Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 57 (1982)

Artikel: On the structure of 5-dimensional Poincaré duality spaces.
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DOI: https://doi.org/10.5169/seals-43897

Comment. Math. Helvetici 57 (1982) 481–510 0010-2571/82/003481-30$01.50 + 0.20/0

On the structure of 5-dimensional Poincaré duality spaces

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Abstract. We give a complete classification of simply connected 5-dimensional Poincaré duality spaces up to oriented homotopy type. The most important step is a method for describing the Spivak normal fibration and hence the exotic characteristic class.

1. Introduction

In the last two decades there have been developed many and powerful methods to reduce problems in differential topology to questions in homotopy theory. If, for example, you want to classify differentiable manifolds with certain properties up to diffeomorphism, you may try it as follows. First, you classify Poincaré duality spaces with the corresponding properties up to homotopy type. Then you decide which of these spaces have the homotopy type of a manifold. And finally you look if this manifold is unique, i.e. you study the connection between diffeomorphism and homotopy type in the given class of manifolds. Each step leads to purely homotopy theoretical questions.

In this paper we present the first two steps of this program for the class of closed simply connected 5-dimensional differentiable manifolds. Thus we classify the corresponding Poincaré duality spaces and we decide which of them have the homotopy type of a closed manifold. This especially gives the homotopy classification of these manifolds. In a subsequent paper [15] we shall present the third step and hence a new and purely homotopy theoretical proof of Barden's classification theorem [1].

Of course the steps above give more than technical methods for solving problems in differential topology. First, the understanding of the underlying homotopy theory is necessary to understand the topology of manifolds. The reason, for example, that diffeomorphism and homotopy type coincide for the 5-manifolds above, is not that the diffeomorphism invariants in [1] are also homotopy invariants. The real reason is that these spaces admit sufficiently many self-equivalences (which together with the exact sequence of surgery gives the result). Second, Poincaré duality spaces are of own interest. The most intrinsic invariants of these spaces are their "tangential invariants" (the Spivak fibration and derived invariants, e.g. exotic characteristic classes). In general, they are difficult to compute, and their geometric interpretation is not obvious. In this paper too the calculation of the exotic class is the most difficult part. But the results and examples show clearly its geometric meaning, so from the exotic only the fascination remains, but no mystery.

Recall that an n-dimensional Poincaré duality space is a topological space P, of the homotopy type of a compact n-dimensional polyhedron, together with a class $[P]\in H_n(P)$ such that the cap product $\cap[P]: H^q(P)\to H_{n-q}(P)$ is an isomorphism for all q. Two such spaces P and P' are of the same oriented homotopy type if there exists a homotopy equivalence $P\to P'$ sending $[P]$ to $[P']$. We denote by OHP^n the set of oriented homotopy types of simply connected n-dimensional Poincaré duality spaces. This is a semigroup under connected sum with zero element the class of the n-sphere. Of course $OHP^1=\varnothing$, $OHP^n=0$ for $n=2,3$, and there is a bijection between OHP^4 and the set of isomorphy classes of nonsingular symmetric bilinear forms on free abelian groups of finite rank (induced by intersection numbers; this is an easy exercise). So the first nontrivial example is to describe the structure of OHP^5, and this will be presented here.

The paper is organized as follows. In Section 2 we describe the classifying invariants and we formulate the classification theorem (Theorem 2.2): OHP^5 is isomorphic to a certain algebraically defined semigroup J. The structure of J will be studied in Section 3. With that result an alternative formulation of the classification theorem is given in Section 4 (Theorem 4.1); it says that the elements of OHP^5 may be uniquely described by certain integers (including ∞). The most intrinsic invariant is what we call the linking order: it tells whether or not the Stiefel–Whitney characteristic cycle and the exotic cycle are linked in the whole space. The relations between the classifying invariants are proved in Section 5 and Section 6. In Section 7 we do the necessary calculations in homotopy groups; some very helpful remarks of the referee made this section much more readable than in the first version of the paper. The calculation of the exotic class, depending on a cell decomposition of the given space, is presented in Section 8; it uses the results of [14]. The proof of the classification theorem 2.2 is given in Section 9; here we construct models for the generators of OHP^5, and we prove that the elements of that semigroup are uniquely determined by the invariants described in Section 2. In Section 10 we give a third version of the classification theorem, namely a complete list of all simply connected 5-dimensional Poincaré duality spaces.

A first step for proving Theorem 10.1 was done in [6] where it was shown that the spaces described in Section 9 generate the semigroup OHP^5. The complete structure of that semigroup was first given in [13], but with an unsatisfactory proof since it used Bardens classification of 5-manifolds and hence methods of differential topology. A homotopy theoretical proof failed because of the mystery of the exotic class. Now it is possible, using the theory developed in [14].

Remarks on notations. They are as usual, but the following should be noted. $\gamma\in\pi_3(S^2)$ is the Hopf class. $X^{(n)}$ is the n-skeleton of the CW complex X. If $e^n\subset X$ is a n-cell, then $e^n\in\pi_n(X^{(n)},X^{(n-1)})$, $e^n\in H_n(X)$ and $\tilde{e}^n\in H^n(X)$ are the elements corresponding to a fixed characteristic map of that cell (if they are defined). If it happens that the boundary of e^n is the base point, we also write ι^n, $\tilde{\iota}^n$ insted of e^n, $\tilde{e}^n$; then also $\iota^n\in\pi_n(X)$. The map $\check{e}^n:X^{(n)}\to X^{(n)}\vee S^n$ pinches the boundary of a n-ball in e^n to the base point and $\hat{e}^n:X^{(n)}\to S^n$ is its composite with $X^{(n)}\vee S^n\to S^n$.

I thank the referee for his helpful suggestions.

2. The invariants and the clasification theorem

The invariants which classify simply connected 5-dimensional Poincaré duality spaces are the second homology group, the linking numbers, the second Stiefel–Whitney-class and, finally, the first exotic characteristic class.

Let P be a simply connected 5-dimensional Poincaré duality space.

The linking number of $x,y\in\operatorname{Tor}H_2(P)$, where $\operatorname{Tor}G$ denotes the torsion subgroup of the abelian group G, is defined to be Kronecker product $b(x,y)=\langle x',y\rangle\in\mathbf{Q}/\mathbf{Z}$, where $x'\in H^2(P;\mathbf{Q}/\mathbf{Z})$ is such that $\beta^*(x')\cap[P]=x$, with $\beta^*:H^2(P;\mathbf{Q}/\mathbf{Z})\to H^3(P;\mathbf{Z})$ the Bockstein corresponding to $0\to\mathbf{Z}\to\mathbf{Q}\to\mathbf{Q}/\mathbf{Z}\to 0$. This defines a nonsingular skew-symmetric bilinear form (see e.g. [1])

$$b:\operatorname{Tor}H_2(P)\times\operatorname{Tor}H_2(P)\to\mathbf{Q}/\mathbf{Z}.$$

Let ν_P be the Spivak normal fibration of P [12]. This is a spherical fibration over P, so its second Stiefel–Whitney-class

$$w=w_2\in H^2(P;\mathbf{Z})=\operatorname{Hom}(H_2(P),\mathbf{Z}_2)$$

is defined. Let $g:P\to BG$ be the classifying map of ν_P. There is a unique obstruction $e\in H^3(P;\mathbf{Z}_2)$ to lifting g to BO with respect to the canonical map $j:BO\to BG$. Since $H^3(P;\mathbf{Z}_2)\cong H_2(P;\mathbf{Z}_2)\cong H_2(P)\otimes\mathbf{Z}_2$, we may view e as an element $e\in H_2(P)\otimes\mathbf{Z}_2$.

