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## Proper holomorphic mappings between circular domains

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### 1. Introduction

A now classical theorem of H. Cartan [3] states that if  $f: \Omega_1 \rightarrow \Omega_2$  is a biholomorphic mapping between bounded circular domains in  $\mathbf{C}^n$  which contain the origin, and if  $f(0) = 0$ , then  $f$  is a linear mapping. Cartan's theorem was later generalized by W. Kaup [4] to biholomorphic mappings between domains in a much wider class. In this note, we prove a generalization of Cartan's theorem which allows the mapping  $f$  to be proper and non-biholomorphic. To be precise, we prove

**THEOREM 1.** *Suppose that  $f: \Omega_1 \rightarrow \Omega_2$  is a proper holomorphic mapping between bounded circular domains in  $\mathbf{C}^n$  which contain the origin, and suppose that  $f^{-1}(0) = \{0\}$ . Then the mapping  $f$  is a polynomial mapping.*

The proof of this theorem uses only elementary properties of the Bergman projection associated to a bounded circular domain. Therefore, before we attempt to prove Theorem 1, it seems worthwhile to recall some basic definitions and to list the rudimental properties of the Bergman projection.

### 2. Basic definitions and facts

A circular domain contained in  $\mathbf{C}^n$  is a connected open set such that if  $z = (z_1, \dots, z_n)$  is in the set, then for any real number  $\theta$ , the point  $e^{i\theta}z = (e^{i\theta}z_1, \dots, e^{i\theta}z_n)$  is also in the set.

The Bergman projection  $P$  associated to a bounded domain  $D$  contained in  $\mathbf{C}^n$  is the orthogonal projection of  $L^2(D)$  onto its closed subspace  $H(D)$  consisting of  $L^2$  holomorphic functions. Connected to the projection  $P$  is the Bergman kernel function  $K(w, z)$ . This kernel is determined by the property that

$$P\phi(w) = \int_D K(w, z)\phi(z) dV_z$$

for all  $\phi$  in  $L^2(D)$ . The kernel  $K(w, z)$  is defined on  $D \times D$ , and is holomorphic in  $w$  and antiholomorphic in  $z$ , and  $K(w, z) = \overline{K(z, w)}$ . Proofs of these elementary facts can be found in [2] and [5].

We shall require a lemma which is proved in [1].

**LEMMA A.** *Suppose that  $f: \Omega_1 \rightarrow \Omega_2$  is a proper holomorphic mapping between bounded domains  $\Omega_1$  and  $\Omega_2$  contained in  $\mathbf{C}^n$ . Let  $P_i$  denote the Bergman projection associated to  $\Omega_i$ ,  $i = 1, 2$ , and let  $u = \text{Det}[f']$ . The Bergman projections transform according to*

$$P_1(u \cdot (\phi \circ f)) = u \cdot ((P_2\phi) \circ f)$$

for all  $\phi$  in  $L^2(\Omega_2)$ .

The classical Remmert proper mapping theorem states that the mapping  $f$  in Lemma A is a branched cover of some finite order  $m$ . Since  $|u|^2$  is equal to the real Jacobian determinant of  $f$  viewed as a mapping of  $\mathbf{R}^{2n}$ , it follows from a simple change of variables that

$$\|u \cdot (\phi \circ f)\|_{L^2(\Omega_1)} = m^{1/2} \|\phi\|_{L^2(\Omega_2)}$$

for all  $\phi$  in  $L^2(\Omega_2)$ . This fact will be used at a crucial step in the proof of Theorem 1.

It is well known ([2, 5]) that if the mapping  $f$  in Lemma A is biholomorphic, then the Bergman kernel functions transform according to

$$u(w)K_2(f(w), f(z))\overline{u(z)} = K_1(w, z)$$

where  $K_i$  denotes the kernel function associated to  $\Omega_i$ ,  $i = 1, 2$ .

Finally, before we proceed to prove Theorem 1, it is instructive to take a glance at the original proof of Cartan's theorem. Suppose  $f: \Omega_1 \rightarrow \Omega_2$  is a biholomorphic mapping between bounded circular domains which contain the origin, and suppose  $f(0) = 0$ . For each real number  $\theta$ , the mapping  $F_\theta$  defined by

$$F_\theta(z) = f^{-1}(e^{-i\theta}f(e^{i\theta}z)) \tag{2.1}$$

is an automorphism of  $\Omega_1$  such that  $F_\theta(0) = 0$  and such that the Jacobian matrix  $F'_\theta(0)$  is equal to the identity matrix. Therefore, according to Cartan's lemma, the mapping  $F_\theta$  is the identity. Now, writing equation (2.1) out in terms of power series reveals that  $f$  must be a linear mapping. It is interesting that a very similar argument must be used at a key point in the proof of Theorem 1.

