. *Assume that $\bar{C}_D$ contains no conjugacy classes of codimension 2. Then the variety $Z$ is normal and so $\bar{C}_D$ is normal too.*

(Use theorem 5.3(ii) and 0.11 for the last statement.)

**8.4** *Remark.* One can show that *$Z$ is normal if and only if the only codimension 2 degenerations of $C_D$ are of type a* (cf. 3.4). In all other cases $Z$ is singular in some stratum $Z_\lambda$ of codimension 1.

E.g. let $D \in \mathfrak{so}_3$ be a regular nilpotent element, i.e. $D \in C_{1,\eta}$ with $\eta = aaa$. Then the stratum $Z_\lambda$ with

$$\lambda := \begin{pmatrix} bab & \\ a & , bab \\ a & \end{pmatrix}$$

is of codimension 1 in $Z$ (cf. formula (*) in 8.2), and is in fact the only possible such stratum. We claim that $Z$ is singular in $Z_\lambda$. Since $\bar{O}_\tau$ with $\tau = ababa$ is a quotient of $Z$ it is enough to show that $\bar{O}_\tau$ is not normal. To see this one remarks that the map $\pi$ is not smooth in $O_{\tau'} \subseteq \overline{O_\tau}$, $\tau' := \begin{matrix} bab \\ a \\ a \end{matrix}$, and that $O_{\tau'}$ has codimension 1.

## 9. Functions on orbit closures

**9.1** For any variety $Y$ let us denote by $\mathcal{O}(Y)$ the ring of global regular functions on $Y$. We need a general lemma which seems to be known by the specialists but for which we could not find a reference.

LEMMA. *Let $Z$ be an affine Cohen–Macaulay variety, $W \subseteq Z$ a closed subset of codimension $\geq 2$. Then every regular function on $Z - W$ extends to a regular function on $Z$, i.e. $\mathcal{O}(Z - W) = \mathcal{O}(Z)$.*

*Proof.* Let $S \subset R := \mathcal{O}(Z)$ be the set of non zero divisor, $K := R_S$ and $f \in \mathcal{O}(Z - W) \subseteq K$. Consider the ideal $I := \{r \in R \mid r \cdot f \in R\}$. By assumption the zero set $\mathcal{V}(I)$ of the ideal $I$ is contained in $W$. Hence there is an $s \in I \cap S$ (i.e. an $s \in I$ not vanishing identically on the irreducible components of $Z$). Since $R/sR$ is Cohen–Macaulay, the ideal $sR$ has no embedded primes. Let $sR = \bigcap_i \mathfrak{q}_i$ be the primary decomposition, $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$. It follows that $I \not\subseteq \mathfrak{p}_i$ for any $i$, since $\operatorname{codim}_X \mathcal{V}(I) \geq \operatorname{codim}_X W \geq 2$. Hence $f \in R_{\mathfrak{p}_i}$ for all $i$. If we write $f = r/s$ for some $r \in R$ this implies $r \in sR_{\mathfrak{p}_i} = \mathfrak{q}_i R_{\mathfrak{p}_i}$ and therefore $r \in \mathfrak{q}_i R_{\mathfrak{p}_i} \cap R = \mathfrak{q}_i$ for all $i$. Thus $r \in \bigcap_i \mathfrak{q}_i = sR$ and so $f = r/s \in R$. qed.

*Remark.* In the setting of the lemma *every regular map $\varphi: Z - W \to Y$ into an affine variety $Y$ extends to the whole variety $Z$.* This implies for instance that every connected Cohen-Macaulay variety is *connected in codimension* 1.

**9.2** For any conjugacy class $C \subset \mathfrak{g}(V)$ we denote by $\check{C}$ the complement in $\bar{C}$ of the union of all conjugacy classes of codimension $\geq 4$. $\check{C}$ is open in $\bar{C}$ and it is the union of $C$ with the codimension 2 classes:

$$\tilde{C}=C\cup\bigcup_i C_i, \qquad \operatorname{codim}_{\bar{C}}C_i=2.$$

THEOREM. (i) *Every regular function on* $\tilde{C}$ *extends to* $\bar{C}$.
(ii) $\bar{C}$ *is normal if and only if it is normal in the conjugacy classes of codimension* 2.

*Proof.* (i) Let $f\in\mathcal{O}(\tilde{C})$ and consider the quotient map $\vartheta: Z\to\bar{C}$ (5.2, 5.3). We know from lemma 5.4 that $\operatorname{codim}_Z\vartheta^{-1}(\bar{C}-\tilde{C})\geq 2$. Hence the composed function $F:=f\circ\vartheta$, defined on $\vartheta^{-1}(\tilde{C})$, extends to $Z$ by the previous lemma 9.1. On the other hand $F$ is invariant on $\vartheta^{-1}(\tilde{C})$ and so also on $Z$. Thus $F$ defines an extension of $f$ to the whole $\bar{C}$.
(ii) Since $\operatorname{codim}_{\bar{C}}(\bar{C}-C)\geq 2$ the variety $\bar{C}$ is normal if and only if every regular function on $C$ extends to $\bar{C}$. Now if $\tilde{C}$ is normal every regular function on $C$ extends to $\tilde{C}$ by the same reason, and so, by (i), to the whole $\bar{C}$. qed.

**9.3** The previous theorem reduces the problem of normality for $\bar{C}$ to the study of the singularity in a codimension 2 class $C'$. This will be the main object of part II, where we will prove, as a consequence of a more precise description, that $\bar{C}$ *is not normal in* $C'$ *if and only if this is a degeneration of type e* (3.4; cf. theorem 16.2).

## 10. Polarization and Cohen–Macaulay property

**10.1** Let $C$ be a nilpotent conjugacy class in a semisimple Lie algebra $\mathfrak{g}=\operatorname{Lie} G$. If $\bar{C}$ is a Cohen–Macaulay variety then $\bar{C}$ is also normal. This follows from Serre's criterion since $\operatorname{codim}_{\bar{C}}(\bar{C}-C)\geq 2$. The converse is not known in general, but only for the so called "*polarizable*" *classes* (and also some special cases, cf. section 18).

DEFINITION. A nilpotent conjugacy class $C$ is called *polarizable* if there is a parabolic subalgebra $\mathfrak{p}\subseteq\mathfrak{g}$ with nilradical $\mathfrak{n}$ such that $\mathfrak{n}\cap C$ is dense in $\mathfrak{n}$ (cf. [H2]). Such a parabolic is called a *polarization* of $C$.

**10.2** The following result is due to R. Elkik.

PROPOSITION. *If a nilpotent conjugacy class* $C$ *admits a polarization, the normalization of* $\bar{C}$ *is Cohen-Macaulay with rational singularities* (0.4).

*Proof.* Let $\mathfrak{p} = \operatorname{Lie} P$ be a polarization of $C$, $\mathfrak{n}$ the nilradial of $\mathfrak{p}$ and $z \in \mathfrak{n} \cap C$. For the stabilizers $G_z$ and $P_z$ of $z$ in $G$ and $P$ one has $G_z \supset P_z \supset G_z^0$. The natural map $\varphi: G \times^P \mathfrak{n} \to \tilde{\bar{C}}$, $\tilde{\bar{C}}$ the normalization of $\bar{C}$, is proper, surjective and of degree $[G_z : P_z]$ and $G \times^P \mathfrak{n}$ is the cotangent bundle over $G/P$ (see for example [BK] §7). Consider the Stein factorization

$$Y := G \times^P \mathfrak{n} \xrightarrow{\varphi'} X \xrightarrow{\bar{\varphi}} \tilde{\bar{C}}$$

i.e. $X$ is affine with coordinate ring $\mathcal{O}(X) = \mathcal{O}(G \times^P \mathfrak{n})$. Now $\varphi'$ is a resolution of singularities. Since $Y$ is the cotangent bundle over $G/P$ the canonical divisor of $Y$ is trivial, hence $R^i\varphi'_*\mathcal{O}_Y = 0$ for $i > 0$ by the theorem of Grauert–Riemenschneider ([GR] Satz 2.3, cf. [HO] proposition 2.2). Thus $X$ has rational singularities.

For $x := \varphi'((1_G, z)) \in X$ we have $X = \overline{Gx}$, $G_x = P_z$ and $\operatorname{codim}_X (X - Gx) \geq 2$. Since $X$ and $\tilde{\bar{C}}$ are normal, this implies $\mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(G)^{P_z}$, $\mathcal{O}(\tilde{\bar{C}}) \xrightarrow{\sim} \mathcal{O}(G)^{G_z}$ and $\varphi: \mathcal{O}(\tilde{\bar{C}}) \to \mathcal{O}(X)$ is identified with the inclusion.

$$\mathcal{O}(G)^{G_z} \hookrightarrow \mathcal{O}(G)^{P_z}.$$

It follows that $\mathcal{O}(\tilde{\bar{C}})$ is a direct summand of $\mathcal{O}(X)$ as $\mathcal{O}(\tilde{\bar{C}})$-module. Hence by Boutot's theorem (cf. 0.11) $\tilde{\bar{C}}$ has rational singularities too. qed.

**10.3** In order to apply the previous proposition one has to determine the polarizable nilpotent conjugacy classes. For classical Lie algebras this is done in [H2]. We only state the following partial result which is sufficient for our purpose (cf. [H2] theorem 7.1(a) and 6.2 or [Ke1]).

PROPOSITION. *Let $C$ be an orthogonal or symplectic nilpotent conjugacy class with associated Young-diagram $\eta$. If all rows of $\eta$ have even length or all rows odd length, then $C$ is polarizable.*

# Part II. Minimal singularities

## 11. Geometry of $\pi$ and $\rho$

**11.1** Let us go back to our basic set up (1.2)

$$\begin{array}{ccc} L(V, U) & \xrightarrow{\pi} & \mathfrak{g}(U) \\ \big\downarrow{\scriptstyle \rho} & & \\ \mathfrak{g}(V) & & \end{array}$$

$V$ a quadratic space of type $\varepsilon$, $U$ a quadratic space of type $-\varepsilon$, $\dim V=:n\geq m:=\dim U$. As in 4.2 we set

$$\mathrm{L}':=\mathrm{L}'(V,U)=\{X\in\mathrm{L}(V,U)\mid X \text{ surjective}\}.$$

PROPOSITION. (i) *$\pi$ is smooth in $\mathrm{L}'$ and* $\pi(\mathrm{L}')=\{D\in\mathfrak{g}(U)\mid \mathrm{rk}\, D\geq 2m-n\}$.

(ii) $\rho(\mathrm{L}')=\{D\in\mathfrak{g}(V)\mid \mathrm{rk}\, D=m\}$ *and* $\rho|_{\mathrm{L}'}:\mathrm{L}'\to\rho(\mathrm{L}')$ *is a fibration with typical fibre* $G(U)$.

(Fibration here means "locally trivial in the étale topology".)

*Proof.* (i) By definition we have $(d\pi)_X(P)=PX^*+XP^*$. We want to show that $(d\pi)_X$ is surjective for $X\in\mathrm{L}'$. To solve $PX^*+XP^*=Q$ for given $Q\in\mathfrak{g}(U)$ it is enough to solve $PX^*=\frac{1}{2}Q$, since then $PX^*+XP^*=PX^*-(PX^*)^*=\frac{1}{2}Q-\frac{1}{2}Q^*=Q$. This is always possible since $X^*$ is injective. Furthermore it is clear that if $D\in\pi(\mathrm{L}')$, $D=XX^*$ with $X$ surjective, $X^*$ injective, and so rk $D\geq 2m-n$. The converse can be proved by an easy matrix argument: Given a symmetric (or skew symmetric) $n\times n$ matrix $S$ of rank $\geq 2m-n$, one must write $S=Y^tY$ (or $S=Y^tJY$, $J$ a non degenerate skew matrix) where $Y$ is an $m\times n$ matrix of rank $m$. (One can verify it also by the classification of ortho-symplectic pairs given in section 6.)

(ii) $\mathrm{L}'$ is an orbit under $GL(V)$ acting by left multiplication, and $\rho$ is equivariant under $GL(V)$ with respect to the action $D\mapsto g^*Dg$ on $\mathfrak{g}(V)$. Thus $\rho$ is of the form $H\backslash GL(V)\to H'\backslash GL(V)$, hence locally trivial. Since the actions of $G(U)$ and $GL(V)$ on $\mathrm{L}(V,U)$ commute, the claim follows from lemma 4.2. qed.

*Remark.* We will later use the second statement of the proposition in the following way: *For any locally closed $G(U)$-stable subset $W\subseteq\mathrm{L}'$ the image $\rho(W)$ is locally closed in $\mathfrak{g}(V)$ and $\rho|_W:W\to\rho(W)$ is smooth.* (Since $W$ is $G(U)$-stable we have $W=\rho^{-1}(\rho(W))$, hence $\rho|_W:W\to\rho(W)$ is a fibration.)