These invariants are related as follows:

2.1 LEMMA. (a) *If* $x\in\operatorname{Tor}H_2(P)$, *then* $b(x,x)=\langle w,x\rangle$, *where* $\{0,\frac{1}{2}\}\subset\mathbf{Q}/\mathbf{Z}$ *is identified with* $\mathbf{Z}_2$.

(b) $(w \otimes id)(e) = 0$, *where* $w \otimes id: H_2(P) \otimes \mathbf{Z}_2 \to \mathbf{Z}_2 \otimes \mathbf{Z}_2 = \mathbf{Z}_2$.

The proof will be given in Sections 5 and 6.

Suppose given a finitely generated abelian group G, a nonsingular skew-symmetric bilinear form $b: \operatorname{Tor} G \times \operatorname{Tor} G \to \mathbf{Q}/\mathbf{Z}$, a homomorphism $w: G \to \mathbf{Z}_2$ and an element $e \in G \otimes \mathbf{Z}_2$. If these dates satisfy $w(x) = b(x, x)$ for $x \in \operatorname{Tor} G$ and $(w \otimes id)(e) = 0$, then the system $I = (G, b, w, e)$ is called a system of invariants. It is obvious how to define isomorphism and direct sums of systems of invariants: let J be the semigroup of isomorphism classes of systems of invariants.

It follows from 2.1 that to each simply-connected 5-dimensional Poincaré duality space there corresponds a system of invariants $I(P) = (H_2(P), b, w, e)$. The main result of this paper is the following:

2.2 CLASSIFICATION THEOREM. *The assignment* $P \to I(P)$ *induces an isomorphism of semigroups* $OHP^5 \to J$.

It is straightforward that this assignment is well defined and homomorphic; bijectivity will be proved in Section 9.

3. The algebraic classification of systems of invariants

Let T be a finite abelian group and let $b: T \times T \to \mathbf{Q}/\mathbf{Z}$ be a nonsingular skew-symmetric bilinear form. Let $|x|$ be the order of $x \in T$. A subset $B = \{x_1, x_2, \ldots, x_{2n-1}, x_{2n}\} \subset T$ is called symplectic if $|x_i| = |x_{i+1}|$ and $b(x_i, x_{i+1}) = -b(x_{i+1}, x_i) = 1/|x_i|$ for $i = 1, 3, \ldots, 2n-1$, and if $b(u, v) = 0$ for all other pairs $(u, v) \in B \times B$. If the same is true except $b(x_j, x_j) = \frac{1}{2}$ for some fixed j, then B is called almost-symplectic with b-exceptional element x_j. A subset of the form $B \cup \{z\} \subset T$ with $|z| = 2$, $b(z, z) = \frac{1}{2}$ and $b(z, x) = 0$ for $x \in B$ is called quasi-symplectic if either $B = \varnothing$ or B is symplectic.

3.1 PROPOSITION. *There exists a maximal basis of T which is symplectic or almost-symplectic or quasi-symplectic.*

For a proof see [1]; it follows that $T = T_1 \oplus T_1$ or $T = T_1 \oplus T_1 \oplus \mathbf{Z}_2$ for some subgroup $T_1 \subset T$ (compare [17]).

Now let $I = (G, b, w, e)$ be a system of invariants. A basis $B \subset G$ is called special if the following holds:

(a) B contains a basis of Tor G as in 3.1.

(b) If $w \neq 0$, then there exists $z_w \in B$ (the w-exceptional element) such that $w(z_w) = 1$ and $w(b) = 0$ for $z_w \neq b \in B$.

(c) If $e \neq 0$, then there exists $z_e \in B$ (the e-exceptional element) such that $e = z_e \otimes 1$.

3.2. PROPOSITION. *There exists a special basis of G. If B resp. B′ is a special basis of G with exceptional elements* z_w, z'_w *and* z_e, z'_e, *then*

(i) $|z_w| = |z'_w|$ *and* $|z_e| = |z'_e|$.

(ii) *If these orders are finite, then* $b(z_w, z_e) = 0$ *iff* $b(z'_w, z'_e) = 0$.

The existence proof is similar to the proof of Lemma E in [1]. Take a basis B of G containing a basis of $\operatorname{Tor} G$ as in 3.1 such that (b) holds. Then by appropriate change of basis elements (using $(w \otimes id)(e) = 0$) one gets a new basis which is special. The proof of (i) and (ii) is the same as the proof of Lemma C in [1].

To each system of invariants $I = (G, b, w, e)$ we assign numerical invariants i, j and k as follows. For $w = 0$ let $i = k = 0$. For $e = 0$ let $j = k = 0$. If $w \neq 0$ resp. $e \neq 0$, and if z_w resp. z_e is the exceptional element of some special basis of G, let

$$i = \begin{cases} \infty & \text{if } |z_w| = \infty \\ m & \text{if } |z_w| = 2^m \end{cases} \qquad j = \begin{cases} \infty & \text{if } |z_e| = \infty \\ n & \text{if } |z_e| = 2^n \end{cases}.$$

By 3.1 and 3.2 this is well defined. Finally let $k = 1$ if $b(z_w, z_e)$ is defined and not zero; in all other cases let $k = 0$. Thus we have $0 \leq i, j \leq \infty$ and $k = 0, 1$, and if $k = 1$ then $0 < i = j < \infty$. (A more intrinsic definition of these invariants will be given in section 4.)

3.3 PROPOSITION. *Two systems of invariants* $I = (G, b, w, e)$ *and* $I' = (G', b', w', e')$ *are isomorphic if and only if* $G \cong G'$ *and they have the same numerical invariants i, j and k.*

Proof. Since $G \cong G'$ there exists a special basis

$$B = \{a_1, \ldots, a_s, x_1, y_1, \ldots, x_r, y_r, z\}$$
$$B' = \{a'_1, \ldots, a'_s, x'_1, y'_1, \ldots, x'_r, y'_r, z'\}$$

of G resp. G' such that $|a_i| = |a'_i| = \infty$, and $|x_j| = |y_j| = |x'_j| = |y'_j|$, where the elements z, z' with $|z| = |z'| = 2$ only occur in the quasi-symplectic case. Let $f: G \to G'$ be the obvious isomorphism. We show that we may choose B and B' such that $b' \circ (f \times f) = b$, $w' \circ f = w$ and $(f \otimes id)(e) = e'$; then f is an isomorphism of systems of invariants. The first two conditions are satisfied in the quasi-symplectic case, and the third holds after an appropriate change of basis elements (which is possible since $j=j'$). If I is symplectic, then $i=0$ or ∞, hence I' is symplectic. Therefore $b'\circ(f\times f)=b$, and by changing basis elements one gets $w'\circ f=w$ and $(f\otimes id)(e)=e'$. In the almost-symplectic case one gets similarly $b'\circ(f\times f)=b$, hence also $w'\circ f=w$. If $j=0$ or ∞, then $(f\otimes id)e=e'$ is no problem. Let $1\leq j\leq\infty$, and let y_1, y_1' be the w-exceptional elements. If $k=k'=1$, then x_1, x_1' are the e-exceptional elements, thus $(f\otimes id)(e)=e'$ is true. If $k=k'=0$, then we may assume that the e-exceptional elements are x_ν and x'_μ with $\nu, \mu\neq 1$. Then $(f\otimes id)(e)=e'$ holds after interchanging (x'_ν, y'_ν) and x'_μ, y'_μ).

