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On the Gauss map of complete surfaces of constant mean curvature in $\mathbf{R}^3$ and $\mathbf{R}^4$

D. A. Hoffman, R. Osserman and R. Schoen*

1. Introduction

The Gauss map for complete minimal surfaces in $\mathbf{R}^n$ has been the object of extensive study over the past twenty years. The gist of the results is that if the set of tangent planes to a complete minimal surface S is sufficiently restricted, then S must be a plane. The first such result, conjectured by Nirenberg, was that if a complete minimal surface S in $\mathbf{R}^3$ is not a plane, then its normals must be everywhere dense on the unit sphere (Osserman [16]). This was later extended by Chern [4] to minimal surfaces in $\mathbf{R}^4$, using the structure of the Grassmannian of oriented 2-planes in $\mathbf{R}^4$ as a product of 2-spheres, and in a somewhat different form, to surfaces in $\mathbf{R}^n$. Chern's results were further refined in Osserman [17] and Chern–Osserman [6]. In a surprising recent development, Xavier [22] obtained a much stronger version of the theorem in $\mathbf{R}^3$. He showed that the normals to a complete non-planar minimal surface S in $\mathbf{R}^3$ can omit only a finite number of points.† His method carries over also to surfaces in $\mathbf{R}^4$, using the product decomposition referred to above (Chen [3]).

It is natural to ask whether analogous results hold for complete surfaces of constant mean curvature in $\mathbf{R}^3$, or more generally for surfaces with parallel mean curvature vector in $\mathbf{R}^n$. By way of background, we note that Bernstein ([1], pp. 242–244) proved that there are no complete graphs of constant mean curvature in $\mathbf{R}^3$. In fact he gave a specific upper bound to the radius R of the largest disk over which there can lie a surface of constant mean curvature $H>0$. Heinz [10] gave a very simple proof of the sharp bound $R \le 1/H$. A more intrinsic form of this result was given by Chern ([5], p. 82) who considered hypersurfaces of constant mean curvature in $\mathbf{R}^n$ with the property that the Gauss map lies in a closed hemisphere. Chern ([5], p. 83) also raised the question whether the

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† Specifically, Xavier shows that no more than six points may be omitted. The last step in his proof is not correct as it stands, but a somewhat different estimation leads to the desired result.

property of the Gauss map being everywhere dense holds also for complete surfaces with non-zero constant mean curvature. However, no such result can hold, since there are complete surfaces of revolution with constant mean curvature (the "unduloids") whose Gauss maps lie in an arbitrarily narrow strip about a great circle on the sphere. On the other hand, in a conversation with one of the authors, M. do Carmo suggested that on the basis of the known examples one might well conjecture that some limitation on the Gauss map would still suffice to prove that the surface must be a plane–for example, the assumption that the image under the Gauss map lie in a sufficiently small neighborhood of a point. We are grateful to do Carmo for his comments, which provided the impetus for the results presented here.

Our principal goal is to prove the following two theorems.

THEOREM 1. *Let S be a complete oriented surface of constant mean curvature in $\mathbf{R}^3$. If the image of S under the Gauss map lies in some open hemisphere, then S is a plane. If the image under the Gauss map lies in a closed hemisphere, then S is a plane or a right circular cylinder.*

THEOREM 2. *Let S be a complete oriented surface in $\mathbf{R}^4$ whose mean curvature vector is parallel and non-zero. Let the Grassmannian of oriented two-planes in $\mathbf{R}^4$ be represented as the product of spheres $S_1 \times S_2$. Then the image of S under the generalized Gauss map has the property that neither of its projections onto S_1 or S_2 can lie in an open hemisphere; if either projection lies in a closed hemisphere, then S is a right circular cylinder in some $\mathbf{R}^3 \subset \mathbf{R}^4$, or a product of circles.*

We note that if a surface S of constant mean curvature in a sphere $S^3(r)$ is considered as lying in $\mathbf{R}^4$ under the natural embedding of $S^3(r)$ in $\mathbf{R}^4$, then S has mean curvature vector in $\mathbf{R}^4$ that is parallel and non-zero. An immediate consequence of Theorem 2 is therefore:

COROLLARY. *Let S be a complete surface in $S^3(r)$ such that, when considered as lying in $\mathbf{R}^4$, its Gauss map has at least one projection lying in a closed hemisphere. If S has constant mean curvature, then it is a product of circles. In particular, if S is minimal in $S^3(r)$, then it is a Clifford torus.*

We make the following observations concerning the above theorems.

