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# On the existence of extremal Teichmüller mappings

LI ZHONG

## 1. Introduction

Let $\mu$ be an orientation preserving homeomorphism of the unit circumference $|z|=1$ onto $|w|=1$ which admits a quasiconformal extension into the disk $D:|z|<1$. Let $K_0$ be the maximal dilatation of an extremal q.c. extension of $\mu$ into $D$, while $H$ denotes the dilatation of $\mu$, i.e. the infimum of the dilatations of all the extensions of $\mu$ into arbitrarily small annuli $1-\varepsilon<|z|\leq 1$, $\varepsilon>0$. If $H<K_0$, then there exists a unique extremal Teichmüller mapping associated with a quadratic differential of finite norm ([9], [10]). To estimate $H$ one can introduce a local dilatation $H(\zeta)$ of $\mu$ at a point $\zeta\in\partial D$: It is the infimum of the maximal dilatations of all extensions of $\mu$ into arbitrarily small neighborhoods of $\zeta$ with respect to $\bar{D}$ (see [5]). R. Fehlmann recently showed that $H=\max_{|\zeta|=1} H(\zeta)$. In the present paper we use this result to give an upper estimate of $H$ which, together with a lower estimate of $K_0$, allows the conclusion $H<K_0$. (So far, in the literature, this has been done only in the case $H=1$, see [9] and [10].)

In order to carry out the program, two quantities are introduced in sections (2) and (3) respecively. The first one is motivated by the maximal dilatation $q(\rho)$ of the extremal selfmapping of the upper halfplane with boundary values $\mu(x)=x$ for $x\geq 0$ and $\mu(x)=\rho x$, $\rho\geq 1$ for $x<0$, which is easily computed from [7]:

$$q(\rho)=1+\frac{1}{2\pi^2}\log^2\rho+\frac{1}{\pi}(\log\rho)\sqrt{1+\frac{1}{4\pi^2}\log^2\rho}.$$

The second one (section (3)) is obtained in the usual way (see [1]) as the ratio of the moduli of certain quadrilaterals in $|z|<1$ and $|w|<1$ respectively.

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## 2. An estimate of the local dilatation of $\mu$

In what follows, we will write the given boundary correspondence $\mu$ in a real form: $\vartheta = h_\mu(\theta)$, where $\vartheta = \arg w$ and $\theta = \arg z$: namely $h_\mu(\theta) := \arg \mu(e^{i\theta})$. Then $\vartheta = h_\mu(\theta)$ is a $\rho$-quasisymmetric homeomorphism of $\mathbf{R}$ onto itself. Assume $\mu$ is absolutely continuous and satisfies the condition:

$$m \leq h'_\mu(\theta) \leq M, \quad \text{almost everywhere,} \tag{1}$$

where $m$ and $M$ are constant, $0 < m < M < \infty$. We introduce the functions:

$$\begin{aligned} q(\lambda, \omega) :=& \tfrac{1}{2}(\omega + \omega^{-1}) + \frac{\omega}{2\pi^2} \log^2 \lambda \\ &+ \frac{\omega}{2} \sqrt{\left[(\omega - 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]\left[(\omega + 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]}, \end{aligned} \tag{2}$$

$$\lambda(\theta) := \lim_{\varepsilon \to 0} \sup_{0 < |t| < \varepsilon} \left\{\frac{h_\mu(\theta + t) - h_\mu(\theta)}{h_\mu(\theta) - h_\mu(\theta - t)}\right\}, \tag{3}$$

and

$$\omega(\theta) := \lim_{\varepsilon \to 0+} \max\left(\frac{M(\theta, \theta + \varepsilon)}{m(\theta, \theta + \varepsilon)}, \frac{M(\theta - \varepsilon, \theta)}{m(\theta - \varepsilon, \theta)}\right) \tag{4}$$

where

$$m(x, y) := \operatorname*{ess\,inf}_{(x,y)} \{h'_\mu\}, \qquad M(x, y) := \operatorname*{ess\,sup}_{(x,y)} \{h'_\mu\}.$$