4. Classification of the spaces by numerical invariants

Let P be a simply connected 5-dimensional Poincaré duality space, with Stiefel–Whitney class $w\in\operatorname{Hom}(H_2(P), \mathbf{Z}_2)$ and exotic class $e\in H_2(P)\otimes\mathbf{Z}_2$. We define numerical invariants of P, the Stiefel–Whitney order, the exotic order and the linking order, as follows.

The Stiefel–Whitney order is zero if $w=0$, and it is ∞ if $w\neq 0$, but $w=0$ on $\operatorname{Tor} H_2(P)$. If $w\neq 0$ on $\operatorname{Tor} H_2(P)$, then it is the largest integer n such that w is zero on the subgroup G_n of $\operatorname{Tor} H_2(P)$ consisting of all x such that $2^{n-1}x=0$. Similarly, the exotic order is zero if $e=0$, and it is ∞ if $e\neq 0$ and $e\notin(\operatorname{Tor} H_2(P))\otimes\mathbf{Z}_2$. If $0\neq e\in(\operatorname{Tor} H_2(P))\otimes\mathbf{Z}_2$, then it is the largest integer m such that e is not contained in the image of $G_m\otimes\mathbf{Z}_2\to H_2(P)\otimes\mathbf{Z}_2$.

Suppose that the Stiefel–Whitney order and the exotic order are both equal to n, where $1\leq n<\infty$, and suppose further that for all elements $x, y\in H_2(P)$ such that $\langle w, x\rangle=1$, $y\otimes 1=e$ and $|x|=|y|=2^n$ the linking number $b(x, y)$ has order 2^n. Then the linking order of P is defined to be 1; in all other cases it is defined to be zero.

Choosing a special basis of $H_2(P)$ it is not difficult to prove that the invariants i, j and k of the system of invariants $(H_2(P), b, w, e)$ are just the Stiefel–Whitney order, the exotic order and the linking order of P, respectively. Therefore we get from 2.2 and 3.3 the following formulation of the main theorem:

4.1 CLASSIFICATION THEOREM. *Two simply connected 5-dimensional Poincaré duality spaces are of the same oriented homotopy type if and only if they have the same second Betti number, the same two-dimensional torsion coefficients, the same Stiefel–Whitney order, the same exotic order and the same linking order.*

4.2 *Remarks.* (a) Since these invariants do not depend on the orientation, we see that homotopy type and oriented homotopy type coincide.

(b) A simply connected 5-dimensional Poincaré duality space P has the homotopy type of a closed smooth manifold if and only if its exotic order is zero. For this is equivalent to $e=0$ in $H^3(P;\mathbf{Z}_2)$, hence to the existence of an orthogonal sphere bundle structure on the Spivak fibration (compare 5.2 below), and the result follows from Browder–Novikov theory [2].

(c) From the preceeding remarks it follows that (oriented) homotopy types of closed smooth simply connected 5-manifolds are classified by $H_2(P)$ and the second Stiefel–Whitney class, or, equivalently, by $H_2(P)$ and the Stiefel–Whitney order $i=i_P$.

(d) The numerical invariants i, j and k above have the following geometric interpretation. The w-exceptional element z_w (of some special basis) which may be called the "Stiefel–Whitney" cycle of P, has order 2^i. Similarly, the "exotic cycle" z_e has order 2^j. The linking order k describes the connection between these cycles: it is 1 if they are linked and 0 otherwise (compare remark (b) in 10.2).

5. The relation between the Stiefel–Whitney class and the exotic class

In the following we denote by X a simply connected CW complex of dimension ≤ 3. Let $\widetilde{KO}(X)\cong[X, BO]$ resp. $\widetilde{KG}(X)\cong[X, BG]$ be the group of stable orthogonal sphere bundles resp. stable spherical fibrations over X, and let $j:BO\to BG$ be the natural map. The second Stiefel–Whitney class $w=w_2$ defines homomorphisms $w:\widetilde{KO}(X)\to H^2(X;\mathbf{Z}_2)$ and $w:\widetilde{KG}(X)\to H^2(X;\mathbf{Z}_2)$. If $g:X\to BG$ is the classifying map of $\xi\in\widetilde{KG}(X)$, then there exists $g':X\to BO$ such that $jg'\mid X^{(2)}=g\mid X^{(2)}$. The difference cochain of the maps $jg', g:X\to BG$ represents an element $e(\xi)\in H^3(X;\mathbf{Z}_2)$, called the first exotic class of ξ [5]. This defines a homomorphism $e:\widetilde{KG}(X)\to H^3(X;\mathbf{Z}_2)$.

5.1 PROPOSITION. *The following homomorphisms are isomorphims:*
(a) $w:\widetilde{KO}(X)\to H^2(X;\mathbf{Z}_2)$,
(b) $w+e:\widetilde{KG}(X)\to H^2(X;\mathbf{Z}_2)\oplus H^3(X;\mathbf{Z}_2)$.

Proof. By well known facts on $\pi_n(BO)\to\pi_n(BG)$ for $n\leq 3$, this is true for S^2, S^3 and $S^2\cup_k e^3$. Since X is a wedge of these spaces, it is true in general.

5.2 PROPOSITION. *Let* $A=X\cup_\alpha e^5$ *with* $\alpha\in\pi_4(X)$. *Then the following homomorphisms are isomorphims:*
(a) $w:\widetilde{KO}(A)\to H^2(A;\mathbf{Z}_2)$
(b) $w+e:\widetilde{KG}(A)\to H^2(A,\mathbf{Z}_2)\oplus \operatorname{Ker}(Sq^2:H^3(A;\mathbf{Z}_2)\to H^5(A;\mathbf{Z}_2))$.

Proof. The fourth homotopy group of S^2, S^3 and $S^2\cup_k e^3$ is finite (the last one by the Hurewicz theorem modulo the class of finite groups). Therefore, by the

Hilton–Milnor theorem, $\pi_4(X)$ is a finite group modulo Whitehead products. Since these products are zero in the H-space BO, and since $\pi_4(BO)=\mathbb{Z}$, any map $X\to BO$ extends to $A\to BO$, unique up to homotopy since $\pi_5(BO)=0$. Therefore $\widetilde{KO}(A)\cong\widetilde{KO}(X)$, and 5.2(a) follows from 5.1(a).

Since $\pi_5(BG)=0$, the restriction $\widetilde{KG}(A)\to\widetilde{KG}(X)$ is injective. Therefore $w+e$ in 5.2(b) is injective by 5.1(b).