### 3. Proof of Theorem 1

Theorem 1 is a relatively simple consequence of two basic lemmas which we now list.

**LEMMA B.** *Suppose that  $K(w, z)$  is the Bergman kernel function associated to a bounded circular domain  $\Omega$ . Suppose that  $w$  and  $z$  are points in  $\Omega$  and that  $U$  is any connected neighborhood of the unit circle in  $\mathbf{C}$  such that  $tw$  and  $\bar{t}z$  are in  $\Omega$  for each  $t$  in  $U$ . Then  $K(tw, z) = K(w, \bar{t}z)$  for all  $t$  in  $U$ .*

Let  $B_R$  denote the ball in  $\mathbf{C}^n$  of radius  $R$  centered at the origin.

**LEMMA C.** *Suppose that  $\Omega$  is a bounded circular domain in  $\mathbf{C}^n$  which contains the unit ball  $B_1$ . Let  $P$  denote the Bergman projection associated to  $\Omega$ . For each multi-index  $\alpha$ , there is a function  $\phi_\alpha$  in  $C_0^\infty(B_1)$  such that  $P\phi_\alpha = z^\alpha$ . Furthermore,  $\phi_\alpha$  can be chosen so that if  $\phi_{\alpha,\varepsilon}$  is defined via  $\phi_{\alpha,\varepsilon}(z) = \varepsilon^{-2n-|\alpha|}\phi_\alpha(z/\varepsilon)$ , then  $P\phi_{\alpha,\varepsilon} = z^\alpha$  if  $0 < \varepsilon < 1$ .*

We shall now prove Theorem 1, assuming the truth of the lemmas. We may suppose, without loss of generality, that  $\Omega_1$  and  $\Omega_2$  both contain  $\bar{B}_1$ , the closure of the unit ball.

Let  $K(w, z)$  denote the Bergman kernel function associated to  $\Omega_1$ . Lemma B has as an important consequence the fact that for  $z$  close to the origin, the function  $K(w, z)$  extends to be holomorphic in  $w$  on a large neighborhood of  $\bar{\Omega}_1$ . Indeed, if  $R$  is a large positive number, then  $K(w, z)$  extends holomorphically as a function of  $w$  to  $B_R$  whenever  $z$  is in  $B_{1/R}$ . This follows from the formula

$$K(w, z) = K\left(\frac{w}{R}, Rz\right)$$

which holds for  $(w, z)$  in  $B_1 \times B_{1/R}$ , and which extends to hold for  $(w, z)$  in  $B_R \times B_{1/R}$  by analytic continuation.

Now notice that if  $\phi_{\alpha,\varepsilon}$  is the function of Lemma C associated to  $\Omega_2$ , and if  $u = \text{Det}[f']$ , then Lemma A yields that

$$u \cdot f^\alpha = u \cdot (z^\alpha \circ f) = P_1(u \cdot (\phi_{\alpha,\varepsilon} \circ f)) \quad (3.1)$$

where  $P_1$  denotes the Bergman projection associated to  $\Omega_1$ . We may rewrite (3.1)

in integral form:

$$u(w)f(w)^\alpha = \int_{\Omega_1} K(w, z)u(z)\phi_{\alpha,\varepsilon}(f(z)) dV_z. \tag{3.2}$$

Equation (3.2) contains the core of the proof of Theorem 1.

We shall now prove that  $u \cdot f^\alpha$  is a polynomial for each  $\alpha$ , including  $\alpha = (0, 0, \dots, 0)$ , by showing that the functions  $u \cdot f^\alpha$  are entire functions which satisfy an estimate of the form  $|u(w)f(w)^\alpha| \leq C|w|^q$ . Then, since  $u$  is a polynomial and  $u \cdot f^\alpha$  is a polynomial for each  $\alpha$ , and since the ring of polynomials is a unique factorization domain, we will conclude that  $f$  must be a polynomial mapping.