Because the function $h_\mu$ is $\rho$-quasisymmetric, there always exists the limit (3). We denote by $H(\theta)$ the local dilatation of $\mu$ at the point $e^{i\theta}$. We have

THEOREM 1. *Under the above assumption we have*

$$H(\theta) \leq q(\lambda(\theta), \omega(\theta)), \quad \textit{for all } \theta. \tag{5}$$

*Proof.* Without loss of generality, we only look at $\theta = 0$ and assume that $h_\mu(0) = 0$. We are going to show the estimate (5) for the point $\theta = 0$. Applying the mapping $\zeta = \log z$, we pass to the strip $\Sigma$:

$$\Sigma := \{\zeta = \xi + i\eta : 0 < \eta < \pi\} \tag{6}$$

which is the image of the upper half-plane $U$. The boundary correspondence $h_\mu$

of $U$ becomes a boundary correspondence of $\Sigma$ as follows:

$$\begin{cases} \xi \to \log(h_\mu(e^\xi)), & \text{for the lower boundary,} \\ \xi+\pi i \to \log(-h_\mu(-e^\xi))+\pi i, & \text{for the upper boundary.} \end{cases} \tag{7}$$

We construct a function $f(\zeta)$ in $\Sigma$ with the boundary correspondence (7):

$$f(\zeta):=\left(1-\frac{\eta}{\pi}\right)\log(h_\mu(e^\xi))+\frac{\eta}{\pi}\log(-h_\mu(-e^\xi))+i\eta. \tag{8}$$

It is easily seen that $f$ ia a $1-1$ mapping of $\Sigma$ onto itself. Since $h_\mu$ is absolutely continuous, $f$ is absolutely continuous along every line $\xi=\text{const.}$ or $\eta=\text{const.}$ in $\Sigma$. A simple computation shows that

$$2f_{\bar\zeta}=E(\zeta)-1+iL(\zeta), \tag{9}$$

and

$$2f_\zeta=E(\zeta)+1-iL(\zeta) \tag{10}$$

where $E$ and $L$ are real functions in $\Sigma$:

$$E(\zeta):=\left(1-\frac{\eta}{\pi}\right)\frac{e^\xi h'_\mu(e^\xi)}{h_\mu(e^\xi)}+\frac{\eta}{\pi}\cdot\frac{e^\xi h'_\mu(-e^\xi)}{-h_\mu(-e^\xi)} \tag{11}$$

and

$$L(\zeta):=\frac{1}{\pi}\log\frac{-h_\mu(-e^\xi)}{h_\mu(e^\xi)}. \tag{12}$$

Therefore we have

$$k(\zeta)^2=\left|\frac{f_{\bar\zeta}}{f_\zeta}(\zeta)\right|^2=\frac{(E(\zeta)-1)^2+L(\zeta)^2}{(E(\zeta)+1)^2+L(\zeta)^2}=1-\frac{4E(\zeta)}{(E(\zeta)+1)^2+L(\zeta)^2}.$$

By the condition (1) it is easy to check that $\|k(\zeta)\|_\infty<1$ and hence $f$ is a quasiconformal mapping of $\Sigma$ onto itself. It is easily seen that

$$\begin{aligned} H(0) &\le \sup_{\operatorname{Re}\zeta<l}\frac{1+k(\zeta)}{1-k(\zeta)} \\ &= \sup_{\operatorname{Re}\zeta<l}\frac{1}{2E}(1+E^2+L^2+\sqrt{[(E+1)^2+L^2][(E-1)^2+L^2]}) \end{aligned} \tag{13}$$

for any real number $l$. On the other hand, a simple computation shows that

$$\frac{1}{\omega(0)}+o(1)\leq E(\zeta)\leq\omega(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty \tag{14}$$

and

$$|L(\zeta)|\leq\frac{1}{\pi}\log\lambda(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty. \tag{15}$$

Setting $l\to-\infty$, from (13), (14) and (15) we prove the inequality (5) for $\theta=0$.