**11.2** PROPOSITION. *Let $D\in\mathfrak{g}(U)$ be nilpotent with* $\dim\mathrm{Ker}\, D\leq n-m$. *Assume that $\bar{C}_D$ is Cohen-Macaulay. Then $\pi^{-1}(\bar{C}_D)$ is reduced and Cohen-Macaulay.*

*Proof.* The assumption implies that the first column of the Young-diagram $\eta$ of $D$ has length $\leq n-m$, hence $\rho(\pi^{-1}(\bar{C}_D))=\bar{C}_{\varepsilon,\tilde{\eta}}$, where $\tilde{\eta}$ is obtained from $\tilde{\eta}$ by adding one column of length $n-m$ (4.1). Since $\pi$ is smooth in $\mathrm{L}'$ (proposition 11.1(i)), $N:=\pi^{-1}(\bar{C}_D)$, as a scheme, is smooth in the orthosymplectic orbit $O_\tau:=\rho^{-1}(C_{\varepsilon,\tilde{\eta}})$ by lemma 4.3(ii) and (iii). The claim will follow if we show that

$\dim O \leq \dim O_\tau - 1$ for all other orthosymplectic orbits $O \subseteq N$, since this implies first that $\mathrm{codim}_L N = \mathrm{codim}_{\mathfrak{g}(U)} \overline{C_D}$, hence $N$ is Cohen–Macaulay ([EGA] IV, 15.4.2, a) $\Rightarrow e'$)), then that $N = \bar{O}_\tau$ and finally that $N$ is smooth in codimension 0, hence reduced ([EGA] IV, 5.8.5). This inequality is a consequence of the dimension formula for orthosymplectic orbits (6.8) plus the remark that $O_\tau$ is the unique orbit on top of $C_{\varepsilon,\bar{\eta}}$ (lemma 4.3(ii)) and that $\Delta_\tau = 0$. qed.

*Remark. Under the assumptions of the proposition above* $\pi^{-1}(\bar{C}_D)$ *contains a dense orthosymplectic orbit, i.e.* $O_\tau = \rho^{-1}(C_{\varepsilon,\bar{\eta}})$ (see proof).

**11.3** To complete the picture we state some remarks which can be deduced from the previous analysis using [EGA] IV, 15.4.1 and 12.1.1, the Serre criterion ([EGA] IV, 5.8.6) and the fact that an orthogonal space of dimension $2m$ has two rulings of isotropic subspaces of dimension $m$, inequivalent under $SO_{2m}$.

*Remark.* Assume $\dim V \geq 2 \dim U$. Then the map

$$\pi : \mathrm{L}(V, U) \to \mathfrak{g}(U)$$

is *flat, Cohen–Macaulay and reduced.* If in addition $U$ is *orthogonal or* $\dim V > 2 \cdot \dim U$ the map $\pi$ is even *normal.* If $U$ is *symplectic and* $\dim V = 2 \cdot \dim U$ the zero fibre of $\pi$ has *two components intersecting in codimension* 1.

**11.4** The first assertion of proposition 11.1 can be improved if $U$ is an *orthogonal space.*

PROPOSITION. *In the setting* 11.1 *assume that* $U$ *is orthogonal. Then* $\pi$ *is smooth in* $\mathrm{L}^0 := \{x \in \mathrm{L}(V, U) \mid \mathrm{codim}\,\mathrm{Im}\, X \leq 1\}$.

*Proof.* Let $X \in \mathrm{L}^0$ and $Q = \mathfrak{g}(U)$. As in the proof of proposition 11.1 (i) we have to solve $PX^* = T$ for some $T \in \mathrm{End}\,(U)$ with $T - T^* = Q$. If $\mathrm{Ker}\, T \supseteq \mathrm{Ker}\, X^*$ this is obviously possible. If not let $u \in \mathrm{Ker}\, X^*$, $u \neq 0$ and put $v := Tu$. Then there is an $S \in \mathrm{End}\,(U)$ such that $S^* = S$ and $Su = v$. (In fact choosing an orthonormal basis in $U$ it is easy to see that for given vectors $u, v \in U$, $u \neq 0$, one always finds a symmetric matrix $S$ such that $Su = v$.) Replacing $T$ by $T' := T - S$ we get $u \in \mathrm{Ker}\, T'$ and we still have $T' - T'^* = Q$. qed.

*Remark.* One can show by a similar argument that $\mathrm{L}'$ and $\mathrm{L}^0$ in the cases $U$ symplectic and $U$ orthogonal respectively are *exactly the smooth points of the map* $\pi$.

**11.5** In the setting of 11.4 (i.e. $U$ orthogonal) consider the following decomposition of the map $\rho$:

$$\begin{array}{ccc} & & \mathrm{L}:=\mathrm{L}(V,U) \\ & {\scriptstyle\rho^0}\swarrow & \big\downarrow {\scriptstyle\rho} \\ \mathrm{L}/SO(U) & & \\ & {\scriptstyle\bar\rho}\searrow & \\ & & \mathfrak{sp}(V) \end{array}$$

where $\rho^0$ is the quotient under $SO(U)$ and $\bar\rho$ the quotient under $\mathbb{Z}/2\mathbb{Z} = O(U)/SO(U)$. Define $\mathrm{L}'':=\{X\in \mathrm{L}(V,U) \mid \mathrm{rk}\,\rho(X)\geq \dim U-1\}$. Of course $\mathrm{L}''\supseteq \mathrm{L}'$.

PROPOSITION. (a) $\rho^0|_{\mathrm{L}''}:\mathrm{L}''\to\rho^0(\mathrm{L}'')$ *is a fibration with typical fibre* $SO(U)$.
(b) $\mathbb{Z}/2\mathbb{Z}$ *acts trivially on* $\rho^0(\mathrm{L}''-\mathrm{L}')$.

*Proof.* For (a) we want to use Luna's criterion ([Lu], III. Corollaire 1) for principal fibrations, i.e. prove that the stabilizer in $SO(U)$ of any point $X\in\mathrm{L}''$ is trivial. We already know this if $X\in\mathrm{L}'$ (11.1(ii)). So we may assume that $\rho(X)$ is a matrix of rank $m-1$, $m=\dim U$. Choosing a basis of $V$ we may identify $\mathrm{L}(V,U)$ with the set of $n$-tuples $(u_1,\dots,u_n)$ of vectors in $U$, $n=\dim V$. Then $\rho$ can also be thought as mapping $(u_1,\dots,u_n)$ into the symmetric matrix $\rho(X)=((u_i,u_j))^n$ of scalar products (cf. remark 1.2). Using the action of $GL_n$ we may assume that $\rho(X)$ has the form $\left(\begin{array}{c|c}\mathbb{1}_{m-1} & 0\\ \hline 0 & 0\end{array}\right)$. This means that $u_1,\dots,u_{m-1}$ are an orthonormal basis of a subspace $U'\subseteq U$ of codimension 1. The remaining vectors $u_m, u_{m+1},\dots,u_n$ must be 0 being isotropic in the non degenerate one dimensional space $U'^{\perp}$. Now it is clear that the stabilizer of $X$ in $SO(U)$ is trivial, proving (a), and that the stabilizer of $X$ in $O(U)$ is $\mathbb{Z}/2\mathbb{Z}$, proving also (b). qed.

## 12. Smoothly equivalent singularities, cross sections

**12.1** DEFINITION (cf. [H1] 1.7). Consider two varieties $X$, $Y$ and two points $x\in X$, $y\in Y$. The singularity of $X$ in $x$ is called *smoothly equivalent* to the singularity of $Y$ in $y$ if there is a variety $Z$, a point $z\in Z$ and two maps

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ {\scriptstyle\psi}\big\downarrow & & \\ Y & & \end{array}$$

such that $\varphi(z)=x$, $\psi(z)=y$, and $\varphi$ and $\psi$ are smooth in $z$. This clearly defines an *equivalence relation* between pointed varieties $(X, x)$. We denote the *equivalence* class of $(X, x)$ by $\mathrm{Sing}(X, x)$.

Assume that an algebraic group $G$ acts regularly on the variety $X$. Then $\mathrm{Sing}(X, x)=\mathrm{Sing}(X, x')$ if $x$ and $x'$ belong to the same orbit $O$. In this case we denote the equivalence class also by $\mathrm{Sing}(X, O)$.

**12.2** *Remark.* The smooth equivalence of two singularities $x \in X$, $y \in Y$ means that, after multiplication by affine spaces, they are analytically isomorphic. This implies that various geometric properties of $X$ in $x$ depend only on the equivalence class $\mathrm{Sing}(X, x)$, for example: *Smoothness, normality, seminormality* (cf. 16.1), *unibranchness, Cohen–Macaulay, rational singularities* ([E1] théorème 5). A typical example of a property which is not preserved, since it has not an analytic meaning, is irreducibility in $x$.

**12.3** Now we can formulate the main result of this section. We use the notations introduced in section 3.

THEOREM. *Let the $\varepsilon$-degeneration $\sigma \leq \eta$ be obtained from the $\varepsilon'$-degeneration $\sigma' \leq \eta'$ by adding rows and columns. Then* $\mathrm{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})=\mathrm{Sing}(\bar{C}_{\varepsilon',\eta'}, C_{\varepsilon',\sigma'})$.

The proof is similar to the one in the linear case (cf. [KP2]). We must treat separately the two steps "cancelling rows" and "cancelling columns". In the second case (proposition 13.5) we will use the analysis carried out in section 11, while the first case (proposition 13.4) will be handled with the method of cross sections, which we now describe. There is a difficulty in this case that did not appear in the linear case and is due to the possible lack of normality of the closure of a conjugacy class. This is overcome by a suitable reduction to the linear case (cf. 13.1).

**12.4** DEFINITION. Let $X$ be a variety with a regular action of an algebraic group $G$. A *cross section* at a point $x \in X$ is defined to be a locally closed subvariety $S \subseteq X$ such that $x \in S$ and the map $G \times S \to X$, $(g, s) \mapsto gs$, is smooth at the point $(e, x)$.

Of course we have $\mathrm{Sing}(S, x)=\mathrm{Sing}(X, x)$. There is a natural way to construct cross sections for affine $G$-varieties $X$. Choose a $G$-equivariant closed embedding $X \hookrightarrow V$ in some vector space $V$ with a linear $G$-action and a complement $N$ of the tangent space $T_x(Gx)$ in $V$. Define $S:=(N+x) \cap X$ (schematic intersection). Then $G \times S \to X$, $(g, s) \mapsto gs$, is smooth at the point $(e, x)$, since $G \times (N+x) \to V$,

$(g, n+x) \mapsto g(n+x)$ is smooth at $(e, x)$ ([EGA] IV, 17.11.1) and

$$\begin{array}{ccc} G\times(N+x) & \longrightarrow & V \\ \uparrow & & \uparrow \\ G\times S & \longrightarrow & X \end{array}$$

is a fibre product. Hence $S$ is reduced in $x$ and so $S$, as a variety, is a cross section at $x$. The construction implies that $x$ is an isolated point in $S\cap Gx$. Assuming $X$ irreducible (or equidimensional) we get $\dim_x S = \operatorname{codim}_{Gx} X$.

**12.5** Another useful fact on singularities is the following result.

LEMMA. *Let $X$, $Y$ be varieties with an action of an algebraic group $G$ and $\varphi: X \to Y$ an equivariant map. Assume that $Y$ is an orbit under $G$. Then $\varphi$ is a locally trivial fibration* (in the étale topology). *In particular for each $x\in X$ we have*

$$\operatorname{Sing}(X, x) = \operatorname{Sing}(\varphi^{-1}(\varphi(x)), x).$$

*Proof.* Consider a point $y_0\in Y$, the orbit map $\psi: G \to Y$ and the fibre product

$$\begin{array}{ccc} G\times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\varphi} \\ G & \xrightarrow[\psi]{} & Y \end{array}$$

Since $\psi$ is smooth, $G\times_Y X$ is the subvariety of $G\times X$ given by

$$G\times_Y X = \{(g, x) \mid gy_0 = \varphi(x)\}.$$

The image of $G\times_Y X$ under the isomorphism $G\times X \xrightarrow{\sim} G\times X$, $(g, x) \to (g, g^{-1}x)$, is clearly $G\times\varphi^{-1}(y_0)$. qed.

*Remark.* If in the setting of the lemma we do not assume that $Y$ is an orbit we still have the following result: *If $S\subset Y$ is a cross section in the point $\varphi(x)$, then* $\operatorname{Sing}(X, x) = \operatorname{Sing}(\varphi^{-1}(S), x)$.

## 13. Cancelling rows and columns

**13.1** Let $G$ be an algebraic group. As usual we denote its Lie algebra by the corresponding german letter $\mathfrak{g}$. For $x\in\mathfrak{g}$ we write $Gx$ for its conjugacy class in $\mathfrak{g}$ (i.e. its orbit under the adjoint action).

PROPOSITION. *Let $G$ be an algebraic group, $G'$, $H \subseteq G$ closed subgroups such that $H' := G' \cap H$ is reductive and let $x \in \mathfrak{h}'$, $y \in \overline{H'x}$. Assume*

(i) $\operatorname{codim}_{\overline{G'x}} \overline{G'y} = \operatorname{codim}_{\overline{Gx}} \overline{Gy}$,

(ii) $\overline{G'x} \cap \mathfrak{h}' = \overline{H'x}$,

(iii) *$\overline{Gx}$ is normal in $y$.*

*Then* $\operatorname{Sing}(\overline{Hx}, y) = \operatorname{Sing}(\overline{H'x}, y)$.

