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On the existence of extremal Teichmüller mappings

LI ZHONG

1. Introduction

Let μ be an orientation preserving homeomorphism of the unit circumference $|z| = 1$ onto $|w| = 1$ which admits a quasiconformal extension into the disk $D: |z| < 1$. Let K_0 be the maximal dilatation of an extremal q.c. extension of μ into D , while H denotes the dilatation of μ , i.e. the infimum of the dilatations of all the extensions of μ into arbitrarily small annuli $1 - \varepsilon < |z| \leq 1$, $\varepsilon > 0$. If $H < K_0$, then there exists a unique extremal Teichmüller mapping associated with a quadratic differential of finite norm ([9], [10]). To estimate H one can introduce a local dilatation $H(\zeta)$ of μ at a point $\zeta \in \partial D$: It is the infimum of the maximal dilatations of all extensions of μ into arbitrarily small neighborhoods of ζ with respect to \bar{D} (see [5]). R. Fehlmann recently showed that $H = \max_{|\zeta|=1} H(\zeta)$. In the present paper we use this result to give an upper estimate of H which, together with a lower estimate of K_0 , allows the conclusion $H < K_0$. (So far, in the literature, this has been done only in the case $H = 1$, see [9] and [10].)

In order to carry out the program, two quantities are introduced in sections (2) and (3) respectively. The first one is motivated by the maximal dilatation $q(\rho)$ of the extremal selfmapping of the upper halfplane with boundary values $\mu(x) = x$ for $x \geq 0$ and $\mu(x) = \rho x$, $\rho \geq 1$ for $x < 0$, which is easily computed from [7]:

$$q(\rho) = 1 + \frac{1}{2\pi^2} \log^2 \rho + \frac{1}{\pi} (\log \rho) \sqrt{1 + \frac{1}{4\pi^2} \log^2 \rho}.$$

The second one (section (3)) is obtained in the usual way (see [1]) as the ratio of the moduli of certain quadrilaterals in $|z| < 1$ and $|w| < 1$ respectively.

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2. An estimate of the local dilatation of μ

In what follows, we will write the given boundary correspondence μ in a real form: $\vartheta = h_\mu(\theta)$, where $\vartheta = \arg w$ and $\theta = \arg z$: namely $h_\mu(\theta) := \arg \mu(e^{i\theta})$. Then $\vartheta = h_\mu(\theta)$ is a ρ -quasisymmetric homeomorphism of \mathbf{R} onto itself. Assume μ is absolutely continuous and satisfies the condition:

$$m \leq h'_\mu(\theta) \leq M, \quad \text{almost everywhere,} \quad (1)$$

where m and M are constant, $0 < m < M < \infty$. We introduce the functions:

$$q(\lambda, \omega) := \frac{1}{2}(\omega + \omega^{-1}) + \frac{\omega}{2\pi^2} \log^2 \lambda + \frac{\omega}{2} \sqrt{\left[(\omega - 1)^2 + \frac{1}{\pi^2} \log^2 \lambda \right] \left[(\omega + 1)^2 + \frac{1}{\pi^2} \log^2 \lambda \right]}, \quad (2)$$

$$\lambda(\theta) := \lim_{\varepsilon \rightarrow 0} \sup_{0 < |t| < \varepsilon} \left\{ \frac{h_\mu(\theta + t) - h_\mu(\theta)}{h_\mu(\theta) - h_\mu(\theta - t)} \right\}, \quad (3)$$

and

$$\omega(\theta) := \lim_{\varepsilon \rightarrow 0^+} \max \left(\frac{M(\theta, \theta + \varepsilon)}{m(\theta, \theta + \varepsilon)}, \frac{M(\theta - \varepsilon, \theta)}{m(\theta - \varepsilon, \theta)} \right) \quad (4)$$

where

$$m(x, y) := \operatorname{ess\,inf}_{(x,y)} \{h'_\mu\}, \quad M(x, y) := \operatorname{ess\,sup}_{(x,y)} \{h'_\mu\}.$$