Next we prove $Sq^2e(\xi)=0$ if $\xi\in\widetilde{KG}(A)$. We may assume $w(\xi)=0$ (if not, replace ξ by $\xi+\xi'$, where $\xi'\in\widetilde{KO}(A)$ is such that $w(\xi')=w(\xi)$; it exists by 5.2(a)). Then $\xi\mid X^{(2)}=0$ by 5.1(b), and this implies $\xi\mid X=f^*\xi_0$ for some $f:X\to S^3$, where ξ_0 is the non zero element in $\widetilde{KG}(S^3)=\mathbf{Z}_2$. Let $g:A\to BG$ and $g_0:S^3\to BG$ be the classifying maps of ξ and ξ_0, respectively. Then $0=(g\mid X)\circ\alpha=g_0\circ f\circ\alpha$. Since $f\circ\alpha\in\pi_4(S^3)=\mathbf{Z}_2(S\gamma)$ and $g_0\circ S\gamma\neq 0$ in $\pi_4(BG)$, it follows that $f\circ\alpha=0$. Then $f:X\to S^3$ extends to $h:A\to S^3$, and $\xi=h^*\xi_0$ since $\widetilde{KG}(A)\to\widetilde{KG}(X)$ is injective. Therefore $Sq^2e(\xi)=Sq^2e(h^*\xi_0)=h^*Sq^2e(\xi_0)=0$.

It remains to prove that to $u\in H^3(A;\mathbf{Z}_2)$ with $Sq^2u=0$ there exists $\xi\in\widetilde{KG}(A)$ such that $e(\xi)=u$. Choose $g:A\to S^3\cup e^5$ with $g^*(\bar\iota^3)=u$ and with $g_*:H_5(A)\cong H_5(S^3\cup e^5)$. Then $Sq^2=0$ in $H^*(S^3\cup e^5;\mathbf{Z}_2)$, and since Sq^2 detects the non zero element of $\pi_4(S^3)$, this implies $S^3\cup e^5=S^3\vee S^5$. Thus there exists $f:A\to S^3$ such that $u=f^*(\bar\iota^3)$, and we may take $\xi=f^*\xi_0$.

Now we are ready to prove (b) of Lemma 2.1. Let P be a simply connected 5-dimensional Poincaré duality space with Spivak fibration $\nu_p\in\widetilde{KG}(P)$. Recall from Section 2 that the invariants of P are defined by $w=w(\nu_p)$ and $e=e(\nu_p)$. The following is a special case of the Wu formula:

5.3 PROPOSITION. $w\cup u=Sq^2u$ *for all* $u\in H^3(P;\mathbf{Z}_2)$, *and* $w\in H^2(P;\mathbf{Z}_2)$ *is uniquely determined by this property.*

We may assume $P=X\cup_\alpha e^5$ for some simply connected CW complex X of dimension ≤ 3 and some $\alpha\in\pi_4(X)$. Then $w\cup e(\xi)=Sq^2e(\xi)=0$ for all $\xi\in\widetilde{KG}(P)$, by 5.2(b). Especially, for $\xi=\nu_p$, we get $w\cup e=0$ which is just (b) of Lemma 2.1.

5.4 *Remarks.* (a) The proof above shows that the relation $w\cup e(\xi)=0$ is not a special property of the Spivak fibration, but holds for all stable spherical fibrations over P. And (b) of Lemma 2.1 is essentially the Wu formula.

(b) If $e\in H^3(BG;\mathbf{Z}_2)$ is the universal first exotic class, then $Sq^2e\neq 0$ [10]. Thus the relation $Sq^2e(\xi)=0$ for all $\xi\in\widetilde{KG}(A)$ is a special property of A in 5.2.

(c) Observe that the full statements of 5.1 and 5.2 are not necessary for the proof of 2.1(b). However, we'll need them in Section 8.

6. The relation between the Stiefel–Whitney class and the linking numbers

Let us first recall some wellknown facts on homotopy groups. We have $\pi_4(S^2)=\mathbf{Z}_2(\gamma\circ S\gamma)$ and $\pi_4(S^3)=\mathbf{Z}_2(S\gamma)$. I don't know an explicit reference for the following proposition, but it is easily proved using 2.1 in [8], 5.4 in [7], and [11], 3.2 and page 261.

6.1 PROPOSITION. *Let* $X(k)=S^2\cup_k e^3$ *with* $k\geq 2$. *Then* $\pi_4(X(k))\cong \pi_4^s(X(k))$ *under suspension and*

(a) $\pi_4(X(k))=0$ *if* k *is odd.*

(b) $\pi_4(X(k))=\mathbf{Z}_2\oplus\mathbf{Z}_2$ *if* $k\equiv 0 \bmod 4$ *and* $=\mathbb{Z}_4$ *for* $k\equiv 2 \bmod 4$. *In both cases, the following sequence is exact*

$$0\longrightarrow \pi_4(S^2)\xrightarrow{\iota^2}\pi_4(X(k))\xrightarrow{\hat{e}^3}\pi_4(S^3)\longrightarrow 0.$$

As in Section 5, we denote by X a simply connected CW complex of dimension ≤ 3. Then to $v\in H^3(X)$ there exists a map $\hat{v}:X\to S^3$, unique up to homotopy, such that $\hat{v}^*:H^3(S^3)\to H^3(X)$ maps the generator of $H^3(S^3)$ onto v. Let $j_*:H^3(X)\to H^3(X;\mathbf{Z}_2)$ be the obvious homomorphism.

6.2 PROPOSITION. *There is a natural homorphism* $\lambda:\pi_4^s(X)\to H_3(X;\mathbf{Z}_2)$, *defined by* $\langle j_*, v, \lambda\alpha\rangle=\{\hat{v}\}\circ\alpha\in\pi_4^s(S^3)=\mathbf{Z}_2$ *for all* $v\in H^3(X)$ *and* $\alpha\in\pi_4^s(X)$. *If* $X=S^3$ *then* $\lambda\{S\gamma\}=\iota^3$. *If* $X=X(k)$ *and* k *is even, then* $\lambda(\{\alpha\})\neq 0$ *if and only if* $\hat{e}^3\circ\alpha\neq 0$.

The proof is obvious (use 6.1 for the last part). In the following, the composite $\pi_4(X)\to\pi_4^s(X)\to H_3(X;\mathbf{Z}_2)$ is also denoted by $\lambda:\pi_4(X)\to H_3(X;\mathbf{Z}_2)$.

Next we define bilinear forms

$$H^3(X;G)\times\pi_4(X)\to H_2(X;G)\quad\text{and}\quad H^2(X;G)\times\pi_4(X,G)\to H_3(X,G),$$

both denoted by $(x,\alpha)\to x\cap\alpha$, as follows. Let $A=X\cup_\alpha e^5$. If $x\in H^i(X;G)=H^i(A;G)$, where $i=2,3$, then $x\cap\alpha\in H_{5-i}(X;G)=H_{5-i}(A;G)$ is the cap product $x\cap e^5$ of x and the generator $e^5\in H_5(A)$. Here, the coefficient group G is arbitrary, and the cap product is with respect to $G\otimes\mathbf{Z}\to G$. Observe that $A=X\cup_\alpha e^5$ is a Poincaré duality space if and only if $\cap\alpha:H^3(X)\to H_2(X)$ is an isomorphism, and then $\cap\alpha=\cap[A]$.

