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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **57 (1982)**

PDF erstellt am: **21.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-43898

Comment. Math. Helvetici 57 (1982) 511–517 0010-2571/82/003511-07$01.50+0.20/0

On the existence of extremal Teichmüller mappings

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1. Introduction

Let μ be an orientation preserving homeomorphism of the unit circumference $|z|=1$ onto $|w|=1$ which admits a quasiconformal extension into the disk $D:|z|<1$. Let K_0 be the maximal dilatation of an extremal q.c. extension of μ into D, while H denotes the dilatation of μ, i.e. the infimum of the dilatations of all the extensions of μ into arbitrarily small annuli $1-\varepsilon<|z|\leq 1$, $\varepsilon>0$. If $H<K_0$, then there exists a unique extremal Teichmüller mapping associated with a quadratic differential of finite norm ([9], [10]). To estimate H one can introduce a local dilatation $H(\zeta)$ of μ at a point $\zeta\in\partial D$: It is the infimum of the maximal dilatations of all extensions of μ into arbitrarily small neighborhoods of ζ with respect to $\bar{D}$ (see [5]). R. Fehlmann recently showed that $H=\max_{|\zeta|=1} H(\zeta)$. In the present paper we use this result to give an upper estimate of H which, together with a lower estimate of K_0, allows the conclusion $H<K_0$. (So far, in the literature, this has been done only in the case $H=1$, see [9] and [10].)

In order to carry out the program, two quantities are introduced in sections (2) and (3) respecively. The first one is motivated by the maximal dilatation $q(\rho)$ of the extremal selfmapping of the upper halfplane with boundary values $\mu(x)=x$ for $x\geq 0$ and $\mu(x)=\rho x$, $\rho\geq 1$ for $x<0$, which is easily computed from [7]:

$$q(\rho)=1+\frac{1}{2\pi^2}\log^2\rho+\frac{1}{\pi}(\log\rho)\sqrt{1+\frac{1}{4\pi^2}\log^2\rho}.$$

The second one (section (3)) is obtained in the usual way (see [1]) as the ratio of the moduli of certain quadrilaterals in $|z|<1$ and $|w|<1$ respectively.

This paper was written at the University of Zürich under the direction of Professor K. Strebel, whose help is greatly appreciated. The author wishes to thank Professor A. Pfluger, who read the manuscript and suggested a simple proof of the main theorem.

2. An estimate of the local dilatation of μ

In what follows, we will write the given boundary correspondence μ in a real form: $\vartheta = h_\mu(\theta)$, where $\vartheta = \arg w$ and $\theta = \arg z$: namely $h_\mu(\theta) := \arg \mu(e^{i\theta})$. Then $\vartheta = h_\mu(\theta)$ is a ρ-quasisymmetric homeomorphism of $\mathbf{R}$ onto itself. Assume μ is absolutely continuous and satisfies the condition:

$$m \leq h_\mu'(\theta) \leq M, \quad \text{almost everywhere,} \tag{1}$$

where m and M are constant, $0 < m < M < \infty$. We introduce the functions:

$$q(\lambda, \omega) := \tfrac{1}{2}(\omega + \omega^{-1}) + \frac{\omega}{2\pi^2} \log^2 \lambda + \frac{\omega}{2} \sqrt{\left[(\omega - 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]\left[(\omega + 1)^2 + \frac{1}{\pi^2} \log^2 \lambda\right]}, \tag{2}$$

$$\lambda(\theta) := \lim_{\varepsilon \to 0} \sup_{0<|t|<\varepsilon} \left\{\frac{h_\mu(\theta + t) - h_\mu(\theta)}{h_\mu(\theta) - h_\mu(\theta - t)}\right\}, \tag{3}$$

and

$$\omega(\theta) := \lim_{\varepsilon \to 0+} \max \left(\frac{M(\theta, \theta + \varepsilon)}{m(\theta, \theta + \varepsilon)}, \frac{M(\theta - \varepsilon, \theta)}{m(\theta - \varepsilon, \theta)}\right) \tag{4}$$

where

$$m(x, y) := \operatorname*{ess\,inf}_{(x,y)} \{h_\mu'\}, \qquad M(x, y) := \operatorname*{ess\,sup}_{(x,y)} \{h_\mu'\}.$$