Because the function h_μ is ρ -quasisymmetric, there always exists the limit (3). We denote by $H(\theta)$ the local dilatation of μ at the point $e^{i\theta}$. We have

THEOREM 1. *Under the above assumption we have*

$$H(\theta) \leq q(\lambda(\theta), \omega(\theta)), \quad \text{for all } \theta. \quad (5)$$

Proof. Without loss of generality, we only look at $\theta = 0$ and assume that $h_\mu(0) = 0$. We are going to show the estimate (5) for the point $\theta = 0$. Applying the mapping $\zeta = \log z$, we pass to the strip Σ :

$$\Sigma := \{\zeta = \xi + i\eta : 0 < \eta < \pi\} \quad (6)$$

which is the image of the upper half-plane U . The boundary correspondence h_μ

of U becomes a boundary correspondence of Σ as follows:

$$\begin{cases} \xi \rightarrow \log(h_\mu(e^\xi)), & \text{for the lower boundary,} \\ \xi + \pi i \rightarrow \log(-h_\mu(-e^\xi)) + \pi i, & \text{for the upper boundary.} \end{cases} \quad (7)$$

We construct a function $f(\zeta)$ in Σ with the boundary correspondence (7):

$$f(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \log(h_\mu(e^\xi)) + \frac{\eta}{\pi} \log(-h_\mu(-e^\xi)) + i\eta. \quad (8)$$

It is easily seen that f is a 1-1 mapping of Σ onto itself. Since h_μ is absolutely continuous, f is absolutely continuous along every line $\xi = \text{const.}$ or $\eta = \text{const.}$ in Σ . A simple computation shows that

$$2f_{\bar{\zeta}} = E(\zeta) - 1 + iL(\zeta), \quad (9)$$

and

$$2f_{\zeta} = E(\zeta) + 1 - iL(\zeta) \quad (10)$$

where E and L are real functions in Σ :

$$E(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \frac{e^\xi h'_\mu(e^\xi)}{h_\mu(e^\xi)} + \frac{\eta}{\pi} \cdot \frac{e^\xi h'_\mu(-e^\xi)}{-h_\mu(-e^\xi)} \quad (11)$$

and

$$L(\zeta) := \frac{1}{\pi} \log \frac{-h_\mu(-e^\xi)}{h_\mu(e^\xi)}. \quad (12)$$

Therefore we have

$$k(\zeta)^2 = \left| \frac{f_{\bar{\zeta}}}{f_{\zeta}}(\zeta) \right|^2 = \frac{(E(\zeta) - 1)^2 + L(\zeta)^2}{(E(\zeta) + 1)^2 + L(\zeta)^2} = 1 - \frac{4E(\zeta)}{(E(\zeta) + 1)^2 + L(\zeta)^2}.$$

By the condition (1) it is easy to check that $\|k(\zeta)\|_\infty < 1$ and hence f is a quasiconformal mapping of Σ onto itself. It is easily seen that

$$\begin{aligned} H(0) &\leq \sup_{\text{Re } \zeta < l} \frac{1 + k(\zeta)}{1 - k(\zeta)} \\ &= \sup_{\text{Re } \zeta < l} \frac{1}{2E} (1 + E^2 + L^2 + \sqrt{[(E + 1)^2 + L^2][(E - 1)^2 + L^2]}) \end{aligned} \quad (13)$$

for any real number l . On the other hand, a simple computation shows that

$$\frac{1}{\omega(0)} + o(1) \leq E(\zeta) \leq \omega(0) + o(1), \quad \text{as } \operatorname{Re} \zeta \rightarrow -\infty \tag{14}$$

and

$$|L(\zeta)| \leq \frac{1}{\pi} \log \lambda(0) + o(1), \quad \text{as } \operatorname{Re} \zeta \rightarrow -\infty. \tag{15}$$

Setting $l \rightarrow -\infty$, from (13), (14) and (15) we prove the inequality (5) for $\theta = 0$.