For any $\alpha\in\pi_4(X)$ we define a bilinear form $b_\alpha:\operatorname{Tor}H^3(X)\times\operatorname{Tor}H^3(X)\to\mathbf{Q}/\mathbf{Z}$ by $b_\alpha(x,y)=\langle x', y\cap\alpha\rangle$, where $x'\in H^2(X;\mathbf{Q}/\mathbf{Z})$ is such that $\beta^*x'=x$. This is motivated as follows. If $P=X\cup_\alpha e^5$ is a Poincaré duality space, then $b_\alpha(x,y)$ is just the linking number of the elements $x\cap[P]$ and $y\cap[P]$. As for linking numbers one proves that b_α is well defined, homomorphic and skew-symmetric. Furthermore, it is natural: if $f:X\to X'$ and $\alpha\in\pi_4(X)$ and $u,v\in\operatorname{Tor}H^3(X')$, then $b_{f_*\alpha}(u,v)=b_\alpha(f^*u,f^*v)$. In the following proposition we identify $\{0,\frac{1}{2}\}\subset\mathbf{Q}/\mathbf{Z}$ with $\mathbf{Z}_2$.

6.3 PROPOSITION. *$b_\alpha(z,z)=\langle j_*(z),\lambda(\alpha)\rangle$ for all $z\in\operatorname{Tor}H^3(X)$.*

Proof. Since b_α is skew-symmetric, the function $z\to b_\alpha(z,z)$ is a homomorphism. From this and from naturality it follows that it is enough to prove 6.3 in the cases $X=S^2$, S^3 and $X(k)$. The only non trivial case is $X(k)$ with k even, and here it is enough to consider $z=\tilde{e}^3\in H^3(X)$ and $\alpha\in\pi_4(X)$ with $\hat{e}^3\circ\alpha=S\gamma$, see 6.1. Then the right had side of 6.3 is not zero (see 6.2), and so we have to show that $b_\alpha(\tilde{e}^3,\tilde{e}^3)=\frac{1}{2}$. Since $\beta^*((1/k)\tilde{\iota}^2)=\tilde{e}^3$, this linking number is $(1/k)\langle\tilde{\iota}^2,\tilde{e}^3\cap\alpha\rangle$, and it is enough to prove the following

Assertion. If $\alpha\in\pi_4(X(k))$ and $\hat{e}^3\circ\alpha=S\gamma$, then $\tilde{e}^3\cap\alpha=(k/2)\iota^2$.

Proof. Let $[e^3,\iota^2]\in\pi_4(X(k),S^2)$ be the relative Whitehead product of $e^3\in\pi_3(X(k),S^2)$ and $\iota^2\in\pi_2(S^2)$. Let $b=(k/2)[e^3,\iota^2]-e^3\circ\partial^{-1}\gamma\in\pi_4(X(k),S^2)$ where $\partial:\pi_4(D^3,S^2)\cong\pi_3(S^2)$. Then $\partial b=(k^2/2)[\iota^2,\iota^2]-(k\iota^2)\circ\gamma=0$ in $\pi_3(S^2)$, therefore b has a counterimage β in $\pi_4(X(k))$. From 6.1 it follows that $\beta\equiv\alpha \bmod \pi^4(S^2)$, and since $\bigcap(\iota^2\circ\gamma\circ S\gamma)=0$, we may assume $\beta=\alpha$. Thus α has image b in $\pi_4(X(k),S^2)$. This easily implies the formula

$$\check{e}^3\circ\alpha=\alpha+\frac{k}{2}[\iota^3,\iota^2]+\iota^3\circ S\gamma$$

where $\check{e}^3:X(k)\to X(k)\vee S^3$, from which the assertion follows by naturality (since $\cap[\iota^3,\iota^2]:H^3(S^2\vee S^3)\to H_2(S^2\vee S^3)$ maps $\tilde{\iota}^3$ onto ι^2).

Now we return to Poincaré duality spaces.

6.4 PROPOSITION. *Let $\alpha\in\pi_4(X)$ be such that $P=X\cup_\alpha e^5$ is a Poincaré duality space, with second Stiefel–Whitney class $w\in H^2(X;\mathbf{Z}_2)$. Then $w\cap\alpha=\lambda(\alpha)$.*

Proof. By definition of λ we must show that $\langle j_*v,w\cap\alpha\rangle=\hat{v}\circ\alpha$ for all $v\in H^3(X)$. We have

$$\langle j_*v,w\cap\alpha\rangle=\langle j_*v,w\cap[P]\rangle=\langle j_*v\cup w,[P]\rangle=\langle Sq^2j_*v,[P]\rangle,$$

using the Wu formula 5.3. Extend $\hat{v}: X \to S^3$ to $f: P \to S^3 \cup_\beta e^5$ with $\beta = \hat{v} \circ \alpha$ such that $f_*: H_5(P) \cong H_5(S^2 \cup_\beta e^5)$. Then $Sq^2 j_* v = Sq^2 f^*(\tilde{\iota}^3) = f^* Sq^2(\tau^3)$ and $\langle j_* v, w \cap \alpha\rangle = \langle Sq^2(\tilde{\iota}^3), e^5\rangle$. Thus we have to prove that $\beta = \hat{v} \circ \alpha = 0$ if and only if $Sq^2 = 0$ in $H^*(S^3 \cup_\beta e^5; \mathbf{Z}_2)$. This is true since Sq^2 detects the non zero element in $\pi_4(S^3)$.

Now we are ready to prove (a) of Lemma 2.1. Let P be a simply connected 5-dimensional Poincaré duality space, as usual $P = X \cup_\alpha e^5$. If $x \in \operatorname{Tor} H_2(P)$, let $z \in \operatorname{Tor} H^3(P)$ be the element with $z \cap [P] = x$. Then, using 6.3 and 6.4, we get (a) of 2.1 as follows:

$$\begin{aligned} b(x, x) &= b_\alpha(z, z) = \langle j_* z, \lambda\alpha\rangle = \langle j_* z, w \cap \alpha\rangle = \langle j_* z, w \cap [P]\rangle \\ &= \langle j_* z \cup w, [P]\rangle = \langle w, z \cap [P]\rangle = \langle w, x\rangle. \end{aligned}$$

6.5 *Remarks.* (a) Observe that 6.4 may be formulated as follows. If $P = X \cup_\alpha e^5$ is a Poincaré duality space and $D = \cap [P]: H^2(X; \mathbf{Z}_2) \cong H_3(X; \mathbf{Z}_2)$, then the Stiefel–Whitney class is given by $w = D^{-1}\lambda(\alpha)$. Especially, w only depends on the stable class $\{\alpha\} \in \pi_4^s(X)$.

(b) For closed smooth simply connected 5-manifolds Lemma 2.1(a) is proved in [17], but the proof does not generalize to Poincaré duality spaces.

7. Calculations in homotopy groups

In this section we study the group $\pi_4(X)$, where X is a simply connected CW complex of dimension ≤ 3. We start with a definition. If $\alpha \in \pi_4(X)$, then, by 6.3, the homomorphism $\cap\, \alpha: H^3(X) \to H_2(X)$ and the element $\{\alpha\} \in \pi_4^s(X)$ are related by $b_\alpha(z, z) = \langle j_* z, \lambda\{\alpha\}\rangle$ for all $z \in \operatorname{Tor} H^3(X)$. By definition of b_α this is equivalent to $\langle x, \beta^*(x) \cap \alpha\rangle = \langle j_* \beta^* x, \lambda\{\alpha\}\rangle = \langle x, \beta_* \lambda\{\alpha\}\rangle$ for all $x \in H^2(Y; \mathbf{Q}/\mathbf{Z})$, where $\beta_*: H_3(X; \mathbf{Z}_2) \to H_2(X)$ is the Bockstein corresponding to $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$.