First, notice that if  $\varepsilon > 0$  is taken to be very small, the formula (3.2) implies that  $u \cdot f^\alpha$  extends to be holomorphic in a large neighborhood of  $\bar{\Omega}_1$ . Indeed, since  $f^{-1}(0) = \{0\}$ , the nullstellensatz implies that there are holomorphic functions  $a_{ij}(z)$  and positive integers  $k_i$  such that  $z_i^{k_i} = \sum_{j=1}^n a_{ij}(z)f_j(z)$  near  $z = 0$ . Hence, there are positive constants  $m$  and  $c$  such that  $f$  satisfies an estimate of the form  $|z|^m \leq c|f(z)|$  for all  $z$  in  $\bar{\Omega}_1$ . Therefore,

$$\text{Supp}(\phi_{\alpha,\varepsilon} \circ f) \subset \{z : |f(z)| \leq \varepsilon\} \subset \{z : |z| \leq (c\varepsilon)^{1/m}\}.$$

Hence, if  $R$  is a large positive number and if  $\varepsilon$  is chosen so that  $(c\varepsilon)^{1/m} < 1/R$ , formula (3.2) in conjunction with the fact that  $K(w, z)$  extends to  $B_R \times B_{1/R}$  reveals that  $u \cdot f^\alpha$  extends to be holomorphic on  $B_R$ . Therefore, we conclude that  $u \cdot f^\alpha$  is an entire function.

We must now show that  $|u(w)f(w)^\alpha| < C|w|^q$ . Fix a point  $w$  in  $\mathbf{C}^n$ . Pick  $\varepsilon$  so that  $(c\varepsilon)^{1/m} = |w|^{-1}$ , i.e., let  $\varepsilon = c^{-1}|w|^{-m}$ . Note that  $\text{supp}(\phi_{\alpha,\varepsilon} \circ f) \subset B_{|w|^{-1}}$  and that

$$\begin{aligned} \|u \cdot (\phi_{\alpha,\varepsilon} \circ f)\|_{L^2(\Omega_1)} &= (\text{constant}) \|\phi_{\alpha,\varepsilon}\|_{L^2(\Omega_2)} \\ &= (\text{const.}) \varepsilon^{-n-|\alpha|} \|\phi_\alpha\|_{L^2(B_1)} \\ &= (\text{const.}) |w|^{m(n+|\alpha|)}. \end{aligned}$$

We now use formula (3.2) and Lemma B to obtain that

$$\begin{aligned} |u(w)f(w)^\alpha| &= \left| \int_{\Omega_1} K\left(\frac{w}{|w|}, |w|z\right) u \cdot (\phi_{\alpha,\varepsilon} \circ f) dV_z \right| \\ &\leq (\text{const.}) \left( \text{Sup}_{\bar{B}_1 \times \bar{B}_1} |K(\zeta, \xi)| \right) \|u \cdot (\phi_{\alpha,\varepsilon} \circ f)\|_{L^2(\Omega_1)} \\ &\leq (\text{const.}) |w|^{m(n+|\alpha|)} \end{aligned}$$

where the constant is independent of  $w$ . This completes the proof of Theorem 1.

#### 4. Proofs of the lemmas

Lemma B is well known. We shall present a proof for the sake of completeness.

*Proof of Lemma B.* If  $\theta$  is a real number, the mapping  $\Phi$  defined via  $\Phi(z) = e^{i\theta}z$  is an automorphism of the domain  $\Omega$ . Therefore, the Bergman kernel function  $K(w, z)$  satisfies the identity

$$\text{Det}[\Phi'(w)]K(\Phi(w), \Phi(z))\text{Det}[\overline{\Phi'(z)}] = K(w, z).$$

If we replace  $z$  by  $e^{-i\theta}z$  in this formula we obtain

$$K(e^{i\theta}w, z) = K(w, e^{-i\theta}z).$$

Now  $K(tw, z)$  and  $K(w, \bar{t}z)$  are holomorphic functions of  $t$  on  $U$  which agree on the unit circle. Hence,  $K(tw, z) = K(w, \bar{t}z)$  for all  $t$  in  $U$ .