3. An estimate of the smallest maximal dilatation K_0 from below

Apply the mapping $g : z \mapsto i(1-z)/(1+z)$ to the disk $|z| < 1$ and a fractional linear transformation G to the disk $|w| < 1$, which maps $|w| < 1$ onto a upper half-plane with $G(\mu(-1)) = \infty$. Then the boundary homeomorphism $G \circ \mu \circ g^{-1}$ of \mathbf{R} onto \mathbf{R} is a ρ -quasisymmetric function, namely, there is a number ρ such that $G \circ \mu \circ g^{-1}$ satisfies the ρ -condition. The infimum of all such numbers ρ is denoted by ρ_0 . Denote by $U(z_1, z_2, z_3, z_4)$ the quadrilateral formed by the upper half-plane and the vertexes z_1, z_2, z_3 , and z_4 . We introduce a function

$$p(\rho) := M\{U(\infty, -1, 0, \rho)\}, \quad \text{for } \rho > 0, \tag{16}$$

where $M\{U(\infty, -1, 0, \rho)\}$ is the modulus of $U(\infty, -1, 0, \rho)$, the a -side and b -side of which are chosen such that $p(\rho)$ is an increasing function of ρ . It is known that $p(\rho) = 1 + r(\rho) \log \rho$, where $r(\rho)$ is a monotone function of ρ and

$$0.2284 \cdots < r(\rho) < \frac{1}{\pi}. \tag{17}$$

(See [1] by A. Beurling and L. Ahlfors.)

We are now going to show the inequality:

$$p(\rho_0) \leq K_0. \tag{18}$$

Obviously, for any $x \in \mathbf{R}$ and $t > 0$, $M\{U(\infty, x-t, x, x+t)\} = 1$ and

$$M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\} = M\{U(\infty, -1, 0, B(x, t))\} = p(B(x, t)), \tag{19}$$

where $\tilde{\mu} := G \circ \mu \circ g^{-1}$ and

$$B(x, t) := \frac{\tilde{\mu}(x+t) - \tilde{\mu}(x)}{\tilde{\mu}(x) - \tilde{\mu}(x-t)}. \tag{20}$$

On the other hand, if f_0 is an extremal mapping of $|z| < 1$ onto $|w| < 1$ with the given boundary correspondence μ , then the mapping $G \circ f_0 \circ g^{-1}$ is an extremal mapping of U onto itself with the boundary correspondence $\tilde{\mu} = G \circ \mu \circ g^{-1}$ and hence we have

$$p(B(x, t)) = \frac{M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}}{M\{U(\infty, x-t, x, x+t)\}} \leq K_0. \tag{21}$$

Similarly, one can show that the inequality (21) is true for $t < 0$. Noting that ρ_0 is the supremum of $B(x, t)$ for all $x \in \mathbf{R}$ and $t \neq 0$, we get the estimate (18).

One can easily prove that the estimate is sharp in the sense that for any K_0 , there is a boundary correspondence μ such that the equality in (18) holds.

4. The main theorem

For a given homeomorphism μ of $|z|=1$ onto $|w|=1$, we call a boundary point $z = e^{i\theta}$ an essential boundary point if $H(\theta) = K_0$. R. Fehlmann proved that if there is a degenerating Hamilton sequence, then there exists an essential boundary point on the circle $|z|=1$ (See [2], p. 567). On the other hand, K. Strebel proved that if there is no degenerating Hamilton sequence for a complex dilatation of an extremal mapping f_0 then f_0 is a Teichmüller mapping associated with a quadratic differential of finite norm. Therefore one can conclude that if there is no essential boundary point, then the extremal Teichmüller mapping exists. By the inequality (18) and Theorem 1 we see that if $q(\lambda(\theta), \omega(\theta)) < p(\rho_0)$ for all θ , then there is no essential boundary point. We have proved