More generally, we consider pairs (f, α) with $f: H^3(X) \to H_2(X)$ a homomorphism and with $\alpha \in \pi_4^s(X)$, such that

$$\langle x, f\beta^*(x)\rangle = \langle x, \beta_* \lambda\alpha\rangle \quad \text{for all } x \in H^2(X; \mathbf{Q}/\mathbf{Z}). \tag{7.1}$$

These pairs form a subgroup of $\operatorname{Hom}(H^3(X), H_2(X)) \oplus \pi_4^s(X)$ which we denote by $A(X)$. Then we have a homomorphism $\phi: \pi_4(X) \to A(X)$ by $\alpha \to (\cap\, \alpha, \{\alpha\})$.

7.2 PROPOSITION. *There is an exact sequence*

$$\pi_2(X) \otimes \pi_2(X) \otimes \mathbf{Z}_2 \oplus \pi_2(X) \otimes \pi_2(X) \otimes \pi_2(X) \xrightarrow{\phi'} \pi_4(X) \xrightarrow{\phi} A(X) \longrightarrow 0$$

where the homomorphism ϕ' is induced by $a\otimes b\otimes 1\mapsto[a,b]\circ S\gamma$ *and* $a\otimes b\otimes c\mapsto [a,[b,c]]$ *for* $a,b,c\in\pi_2(X)$.

The proof will be finished after 7.9 below. Let us first show that 7.2 is true if $X=S^2$, S^3 or $X(k)$. From $[\iota^2,\iota^2]\circ S\gamma=[\iota^2,[\iota^2,\iota^2]]=0$ in $\pi_4(S^2)$ it follows that $\phi'=0$ in these cases, and we must prove that ϕ is bijective. This is trivial in the first two cases, so let $X=X(k)$. By 6.1 it is enough to show that the projection $A(X)\to\pi_4^s(X)$ is injective. Given $f:H^3(X)\to H_2(X)$, we have $f(\tilde{e}^3)=m\iota^2$ for some $m\in\mathbf{Z}_k$. If $(f,0)\in A(X)$, then, by taking $x=(1/k)\tilde{\iota}^2$ in 7.1, we get $0=\langle(1/k)\tilde{\iota}^2,m\iota^2\rangle=m/k$ in $\mathbf{Q}/\mathbf{Z}$, hence $m=0$ in $\mathbf{Z}_k$, which implies $f=0$.

In the following we'll prove that 7.2 is true for $X\vee Y$, if it is true for X and Y; then 7.2 is true in general. Thus we have to study how the groups in 7.2 change if X is replaced by $X\vee Y$.

Let $B(X,Y)$ be the group of pairs (f',f''), with $f':H^3(Y)\to H_2(X)$ and $f'':H^3(X)\to H_2(Y)$ homomorphisms such that

$$\langle x,f'\beta^*y\rangle=\langle y,f''\beta^*x\rangle\quad\text{for all}\quad x\in H^2(X;\mathbf{Q}/\mathbf{Z}),\qquad y\in H^2(Y;\mathbf{Q}/\mathbf{Z}).\tag{7.3}$$

From 7.1 it follows that $A(X\vee Y)=A(X)\oplus A(Y)\oplus B(X,Y)$, with the last summand imbedded by $(f',f'')\mapsto(f''+f',0)$. Given $v\in H_5(X\wedge Y)$, consider the homomorphisms induced by the (cohomology) slant product [4]

$$\begin{aligned}\backslash v&:H^3(Y)\to H_2(X)\quad && b\mapsto b\backslash v\\ \backslash t_*v&:H^3(X)\to H_2(Y)\quad && a\mapsto a\backslash t_*v\end{aligned}$$

where $t:X\wedge Y\to Y\wedge X$ permutes the factors. The pair $(\backslash v,\backslash t_*v)$ satisfies 7.3, and so we get $H_5(X\wedge Y)\to B(X,Y)\subset A(X\vee Y)$.

7.4 PROPOSITION. $A(X\vee Y)=A(X)\oplus A(Y)\oplus H_5(X\wedge Y)$, *with the last summand imbedded by* $v\mapsto(\backslash t_*v+\backslash v,0)$.

Proof. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}0\to&\sum_{i=2,3}H_i(X)\otimes H_{5-i}(Y)&\to&H_5(X\wedge Y)&\xrightarrow{s}&\operatorname{Tor}(H_2X,H_2Y)&\to 0\\ &\downarrow{\scriptstyle pr_1}&&\downarrow{\scriptstyle\backslash}&&\downarrow{\scriptstyle id}&\\ 0\to&H_2(X)\otimes H_3(Y)&\to&\operatorname{Hom}(H^3Y,H_2X)&\to&\operatorname{Tor}(H_2X,H_2Y)&\to 0.\end{array}$$

The first row is exact by the Künneth formula. The second is obtained by applying the cofunctor Hom $(-, H_2X)$ to the split exact sequence

$$0 \to \operatorname{Ext}(H_2Y, \mathbf{Z}) \xrightarrow{\mu} H^3(Y) \to \operatorname{Hom}(H_3Y, \mathbf{Z}) \to 0,$$

using the identifications (observe that $H_3(Y)$ is free abelian)

$$\operatorname{Hom}(\operatorname{Hom}(H_3Y, \mathbf{Z}), H_2(X)) = H_2(X) \otimes H_3(Y)$$
$$\operatorname{Hom}(\operatorname{Ext}(H_2Y, \mathbf{Z}), H_2(X)) = \operatorname{Tor}(H_2X, H_2Y).$$

If $v \in H_5(X \wedge Y)$ and $\backslash v = 0 : H^3(Y) \to H_2(X)$, then v is in the image of $H_3(X) \otimes H_2(Y)$ in $H_5(X \wedge Y)$. Similarly if $\backslash t_* v = 0 : H^3(X) \to H_2(Y)$, then $t_* v$ is in the image of $H_3(Y) \otimes H_2(X)$ in $H_5(Y \wedge X)$, and v is therefore also in the image of $H_2(X) \otimes H_3(Y)$ in $H_5(X \wedge Y)$. Both facts imply $v = 0$ and so $H_5(X \wedge Y) \to A(X \vee Y)$ is injective.