## 3. An estimate of the smallest maximal dilatation $K_0$ from below

Apply the mapping $g:z\mapsto i(1-z)/(1+z)$ to the disk $|z|<1$ and a fractional linear transformation $G$ to the disk $|w|<1$, which maps $|w|<1$ onto a upper half-plane with $G(\mu(-1))=\infty$. Then the boundary homeomorphism $G\circ\mu\circ g^{-1}$ of $\mathbf{R}$ onto $\mathbf{R}$ is a $\rho$-quasisymmetric function, namely, there is a number $\rho$ such that $G\circ\mu\circ g^{-1}$ satisfies the $\rho$-condition. The infimum of all such numbers $\rho$ is denoted by $\rho_0$. Denote by $U(z_1, z_2, z_3, z_4)$ the quadrilateral formed by the upper half-plane and the vertexes $z_1$, $z_2$, $z_3$, and $z_4$. We introduce a function

$$p(\rho):=M\{U(\infty,-1,0,\rho)\}, \quad \text{for} \quad \rho>0, \tag{16}$$

where $M\{U(\infty,-1,0,\rho)\}$ is the modulus of $U(\infty,-1,0,\rho)$, the $a$-side and $b$-side of which are chosen such that $p(\rho)$ is an increasing function of $\rho$. It is known that $p(\rho)=1+r(\rho)\log\rho$, where $r(\rho)$ is a monotone function of $\rho$ and

$$0.2284\cdots<r(\rho)<\frac{1}{\pi}. \tag{17}$$

(See [1] by A. Beurling and L. Ahlfors.)

We are now going to show the inequality:

$$p(\rho_0)\leq K_0. \tag{18}$$

Obviously, for any $x\in\mathbf{R}$ and $t>0$, $M\{U(\infty, x-t, x, x+t)\}=1$ and

$$M\{U(\infty,\tilde{\mu}(x-t),\tilde{\mu}(x),\tilde{\mu}(x+t))\}=M\{U(\infty,-1,0,B(x,t))\}=p(B(x,t)), \tag{19}$$

where $\tilde{\mu} := G \circ \mu \circ g^{-1}$ and

$$B(x, t) := \frac{\tilde{\mu}(x+t) - \tilde{\mu}(x)}{\tilde{\mu}(x) - \tilde{\mu}(x-t)}. \tag{20}$$

On the other hand, if $f_0$ is an extremal mapping of $|z|<1$ onto $|w|<1$ with the given boundary correspondence $\mu$, then the mapping $G \circ f_0 \circ g^{-1}$ is an extremal mapping of $U$ onto itself with the boundary correspondence $\tilde{\mu} = G \circ \mu \circ g^{-1}$ and hence we have

$$p(B(x, t)) = \frac{M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}}{M\{U(\infty, x-t, x, x+t)\}} \leq K_0. \tag{21}$$

Similarly, one can show that the inequality (21) is true for $t<0$. Noting that $\rho_0$ is the supremum of $B(x, t)$ for all $x \in \mathbf{R}$ and $t \neq 0$, we get the estimate (18).

One can easily prove that the estimate is sharp in the sense that for any $K_0$, there is a boundary correspondence $\mu$ such that the equality in (18) holds.

## 4. The main theorem

For a given homeomorphism $\mu$ of $|z|=1$ onto $|w|=1$, we call a boundary point $z = e^{i\theta}$ an essential boundary point if $H(\theta) = K_0$. R. Fehlmann proved that if there is a degenerating Hamilton sequence, then there exists an essential boundary point on the circle $|z|=1$ (See [2], p. 567). On the other hand, K. Strebel proved that if there is no degenerating Hamilton sequence for a complex dilatation of an extremal mapping $f_0$ then $f_0$ is a Teichmüller mapping associated with a quadratic differential of finite norm. Therefore one can conclude that if there is no essential boundary point, then the extremal Teichmüller mapping exists. By the inequality (18) and Theorem 1 we see that if $q(\lambda(\theta), \omega(\theta)) < p(\rho_0)$ for all $\theta$, then there is no essential boundary point. We have proved