Because the function h_μ is ρ-quasisymmetric, there always exists the limit (3). We denote by $H(\theta)$ the local dilatation of μ at the point $e^{i\theta}$. We have

THEOREM 1. *Under the above assumption we have*

$$H(\theta) \leq q(\lambda(\theta), \omega(\theta)), \quad \textit{for all } \theta. \tag{5}$$

Proof. Without loss of generality, we only look at $\theta = 0$ and assume that $h_\mu(0) = 0$. We are going to show the estimate (5) for the point $\theta = 0$. Applying the mapping $\zeta = \log z$, we pass to the strip Σ:

$$\Sigma := \{\zeta = \xi + i\eta : 0 < \eta < \pi\} \tag{6}$$

which is the image of the upper half-plane U. The boundary correspondence h_μ

of U becomes a boundary correspondence of Σ as follows:

$$\begin{cases} \xi \to \log (h_\mu(e^\xi)), & \text{for the lower boundary,} \\ \xi + \pi i \to \log(-h_\mu(-e^\xi)) + \pi i, & \text{for the upper boundary.} \end{cases} \tag{7}$$

We construct a function $f(\zeta)$ in Σ with the boundary correspondence (7):

$$f(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \log (h_\mu(e^\xi)) + \frac{\eta}{\pi} \log(-h_\mu(-e^\xi)) + i\eta. \tag{8}$$

It is easily seen that f ia a $1-1$ mapping of Σ onto itself. Since h_μ is absolutely continuous, f is absolutely continuous along every line $\xi = \text{const.}$ or $\eta = \text{const.}$ in Σ. A simple computation shows that

$$2f_{\bar\zeta} = E(\zeta) - 1 + iL(\zeta), \tag{9}$$

and

$$2f_\zeta = E(\zeta) + 1 - iL(\zeta) \tag{10}$$

where E and L are real functions in Σ:

$$E(\zeta) := \left(1 - \frac{\eta}{\pi}\right) \frac{e^\xi h_\mu'(e^\xi)}{h_\mu(e^\xi)} + \frac{\eta}{\pi} \cdot \frac{e^\xi h_\mu'(-e^\xi)}{-h_\mu(-e^\xi)} \tag{11}$$

and

$$L(\zeta) := \frac{1}{\pi} \log \frac{-h_\mu(-e^\xi)}{h_\mu(e^\xi)}. \tag{12}$$

Therefore we have

$$k(\zeta)^2 = \left| \frac{f_{\bar\zeta}}{f_\zeta}(\zeta) \right|^2 = \frac{(E(\zeta)-1)^2 + L(\zeta)^2}{(E(\zeta)+1)^2 + L(\zeta)^2} = 1 - \frac{4E(\zeta)}{(E(\zeta)+1)^2 + L(\zeta)^2}.$$

By the condition (1) it is easy to check that $\|k(\zeta)\|_\infty < 1$ and hence f is a quasiconformal mapping of Σ onto itself. It is easily seen that

$$\begin{aligned} H(0) &\leq \sup_{\mathrm{Re}\,\zeta < l} \frac{1+k(\zeta)}{1-k(\zeta)} \\ &= \sup_{\mathrm{Re}\,\zeta < l} \frac{1}{2E} (1 + E^2 + L^2 + \sqrt{[(E+1)^2 + L^2][(E-1)^2 + L^2]}) \end{aligned} \tag{13}$$

for any real number l. On the other hand, a simple computation shows that

$$\frac{1}{\omega(0)}+o(1)\leqslant E(\zeta)\leqslant\omega(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty \tag{14}$$

and

$$|L(\zeta)|\leqslant\frac{1}{\pi}\log\lambda(0)+o(1), \quad \text{as} \quad \operatorname{Re}\zeta\to-\infty. \tag{15}$$

Setting $l\to-\infty$, from (13), (14) and (15) we prove the inequality (5) for $\theta=0$.