THEOREM 2. *Let μ be an orientation preserving homeomorphism of $|z|=1$ onto $|w|=1$ which admits a quasiconformal extension into the disk $|z| < 1$. Suppose that μ is absolutely continuous and satisfies the condition (1). If*

$$q(\lambda(\theta), \omega(\theta)) < p(\rho_0), \quad \text{for all } \theta, \tag{22}$$

then there exists an extremal Teichmüller mapping associated with a quadratic differential of finite norm.

5. Applications

Applying Theorem 1 and Theorem 2 to the special case that the given boundary homeomorphism μ is piecewise smooth, we get some interesting results. In this case the condition (1) requires

$$h'_+(\theta) \neq 0 \quad \text{and} \quad h'_-(\theta) \neq 0 \quad \text{for all } \theta, \quad (23)$$

where $h'_+(\theta)$ and $h'_-(\theta)$ are the right-derivative and the left-derivative of h_μ , respectively. It is easily seen that

$$\lambda(\theta) = \max \left\{ \frac{h'_+(\theta)}{h'_-(\theta)}, \frac{h'_-(\theta)}{h'_+(\theta)} \right\} \quad \text{and} \quad \omega(\theta) = 1. \quad (24)$$

By Theorem 1, we have

$$H(\theta) \leq q(\lambda(\theta), 1) := q(\lambda(\theta)). \quad (25)$$

Moreover, by a normal family argument, one can prove that the equality $H(\theta) = q(\lambda(\theta))$ for every point. Here the function $q(\lambda)$ is given by the expression

$$q(\lambda) = 1 + \frac{1}{2\pi^2} \log^2 \lambda + \frac{1}{\pi} (\log \lambda) \sqrt{1 + \frac{1}{4\pi^2} \log^2 \lambda}. \quad (26)$$

From (24), (25) and Theorem 2, we have

THEOREM 3. *If the given boundary homeomorphism μ is assumed as above and*

$$\frac{1}{q^{-1} \circ p(\rho_0)} < \frac{h'_+(\theta)}{h'_-(\theta)} < q^{-1} \circ p(\rho_0), \quad \text{for all } \theta, \quad (27)$$

then there exists an extremal Teichmüller mapping.

COROLLARY. *If there are three points $z_1, z_2,$ and z_3 on the circle $|z| = 1$ such that the cross ratio $D(-1, z_1, z_2, z_3) = -1$, $D(\mu(-1), \mu(z_1), \mu(z_2), \mu(z_3)) = -A$, $A \geq 1$, and*

$$\max(5^{-1}, A^{-1/2}) \leq \frac{h'_+(\theta)}{h'_-(\theta)} \leq \min(5, A^{1/2}), \quad \text{for all } \theta, \quad (28)$$

then there exists an extremal Teichmüller mapping.

Proof. If $A = 1$, then the condition (28) implies that μ is smooth everywhere and hence the dilatation of μ is equal to one. By the result of K. Strebel [9], there is an extremal Teichmüller mapping. We assume $A > 1$. A simple computation shows that $p(\rho_0) \geq p(A)$. By the inequality (17), we have

$$p(A) > 1 + 0.2284(\log A). \quad (29)$$

On the other hand, by (28) and (26), $\lambda(\theta) \leq \min(5, A^{1/2})$, and

$$\begin{aligned} q(\lambda(\theta)) &\leq 1 + \left[\frac{1}{4\pi^2} \log 5 + \frac{1}{2\pi} \left(1 + \frac{1}{8\pi^2} \log^2 5 \right) \right] \log A \\ &\leq 1 + 0.2052(\log A). \end{aligned} \quad (30)$$

Therefore $q(\lambda(\theta)) < p(\rho_0)$ for all θ and hence the corollary is proved.

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