*Proof.* By assumption we have $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$. We claim that there is a complement $N$ of $[\mathfrak{g}, y]$ in $\mathfrak{g}$ such that

$$\begin{aligned} \mathfrak{g}' &= [\mathfrak{g}', y] \oplus N', & N' &:= N \cap \mathfrak{g}', \\ \mathfrak{h} &= [\mathfrak{h}, y] \oplus N_0, & N_0 &:= N \cap \mathfrak{h}, \\ \mathfrak{h}' &= [\mathfrak{h}', y] \oplus N_0', & N_0' &:= N \cap \mathfrak{h}' = N_0 \cap \mathfrak{g}'. \end{aligned}$$

Since $H'$ is reductive, we can find an $H'$-stable decomposition $\mathfrak{g} = \mathfrak{h}' \oplus M' \oplus M_0 \oplus D$ such that $\mathfrak{g}' = \mathfrak{h}' \oplus M'$, $\mathfrak{h} = \mathfrak{h}' \oplus M_0$. Hence $[\mathfrak{g}, y] = [\mathfrak{h}', y] \oplus [M', y] \oplus [M_0, y] \oplus [D, y]$ since $y \in \mathfrak{h}'$, and so $[\mathfrak{h}', y] \subseteq \mathfrak{h}'$, $[M', y] \subseteq M'$, $[M_0, y] \subseteq M_0$ and $[D, y] \subseteq D$. This implies the existence of decompositions $\mathfrak{h}' = [\mathfrak{h}', y] \oplus N_0'$, $M' = [M', y] \oplus \bar{N}'$, $M_0 = [M_0, y] \oplus \bar{N}_0$ and $D = [D, y] \oplus \bar{D}$. It follows that $N := N_0' \oplus \bar{N}' \oplus \bar{N}_0 \oplus \bar{D}$ has the required property, since $N_0' = N \cap \mathfrak{h}'$, $N' = N \cap \mathfrak{g}' = N_0' \oplus \bar{N}'$ and $N_0 = N \cap \mathfrak{h} = N_0' \oplus \bar{N}_0$. Now define $S := (N + y) \cap \overline{Gx}$, $S' := (N' + y) \cap \overline{G'x}$, $S_0 := (N_0 + y) \cap \overline{Hx}$ and $S_0' := (N_0' + y) \cap \overline{H'x}$. These are all cross sections in $y$ (12.4) and we have $S' \cap \mathfrak{h}' = ((N' + y) \cap \mathfrak{h}') \cap (\overline{G'x} \cap \mathfrak{h}') = (N_0' + y) \cap \overline{H'x} = S_0'$ and $S' \subseteq \mathfrak{g}'$, hence $S_0' = S' \cap \mathfrak{h}$. From assumption (i) we get $\dim_y S = \dim_y S'$ (12.4). Since $S$ is normal in $y$ by assumption (iii) and remark 12.2, this implies that $S$ and $S'$ coincide in a suitable neighbourhood of $y$, and so the same holds for $S \cap \mathfrak{h}$ and $S' \cap \mathfrak{h}$. But $S \cap \mathfrak{h} \supseteq S_0 \supseteq S_0' = S' \cap \mathfrak{h}$ by construction, hence $S_0$ and $S_0'$ coincide in a suitable neighbourhood of $y$ too. Thus finally $\operatorname{Sing}(\overline{H'x}, y) = \operatorname{Sing}(S_0', y) = \operatorname{Sing}(S_0, y) = \operatorname{Sing}(\overline{Hx}, y)$. qed.

**13.2** *Remark.* The proposition remains true if we replace the normality condition (iii) by the slightly weaker assumption:

(iii)′ *$\overline{Gx}$ is unibranch in $y$.*

In fact the assumption (iii) was used to show that $S$ and $S'$ coincide in a suitable neighbourhood of $y$ (notations of the proof). Now (iii)′ implies that $S$ is unibranch in $y$ (being a cross section in a neighbourhood of $y$, cf. 12.2) and in particular irreducible in $y$. Hence the equality $\dim_y S = \dim_y S'$ is enough to insure that $S$ and $S'$ coincide in a neighbourhood of $y$.

**13.3** Using this remark, we get the following corollary (put $H = G$, $H' = G'$).

COROLLARY. *Let $G$ be an algebraic group, $G' \subseteq G$ a closed reductive subgroup and let $x \in \mathfrak{g}'$, $y \in \overline{G'x}$.* Assume

(i) $\operatorname{codim}_{\overline{G'x}} G'y = \operatorname{codim}_{\overline{Gx}} Gy$,

(ii) *$\overline{Gx}$ is unibranch in $y$.*

*Then* $\operatorname{Sing}(\overline{G'x}, G'y) = \operatorname{Sing}(\overline{Gx}, Gy)$.

**13.4** We now can prove one part of theorem 12.3.

PROPOSITION. *Assume that the $\varepsilon$-degeneration $\sigma \lesssim \eta$ is obtained from the $\varepsilon$-degeneration $\sigma' \leq \eta'$ by adding rows* (3.3). *Then* $\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \operatorname{Sing}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'})$.

*Proof.* Let $V$ be a quadratic space of type $\varepsilon$ of dimension $|\eta|$ and $D \in C_{\varepsilon,\eta} \subseteq \mathfrak{g}(V)$. By assumption the diagrams $\eta$ and $\sigma$ are decomposed, $\eta = \nu + \eta'$ and $\sigma = \nu + \sigma'$, $\nu$ also an $\varepsilon$-diagram. These decompositions correspond to an orthogonal decomposition $V = W \oplus V'$ such that $D = (F, D') \in \mathfrak{g}(W) \oplus \mathfrak{g}(V') \subseteq \mathfrak{g}(V)$, $D' \in C_{\varepsilon,\eta'}$, and there exists $E = (F, E') \in C_{\varepsilon,\sigma}$ with $E' \in C_{\varepsilon,\sigma'}$. To apply proposition 13.1 we define $G := GL(V)$, $G' := GL(W) \times GL(V')$, $H := G(V)$ and $H' := G(W) \times G(V')$. Now condition (i) follows from the dimension formula for linear conjugacy classes (cf. remark 3.2), (ii) from theorem 2.2 (iib) and (iii) from the normality of conjugacy classes in $\mathfrak{gl}_n$ ([KP1]). Hence $\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, E) = \operatorname{Sing}(\overline{H.D}, E) = \operatorname{Sing}(\overline{H'.D}, E)$. Since $\overline{H'.D} = \overline{G(W).F} \times \overline{G(V').D'}$ and $E = (F, E')$ is contained in the open subset $G(W).F \times \overline{G(V').D'}$, we get $\operatorname{Sing}(\overline{H'.D}, E) = \operatorname{Sing}(\overline{G(V').D'}, E') = \operatorname{Sing}(\bar{C}_{\varepsilon,\eta'}, E')$. qed.

**13.5** *Proposition*: *Assume that the $\varepsilon$-degeneration $\sigma \leq \eta$ is obtained from the $\varepsilon'$-degeneration $\sigma' \lesssim \eta'$ by adding columns* (3.3). *Then* $\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \operatorname{Sing}(\bar{C}_{\varepsilon',\eta'}, C_{\varepsilon',\sigma'})$.

*Proof.* It is enough to treat the case where $\sigma \leq \eta$ is obtained from $\sigma' \leq \eta'$ by adding a single column. Let $V$ be a quadratic space of type $\varepsilon$ and dimension $|\eta|$ and $U$ be a quadratic space of type $-\varepsilon$ and dimension $|\eta'|$. Consider the basic set up (1.2):

$$
\begin{array}{ccc}
L(V, U) & \xrightarrow{\ \pi\ } & \mathfrak{g}(U) \\
\downarrow{\scriptstyle \rho} & & \\
\mathfrak{g}(V) & &
\end{array}
$$

and the induced diagram of maps (4.3)

$$\begin{array}{ccc} \pi^{-1}(\bar{C}_{-\varepsilon,\eta'}) =: N_{\varepsilon,\eta} & \xrightarrow{\pi} & \bar{C}_{-\varepsilon,\eta'} \\ \downarrow \rho & & \\ \bar{C}_{\varepsilon,\eta} & & \end{array}$$

As a consequence of proposition 11.1 and its remark we have that these two maps are smooth in the open set $N'_{\varepsilon,\eta} := N_{\varepsilon,\eta} \cap L'(V, U)$. Thus it is sufficient to show that there is a point $X \in N'_{\varepsilon,\eta}$ with $\pi(X) \in C_{-\varepsilon,\sigma'}$ and $\rho(X) \in C_{\varepsilon,\sigma}$ (12.1). From 4.3(ii), (iii) (and remark) we have $\rho^{-1}(C_{\varepsilon,\sigma}) \subseteq N'_{\varepsilon,\eta}$ and $\pi(\rho^{-1}(C_{\varepsilon,\sigma})) = C_{-\varepsilon,\sigma'}$ and so we can choose any $X \in \rho^{-1}(C_{\varepsilon,\sigma})$. qed.

**13.6** Let $V = V_1 \oplus V_2$ be an orthogonal decomposition of a quadratic space $V$. Consider nilpotent conjugacy classes $C_i$ in $\mathfrak{g}(V_i)$ and degenerations $C'_i \subseteq \bar{C}_i$, $i = 1, 2$, and denote by $C$ and $C'$ the conjugacy class in $\mathfrak{g}(V)$ generated by $C_1 \times C_2$ and $C'_1 \times C'_2$ respectively. Generalizing 13.4 we give a simple condition under which $\mathrm{Sing}(\bar{C}, C') = \mathrm{Sing}(\bar{C}_1, C'_1) \times \mathrm{Sing}(\bar{C}_2, C'_2) := \mathrm{Sing}(\bar{C}_1 \times \bar{C}_2, C'_1 \times C'_2)$. For this let $\eta_i$ be the diagram of $C_i$ and $\sigma_i$ that of $C'_i$.

PROPOSITION. *Assume that $\eta_1$ and $\sigma_1$ have the same number of rows and that the last row of $\eta_1$ is larger than the first row of $\eta_2$. Then*

$$\mathrm{Sing}(\bar{C}, C') = \mathrm{Sing}(\bar{C}_1, C'_1) \times \mathrm{Sing}(\bar{C}_2, C'_2)$$

*Proof.* We proceed as in 13.4 applying proposition 13.1. The only point is to verify the codimension condition which is easily seen to be a consequence of the hypotheses made. qed.

*Remark.* One can easily extend the statement to any decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$.

## 14. Singularities of minimal degenerations

**14.1** In this section we give the classification of the singularities $\mathrm{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$ for a minimal $\varepsilon$-degeneration $\sigma \leq \eta$ (3.1). By theorem 12.3 we are reduced to study the irreducible ones given in table I (3.3, 3.4). We distinguish the two cases $\mathrm{codim}_{\bar{C}_{\varepsilon,\eta}} C_{\varepsilon,\sigma} = 2$ and $>2$. For the first case we need to recall part of Brieskorn's theory on subregular singularities in simple groups (cf. 0.6). The nilpotent cone $\mathcal{N}$

of a simple Lie algebra is the closure of a unique conjugacy class $C_{\text{reg}}$, the *regular class*; its boundary $\partial C_{\text{reg}} = \mathcal{N} - C_{\text{reg}}$ is itself the closure of a unique conjugacy class $C_{\text{subreg}}$, the *subregular class*, and $\operatorname{codim}_{\mathcal{N}} C_{\text{subreg}} = 2$.

THEOREM (cf. [S1], 6.4 Theorem): *Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ $(n \geq 1)$ or $\mathfrak{so}_{2n}$ $(n \geq 3)$. Then the singularity of $\mathcal{N}$ in the subregular class $C_{\text{subreg}}$ is smoothly equivalent to the simple surface singularity of type $A_n$, $A_{2n-1}$, $D_{n+1}$ and $D_n$ respectively.*

**14.2** If we now look at table I (3.4) we can immediately recognize that for the types *a*, *b* and *c* the diagrams $\eta$ and $\sigma$ are those of the regular and the sub-regular class in the corresponding Lie algebra. In case *e* the conjugacy class $C_{1,\eta}$ has two components $C^{(1)}$ and $C^{(2)}$ and $\overline{C^{(1)}} \cap \overline{C^{(2)}} = \overline{C_{1,\sigma}}$ (remark 2.3). In particular $\bar{C}_{1,\eta}$ is *not normal* in this case. We will describe more precisely this singularity and show in particular (15.4(a), 15.1) that the intersection $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is reduced and $\operatorname{Sing}(\overline{C^{(i)}}, C_{1,\sigma}) = A_{2n-1}$. We will indicate such singularity by $\operatorname{Sing}(\bar{C}_{1,\eta}, C_{1,\sigma}) = A_{2n-1} \cup A_{2n-1}$.

The remaining case *d* is related to the exceptional case *e*; we will prove that it gives rise to a singularity of type $A_{2n-1}$ also (15.4(b)). We set aside to the next section these two cases and first complete the study of minimal degenerations of codimension >2.

**14.3** Inspecting table I in the cases *f*, *g*, *h* we see that $\sigma$ is the diagram of the zero class while $\eta$ is the diagram of the *unique minimal non zero class*. It is well known that this is the orbit of a highest weight vector in the Lie algebra, i.e. *the conjugacy class of a long root vector x*. This singularity is usually described as a "collapsing" of a line bundle: One considers the line $L := kx$, the parabolic $P$ stabilizing $L$ in the corresponding group $G$ and the line bundle $G \times^P L$ over $G/P$. The natural map $\varphi : G \times^P L \to \bar{C}_{\varepsilon,\eta}$ is a resolution of singularities and $\varphi^{-1}(0)$ is the zero section of this bundle (cf. 0.7).