3. An estimate of the smallest maximal dilatation K_0 from below

Apply the mapping $g: z\mapsto i(1-z)/(1+z)$ to the disk $|z|<1$ and a fractional linear transformation G to the disk $|w|<1$, which maps $|w|<1$ onto a upper half-plane with $G(\mu(-1))=\infty$. Then the boundary homeomorphism $G\circ\mu\circ g^{-1}$ of $\mathbf{R}$ onto $\mathbf{R}$ is a ρ-quasisymmetric function, namely, there is a number ρ such that $G\circ\mu\circ g^{-1}$ satisfies the ρ-condition. The infimum of all such numbers ρ is denoted by ρ_0. Denote by $U(z_1, z_2, z_3, z_4)$ the quadrilateral formed by the upper half-plane and the vertexes z_1, z_2, z_3, and z_4. We introduce a function

$$p(\rho):=M\{U(\infty,-1,0,\rho)\}, \quad \text{for} \quad \rho>0, \tag{16}$$

where $M\{U(\infty,-1,0,\rho)\}$ is the modulus of $U(\infty,-1,0,\rho)$, the a-side and b-side of which are chosen such that $p(\rho)$ is an increasing function of ρ. It is known that $p(\rho)=1+r(\rho)\log\rho$, where $r(\rho)$ is a monotone function of ρ and

$$0.2284\cdots<r(\rho)<\frac{1}{\pi}. \tag{17}$$

(See [1] by A. Beurling and L. Ahlfors.)

We are now going to show the inequality:

$$p(\rho_0)\leqslant K_0. \tag{18}$$

Obviously, for any $x\in\mathbf{R}$ and $t>0$, $M\{U(\infty, x-t, x, x+t)\}=1$ and

$$M\{U(\infty,\tilde{\mu}(x-t),\tilde{\mu}(x),\tilde{\mu}(x+t))\}=M\{U(\infty,-1,0,B(x,t))\}=p(B(x,t)), \tag{19}$$

where $\tilde{\mu} := G \circ \mu \circ g^{-1}$ and

$$B(x, t) := \frac{\tilde{\mu}(x+t) - \tilde{\mu}(x)}{\tilde{\mu}(x) - \tilde{\mu}(x-t)}. \tag{20}$$

On the other hand, if f_0 is an extremal mapping of $|z|<1$ onto $|w|<1$ with the given boundary correspondence μ, then the mapping $G \circ f_0 \circ g^{-1}$ is an extremal mapping of U onto itself with the boundary correspondence $\tilde{\mu} = G \circ \mu \circ g^{-1}$ and hence we have

$$p(B(x, t)) = \frac{M\{U(\infty, \tilde{\mu}(x-t), \tilde{\mu}(x), \tilde{\mu}(x+t))\}}{M\{U(\infty, x-t, x, x+t)\}} \leq K_0. \tag{21}$$

Similarly, one can show that the inequality (21) is true for $t<0$. Noting that ρ_0 is the supremum of $B(x, t)$ for all $x \in \mathbf{R}$ and $t \neq 0$, we get the estimate (18).

One can easily prove that the estimate is sharp in the sense that for any K_0, there is a boundary correspondence μ such that the equality in (18) holds.

4. The main theorem

For a given homeomorphism μ of $|z|=1$ onto $|w|=1$, we call a boundary point $z = e^{i\theta}$ an essential boundary point if $H(\theta) = K_0$. R. Fehlmann proved that if there is a degenerating Hamilton sequence, then there exists an essential boundary point on the circle $|z|=1$ (See [2], p. 567). On the other hand, K. Strebel proved that if there is no degenerating Hamilton sequence for a complex dilatation of an extremal mapping f_0 then f_0 is a Teichmüller mapping associated with a quadratic differential of finite norm. Therefore one can conclude that if there is no essential boundary point, then the extremal Teichmüller mapping exists. By the inequality (18) and Theorem 1 we see that if $q(\lambda(\theta), \omega(\theta)) < p(\rho_0)$ for all θ, then there is no essential boundary point. We have proved