First, we note that a theorem of Yau ([23], p. 358) states that every surface of parallel mean curvature in $\mathbf{R}^4$ is in fact a surface of constant mean curvature either in some affine 3-space in $\mathbf{R}^4$ or else in some 3-sphere $S^3(r)$. In view of

Yau's theorem, Theorem 2 is equivalent to the combination of Theorem 1 and the Corollary to Theorem 2.

Second, the example of the unduloid referred to above shows that Theorem 1 is sharp, in the sense that the conclusion fails if one allows the Gauss image to lie in any open set containing a closed hemisphere. Thus, for complete surfaces of arbitrary constant mean curvature, a much greater restriction is needed on the Gauss map to force it to be a plane than in the special case of minimal surfaces.

For surfaces in $\mathbf{R}^4$ the situation is in a sense reversed. There one does not have a stronger version of Theorem 2 for minimal surfaces, and one must in fact specifically exclude zero mean curvature for the theorem to be valid. In the case of a minimal surface, no restriction on a single projection of the Gauss map can force the surface to be a plane, since one of the projections may even be constant, as happens for those minimal surfaces that correspond to holomorphic curves with respect to some complex structure on $\mathbf{R}^4$. One can also construct nonholomorphic complete minimal immersions of the disk into $\mathbf{R}^4$, where one of the projections of the Gauss map lies in an arbitrarily small neighborhood of a point, and neither projection is constant. The details are given in Section 4 below.

The proof of Theorems 1 and 2 are given in Sections 2 and 3, respectively. Before proceeding with the details, we make a few general comments concerning the proofs.

We note first that a vital tool in the case of minimal surfaces is the fact that the Gauss map is anti-holomorphic. The corresponding property of surfaces of parallel mean curvature is that the Gauss map is harmonic (Ruh–Vilms [19]). We do not actually make explicit use of harmonic maps in this paper, although they play an important background role. In particular, the Liouville theorem for harmonic maps due to Hildebrandt, Jost and Widman [11] led us to conjecture that the hemisphere was the correct domain to consider in the formulation of Theorem 1.

The main tool that we do use in the proof of Theorem 1 is the well-known equation

$$\Delta\nu + \|d\nu\|^2\,\nu = 0 \tag{1.1}$$

for the unit normal ν to a surface S of constant mean curvature in $\mathbf{R}^3$, where the coefficient $\|d\nu\|^2$ may be viewed equivalently as the square norm of the Weingarten map $d\nu$ or of the second fundamental form of S. One interpretation of equation (1.1) is that the Gauss map is harmonic. (See Remark 3 at the end of Section 3.) The classical theorem of Rodrigues shows that the mean curvature of a surface in $\mathbf{R}^3$ vanishes at a point if and only if the Gauss map is anticonformal at the point. Thus every surface of constant non-zero mean curvature in $\mathbf{R}^3$ induces a harmonic map into the sphere which is nowhere (weakly) anticonformal. A recent result of Kenmotsu [15] gives a remarkable converse to this fact: *let R be a simply-connected Riemann surface, and let h be any harmonic map of R into the unit sphere which is nowhere (weakly) anticonformal. Then for any constant $H \neq 0$, there exists a surface S in $\mathbf{R}^3$ of constant mean curvature H, and a conformal map $f: R \to S$ such that $h = g \circ f$, where g is the Gauss map of S.*

Kenmotsu shows further that if the map h is given in the form $w = h(z)$, where z is a local coordinate on R and w is a complex coordinate on the image sphere obtained by stereographic projection, then the metric on S is given by

$$ds^2 = \left[\frac{2}{H}\frac{1}{1+|h|^2}\left|\frac{\partial h}{\partial \bar{z}}\right|\right]^2 |dz|^2. \tag{1.2}$$

Taking into account the known results for minimal surfaces, it follows that Theorem 1 is equivalent to a result that may be formulated purely in terms of harmonic maps into the sphere:

THEOREM 1A. *Let $h: R \to S^2$ be a harmonic map of a Riemann surface R into the unit sphere. Suppose that h is nowhere weakly anticonformal. If the image $h(R)$ lies in an open hemisphere, then the metric* (1.2) *cannot be complete.* (*Equivalently, assuming that the open hemisphere corresponds to $|h(z)| < 1$, the metric*

$$ds^2 = \left|\frac{\partial h}{\partial \bar{z}}\right|^2 |dz|^2 \tag{1.3}$$

cannot be complete.) *If the image $h(R)$ lies in a closed hemisphere, then the metric* (1.2) (*or* (1.3)) *cannot be complete unless $h(R)$ is a great circle.*

For a surface S in $\mathbf{R}^4$, the theorem of Ruh–Vilms states that S has parallel mean curvature if and only if the Gauss map g is harmonic. But g is harmonic if and only if both projections g_1, g_2 are harmonic. In that case, one cannot assign g_1 and g_2 arbitrarily, as in the Kenmotsu theorem, but rather one has an additional constraint, derived in Hoffman–Osserman [14]. That constraint plays a key role in the proof of Theorem 2.

The other main ingredients in the proof of Theorem 2 are a result of Fischer-Colbrie and Schoen [9] concerning complete conformal metrics on the unit disk (also used in the proof of Theorem 1), a formula for the Gauss curvature of a surface in $\mathbf{R}^4$ due to Blaschke [2], a computation for harmonic maps of surfaces due to Schoen and Yau [20], and a result of Hoffman [12] on flat surfaces of parallel mean curvature in $\mathbf{R}^4$.

2. Proof of Theorem 1

Let S be a complete surface of constant mean curvature in $\mathbf{R}^3$, and let $\hat{S}$ be the universal covering surface of S. If the image of S under the Gauss map lies in a hemisphere, then the same is true of $\hat{S}$. By the uniformization theorem, there are just three possibilities:

Case 1. $\hat{S}$ is conformally a 2-sphere.

This case is clearly impossible, since the image of a sphere under the Gauss map contains every point of the sphere.

Case 2. $\hat{S}$ is conformally the plane.

In this case we get a map ν of the plane into the unit sphere, satisfying equation (1.1). We may assume that the hemisphere containing the image is the lower hemisphere, in which case $-1 \leq \nu_3 \leq 0$. Since by (1.1) we have

$$\Delta \nu_3 = -\|d\nu\|^2 \, \nu_3, \tag{2.1}$$

it follows that ν_3 is a bounded subharmonic function. But when $\hat{S}$ is conformally the plane we must have ν_3 constant. Thus the image under the Gauss map lies on a circle. One may then use any of a number of elementary arguments to conclude that S is either a plane or a right circular cylinder.

For example, one can observe that if ν_3 is a non-zero constant, then by equation (2.1) $\|d\nu\|^2 \equiv 0$, so that $\nu \equiv$ constant, and hence S is a plane. If $\nu_3 \equiv 0$, then the vertical vector e_3 lies in the tangent space to S at every point. This implies that through each point of S there passes a line parallel to e_3 and lying entirely in S. Thus S is a cylinder over a plane curve, and since S has constant mean curvature, that plane curve is a circle or a line implying that S is a plane or a right-circular cylinder.

Case 3. $\hat{S}$ is the unit disk.

We wish to show that this case cannot arise. We again use equation (2.1) and note that

$$\|d\nu\|^2 = \|A\|^2 = 4H^2 - 2K, \tag{2.2}$$

where we may consider A as the Weingarten map (or the second fundamental form of S), H is the mean curvature and K the Gauss curvature of S. Thus ν_3 satisfies the equation

$$\Delta\nu_3 - 2K\nu_3 + 4H^2\nu_3 = 0. \tag{2.3}$$

Again we may assume that the image under the Gauss map lies in the lower hemisphere, so that $-1 \le \nu_3 \le 0$, and by (2.1), ν_3 is subharmonic. By the maximum principle, if $\nu_3 = 0$ at any interior point, then $\nu_3 \equiv 0$. But as noted in Case 2, that would mean that S is a plane or a cylinder, forcing $\hat{S}$ to be the plane, and not a disk. Thus we conclude that ν_3 is strictly negative. But a result of Fischer-Colbrie and Schoen ([9], Corollary 3 on p. 205) states that when K is the Gauss curvature of a complete conformal metric on the unit disk, there can be no positive (or equivalently, negative) solution of equation (2.3). This completes the proof of Theorem 1.