The consequences of this construction for the geometry of $\bar{C}_{\varepsilon,\eta}$ have been studied extensively by several authors. In particular it follows from [K] §2 that $\bar{C}_{\varepsilon,\eta}$ is *normal, Cohen-Macaulay with rational singularities.* We remark that in this case the normality of $\bar{C}_{\varepsilon,\eta}$ can also be deduced from proposition 8.3. For a more precise discussion of these varieties we refer the reader to the previously cited literature. Here we only remark that, in case g, $G/P \cong \mathbb{P}^{2n-1}$ and the line bundle is $\mathcal{O}_{\mathbb{P}^{2n-1}}(-2)$.

Finally, in analogy to the standard notations for simple groups, these singularities are denoted by the symbols $b_n$, $c_n$, $d_n$.

We have now explained the meaning of the symbols on the last line of table I to which we now can refer.

## 15. The types $d$ and $e$

**15.1** Consider the very even conjugacy class $C := C_{(2n,2n)}$ in $\mathfrak{so}_{4n}$ with the two components $C^{(1)}$ and $C^{(2)}$. One has $\overline{C^{(1)}} \cap \overline{C^{(2)}} = \overline{C'}$ with $C'$ associated to the partition $(2n-1, 2n-1, 1, 1)$ (remark 2.3).

PROPOSITION. *The singularity of $\overline{C^{(i)}}$ in $C'$ is smoothly equivalent to the simple surface singularity $A_{2n-1}$.*

*Proof.* Let $U$ be a vectorspace of dimension $2n$, $U^*$ its dual space. Then $V := U \oplus U^*$ is an orthogonal space with respect to the symmetric form $((u, e), (u', e')) := e(u') + e'(u)$. We have the closed immersion $GL(U) \hookrightarrow SO(V)$ given by $g \mapsto (g, g^{*-1})$, which induces the inclusion $\mathfrak{gl}(U) \hookrightarrow \mathfrak{so}(V)$, $D \mapsto \tilde{D} := (D, -D^*)$. If $D \in \mathfrak{gl}(U)$ is nilpotent with partition $\eta = (\eta_1, \ldots, \eta_s)$, its image $\tilde{D}$ has partition $\tilde{\eta} = (\eta_1, \eta_1, \eta_2, \eta_2, \ldots, \eta_s, \eta_s)$. In particular the (connected) regular class $C_0$ of $\mathfrak{gl}_n$ is mapped into one component of $C$, say $C^{(1)}$, the subregular class $C_0' \subset \bar{C}_0$ is mapped into $C'$, and $\operatorname{codim}_{\bar{C}_0} C_0' = 2 = \operatorname{codim}_{C^{(1)}} C'$. This enables us to apply corollary 13.3 and deduce the claim, provided we can show that $\overline{C^{(1)}}$ is unibranch in $C'$.[1]

To see this consider the flag variety $\mathcal{F}$ of isotropic flags $F = (F_1, F_2, \ldots, F_n)$ in the $4n$-dimensional orthogonal space $V$, $F_1 \subset F_2 \subset \cdots \subset F_n$, $F_i$ isotropic of dimension $2i$, and the vectorbundle $\mathcal{V} := \{(F, X) \mid XF_i \subseteq F_{i-1} \text{ for all } i\} \subseteq \mathcal{F} \times \mathfrak{so}(V)$ over $\mathcal{F}$. The projection $pr: \mathcal{F} \times \mathfrak{so}(V) \to \mathfrak{so}(V)$ induces a "desingularisation" $\varphi: \mathcal{V} \to \bar{C}$, i.e. for the two connected components $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ we have $\varphi(\mathcal{V}^{(i)}) = \overline{C^{(i)}}$ and $\varphi|_{\mathcal{V}^{(i)}}: \mathcal{V}^{(i)} \to \overline{C^{(i)}}$ is proper and birational (cf. [H2] §4). So we have to show, that the fibre $P := \varphi^{-1}(D)$ of an element $D \in C'$ has (at most) two connected components. We choose a basis $\{e_1, e_2, \ldots, e_{2n-1}, f_1, f_2, \ldots, f_{2n-1}, g, h\}$ of $V$ such that $Df_i = f_{i-1}$, $De_i = e_{i-1}$, $e_1, f_1, g, h \in \operatorname{Ker} D$ and such that $V = \langle e_1, \ldots, e_{2n-1} \rangle \oplus \langle f_1, \ldots, f_{2n-1} \rangle \oplus \langle g \rangle \oplus \langle h \rangle$ is an orthogonal decomposition ([SS] IV, 2.19). In the non-degenerate orthogonal space $\langle e_n, f_n, g, h \rangle$ we have two types of isotropic planes:

$$E^1_{\lambda/\mu} := \langle \lambda(e_n + \sqrt{-1} f_n) + \mu(g - \sqrt{-1} h),\ \mu(e_n - \sqrt{-1} f_n) - \lambda(g + \sqrt{-1} h) \rangle$$

and

$$E^{-1}_{\lambda/\mu} := \langle \lambda(e_n + \sqrt{-1} f_n) + \mu(g + \sqrt{-1} h),\ \mu(e_n - \sqrt{-1} f_n) - \lambda(g - \sqrt{-1} h) \rangle,$$

$\lambda/\mu \in \mathbb{P}^1$. Consider any flag $F$ in $P (= \varphi^{-1}(D))$. Clearly $F_n \subseteq V' := \langle e_1, \ldots, e_n, f_1, \ldots, f_n, g, h \rangle$ and it is isotropic of dimension $2n$. This implies that

---

[1] The following analysis was indicated to us by N. Spaltenstein.

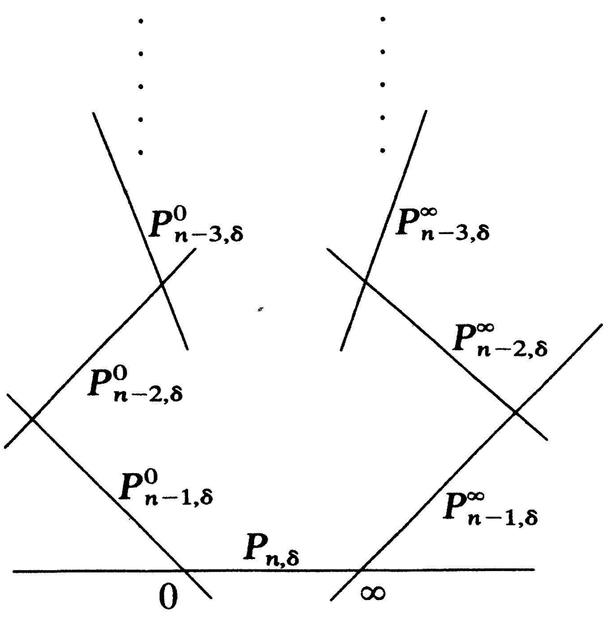
$F_n$ contains $V_{n-1} := \langle e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}\rangle$, the kernel of $V'$. Thus $F_n = V_{n-1} \oplus E^{\delta}_{\lambda/\mu}$ for some $\lambda/\mu \in \mathbb{P}^1$, $\delta = \pm 1$. The condition $DF_i \subseteq F_{i-1}$ implies that for $\lambda/\mu \neq 0, \infty$ we necessarily have $F_i = V_i := \langle e_1, \ldots, e_i, f_1, \ldots, f_i\rangle$ for $i < n$. The flags of this form define a subset $P_{n,\delta}$ of $P$ isomorphic to $\mathbb{P}^1$. Assume now, for instance, $\lambda/\mu = \infty$. Then we may also assume that for some $n > r \geq 0$ we have

$$F_i = V_{i-1} \oplus \langle g + \delta\sqrt{-1}h\rangle \oplus \langle e_i + \sqrt{-1}f_i\rangle \quad \text{for} \quad n \geq i > r$$

and that $F_r$ is not of this form. This implies that $F_{r-1} = V_{r-1}$ and so $F_i = V_i$ for $i \leq r-1$. As for $F_r$ itself it may be chosen arbitrarily of dimension $2r$, such that $V_{r-1} \subset F_r \subset F_{r+1}$, $DF_r \subseteq V_{r-1}$ and $DF_{r+1} \subseteq F_r$. These conditions imply that $F_r$ contains $F_{r-1} \oplus \langle e_r + \sqrt{-1}f_r\rangle$ and is contained in $F_{r-1} \oplus \langle e_r, f_r, g + \delta\sqrt{-1}h\rangle$. Hence these flags form a subset $P^{\infty}_{r,\delta}$ of $P$ isomorphic to $\mathbb{P}^1$. Similarly for $\lambda/\mu = 0$ we find subsets $P^0_{r,\delta}$. The analysis shows that

$$P = P_1 \cup P_{-1}, \qquad P_\delta := \left(\bigcup_r P^0_{r,\delta}\right) \cup P_{n,\delta} \cup \left(\bigcup_r P^{\infty}_{r,\delta}\right)$$

and that the terms are the irreducible components of $P$. One easily determines the intersection properties of these lines and verifies that $P_1$ and $P_{-1}$ are the two connected components of $P$, each consisting of $2n-1$ lines with graph:




qed.

**15.2** To proceed to type *d* and complete type *e* we need a few general facts on ***reduced intersections***. Let $X$ be a variety, $X_1$, $X_2$ two locally closed subvarieties and $x \in X_1 \cap X_2$.

DEFINITION. The intersection $X_1 \cap X_2$ is called *reduced in x*, if in the local ring $\mathcal{O}_{X,x}$ of $X$ in $x$ we have

$$\mathfrak{a}_x = \mathfrak{a}_{1,x} + \mathfrak{a}_{2,x}$$

where $\mathfrak{a}_{1,x}$, $\mathfrak{a}_{2,x}$, $\mathfrak{a}_x$ are the ideals of functions in $\mathcal{O}_{X,x}$ vanishing on $X_1, X_2, X_1 \cap X_2$ respectively. We say that $X_1 \cap X_2$ is *reduced*, if it is reduced in all points $x \in X_1 \cap X_2$.

*Remark.* (1) In the definition one can replace $X$ by any subvariety containing $X_1 \cup X_2$.

(2) This property is equivalent to say that the schematic intersection $X_1 \cap X_2$ is reduced in $x$, i.e. that the sequence

$$\mathcal{O}_{X_1 \cup X_2, x} \to \mathcal{O}_{X_1, x} \times \mathcal{O}_{X_2, x} \rightrightarrows \mathcal{O}_{X_1 \cap X_2, x}$$

is exact.

(3) The set of points $x \in X_1 \cap X_2$, where the intersection is reduced, is open in $X_1 \cap X_2$ ([EGA] IV, 12.1.7).

**15.3** Let us collect some elementary properties on reduced intersections, mostly well known. The setting is as in 15.2; $\mathcal{O}(X)$ indicates the ring of *global regular functions* on $X$.

(a) If $X_1 \cap X_2$ is reduced, we have the following global property (cf. 15.2 remark 2):

(**P**) *For all* $f_1 \in \mathcal{O}(X_1)$, $f_2 \in \mathcal{O}(X_2)$ *such that* $f_1|_{X_1 \cap X_2} = f_2|_{X_1 \cap X_2}$ *there is an* $f \in \mathcal{O}(X_1 \cup X_2)$ *with* $f|_{X_i} = f_i$.

(b) *If $X$ is affine and $X_1, X_2$ closed subsets, then property* (**P**) *is equivalent to $X_1 \cap X_2$ being reduced.*
(In fact for the ideals $\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2 \subseteq \mathcal{O}(X)$ of functions vanishing on $X_1 \cap X_2, X_1, X_2$ we have $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$.)

(c) *If $X_1$ and $X_2$ are smooth in $x$ with normal crossing* (i.e. $\dim(T_x(X_1) \cap T_x(X_2)) = \dim_x(X_1 \cap X_2)$), *then $X_1 \cap X_2$ is reduced in $x$.*
(In this case the schematic intersection is even smooth in $x$.)

(d) *Let $\varphi: Y \to X$ be a regular map, $Y_i := \varphi^{-1}(X_i)$ and $y \in Y_1 \cap Y_2$ a point with $\varphi(y) = x$. Assume that $\varphi$ is smooth in $y$. Then $Y_1 \cap Y_2$ is reduced in $y$ if and only if $X_1 \cap X_2$ is reduced in $x$.*
(The property of being reduced in $x$ is a property in the completion $\hat{\mathcal{O}}_{X,x}$.)

(e) Let $X$ be affine with an action of a reductive group $G$, $\pi: X \to X/G$ the quotient map. *Assume that $X_1, X_2$ are closed and $G$-stable subsets with reduced intersection. Then $\pi(X_1) \cap \pi(X_2)$ is reduced.*
(This follows from general facts of quotient maps; see 0.11.)