THEOREM 2. *Let $\mu$ be an orientation preserving homeomorphism of $|z|=1$ onto $|w|=1$ which admists a quasiconformal extension into the disk $|z|<1$. Suppose that $\mu$ is absolutely continuous and satisfies the condition* (1). *If*

$$q(\lambda(\theta), \omega(\theta)) < p(\rho_0), \quad \textit{for all } \theta, \tag{22}$$

*then there exists an extremal Teichmüller mapping associated with a quadratic differential of finite norm.*

## 5. Applications

Applying Theorem 1 and Theorem 2 to the special case that the given boundary homeomorphism $\mu$ is piecewise smooth, we get some interesting results. In this case the condition (1) requires

$$h'_+(\theta) \neq 0 \quad \text{and} \quad h'_-(\theta) \neq 0 \quad \text{for all } \theta, \tag{23}$$

where $h'_+(\theta)$ and $h'_-(\theta)$ are the right-derivative and the left-derivative of $h_\mu$, respectively. It is easily seen that

$$\lambda(\theta) = \max\left\{\frac{h'_+(\theta)}{h'_-(\theta)}, \frac{h'_-(\theta)}{h'_+(\theta)}\right\} \quad \text{and} \quad \omega(\theta) = 1. \tag{24}$$

By Theorem 1, we have

$$H(\theta) \leqslant q(\lambda(\theta), 1) := q(\lambda(\theta)). \tag{25}$$

Moreover, by a normal family argument, one can prove that the equality $H(\theta) = q(\lambda(\theta))$ for every point. Here the function $q(\lambda)$ is given by the expression

$$q(\lambda) = 1 + \frac{1}{2\pi^2}\log^2\lambda + \frac{1}{\pi}(\log\lambda)\sqrt{1 + \frac{1}{4\pi^2}\log^2\lambda}. \tag{26}$$

From (24), (25) and Theorem 2, we have

THEOREM 3. *If the given boundary homeomorphism $\mu$ is assumed as above and*

$$\frac{1}{q^{-1}\circ p(\rho_0)} < \frac{h'_+(\theta)}{h'_-(\theta)} < q^{-1}\circ p(\rho_0), \quad \textit{for all } \theta, \tag{27}$$

*then there exists an extremal Teichmüller mapping.*

COROLLARY. *If there are three points $z_1$, $z_2$, and $z_3$ on the circle $|z| = 1$ such that the cross ratio $D(-1, z_1, z_2, z_3) = -1$, $D(\mu(-1), \mu(z_1), \mu(z_2), \mu(z_3)) = -A$, $A \geqslant 1$, and*

$$\max(5^{-1}, A^{-1/2}) \leqslant \frac{h'_+(\theta)}{h'_-(\theta)} \leqslant \min(5, A^{1/2}), \quad \text{for all } \theta, \tag{28}$$

*then there exists an extremal Teichmüller mapping.*

*Proof.* If $A=1$, then the condition (28) implies that $\mu$ is smooth everywhere and hence the dilatation of $\mu$ is equal to one. By the result of K. Strebel [9], there is an extremal Teichmüller mapping. We assume $A>1$. A simple computation shows that $p(\rho_0)\geq p(A)$. By the inequality (17), we have

$$p(A)>1+0.2284(\log A). \tag{29}$$

On the other hand, by (28) and (26), $\lambda(\theta)\leq\min(5, A^{1/2})$, and

$$\begin{aligned} q(\lambda(\theta)) &\leq 1+\left[\frac{1}{4\pi^2}\log 5+\frac{1}{2\pi}\left(1+\frac{1}{8\pi^2}\log^2 5\right)\right]\log A \\ &\leq 1+0.2052(\log A). \end{aligned} \tag{30}$$

Therefore $q(\lambda(\theta))<p(\rho_0)$ for all $\theta$ and hence the corollary is proved.

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