3. Surfaces in $\mathbf{R}^4$; proof of Theorem 2

We shall make use of the following facts concerning the Grassmannian and the Gauss map for surfaces in $\mathbf{R}^4$. (See for example Hoffman–Osserman ([13], §§1, 2.))

The Grassmannian of oriented 2-planes in $\mathbf{R}^4$ may be identified with the product $S^2 \times S^2$ where each factor is the standard 2-sphere of radius $1/\sqrt{2}$. The Gauss map g of an oriented surface S in $\mathbf{R}^4$ factors into a pair of maps g_1, g_2, where g_k is a projection of g onto a factor S^2. Let f_k be the complex-valued map produced by composing g_k with stereographic projection. By the theorem of Ruh–Vilms [19], a surface S has parallel mean curvature vector if and only if the Gauss map g is harmonic, which in turn is equivalent to each of the factors g_k being harmonic. The harmonicity of the map g_k may be expressed by the equation

$$\Delta\nu_k + 2\,\|d\nu_k\|^2\,\nu_k = 0, \tag{3.1}$$

where ν_k represents the position vector on the sphere S^2, considered as a standard sphere of radius $1/\sqrt{2}$ in $\mathbf{R}^3$. (The factor 2 in this equation, missing in (1.1), appears because the radius of the sphere in this case is $1/\sqrt{2}$ rather than 1. See Remark 3 at the end of this section.)

In order to prove Theorem 2, we shall make use of a number of facts concerning the maps g_1, g_2, and the corresponding functions f_1, f_2. It will be helpful to introduce the derived functions

$$F_k = \frac{(f_k)_{\bar{z}}}{1+|f_k|^2}, \qquad \hat{F}_k = \frac{(f_k)_z}{1+|f_k|^2}, \tag{3.2}$$

where z is a local isothermal parameter on S. We note that if f_k is composed with a linear fractional transformation corresponding to a rotation of the sphere, then the quantities $|F_k|$, $|\hat{F}_k|$ remain invariant. Thus there are no singularities at points where $f_k = \infty$.

The facts we need are the following:

1. If $e(g)$ denotes the energy density of the Gauss map g, then

$$e(g) = e_1 + e_2 \tag{3.3}$$

where e_k denotes the energy density of g_k.

2. In terms of f_k,

$$\lambda^2 e_k = 2[|F_k|^2 + |\hat{F}_k|^2] \tag{3.4}$$

where the metric on S is given by $ds^2 = \lambda^2 |dz|^2$.

3. If J_k denotes the Jacobian of the map g_k, then

$$\lambda^2 J_k = 2[|\hat{F}_k|^2 - |F_k|^2]. \tag{3.5}$$

4. The Gauss curvature K of the surface S is given by

$$K = J_1 + J_2. \tag{3.6}$$

(Blaschke [2], §4; see also Weiner [21], and Hoffman–Osserman [14].)

5. (Hoffman–Osserman [14])

$$|F_1| \equiv |F_2|. \tag{3.7}$$

6. Let H be the mean curvature vector of S. Then

$$H = 0 \Leftrightarrow F_1 = 0 \quad \text{and} \quad F_2 = 0 \tag{3.8}$$

(Hoffman–Osserman [14]).

7. Suppose the map g_k is harmonic. Then at any point where $F_k \neq 0$,

$$\Delta \log |F_k| = 2J_k, \tag{3.9}$$

Where Δ is the Laplace–Beltrami operator on S. (See Schoen–Yau [20], §1, (17), where $\bar{u}_\theta$ is equal in our notation to $\sqrt{2}\, F/\lambda$. See also Hoffman–Osserman [14].)

We now combine these facts to prove Theorem 2. Since S has parallel mean curvature vector $H \neq 0$, each g_k is a harmonic map, and by virtue of (3.8) we may apply (3.9) for $k = 1, 2$. Combining (3.7) with (3.9) yields

$$J_1 = J_2. \tag{3.10}$$

Another application of (3.7), together with (3.5) and (3.10), gives

$$|\hat{F}_1| = |\hat{F}_2|. \tag{3.11}$$

Inserting (3.7) and (3.11) into (3.4) gives

$$e_1 = e_2. \tag{3.12}$$

Finally, comparing (3.12) with (3.3), we find

$$e = 2e_1 = 2e_2. \tag{3.13}$$

Thus the equations (3.1) take the form

$$\Delta \nu_k + e(g)\nu_k = 0, \qquad k = 1, 2. \tag{3.14}$$

We now invoke our hypothesis that either ν_1 for ν_2 lies in a hemisphere. Say that this holds for ν_1. Then for some fixed unit vector c, the function

$$\mu = c \cdot \nu_1 \tag{3.15}$$

satisfies

$$\mu \leq 0. \tag{3.16}$$

But by (3.14),

$$\Delta \mu + e(g)\mu = 0. \tag{3.17}$$

Since $e(g) \geq 0$, it follows from (3.16) and (3.17) that μ is a subharmonic function. We assert that μ must be constant.