(f) Let $\sigma$ be an automorphism of order 2 of $X$ such that $\sigma(X_1) = X_2$ and $\sigma$ is the identity on $X_1 \cap X_2$. Consider the quotient map $\pi: X \to X/\sigma$. *Then the induced map $\pi_1: X_1 \to \pi(X_1)$ is an isomorphism in $X_1 - (X_1 \cap X_2)$ and in all points $x \in X_1 \cap X_2$ where the intersection is reduced.*
(We can easily reduce to the setting $X = X_1 \cup X_2$, $X$ affine, $X_i \subseteq X$ closed and $X_1 \cap X_2$ reduced. From the exact sequence

$$\mathcal{O}(X) \to \mathcal{O}(X_1) \times \mathcal{O}(X_2) \rightrightarrows \mathcal{O}(X_1 \cap X_2)$$

we get the exact sequence for the invariants

$$\mathcal{O}(X)^\sigma \to (\mathcal{O}(X_1) \times \mathcal{O}(X_2))^\sigma \rightrightarrows \mathcal{O}(X_1 \cap X_2)^\sigma = \mathcal{O}(X_1 \cap X_2).$$

Any function $f \in \mathcal{O}(X_1)$ is transformed by $\sigma$ into a function $f^\sigma \in \mathcal{O}(X_2)$ and the invariants in $\mathcal{O}(X_1) \times \mathcal{O}(X_2)$ are just the pairs $(f, f^\sigma)$. Thus $\mathcal{O}(X)^\sigma \to (\mathcal{O}(X_1) \times \mathcal{O}(X_2))^\sigma$ is an isomorphism and, composed with the projection onto $\mathcal{O}(X_1)$ gives the desired result.)

(g) *Assume that $X = X_1 \cup X_2$ is affine and Cohen-Macaulay and $X_i$ closed in $X$. If there is a closed subset $W \subseteq X_1 \cap X_2$ with* $\operatorname{codim}_X W \geq 2$ *such that $X_1 \cap X_2$ is reduced in $(X_1 \cap X_2) - W$ then $X_1 \cap X_2$ is reduced.*
(This follows immediately from (a) and lemma 9.1.)

(h) *Universal property*: Let $\varphi_i: X_i \to Y$, $i = i, 2$, be regular maps such that $\varphi_1 |_{X_1 \cap X_2} = \varphi_2 |_{X_1 \cap X_2}$. *If $X_1 \cap X_2$ is reduced there is a unique regular map $\varphi: X_1 \cup X_2 \to Y$ such that $\varphi |_{X_i} = \varphi_i$.*

**15.4** We can now formulate the main result of this section and complete the study of singularities in minimal degenerations (cf. 14.2).

PROPOSITION. (a) *The two components of the closure of the very even conjugacy class $C_{(2n,2n)}$ in $\mathfrak{so}_{4n}$ have a reduced intersection and are normal, Cohen–Macaulay with rational singularities.*

(b) *The closure of the conjugacy class $C_{(2n+1,2n+1)}$ in $\mathfrak{sp}_{4n+2}$ is normal, Cohen–Macaulay with rational singularities and*

$$\operatorname{Sing}(\bar{C}_{(2n+1,2n+1)}, C_{(2n,2n,2)}) = A_{2n-1}.$$

We first prove that (a) implies (b) and then (15.6) that (b) for $\mathfrak{sp}_{4k-2}$ implies (a) for $\mathfrak{so}_{4k}$. This proves the result by induction, since (b) is clear for $n=0$.

*Proof of* $(a)\Rightarrow(b)$: Put $m=2n$ and let $U$ be an orthogonal space of dimension $2m$, $V$ a symplectic space of dimension $2m+2$. By assumption the two components $C^{(1)}$ and $C^{(2)}$ of $C:=C_{(m,m)}\subseteq\mathfrak{so}(U)$ have normal closures $\overline{C^{(1)}}$ and $\overline{C^{(2)}}$ with reduced intersection. We have to show that the closure of $D:=C_{(m+1,m+1)}\subseteq\mathfrak{sp}(V)$ is normal and Cohen-Macaulay with rational singularities. We have the maps $(\mathrm{L}:=\mathrm{L}(V,U))$

$$\begin{array}{ccc} & \mathrm{L}(V,U) \xrightarrow{\ \pi\ } \mathfrak{so}(U) & \\ {}^{\rho^0}\swarrow & \downarrow{\scriptstyle\rho} & \\ \mathrm{L}/SO(U) & & \\ {}_{\bar\rho}\searrow & \mathfrak{sp}(V) & \end{array}$$

where $\rho^0$ is the quotient by $SO(U)$ and $\bar\rho$ the quotient by $\mathbb{Z}/2\mathbb{Z}$ $(=O(U)/SO(U)$; cf. 11.5). Put $N:=\pi^{-1}(\bar{C})$. We know that $Q:=\rho^{-1}(D)$ is a single $O(U)\times Sp(V)$-orbit contained in $\mathrm{L}'$ and that $\pi(Q)=C$ (4.3 lemma (ii) and (iii)). Hence $Q$ is the union of two $SO(U)\times Sp(V)$-orbits $Q^{(1)}$ and $Q^{(2)}$ with $\pi(Q^{(i)})=C^{(i)}$. Furthermore $Q':=\rho^{-1}(D')$, $D':=C_{(m,m,2)}$, is a single $O(U)\times Sp(V)$-orbit too, since there is a unique orthosymplectic $ab$-diagram $\tau$ lying on top of $(m,m,2)$, i.e.

$$\tau=\begin{array}{l} bab\cdots ab\\ bab\cdots ab\\ bab\\ a\end{array}$$

(cf. 6.3). From this we see that $Q'\subseteq\mathrm{L}^0=\{X\in\mathrm{L}(V,U)\mid\dim\operatorname{Ker}X\leq 1\}$, $Q'\subseteq\mathrm{L}''=\{X\in\mathrm{L}(V,U)\mid\operatorname{rk}\rho(X)\geq m-1\}$ and $\pi(Q')=C'$. Thus the map $\pi:N\to\bar{C}$ is smooth on $Q\cup Q'$ (11.4) and $\rho^0:Q\cup Q'\to\rho^0(Q\cup Q')$ is a principal $SO(U)$-fibration (11.5). This implies that $\overline{Q^{(1)}}$ and $\overline{Q^{(2)}}$ are normal in $Q'$ with reduced intersection there (15.3), and hence that $\rho^0(\overline{Q^{(1)}})$ and $\rho^0(\overline{Q^{(2)}})$ are normal in $\rho^0(Q')$ with reduced intersection there. Since $Q'\subseteq\mathrm{L}''-\mathrm{L}'$ the action of $\mathbb{Z}/2\mathbb{Z}$ on $\rho^0(Q')$ is trivial (proposition 11.5(b)). Using 15.3(f) we see that the $\bar\rho:\rho^0(\overline{Q^{(1)}})\to\rho(\bar{Q})=\bar{D}$ is an isomorphism on $\rho^0(Q^{(1)}\cup Q')$, hence $\bar{D}$ is normal in $D'$ and $\operatorname{Sing}(\bar{D},D')=\operatorname{Sing}(\rho^0(\overline{Q^{(1)}}),\rho^0(Q'))=\operatorname{Sing}(\overline{Q^{(1)}},Q')=\operatorname{Sing}(\overline{C^{(1)}},C')=A_{2n-1}$. The main theorem 9.2 implies now that $\bar{D}$ is normal and since $D$ admits a polarisation (10.3), it is also Cohen–Macaulay with rational singularities (10.2). qed.

**15.5** For the second implication $(b)\Rightarrow(a)$ we need some preparation. Let $m=2n$ be even, $U$ an orthogonal space of dimension $2m$ and $V$ a symplectic space of dimension $2m-2$. Consider the usual setting

$$\begin{array}{ccc} L(U,V) & \xrightarrow{\pi} & \mathfrak{sp}(V) \\ \downarrow{\scriptstyle\rho} & & \\ \mathfrak{so}(U) & & \end{array}$$

and the two conjugacy classes $D := C_{(m-1,m-1)} \subseteq \mathfrak{sp}(V)$ and $C := C_{(m,m)} \subseteq \mathfrak{so}(U)$. From the classification 6.3 we see the $\pi^{-1}(D)$ consists of three orthosymplectic orbits $P$, $P'$, $P''$ associated to the $ab$-diagrams

| | | |
|---|---|---|
| $abab \cdots aba$ | $ababa \cdots ab$ | $baba \cdots b$ |
| $abab \cdots aba,$ | $baba \cdots\cdot ba,$ | $baba \cdots b.$ |
| | $a$ | $a$ |
| | $a$ | $a$ |
| | | $a$ |
| | | $a$ |

We have (4.3) $\rho^{-1}(C)=P$, $\rho(P)=C=C^{(1)}\cup C^{(2)}$ and $\rho(P')=C':=C_{(m-1,m-1,1,1)}$, the dense orbit in $\overline{C^{(1)}}\cap\overline{C^{(2)}}$. Furthermore $\bar{P}=\pi^{-1}(\bar{D})$ is Cohen–Macaulay (11.2 proposition and remark). Thus $P\cup P'=\pi^{-1}(D)\cap\rho^{-1}(C\cup C')$ is open in $\bar{P}$ and $P=P^{(1)}\cup P^{(2)}$ with $P^{(i)}=\rho^{-1}(C^{(i)})$. In addition, from the dimension formula 7.1, we have $\operatorname{codim}_{\bar{P}} P'=1$.

LEMMA: *$P'\subseteq\overline{P^{(1)}}\cap\overline{P^{(2)}}$, $\overline{P^{(1)}}$ and $\overline{P^{(2)}}$ are smooth in $P'$ with normal crossing in $P'$ and the complement of $P\cup P'$ in $\bar{P}$ has codimension $\geq 2$.*

*Proof.* The last claim follows from the dimension formula 7.1 remarking that, for any other orthosymplectic orbit $O$ in $\bar{P}$ we have

$$\operatorname{codim}_{\bar{C}}\rho(O)+\operatorname{codim}_{\bar{D}}\pi(O)\geq 4.$$

$P^{(1)}$ and $P^{(2)}$ are $SO(U)\times Sp(V)$-orbits, hence connected, and $\overline{P^{(1)}}\cap\overline{P^{(2)}}$ is stable under $O(U)\times Sp(V)$. Since $\bar{P}$ is Cohen–Macaulay it is connected in codimension 1 (cf. remark 9.1). By the previous remark on the complement of $P\cup P'$ we must have $P'\subseteq\overline{P^{(1)}}\cap\overline{P^{(2)}}$. By lemma 12.5 the map $\pi^{-1}(D)\to D$ is a locally trivial fibration. Hence we can verify our claim on a fibre. The map $\pi$ is also equivariant under the larger group $GL(V)$ acting on $L(U,V)$ by $X\mapsto gX$ and on $\mathfrak{sp}(V)$ by $B\mapsto gBg^*$. Thus we can compute the fibre at any point on the $GL(V)$-orbit $\tilde{D}$ generated by $D$. Set for simplicity $d:=\dim V=2m-2$. Choosing a basis of $V$ we can identify $L(U, V)$ with $(U^*)^d = U^d$ and $\mathfrak{sp}(V)$ with the space $\mathrm{Sym}_d$ of symmetric $d \times d$ matrices (cf. 1.2 remark). In this setting the map

$$\pi : U^d \to \mathrm{Sym}_d$$

is given by $(u_1, \ldots, u_d) \mapsto ((u_i, u_j))_{i,j=1}^d$, the matrix of scalarproducts, and the action of $GL_d$ on $U^d$ is by linear combination of the vectors and on $\mathrm{Sym}_d$ the usual $A \mapsto gAg^t$.

Now $\tilde{D}$ is just the set of symmetric matrices of rank $d-2$. For the matrix

$$A = \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & 0 \\ \hline & 0 & \mathbf{1}_{d-2} \end{array}\right) \in \tilde{D}$$

we find

$$\pi^{-1}(A) = \{(u_1, \ldots, u_d) \mid u_3, \ldots, u_d \text{ form an orthonormal set, } \langle u_1, \mu_2\rangle \text{ isotropic in } \langle u_3, \ldots, u_d\rangle^{\perp}\},$$

and so $\pi^{-1}(A) = Q \cup Q' \cup Q''$, where $Q$, $Q'$, $Q''$ are defined by the condition $\dim\langle u_1, u_2\rangle = 2, 1, 0$ respectively. If $A = gBg^t$ for some $B \in D$ we see from the description of the orthosymplectic orbits $P$, $P'$, $P''$ that $Q = g(P \cap \pi^{-1}(B))$, $Q' = g(P' \cap \pi^{-1}(B))$ and $Q'' = g(P'' \cap \pi^{-1}(B))$. Using again lemma 12.5 we get for any $Y \in P' \cap \pi^{-1}(B)$:

$$\mathrm{Sing}(\pi^{-1}(C), P') = \mathrm{Sing}(\pi^{-1}(B), Y) = \mathrm{Sing}(\pi^{-1}(A), X),$$

$X := gY$. To study the singularity of $\pi^{-1}(A)$ we project to the last $d-2$ vectors and obtain a map $\varphi : \pi^{-1}(A) \to \mathrm{St}_{d-2}$ from $\pi^{-1}(A)$ onto the Stiefel variety $\mathrm{St}_{d-2}$ of $d-2$ orthonormal vectors in a $d-2$ dimensional orthogonal space. $\varphi$ is $O(U)$-equivariant and $\mathrm{St}_{d-2}$ is an orbit under $O(U)$, so we can apply again lemma 12.5 and reduce to the study of a fibre of $\varphi$. If $W$ is a four dimensional orthogonal space, each fibre of $\varphi$ is isomorphic to $F := \{(u_1, u_2) \in W^2 | \langle u_1, u_2\rangle \text{ isotropic}\}$, and

$$\mathrm{Sing}(\pi^{-1}(A), X) = \mathrm{Sing}(F, f)$$

where $f = (u_1, u_2)$ is any point of $F$ with $\dim\langle u_1, u_2\rangle = 1$. We can assume $f = (u_0, 0) \in F' := \{(u_1, u_2) \notin F \mid u_1 \neq 0\}$ and study $\mathrm{Sing}(F', f)$. For this consider the projection $\psi : F' \to W$, $(u_1, u_2) \mapsto u_1$. This map $\psi$ is $O(W)$-equivariant and its image is the orbit of isotropic vectors $\neq 0$ in $W$. Thus, always by lemma 12.5,

$$\mathrm{Sing}(F', f) = \mathrm{Sing}(\psi^{-1}(u_0), f).$$

Now

$$\psi^{-1}(u_0) = \{(u_0, u) \mid u \in \langle u_0 \rangle^\perp \text{ and } \langle u_0, u \rangle \text{ isotropic}\}$$
$$\cong \{u \in \langle u_0 \rangle^\perp \mid u \text{ isotropic}\}.$$

Writing $\langle u_0 \rangle^\perp = \langle u_0 \rangle \oplus \bar{W}$, an orthogonal sum with an orthogonal space $\bar{W}$ of dimension 2, we finally get that the last set is isomorphic to

$$\mathbb{A}^1 \times \{\bar{u} \in \bar{W} \mid \bar{u} \text{ isotropic}\}.$$

Of course the set of isotropic vectors in a 2-dimensional orthogonal space is a union of two lines through the origin. qed.