*Proof of Lemma C.* Let  $K(w, z)$  denote the Bergman kernel function associated to  $\Omega$ . We shall use the shorthand notation,

$$K^{\bar{\alpha}}(w, z) = \frac{\partial^\alpha}{\partial \bar{z}^\alpha} K(w, z)$$

and

$$K^\alpha(w, z) = \frac{\partial^\alpha}{\partial w^\alpha} K(w, z)$$

for multi-indices  $\alpha$ . We shall also abbreviate the operators  $\partial^\alpha/\partial z^\alpha$  and  $\partial^\alpha/\partial \bar{z}^\alpha$  by  $\partial^\alpha$  and  $\bar{\partial}^\alpha$ , respectively.

Let  $\theta$  be a radially symmetric function in  $C_0^\infty(B_1)$  such that  $\int \theta = 1$ . For small  $\varepsilon > 0$ , let  $\theta_\varepsilon(z) = \varepsilon^{-2n}\theta(z/\varepsilon)$ . Since holomorphic functions assume their average values, it follows that if  $h(w)$  is a function in  $H(\Omega)$ , then

$$\partial^\alpha h(0) = \int_\Omega \partial^\alpha h \overline{\theta_\varepsilon(z)} dV_z = \int_\Omega h(-1)^{|\alpha|} \bar{\partial}^\alpha \theta_\varepsilon dV = \int_\Omega K^\alpha(0, z) h(z) dV_z.$$

Therefore, the Bergman projection of  $(-1)^{|\alpha|} \bar{\partial}^\alpha \theta_\varepsilon$  is equal to  $K^{\bar{\alpha}}(w, 0)$  as a function of  $w$ .

Now suppose that  $w$  and  $z$  are in  $B_1$ . If we differentiate the formula  $K(tw, z) = K(w, \bar{t}z)$  with respect to  $\bar{z}$ , we obtain that

$$K^{\bar{\alpha}}(tw, z) = t^{|\alpha|} K^{\bar{\alpha}}(w, \bar{t}z). \tag{4.1}$$

The formula (4.1) holds for all  $t$  in the unit disc of  $\mathbf{C}$ . If we set  $z = 0$  in (4.1), we see that

$$K^{\bar{\alpha}}(tw, 0) = t^{|\alpha|} K^{\bar{\alpha}}(w, 0).$$

This implies that  $K^{\bar{\alpha}}(w, 0)$  is a homogeneous polynomial of degree  $|\alpha|$  in  $w$ .

We now claim that the set of homogeneous polynomials  $H^N = \{K^{\bar{\alpha}}(w, 0) : |\alpha| = N\}$  forms a basis for the set of all homogeneous polynomials of degree  $N$ . Indeed, the functions in  $H^N$  are linearly independent because if

$$\sum_{|\alpha|=N} c_{\alpha} K^{\bar{\alpha}}(w, 0) = 0$$

then  $\sum_{|\alpha|=N} \bar{c}_{\alpha} \partial^{\alpha} h(0) = 0$  for every  $h$  in  $H(\Omega)$ , which is absurd. Furthermore, the cardinality of  $H^N$  is equal to the dimension of the vector space of all homogeneous polynomials of degree  $N$ . Hence, each monomial  $z^{\alpha}$  can be written in the form

$$z^{\alpha} = \sum_{|\beta|=|\alpha|} c_{\beta} K^{\bar{\beta}}(z, 0).$$

Therefore,

$$z^{\alpha} = P\left(\sum_{|\beta|=|\alpha|} c_{\beta} (-1)^{|\beta|} \bar{\partial}^{\beta} \theta_{\varepsilon}\right).$$

If we set  $\phi_{\alpha} = \sum_{|\beta|=|\alpha|} c_{\beta} (-1)^{|\beta|} \bar{\partial}^{\beta} \theta$ , then the conditions of Lemma C are met.

*Remark.* Formula (3.2) can be used to prove the following generalization of a result of Kaup [4] on biholomorphic mappings between Reinhardt domains.

**THEOREM 2.** *Suppose  $f: \Omega_1 \rightarrow \Omega_2$  is a proper holomorphic mapping between bounded circular domains in  $\mathbf{C}^n$ . Suppose further that  $\Omega_2$  contains the origin and that the Bergman kernel function  $K(w, z)$  associated to  $\Omega_1$  is such that for each compact subset  $G$  of  $\Omega_1$ , there is an open set  $U = U(G)$  containing  $\bar{\Omega}_1$  such that  $K(w, z)$  extends to be holomorphic on  $U$  as a function of  $w$  for each  $z$  in  $G$ . Then  $f$  extends holomorphically to a neighborhood of  $\bar{\Omega}_1$ .*

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