Let $\tilde{S}$ be the universal covering surface of S, and let $\tilde{\mu}$ be the function μ lifted to $\tilde{S}$. Then $\tilde{\mu}$ is a continuous subharmonic function, bounded above.

Case 1. $\tilde{S}$ is conformally a sphere. Then $\tilde{\mu}$ attains a maximum and hence is constant.

Case 2. $\tilde{S}$ is conformally the plane. Then since $\tilde{\mu}$ is subharmonic and bounded above, it is constant.

Case 3. $\tilde{S}$ is conformally the unit disk. In this case we observe that the coefficient $e(g)$ in (3.17) may also be expressed in terms of the second fundamental form B of the surface S as

$$\begin{aligned} e(g) &= \|B\|^2 = \sum_{i,j=1}^{2} |B_{ij}|^2 \\ &= |B_{11} + B_{22}|^2 + 2(|B_{12}|^2 - B_{11} \cdot B_{22}) \\ &= 4\,|H|^2 - 2K. \end{aligned}$$

Hence (3.17) takes the form

$$\Delta\tilde{\mu} - 2K\tilde{\mu} + 4\,|H|^2\,\tilde{\mu} = 0. \tag{3.18}$$

As in the proof of Theorem 1, we know by the theorem of Fischer-Colbrie and Schoen ([6], p. 205) that (3.18) has no strictly negative solutions for a complete conformal metric on the unit disk. Since $\tilde{\mu} \leq 0$, it follows that $\tilde{\mu} = 0$ somewhere. But then the maximum principle for subharmonic functions implies that $\tilde{\mu} \equiv 0$.

We thus conclude that in all cases, μ must be constant. By the definition (3.15) of μ, ν_1 lies on a circle on the sphere S^2. But then $J_1 \equiv 0$, and by (3.10) also $J_2 \equiv 0$. It then follows from (3.6) that $K \equiv 0$ on S. Finally, a theorem of Hoffman ([12], Theorem 3.1) guarantees that a surface of parallel mean curvature in $\mathbf{R}^4$ with vanishing Gauss curvature lies on a right circular cylinder in $\mathbf{R}^3 \subset \mathbf{R}^4$ or else on a product of circles. This completes the proof of the theorem.

Remark 1. It follows from the theorem that Cases 1 and 3 do not in fact occur; that is, under the hypotheses of Theorem 2, the universal covering surface of S is conformally the plane.

Remark 2. One can give a somewhat different proof of the theorem that does not require equation (3.9). In its place one may use a second equation of Blaschke ([2], §4) complementing (3.6):

$$K_N = J_1 - J_2, \tag{3.19}$$

where K_N is the curvature of the normal bundle. (See also Weiner [21], and Hoffman–Osserman [14], for alternative proofs of (3.19).) If S has non-zero parallel mean curvature vector H, then setting $e_3 = H/|H|$, and e_4 the unit vector orthogonal to H in the normal plane, we find that the normal bundle is flat, and hence $K_N \equiv 0$. Thus equation (3.10) is an immediate consequence of (3.19). The remainder of the proof is as before.