**15.6** *Proof of* $(b) \Rightarrow (a)$: We now assume (b) for $\mathfrak{sp}_{2k-2}$ for all $k \leq n$ and want to prove $(a)$ for $\mathfrak{so}_{4n}$. We use the same notations as before. By assumption $\bar{D}$ $(=\bar{C}_{(2n-1,2n-1)})$ is Cohen-Macaulay and so $\bar{P} = \pi^{-1}(\bar{D})$ is Cohen–Macaulay too (11.2). Thus the previous lemma 15.5 and 15.2 imply that the intersection $\overline{P^{(1)}} \cap \overline{P^{(2)}}$ is reduced, and so, by 15.2(e), the intersection $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is reduced too. Now we claim that $\overline{C'} = \overline{C^{(1)}} \cap \overline{C^{(2)}}$ is normal. This follows from theorem 9.2(ii), since the only codimension 2 degeneration of $C'$ is given by $(2n-1, 2n-3, 3, 1) < (2n-1, 2n-1, 1, 1)$ for $n > 2$ and $(3, 2, 2, 1) < (3, 3, 1, 1)$ for $n = 2$, i.e. are of "normal" type $b$ and a respectively (see Table I and 14.3).

In order to prove that $\overline{C^{(1)}}$ is normal we have to show that each regular function $f_1$ on $C^{(1)}$ extends to a regular function on $\overline{C^{(1)}}$. Since $\overline{C^{(1)}}$ is normal in $C'$ (proposition 15.1), $f_1$ extends to $C^{(1)} \cup C'$. By the normality of $\overline{C'}$ the restriction $f_1|_{C'}$ extends to a regular function $\bar{f}_1$ on $\overline{C'}$. Thus we can find a function $f_2$ on $\overline{C^{(2)}}$ with $f_2|_{\overline{C'}} = \bar{f}_1$. By construction the function $f_2$ on $C^{(2)} \cup C'$ agrees with $f_1$ in the intersection $C'$. Since the intersection $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is reduced in $C'$ (lemma 15.5) we obtain a regular function $f$ on $C^{(1)} \cup C^{(2)} \cup C'$ extending $f_1$ (15.3a). We can now apply the main theorem 9.2(i) saying that $f$ can be extended to the whole variety $\bar{C}$. In particular $f|_{\overline{C^{(1)}}}$ is the required extension of $f_1$. As in the proof of (a) $\Rightarrow$ (b) (15.4) the normality of $\overline{C^{(1)}}$ implies also that this variety is Cohen–Macaulay with rational singularities (10.2 and 10.3). qed.

**15.7** ***Remark***: The results of this section complete the proof of the claims contained in Table I concerning the singularities. In particular we see that *for any* *conjugacy class $C$ the closure $\bar{C}$ is normal in each minimal degeneration $C'$ of type different from e, and not normal in all minimal degenerations of type e.*

## 16. Normality and seminormality results

**16.1** In this section we wish to summarize the results obtained so far and add some more precise information on the geometry of orbit closures. Let us recall (cf. [AB], [T]) that a variety $X$ is said to be *seminormal*, if every homeomorphic regular map $\varphi: Y \to X$ is an isomorphism. This is a local analytic property. A variety $X$ is *normal* if and only if it is *seminormal and unibranch.*

LEMMA. *If a variety $X$ has two components $X_1$ and $X_2$, both normal and with reduced intersection, then $X$ is seminormal.*

*Proof.* Let $\varphi: Y \to X$ be a regular homeomorphic map. Then for $Y_i := \varphi^{-1}(X_i)$ the induced map $\varphi_i: Y_i \to X_i$ is homeomorphic too, hence an isomorphism. If $\psi_i$ is the inverse of $\varphi_i$ the universal property 15.3($h$) implies that there exists a $\psi: X \to Y$ such that $\psi|_{X_i} = \psi_i$. Thus $\psi = \varphi^{-1}$ is regular. qed.

**16.2** *Theorem*: *Let $C$ be an orthogonal or symplectic conjugacy class.*

(i) *$\bar{C}$ is a seminormal variety.*

(ii) *$\bar{C}$ is normal if and only if $C$ has no degenerations of type e* (3.4, Table I).

(iii) *If $C_e$ is the union of $C$ and all conjugacy classes corresponding to degenerations of type e, then any regular function on $C_e$ extends to a regular function on $\bar{C}$.*

*Proof.* We start with (iii). Let $\tilde{C}$ be the complement of the union of all conjugacy classes of codimension $\geq 4$. We cover $\tilde{C}$ with the two open sets $C_e$ and $C'_e$, where $C'_e$ is the complement in $\tilde{C}$ in the classes corresponding to degenerations of type $e$ (i.e. $C'_e = C \cup (\tilde{C} - C_e)$). $C'_e$ is a normal variety (15.7). So if $f$ is a regular function on $C_e$ its restriction to $C$ can be extended to $C'_e$, hence $f$ can be extended to a regular function on $\tilde{C}$. Thus (iii) follows from theorem 9.2 (i).

If $\bar{C}$ is normal, we must have $C_e = C$ (15.7). Conversely if $C_e = C$, any function on $C$ can be extended to $\bar{C}$ by (iii) and so $\bar{C}$ is normal, proving (ii).

For (i) we remark first of all that $C_e$ is seminormal. This follows from the previous lemma 16.1, the fact that seminormality is preserved by smooth equivalence and proposition 15.4(a). Now let $\varphi: Y \to \bar{C}$ be a regular homeomorphism. In particular $Y$ is affine ([EGA] II, 5.2.2). It follows that the induced map $\psi: \varphi^{-1}(C_e) \to C_e$ is an isomorphism, hence by (iii) the composition

$C_e \xrightarrow{\psi^{-1}} \varphi^{-1}(C_e) \to Y$ can be extended to a regular map $\bar{C} \to Y$. This map is necessarily the inverse of $\varphi$. qed.

**16.3** *Remark*: The previous results give some information on the relations between functions on the class $C$ and its closure $\bar{C}$ even in the non normal case. Let us consider in fact the normalization $\bar{\bar{C}}$ of $\bar{C}$. By construction every function on $C$ extends to $\bar{\bar{C}}$. If we look at the preimage $\tilde{C}_e$ of $C_e$ in $\bar{\bar{C}}$ we see that each non normal point is covered by two points in $\tilde{C}_e$. The universal properties proved show now that a regular function $f$ on $\tilde{C}_e$ factors through $C_e$ if and only if $f$ is constant on these fibres. This gives in principle a method to study which functions on $C$ extend to the whole $\bar{C}$.

# Part III. Special Results

## 17. Conjugacy classes under *SO*

**17.1** Recall that the $SO$-conjugacy classes which are not $O$-conjugacy classes are exactly the components of the very even classes (2.3). Up to now the only information on those are contained in the propositions 15.1 and 15.4*a*). Furthermore it is easy to see that for a very even class $C = C^{(1)} \cup C^{(2)}$ the intersection $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is the union of the closures of all codimension 2 degenerations $C_i \leq \bar{C}$, and that all these degenerations are of type $e$ (3.4). This implies the following result:

PROPOSITION. *The closure of a component $C^{(1)}$ of a very even class is normal in codimension 2 with singularities of type $A$ in each codimension 2 degeneration. In addition the intersection $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is reduced in each of these classes.*

**17.2** *Remark.* We shall see that unlike the orthogonal or symplectic classes this result does not imply the normality of $\overline{C^{(i)}}$. On the other hand the classes $C^{(i)}$ are always polarizable (10.3) and so their normalization is Cohen-Macaulay with rational singularities (proposition 10.2).

**17.3** Let $\eta$ be a very even partition. We write $\eta = (\eta_1^{v_1}, \eta_2^{v_2}, \ldots, \eta_t^{v_t})$ with $\eta_1 > \eta_2 > \cdots > \eta_t > 0$ and $v_i \in \mathbb{N}^+$ in place of

$$\eta = (\underbrace{\eta_1, \eta_1, \ldots, \eta_1}_{v_1}, \underbrace{\eta_2, \ldots, \eta_2}_{v_2}, \ldots\ldots, \underbrace{\eta_t, \ldots, \eta_t}_{v_t}).$$

Of course the numbers $\eta_i$ and $v_i$ are all even.

THEOREM. *Let* $\eta=(\eta_1^{v_1},\ldots,\eta_t^{v_t})$ *be a very even partition,* $C_\eta=C^{(1)}\cup C^{(2)}$ *the corresponding class.*

(a) *The intersection* $\overline{C^{(1)}}\cap\overline{C^{(2)}}$ *is reduced.*

(b) *If* $t=1$ *then* $\overline{C^{(i)}}$ *is normal, Cohen–Macaulay with rational singularities.*

(c) *If* $t\geq 3$ *or* $v_2\geq 4$ *then* $\overline{C^{(i)}}$ *is not normal. In fact it is branched in a class of codimension* 4.

*Proof.* (a) Let $S$ be the union of all codimension 2 classes in $\bar{C}_\eta$, $\tilde{C}^{(i)}=C^{(i)}\cup S$ and $\tilde{C}=C\cup S$. Let $f_1, f_2$ be regular functions on $\overline{C^{(1)}}$, $\overline{C^{(2)}}$ which coincide in $\overline{C^{(1)}}\cap\overline{C^{(2)}}=\bar{S}$. We have to show that the function $f$ defined by $f_1$ and $f_2$ is regular on $\bar{C}_\eta$. Since the intersection $\tilde{C}^{(1)}\cap\tilde{C}^{(2)}$ is reduced (17.1), $f$ is regular on $\tilde{C}$. But then by theorem 9.2(i) we know that $f$ is regular on $\bar{C}_\eta$.

(b) In this case we can follow exactly the same argument as in the second part of 15.6, since there is a unique class $C'$ of codimension 2 in $\bar{C}_\eta$, $\overline{C^{(1)}}\cap\overline{C^{(2)}}=\overline{C'}$ is reduced by (a) and $\overline{C'}$ is normal.

(c) We want to apply the result 13.6. We decompose the Young-diagram $\eta=\eta'+\eta''$ with $\eta'=(\eta_1^{v_1},\eta_2,\eta_2)$ and $\eta''$ the rest. Under the assumption $\eta''$ is not empty, and we can perform degenerations $\sigma'<\eta'$, $\sigma''<\eta''$ of type $e$ in such a way that the hypotheses of 13.6 are satisfied. This implies for $\sigma:=\sigma'+\sigma''$:

$$\mathrm{Sing}(\bar{C}_\eta, C_\sigma)=\mathrm{Sing}(\bar{C}_{\eta'}, C_{\sigma'})\times\mathrm{Sing}(\bar{C}_{\eta''}, C_{\sigma''}).$$

This is a non-normal singularity with four branches, hence each component $\overline{C^{(i)}}$ of $\bar{C}_\eta$ has a singularity with two branches in $C_\sigma$. qed.

Note that in these codimension 4 singularities the intersection $\overline{C^{(1)}}\cap\overline{C^{(2)}}$ is reduced.

**17.4** With the previous analysis the following problems remain unsettled.

*Problems.* (i) *If* $\eta$ *is a very even partition of the form* $(\eta_1^{v},\eta_2^{2})$ is $\overline{C_\eta^{(i)}}$ *normal?*

(ii) *If* $\eta$ *is very even, is* $\overline{C_\eta^{(i)}}$ *seminormal?*

(iii) *If* $\eta$ *is very even and* $\tilde{C}_\eta^{(i)}$ *is the union of* $C_\eta^{(i)}$ *and all its degeneration of type e or of two independent steps of type e, can every regular function on* $\tilde{C}_\eta^{(i)}$ *be extended regularly to* $\overline{C_\eta^{(i)}}$*?*

By the previous analysis we can easily show that (iii) ⇒ (ii) ⇒ (i).