Next let $(f', f'') \in B(X, Y)$. By the diagram (and by the same diagram with X and Y permuted) there exists $v', v'' \in H_5(X \wedge Y)$ such that $f' = \backslash v'$ and $f'' = \backslash t_* v''$. Then 7.3 says that

$$\langle x \wedge \mu(y), v' - v'' \rangle = 0 \tag{7.5}$$

for all $x \in H^2(X; \mathbf{Q/Z}) = \operatorname{Hom}(H_2(X), \mathbf{Q/Z})$ and $y \in \operatorname{Ext}(H_2(Y), \mathbf{Z})$. From the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(H_2X, \mathbf{Q/Z}) \otimes \operatorname{Ext}(H_2Y, \mathbf{Z}) \arrow[r, "(\ )\wedge\mu(\ )"] \arrow[d] & \operatorname{Hom}(H_5(X \wedge Y), \mathbf{Q/Z}) \\
\operatorname{Hom}(\operatorname{Hom}(\operatorname{Ext}(H_2Y, \mathbf{Z}), H_2X), \mathbf{Q/Z}) \arrow[r, equal] & \operatorname{Hom}(\operatorname{Tor}(H_2X, H_2Y), \mathbf{Q/Z}) \arrow[u, "\kappa^*"']
\end{tikzcd}$$

and 7.5 we see that $g\kappa(v' - v'') = 0$ for all homomorphisms $g : \operatorname{Tor}(H_2X, H_2Y) \to \mathbf{Q/Z}$. This implies $\kappa(v' - v'') = 0$ and therefore $v' = v'' + a + b$ with $a \in H_2(X) \otimes H_3(Y)$ and $b \in H_3(X) \otimes H_2(Y)$. Now define $v = v' - b$. Then it follows that $\backslash v = f'$ and $\backslash t_* v = f''$. This shows that $H_5(X \wedge Y) \to B(X, Y)$ is surjective and thus proves 7.4.

Next we compute $\pi_4(X \vee Y)$, using the Hilton–Milnor formula. We may assume that $X = SA$ and $Y = SB$ are suspensions, where A and B are connected CW complexes of dimension ≤ 2. Let $[i, j] : S(A \wedge B) \to X \vee Y$ be the Whitehead product of the inclusions $i : X \to X \vee Y$ and $j : Y \to X \vee Y$. The direct summands of $\pi_4(X \vee Y)$ which correspond to the basic triple Whitehead products are easily identified with the groups in (b) below, by connectivity arguments. So we get:

7.6 PROPOSITION. *$\pi_4(X\vee Y)$ is the direct sum of $\pi_4(X)$, $\pi_4(Y)$ and the following subgroups:*

(a) *$\pi_4(S(A\wedge B))$, imbedded by $\alpha\mapsto[i,j]\circ\alpha$.*

(b) *$\pi_2(X)\otimes\pi_2(X)\otimes\pi_2(Y)$ and $\pi_2(Y)\otimes\pi_2(X)\otimes\pi_2(Y)$, both being imbedded by $a\otimes b\otimes c\mapsto[a,[b,c]]\in\pi_4(X\vee Y)$.*

Concerning the summand in (a), we have an exact sequence ([16], page 558)

$$0\to\pi_2(X)\otimes\pi_2(Y)\otimes\mathbf{Z}_2\to[i,j]\circ\pi_4(S(A\wedge B))\to H_5(X\wedge Y)\to 0 \tag{7.7}$$

with the homomorphisms defined by $a\otimes b\otimes 1\mapsto[a,b]\circ S\gamma$ and $[i,j]\circ\alpha\mapsto Sh(\alpha)$ where $\pi_4(S(A\wedge B))\xrightarrow{h}H_4(S(A\wedge B))\xrightarrow{S}H_5(X\wedge Y)$. Using the fact that $X\times Y$ is the mapping cone of $[i,j]$, it is not difficult to prove that the cap product $\bigcap([i,j]\circ\alpha):H^3(X\vee Y)\to H_2(X\vee Y)$ is given by the formulas

$$\begin{cases} x\cap([i,j]\circ\alpha)=x\backslash Sh(\alpha) & x\in H^3(X)\\ y\cap([i,j]\circ\alpha)=y\backslash t_*Sh(\alpha) & y\in H^3(Y)\end{cases}\tag{7.8}$$

With the notations of 7.2 and 7.4 this says

$$\phi([i,j]\circ\alpha)=(\backslash t_*Sh(\alpha)+\backslash Sh(\alpha),0)\in A(X\vee Y).\tag{7.9}$$

Now we are ready to finish the proof of 7.2: from 7.4, 7.6, 7.7 and 7.9 it easily follows that the sequence in 7.2 is exact for $X\vee Y$, if it is exact for X and Y.

7.10 *Remark.* With 7.6 we may calculate the kernel of ϕ' in 7.2, as follows. Let G be an abelian group. Define $L(G)=\bigwedge^2(G\otimes\mathbf{Z}_2)$, the second exterior power on the $\mathbf{Z}_2$ vector space $G\otimes\mathbf{Z}_2$. Define $M(G)$ to be $G\otimes G\otimes G$, with the following relations added (where $a, b, c\in G$):

$a\otimes b\otimes c+b\otimes c\otimes a+c\otimes a\otimes b=0$ (Jacobi identity),

$a\otimes b\otimes c-a\otimes c\otimes b=0$ (commutativity of Whitehead products),

$a\otimes a\otimes a=0$ (triple Whitehead products are zero in $\pi_4(S^2)$).

Then we get the following short exact sequence:

$$0\to L(\pi_2(X))\oplus M(\pi_2(X))\xrightarrow{\phi'}\pi_4(X)\xrightarrow{\phi}A(X)\to 0.$$

In general, this sequence (which will not be used in the following) does not split.

Having calculated the group $\pi_4(X)$, we now study its automorphisms which are induced by homotopy equivalences of X. Given $u \in H^3(X; \pi_3(X))$, there exists a map $f_u : X \to X$, unique up to homotopy, such that $f_u \,|\, X^{(2)} = id$ and such that the difference cochain of the maps f_u, $id : X \to X$ represents the cohomology class u. Let $\alpha \in \pi_4(X)$ and consider

$$u \cap \alpha \in H_2(X; \pi_3(X)) = \pi_2(X) \otimes \pi_3(X) \xrightarrow{\sigma} \pi_4(X)$$

$$u \otimes \lambda\alpha \in H^3(X; \pi_3(X)) \otimes H_3(X; \mathbf{Z}_2) \xrightarrow{\langle\,,\,\rangle} \mathbf{Z}_2 \otimes \pi_3(X) \xrightarrow{\varepsilon} \pi_4(X)$$

where $\sigma(a \otimes b) = [a, b]$ and $\varepsilon(1 \otimes b) = b \circ S\gamma$. With these notations we have:

7.11 PROPOSITION. (a) $f_u \circ \alpha = \alpha + \sigma(u \cap \alpha) + \varepsilon\langle u, \lambda(\alpha)\rangle$.

(b) *If u is contained in the image of the coefficient homomorphism $H^3(X; \pi_3(X^{(2)})) \to H^3(X; \pi_3(X))$, then f_u induces the identity in all homology and cohomology groups. Especially, f_u is a homotopy equivalence.*

7.12 *Remark.* Let $e_1^3, \ldots, e_n^3$ be the 3-cells of X, and $\alpha_1, \ldots, \alpha_n \in \pi_3(X)$. Let $f : X \to X$ be the composite

$$X \xrightarrow{g} X \vee S_1^3 \vee \cdots \vee S_n^3 \xrightarrow{(id, \alpha_1, \ldots, \alpha_n)} X$$

where g pinches the boundary of a 3-ball in e_i^3 to the base point ($i = 1, \ldots, n$). Then $f = f_u$, where $u \in H^3(X; \pi_3(X))$ is the cohomology class represented by the cochain $e_i^3 \mapsto \alpha_i$. In the following, we call f the map which is induced by the assignment $e_i^3 \mapsto \alpha_i$. It is a homotopy equivalence if all $\alpha_i \in \pi_3(X^{(2)})$.