Remark 3. Equations (1.1) and (3.1) are both special cases of the equation

$$\Delta X = -\frac{1}{r^2}|dX|^2 X \tag{3.20}$$

characterizing harmonic maps of a surface S into a sphere of radius r in $\mathbf{R}^n$. Namely, if

$$X : S \to \mathbf{R}^n$$

is a map whose image lies in a submanifold N of $\mathbf{R}^n$, then the map $X : S \to N$ is harmonic if and only if at each point p of S the n-vector ΔX is normal to N at $X(p)$. (See [8], p. 9, and (4.13), p. 16.) In our case, when $N = S^{n-1}(r)$, the condition becomes

$$\Delta X = \lambda X \tag{3.21}$$

for some function λ on S. Thus (3.20) implies that the map $X : S \to S^{n-1}(r)$ is harmonic. Conversely, if $z = x + iy$ is a local conformal parameter on S, then since $X \cdot X \equiv r^2$, we have

$$X_z \cdot X \equiv 0$$

and

$$X_{z\bar{z}} \cdot X + X_z \cdot X_{\bar{z}} \equiv 0.$$

But

$$X_z \cdot X_{\bar{z}} = \tfrac{1}{4}(|X_x|^2 + |X_y|^2).$$

Hence

$$\Delta X \cdot X = 4X_{z\bar{z}} \cdot X = -4X_z \cdot X_{\bar{z}} = -|dX|^2. \tag{3.22}$$

But if (3.21) holds, then $\Delta X \cdot X = \lambda\, |X|^2 = \lambda r^2$, and by (3.22), $\lambda = -|dX|^2/r^2$, so that (3.20) follows.

4. A Counterexample

We present here the example mentioned in the introduction, showing that the hypothesis in Theorem 2 that the mean curvature is different from zero, cannot be dropped.

PROPOSITION 4.1. *Given any $\varepsilon > 0$, there exists a complete regular minimal surface in $\mathbf{R}^4$ whose Gauss map has the property that each projection f_1, f_2 is a non-constant holomorphic map, and the image under f_1 lies in a disk of radius ε.*

Proof: Let $w = g(z)$ be the map of the unit disk $|z| < 1$ onto the universal covering surface of the ζ-plane minus the points $\zeta = 0$ and $\zeta = 1$. Let

$$\begin{aligned} \psi(z) &= g'(z) \\ f_1(z) &= \varepsilon z \\ f_2(z) &= \frac{1}{g(z)[g(z)-1]}. \end{aligned} \tag{4.1}$$

Then $\psi(z) \neq 0$, and the surface S defined by

$$X(z) = \operatorname{Re} \int_0^z \psi(1 + f_1 f_2,\, i(1 - f_1 f_2),\, f_1 - f_2,\, i(f_1 + f_2))\, dz \tag{4.2}$$

is a regular minimal surface in $\mathbf{R}^4$, with f_1 and f_2 the projections of the Gauss map. (See [13], §3.)

Since $X_z = \dfrac{\psi}{2}(1 + f_1 f_2,\, i(1 - f_1 f_2),\, f_1 - f_2,\, i(f_1 + f_2))$,

$$|X_z|^2 = \tfrac{1}{2}|\psi|^2(1 + |f_1|^2)(1 + |f_2|^2).$$

The metric on the surface S is given by $\lambda^2 |dz|^2$ where $\frac{1}{2}\lambda^2(z) = |X_z|^2$. Hence the length of any curve C on S is given by

$$L = \int_\gamma \lambda |dz| = \int_\gamma \sqrt{[(1+|f_1(z)|^2)(1+|f_2(z)|^2)]}\, |\psi(z)|\, |dz|$$

where γ is the path in $|z|<1$ corresponding to C.

Let Γ be the image of γ under the map g. Then

$$\begin{aligned} L &\geq \int_\gamma \sqrt{(1+|f_2(z)|^2)}\, |\psi(z)|\, |dz| \\ &= \int_\Gamma \sqrt{\left(1+\frac{1}{|\zeta|^2\, |\zeta-1|^2}\right)}\, |d\zeta| \\ &\geq \max\left\{\int_\Gamma \frac{|d\zeta|}{|\zeta|\, |\zeta-1|}, \int_\Gamma |d\zeta|\right\}. \end{aligned}$$

Now C is a divergent path on S if and only if γ tends to the boundary of $|z|<1$. There are two cases to consider. If Γ has infinite length, then by (4.3) $L=\infty$. On the other hand, if Γ has finite length, then it tends to a point ζ_0. Since γ tends to $|z|=1$, it follows that $\zeta_0=0$ or $\zeta_0=1$. But then again $L=\infty$ by (4.3). Hence every divergent path on S has infinite length, and S is complete.

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University of Massachusetts
Department of Mathematics
GRL
Amherst, MA 01003

Stanford University
Department of Mathematics
Stanford, CA 94305

University of California at Berkeley
Department of Mathematics
Berkeley, CA 94720

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