The first unsettled case is $\eta=(4,4,2,2)$. In this case we can prove that $\overline{C_\eta^{(i)}}$ is unibranch and that its normalization is the quotient of some irreducible component of the corresponding variety $Z$ under the connected group $Sp_8\times SO_4\times Sp_2$.

**17.5** *Lemma. If* $Y=Y_1\cup Y_2$ *and* $Y_1\cap Y_2$ *are seminormal varieties then* $Y_1, Y_2$ *are seminormal with reduced intersection.*

*Proof.* Let $\pi_i : \bar{Y}_i \to Y_i$ be seminormalizations. The map $\pi_i |_{\pi^{-1}(Y_1 \cap Y_2)}$ is a homeomorphism, hence $\pi_i^{-1}(Y_1 \cap Y_2)$ is isomorphic to $Y_1 \cap Y_2$, since this is seminormal. Thus we can form the cofiber product of $\bar{Y}_1$ and $\bar{Y}_2$ along $Y_1 \cap Y_2$. This cofiber product maps homeomorphically to $Y$ and hence isomorphically. qed.

In our case $Y = \bar{C}_\eta = \overline{C^{(1)}} \cup \overline{C^{(2)}}$ it is hard to verify that $\overline{C^{(1)}} \cap \overline{C^{(2)}}$ is seminormal. This variety is in fact a union of seminormal varieties and we could deduce that it is seminormal if we knew that the intersections are reduced. This can be verified only on some part of the intersection.

## 18. Rational Singularities

**18.1** Let us go back to the discussion on the Cohen–Macaulay property and on rational singularities started in section 10. We saw that for a polarizable conjugacy class $C$ the normalization of $\bar{C}$ has rational singularities (10.2). It is possible that this result is true in general but we are able to indicate only some special methods by which certain non polarizable classes can be treated. Let us say by convention that an $\varepsilon$-diagram $\eta$ is polarizable if the corresponding class $C_{\varepsilon,\eta}$ admits a polarization.

**18.2** One of the first methods which were attempted for the study of singularities of orbit closures is due to Kempf [K] and used successfully by Hesselink [H4] in some special cases of conjugacy classes. His results imply in particular the following (cf. [H4] §5, criterion 2):

PROPOSITION. *If the $\varepsilon$-partition $\eta$ has at most two columns, then $\bar{C}_{\varepsilon,\eta}$ has rational singularities.*

**18.3** The second method is based on the following observation which is already a consequence of 13.4. Let $\eta, \eta'$ be $\varepsilon$-partitions such that $\eta$ is obtained from $\eta'$ by adding some rows (3.3).

*If the normalization $\tilde{\bar{C}}_{\varepsilon,\eta}$ of $\bar{C}_{\varepsilon,\eta}$ has rational singularities so does $\bar{C}_{\varepsilon,\eta'}$.*

One can use this in the situation where a non polarizable $\varepsilon$-diagram $\eta'$ becomes polarizable after adding some suitable rows.

*Example. If $\eta'$ is a symplectic diagram with the first row of even and all other rows of odd length, then $\bar{C}_{\eta'}$ has rational singularities.*

(In fact if $\eta' = (\eta_1, \eta_2, \ldots, \eta_r)$ one can show that $\eta = (\eta_1, \eta_1, \eta_2, \eta_3, \ldots, \eta_r)$ is polarizable; cf. [H2] or use the method of Kempken and Spaltenstein [Ke1].)

**18.4** There is also an inductive method which applies to a very large class of conjugacy classes and which is based on the induction lemma (sections 4 and 11). Let us go back to the basic set up (4.3):

$$\begin{array}{ccc} N:=N_{\varepsilon,\eta} & \xrightarrow{\pi} & \bar{C}_{-\varepsilon,\eta'}\subseteq \mathfrak{g}(U) \\ \downarrow{\scriptstyle\rho} & & \\ \bar{C}_{\varepsilon,\eta}\subseteq \mathfrak{g}(V) & & \end{array}$$

$\dim U=m\leq \dim V=n$. One can try to follow the same strategy as in [ADK]. Assuming that $\bar{C}_{-\varepsilon,\eta'}$ has rational singularities we want to apply Boutot's theorem (0.11) to the map $\rho$ and deduce that $\bar{C}_{\varepsilon,\eta}$ has rational singularities from a similar statement for $N$. Since $N$ is Cohen–Macaulay (11.2) to insure that $N$ has rational singularities it is enough to *find a desingularization* $\varphi: Y\to N$ *and an open set* $A\subset N$ *such that* $N$ *has rational singularities in* $A$ *and* $\operatorname{codim}_Y \varphi^{-1}(N-A)\geq 2$ (cf. [ADK] corollary 1.4).

Now $N$ is the closure of an orthosymplectic orbit $O_\tau=\rho^{-1}(C_{\varepsilon,\eta})$ (remark 11.2), where the $ab$-diagram $\tau$ is obtained from the $a$-diagram $\eta$ by filling in all the $b$'s. Furthermore we have constructed a natural desingularization (cf. lemma 7.6(a))

$$\varphi: Y:=G\times^P \mathfrak{n}_2\to N=\bar{O}_\tau,$$

$G=G(U)\times G(V)$. Let us define the open sets

$$\tilde{\mathrm{L}}:=\{X\in \mathrm{L}(V,U)\mid \pi \text{ is smooth in } X\}\quad\text{and}\quad \mathrm{L}_r:=\{X\in \mathrm{L}(V,U)\mid \operatorname{rk}\pi(X)\geq r\}$$

of L (cf. 11.4).

**18.5** *Lemma*: *If* $\bar{C}_{-\varepsilon,\eta'}$ *has rational singularities, then* $N_{\varepsilon,\eta}$ *has rational singularities in* $A:=N_{\varepsilon,\eta}\cap(\tilde{\mathrm{L}}\cup \mathrm{L}_t)$, *where*

$$t:=\begin{cases} 2m-n & \text{if } \varepsilon=-1 \\ 2m-n+1 & \text{if } \varepsilon=1 \end{cases}.$$

*Proof*. It is clear that $N\cap\tilde{\mathrm{L}}$ has rational singularities. As for $N\cap \mathrm{L}_t$, using the result of Elkik ([E1] IV. theorem 5) it is enough to show that $\pi:\mathrm{L}\to\mathfrak{g}(U)$ is flat with fibers with rational singularities. Following the analysis developed in the proof of lemma 15.5 one easily obtains the following description of a fiber

$F=\pi^{-1}(\pi(X))$ for $X\in \mathrm{L}_r$:

(a) If $U$ is symplectic then

$$F\cong\{(v_1, v_2, \dots, v_m)\in V^m \mid v_1, \dots, v_r \text{ orthonormal and } \langle v_{r+1}, \dots, v_m\rangle \text{ isotropic in } \langle v_1, \dots, v_r\rangle^\perp\}.$$

(b) If $U$ is orthogonal and so $r=2s$ is even then

$$F\cong\{(v_1, \dots, v_m)\in V^m \mid v_1, \dots, v_{2s} \text{ form a symplectic basis and } \langle v_{2s+1}, \dots, v_m\rangle \text{ isotropic in } \langle v_1, \dots, v_{2s}\rangle^\perp\}.$$

For $r\geq 2m-n$ it is easy to see that all fibers have the same dimension, hence $\pi:\mathrm{L}_{2m-n}\to \mathfrak{g}(U)$ is flat. Furthermore the singularities of $F$ are smoothly equivalent to the singularities of the variety $\bar{F}$ of $(m-r)$-tuples of vectors spanning an isotropic space in an orthogonal or symplectic space of dimension $n-r$. This variety can be studied by the method of Kempf [K] and, for $r\geq 2m-n$, it has always rational singularities except when $V$ is orthogonal and $r=2m-n$ (in which case $\bar{F}$ has two components corresponding to the two rulings of maximal isotropic subspaces, both having rational singularities with reduced intersection). qed.

**18.6** In order to verify that $Y-\varphi^{-1}(A)$ has codimension $\geq 2$ in $Y=G\times^P \mathfrak{n}_2$ we have to show that $\mathfrak{n}_2\cap(N-A)$ has codimension $\geq 2$ in the vectorspace $\mathfrak{n}_2$. We have been able to handle many special cases in this way but unfortunately the method also fails many times. (E.g. we have seen in 15.4 that $N$ may be singular in codimension 1 while $\bar{C}_{\varepsilon,\eta}$ has rational singularities.) We have not attempted to give a precise description of all the cases which can be treated this way.

**18.7** We finish with some general questions. Consider a reductive group $G$ acting on an affine variety $V$ with a dense orbit.

*Problem* 1. *Is it true that if V is normal then V has rational singularities*?

There are many examples of such varieties ([ADK], [Ke2]). Moreover, this statement is true when $G$ is a torus ([KK], chap. I §3, theorem 14).

PROPOSITION. *Let $G, V$ be as before, $U\subset G$ the unipotent radical of a parabolic subgroup of $G$ and* $R=\mathcal{O}(V)$. Then we have

(i) *The ring of invariants* $R^U$ *is finitely generated,*
(ii) *If* $V_U := \mathrm{Spec}\ R^U$ *has rational singularities so does* $V$.

Using this and the classification of the affine $SL_2$-embeddings [P] one gets a positive answer for problem 1 also in case $G = SL_2$.

One special case of the first problem is the following:

*Problem* 2. *Let* $M \subset G$ *be a closed subgroup such that the ring* $\mathcal{O}(G)^M$ *of right-invariant functions is finitely generated* (e.g. any observable subgroup). *Is* $G_M := \mathrm{Spec}\, \mathcal{O}(G)^M$ *a variety with rational singularities?*

In order to describe a class of subgroups for which we have a positive answer let us give an inductive definition.

DEFINITION. A unipotent subgroup $U \subset G$ is said to be *of type* $\leq n$ if there is a reductive subgroup $H \subset G$ and parabolic subgroup $P$ of $H$ such that
(i) $U \subseteq H$
(ii) $U$ contains the unipotent radical $U_P$ of $P$ and $U/U_P \subset P/U_P$ is of type $\leq n-1$.
If $U$ is of type $\leq n$ for some $n$ we say $U$ is *of finite type.* (Of course we consider $\{1\}$ of type 0.) Using the proposition above one obtains the following result.

PROPOSITION. *If* $M \subset G$ *is a subgroup such that its unipotent radical is of finite type, then* $G_M := \mathrm{Spec}\, \mathcal{O}(G)^M$ *is a variety with rational singularities.*

## 19. Tables

In this last paragraph we draw tables representing the nilpotent conjugacy classes in $\mathfrak{gl}_n$ for $n \leq 7$ (cf. [KP2]), $\mathfrak{so}_n$ for $n \leq 11$ and $\mathfrak{sp}_{2n}$ for $n \leq 5$. The tables are constructed (following Hesselink [H1]) as follows: Each conjugacy class is represented by a dot, its corresponding partition $\lambda$ and dimension (taken from [H1]) is indicated at its right. For any minimal degeneration of classes we draw an edge and we place the dots from top to bottom according to the containment $\supseteq$ of closures. On each edge we write the type $A_j$, $A_j \cup A_j$, $D_j$, $a_j$, $b_j$, $c_j$ or $d_j$ of the corresponding singularity (cf. 3.4 table I and section 14). We put a question mark on a dot corresponding to any class whose closure is not known to have rational singularities (cf. section 18).