THEOREM 2. *Let μ be an orientation preserving homeomorphism of $|z|=1$ onto $|w|=1$ which admists a quasiconformal extension into the disk $|z|<1$. Suppose that μ is absolutely continuous and satisfies the condition* (1). *If*

$$q(\lambda(\theta), \omega(\theta)) < p(\rho_0), \quad \textit{for all } \theta, \tag{22}$$

then there exists an extremal Teichmüller mapping associated with a quadratic differential of finite norm.

5. Applications

Applying Theorem 1 and Theorem 2 to the special case that the given boundary homeomorphism μ is piecewise smooth, we get some interesting results. In this case the condition (1) requires

$$h'_+(\theta) \neq 0 \quad \text{and} \quad h'_-(\theta) \neq 0 \quad \text{for all } \theta, \tag{23}$$

where $h'_+(\theta)$ and $h'_-(\theta)$ are the right-derivative and the left-derivative of h_μ, respectively. It is easily seen that

$$\lambda(\theta) = \max\left\{\frac{h'_+(\theta)}{h'_-(\theta)}, \frac{h'_-(\theta)}{h'_+(\theta)}\right\} \quad \text{and} \quad \omega(\theta) = 1. \tag{24}$$

By Theorem 1, we have

$$H(\theta) \leqslant q(\lambda(\theta), 1) := q(\lambda(\theta)). \tag{25}$$

Moreover, by a normal family argument, one can prove that the equality $H(\theta) = q(\lambda(\theta))$ for every point. Here the function $q(\lambda)$ is given by the expression

$$q(\lambda) = 1 + \frac{1}{2\pi^2}\log^2\lambda + \frac{1}{\pi}(\log\lambda)\sqrt{1 + \frac{1}{4\pi^2}\log^2\lambda}. \tag{26}$$

From (24), (25) and Theorem 2, we have

THEOREM 3. *If the given boundary homeomorphism μ is assumed as above and*

$$\frac{1}{q^{-1}\circ p(\rho_0)} < \frac{h'_+(\theta)}{h'_-(\theta)} < q^{-1}\circ p(\rho_0), \quad \textit{for all } \theta, \tag{27}$$

then there exists an extremal Teichmüller mapping.

COROLLARY. *If there are three points z_1, z_2, and z_3 on the circle $|z| = 1$ such that the cross ratio $D(-1, z_1, z_2, z_3) = -1$, $D(\mu(-1), \mu(z_1), \mu(z_2), \mu(z_3)) = -A$, $A \geqslant 1$, and*

$$\max(5^{-1}, A^{-1/2}) \leqslant \frac{h'_+(\theta)}{h'_-(\theta)} \leqslant \min(5, A^{1/2}), \quad \text{for all } \theta, \tag{28}$$

then there exists an extremal Teichmüller mapping.

Proof. If $A=1$, then the condition (28) implies that μ is smooth everywhere and hence the dilatation of μ is equal to one. By the result of K. Strebel [9], there is an extremal Teichmüller mapping. We assume $A>1$. A simple computation shows that $p(\rho_0)\geq p(A)$. By the inequality (17), we have

$$p(A)>1+0.2284(\log A). \tag{29}$$

On the other hand, by (28) and (26), $\lambda(\theta)\leq\min(5, A^{1/2})$, and

$$\begin{aligned} q(\lambda(\theta)) &\leq 1+\left[\frac{1}{4\pi^2}\log 5+\frac{1}{2\pi}\left(1+\frac{1}{8\pi^2}\log^2 5\right)\right]\log A \\ &\leq 1+0.2052(\log A). \end{aligned} \tag{30}$$

Therefore $q(\lambda(\theta))<p(\rho_0)$ for all θ and hence the corollary is proved.

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Received November 5, 1981/June 18, 1982

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