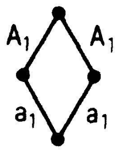
**$A_1$** $\mathfrak{gl}_2$


| λ | dim |
|---|---|
| (2) | 2 |
| (1, 1) | 0 |

**$A_2$** $\mathfrak{gl}_3$


| λ | dim |
|---|---|
| (3) | 6 |
| (2, 1) | 4 |
| (1, 1, 1) | 0 |

**$D_2 = A_1 + A_1$** $\mathfrak{so}_4$




| λ | dim |
|---|---|
| (3, 1) | 4 |
| (2, 2) | 2 |
| $(1^4)$ | 0 |

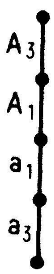
**$B_2 = C_2$** $\mathfrak{so}_5$


| λ | dim |
|---|---|
| (5) | 8 |
| (3, 1, 1) | 6 |
| (2, 2, 1) | 4 |
| $(1^5)$ | 0 |

**$C_2 = B_2$** $\mathfrak{sp}_4$


| λ | dim |
|---|---|
| (4) | 8 |
| (2, 2) | 6 |
| (2, 1, 1) | 4 |
| $(1^4)$ | 0 |

**$A_3 = D_3$** $\mathfrak{gl}_4$

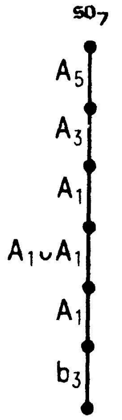



| λ | dim |
|---|---|
| (4) | 12 |
| (3, 1) | 10 |
| (2, 2) | 8 |
| (2, 1, 1) | 6 |
| $(1^4)$ | 0 |

**$D_3 = A_3$** $\mathfrak{so}_6$

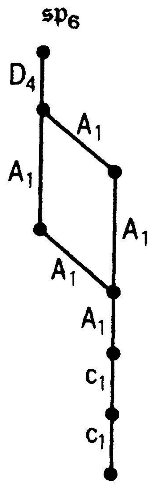

| λ | dim |
|---|---|
| (5, 1) | 12 |
| (3, 3) | 10 |
| (3, 1, 1, 1) | 8 |
| (2, 2, 1, 1) | 6 |
| $(1^6)$ | 0 |

**$B_3$**




| λ | dim |
|---|---|
| (7) | 18 |
| (5, 1, 1) | 16 |
| (3, 3, 1) | 14 |
| (3, 2, 2) | 12 |
| $(3, 1^4)$ | 10 |
| $(2, 2, 1^3)$ | 8 |
| $(1^7)$ | 0 |

**$C_3$**

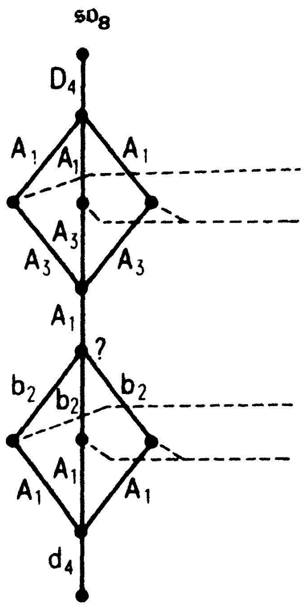



| λ | dim |
|---|---|
| (6) | 18 |
| (4, 2) | 16 |
| (4, 1, 1) | 14 |
| (3, 3) | 14 |
| (2, 2, 2) | 12 |
| (2, 2, 1, 1) | 10 |
| $(2, 1^4)$ | 6 |
| $(1^6)$ | 0 |

**$A_4$**

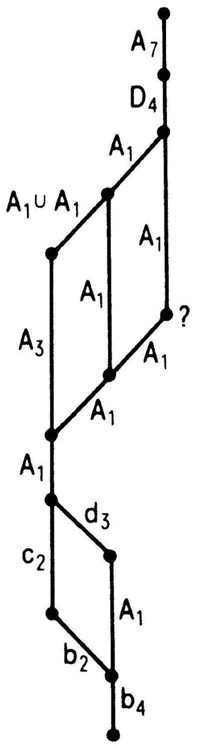

| λ | dim |
|---|---|
| (5) | 20 |
| (4, 1) | 18 |
| (3, 2) | 16 |
| (3, 1, 1) | 14 |
| (2, 2, 1) | 12 |
| $(2, 1^3)$ | 8 |
| $(1^5)$ | 0 |

**$D_4$**



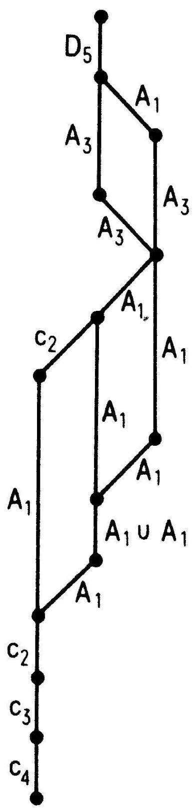

| λ | dim |
|---|---|
| (7, 1) | 24 |
| (5, 3) | 22 |
| (5, 1, 1, 1) | 20 |
| (4, 4) | 20 |
| (3, 3, 1, 1) | 18 |
| (3, 2, 2, 1) | 16 |
| $(3, 1^5)$ | 12 |
| $(2^4)$ | 12 |
| $(2, 2, 1^4)$ | 10 |
| $(1^8)$ | 0 |

**$B_4$** $\mathfrak{so}_9$



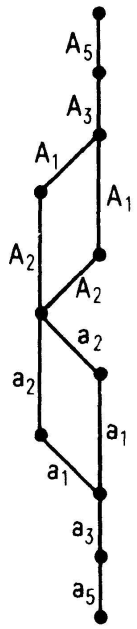

| λ | dim |
|---|---|
| (9) | 32 |
| (7, 1, 1) | 30 |
| (5, 3, 1) | 28 |
| (5, 2, 2) | 26 |
| $(5, 1^4)$ | 24 |
| (4, 4, 1) | 26 |
| (3, 3, 3) | 24 |
| $(3, 3, 1^3)$ | 22 |
| (3, 2, 2, 1, 1) | 20 |
| $(3, 1^6)$ | 14 |
| $(2^4, 1)$ | 16 |
| $(2, 2, 1^5)$ | 12 |
| $(1^9)$ | 0 |

**$C_4$** $\mathfrak{sp}_8$



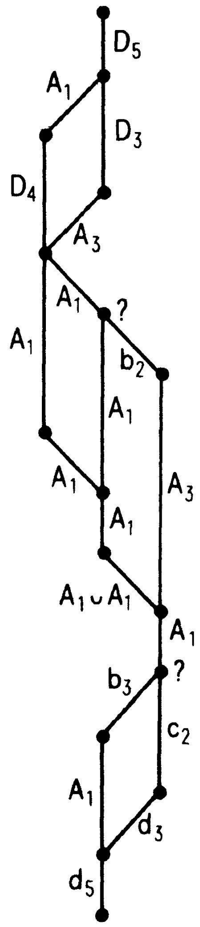

| λ | dim |
|---|---|
| (8) | 32 |
| (6, 2) | 30 |
| (6, 1, 1) | 28 |
| (4, 4) | 28 |
| (4, 2, 2) | 26 |
| (4, 2, 1, 1) | 24 |
| $(4, 1^4)$ | 20 |
| (3, 3, 2) | 24 |
| (3, 3, 1, 1) | 22 |
| (2, 2, 2, 2) | 20 |
| (2, 2, 2, 1, 1) | 18 |
| $(2, 2, 1^4)$ | 14 |
| $(2, 1^6)$ | 8 |
| $(1^8)$ | 0 |

**$A_5$** $\mathfrak{gl}_6$



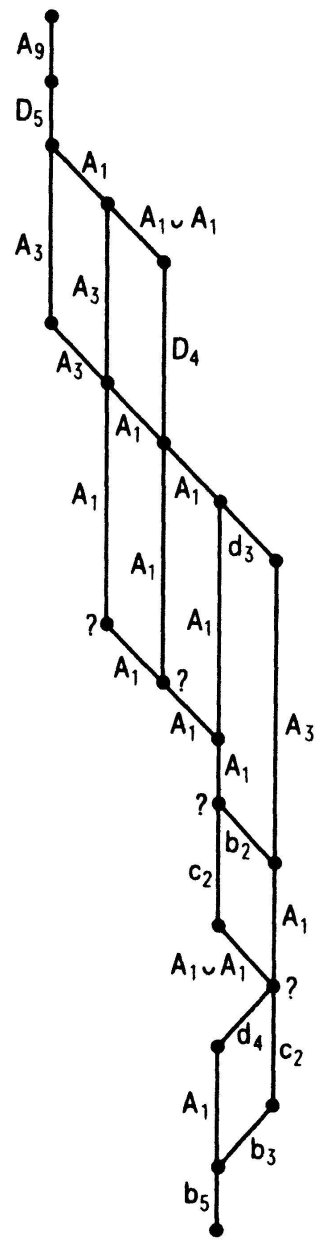

| λ | dim |
|---|---|
| (6) | 30 |
| (4, 2) | 26 |
| (4, 1, 1) | 24 |
| (3, 3) | 24 |
| (3, 2, 1) | 22 |
| (3, 1, 1, 1) | 18 |
| (2, 2, 2) | 18 |
| (2, 2, 1, 1) | 16 |
| $(2, 1^4)$ | 10 |
| $(1^6)$ | 0 |

**$D_5$** $\mathfrak{so}_{10}$



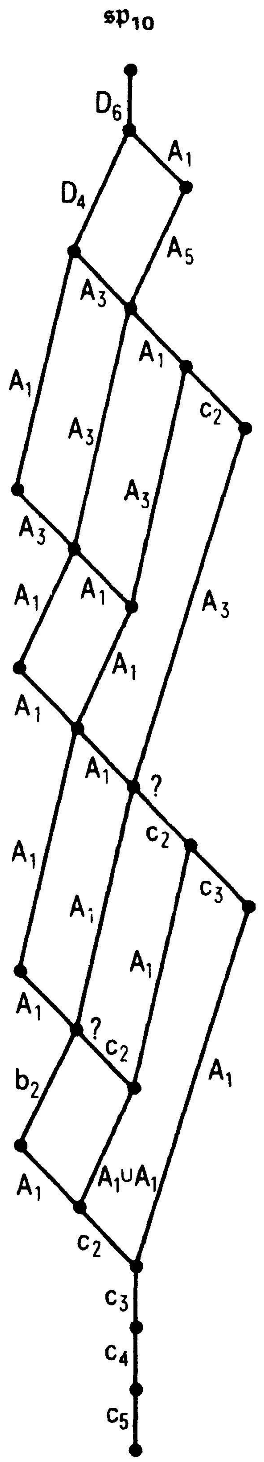

| λ | dim |
|---|---|
| (9, 1) | 40 |
| (7, 3) | 38 |
| (7, 1, 1, 1) | 36 |
| (5, 5) | 36 |
| (5, 3, 1, 1) | 34 |
| (5, 2, 2, 1) | 32 |
| $(5, 1^5)$ | 28 |
| (4, 4, 1, 1) | 32 |
| (3, 3, 3, 1) | 30 |
| (3, 3, 2, 2) | 28 |
| $(3, 3, 1^4)$ | 26 |
| $(3, 2, 2, 1^3)$ | 24 |
| $(3, 1^7)$ | 16 |
| $(2^4, 1, 1)$ | 20 |
| $(2, 2, 1^6)$ | 14 |
| $(1^{10})$ | 0 |

$\mathbf{B_5}$ $\mathfrak{so}_{11}$



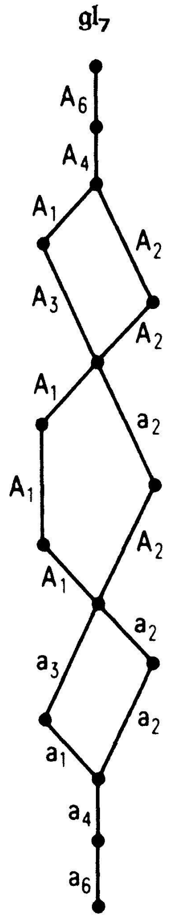

| λ | dim |
|---|---|
| (11) | 50 |
| (9, 1, 1) | 48 |
| (7, 3, 1) | 46 |
| (7, 2, 2) | 44 |
| (7, 1, 1, 1, 1) | 42 |
| (5, 5, 1) | 44 |
| (5, 3, 3) | 42 |
| (5, 3, 1, 1, 1) | 40 |
| (5, 2, 2, 1, 1) | 38 |
| $(5, 1^6)$ | 32 |
| (4, 4, 3) | 40 |
| (4, 4, 1, 1, 1) | 38 |
| (3, 3, 3, 1, 1) | 36 |
| (3, 3, 2, 2, 1) | 34 |
| $(3, 3, 1^5)$ | 30 |
| (3, 2, 2, 2, 2) | 30 |
| $(3, 2, 2, 1^4)$ | 28 |
| $(3, 1^8)$ | 18 |
| $(2, 2, 2, 2, 1^3)$ | 24 |
| $(2, 2, 1^7)$ | 16 |
| $(1^{11})$ | 0 |

$C_5$




| λ | dim |
|---|---|
| (10) | 50 |
| (8, 2) | 48 |
| (8, 1, 1) | 46 |
| (6, 4) | 46 |
| (6, 2, 2) | 44 |
| (6, 2, 1, 1) | 42 |
| (6, 1, 1, 1, 1) | 38 |
| (5, 5) | 44 |
| (4, 4, 2) | 42 |
| (4, 4, 1, 1) | 40 |
| (4, 3, 3) | 40 |
| (4, 2, 2, 2) | 38 |
| (4, 2, 2, 1, 1) | 36 |
| $(4, 2, 1^4)$ | 32 |
| $(4, 1^6)$ | 26 |
| (3, 3, 2, 2) | 36 |
| (3, 3, 2, 1, 1) | 34 |
| $(3, 3, 1^4)$ | 30 |
| (2, 2, 2, 2, 2) | 30 |
| $(2^4, 1, 1)$ | 28 |
| $(2^3, 1^4)$ | 24 |
| $(2, 2, 1^6)$ | 18 |
| $(2, 1^8)$ | 10 |
| $(1^{10})$ | 0 |

$A_6$




| λ | dim |
|---|---|
| (7) | 42 |
| (6, 1) | 40 |
| (5, 2) | 38 |
| (5, 1, 1) | 36 |
| (4, 3) | 36 |
| (4, 2, 1) | 34 |
| (3, 3, 1) | 32 |
| $(4, 1^3)$ | 30 |
| (3, 2, 2) | 30 |
| (3, 2, 1, 1) | 28 |
| (2, 2, 2, 1) | 24 |
| $(3, 1^4)$ | 22 |
| $(2, 2, 1^3)$ | 20 |
| $(2, 1^5)$ | 12 |
| $(1^7)$ | 0 |

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*Mathematisches Institut Universität Basel Rheinsprung 21 CH-4051 Basel*

*Istituto Matematico Guido Castelnuovo Università di Roma I-00100 Roma*

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