Proof of 7.11. As (b) is obvious, we only prove (a). Let $e^3 \subset X$ be a fixed cell and consider $\check{e}^3 : X \to X \vee S^3$. From 7.6 we have

$$\pi_4(X \vee S^3) = \pi_4(X) \oplus \pi_4(S^3) \oplus \pi_2(X)$$

and the image of $a \in \pi_4(X \vee S^3)$ in the last summand is just the image of $\tilde{\iota}^3 \in H^3(X \vee S^3)$ under $\cap\, a : H^3(X \vee S^3) \to H_2(X \vee S^3) = \pi_2(X)$. Applying this to $a = \check{e}^3 \circ \alpha$ (and recalling the definition of $\lambda\alpha$ in 6.2) gives

$$\check{e}^3 \circ \alpha = \alpha + \langle j_* \tilde{e}^3, \lambda\alpha \rangle \tilde{\iota}^3 \circ S\gamma + [\tilde{e}^3 \cap \alpha, \iota^3].$$

By repated application of this formula, we get:

$$g \circ \alpha = \alpha + \sum_i \langle j_*, \tilde{e}_i^3, \lambda\alpha \rangle \tilde{\iota}_i^3 \circ S\gamma + \sum_i [\tilde{e}_i^3 \cap \alpha, \iota_i^3]$$

with g from 7.12. Applying the map $(id, \alpha_1, \ldots, \alpha_n)$ to this equation, where $\alpha_i = \langle u, e_i^3 \rangle$, gives (a) in 7.11.

7.13 PROPOSITION. *Let $\alpha, \alpha' \in \pi_4(X)$ be elements such that $\phi\alpha = \phi\alpha'$ in $A(X)$ and $\lambda\alpha = \lambda\alpha' = 0$. Suppose further that $\bigcap \alpha = \bigcap \alpha'$ is an isomorphism. Then there exists a homotopy equivalence $f: X \to X$ such that $f \circ \alpha = \alpha'$ and $f \mid X^{(2)} = id$.*

Proof. $\alpha' - \alpha$ lies in the image of the homomorphism ϕ' in 7.2. From the relation

$$[a, b] \circ S\gamma = [a, b \circ \gamma] + [b, [a, b]] \qquad (a, b \in \pi_2(X)) \tag{7.14}$$

(which holds since it is true in the universal example $S^2 \vee S^2$) it follows that the image of ϕ' is contained in the image of

$$\pi_2(X) \otimes \pi_3(X^{(2)}) \to \pi_2(X) \otimes \pi_3(X) \xrightarrow{\sigma} \pi_4(X).$$

Therefore $\alpha' = \alpha + \sigma(z)$ for some $z \in \pi_2(X) \otimes \pi_3(X^{(2)})$. Since

$$\bigcap \alpha : H^3(X; \pi_3(X)) \to H_2(X; \pi_3(X)) = \pi_2(X) \otimes \pi_2(X)$$

is an isomorphism too, there exists $u \in H^3(X; \pi_3(X))$ such that $u \cap \alpha = z$, and u lies in the image of $H^3(X; \pi_3(X^{(2)})) \to H^3(X; \pi_3(X))$. Then, by 7.11, we get 7.13 by defining $f = f_u$.

7.15 PROPOSITION. *Let $\alpha, \alpha' \in \pi_4(X)$ be elements such that $\lambda\alpha = \lambda\alpha'$ and $\bigcap \alpha = \bigcap \alpha' : H^3(X) \to H_2(X)$ is an isomorphism. Then there exists a homotopy equivalence $f: X \to X$ with $f \mid X^{(2)} = id$, such that $\alpha' - f \circ \alpha$ is a sum of elements of the form $a \circ \gamma \circ S\gamma$ and $[b, c] \circ S\gamma$, where $a, b, c \in \pi_2(X)$.*

Proof. From $\lambda\alpha = \lambda\alpha'$ and 6.1, 6.2 it follows that $\{\alpha'\} = \{\alpha + \beta\}$ in $\pi_4^s(X)$, where β is a sum of elements of the form $a \circ \gamma \circ S\gamma$. Then $\phi(\alpha') = \phi(\alpha + \beta)$ in $A(X)$, and therefore $\alpha' = \alpha + \beta + \sigma(z)$ for some $z \in \pi_2(X) \otimes \pi_3(X^{(2)})$. If f is defined as in the proof of 7.13, then

$$f \circ \alpha = \alpha + \sigma(u \cap \alpha) + \varepsilon(\langle u, \lambda\alpha \rangle) = \alpha' - \beta + \varepsilon(\langle u, \lambda\alpha \rangle),$$

and $\alpha' - f \circ \alpha$ is therefore a sum as stated above.

We close this section with the following splitting principle which is an important step in the proof of the classification theorem:

7.16 LEMMA. *Let $\alpha \in \pi_4(X \vee Y)$ be an element such that*

(a) $\bigcap \alpha : H^3(X \vee Y) \to H_2(X \vee Y)$ *maps $H^3(X)$ and $H^3(Y)$ isomorphically onto $H_2(X)$ and $H_2(Y)$, respectively.*

(b) $\lambda\alpha \in H_3(X \vee Y; \mathbf{Z}_2)$ *lies in the subgroup $H_3(X; \mathbf{Z}_2)$ of $H_3(X \vee Y; \mathbf{Z}_2)$. Then there exists a homotopy equivalence $f : X \vee Y \to X \vee Y$, restricting to the identity on the 2-skeleton, such that $f \circ \alpha \in \pi_4(X) \oplus \pi_4(Y)$.*

Proof. Let $[i, j] \circ \beta$ with $\beta \in \pi_4(S(A \wedge B))$ be the image of α under the projection onto the summand (a) in the direct sum decomposition in 7.6 (where $X = SA$, $Y = SB$). From assumption (a) and 7.8 it follows that the slant products

$$\backslash Sh(\beta) : H^3(X) \to H_2(Y) \quad \text{and} \quad \backslash t_* Sh(\beta) : H^3(Y) \to H_2(X)$$

are zero. Therefore, by 7.9, the element $[i, j] \circ \beta$ lies in the kernel of $\phi : \pi_4(X \vee Y) \to A(X \vee Y)$, and from the exact sequence 7.2 (with X replaced by $X \vee Y$) and 7.6, 7.7 we get that, in the obvious notation,

$$\alpha \in \pi_4(X) \oplus \pi_4(Y) \oplus [\pi_2 X, \pi_2 Y] \circ S\gamma \oplus [\pi_2 X, [\pi_2 X, \pi_2 Y]] \oplus [\pi_2 Y, [\pi_2 X, \pi_2 Y]].$$

Thus we may write

$$\alpha = \alpha_X + \alpha_Y + r + s + t$$

with the elements on the right hand side lying in the corresponding subgroup of $\pi_4(X \vee Y)$ above. Let $W(X, Y) \subset \pi_3(X \vee Y)$ be the image of the Whitehead product $\pi_2(X) \otimes \pi_2(Y) \to \pi_3(X \vee Y)$. Then we have

$$s = \sigma(s') \quad \text{for some} \quad s' \in \pi_2(X) \otimes W(X, Y)$$
$$t = \sigma(t') \quad \text{for some} \quad t' \in \pi_2(Y) \otimes W(X, Y),$$

where σ is as in 7.11, $\sigma(a \otimes b) = [a, b]$. The assumption (a) implies (compare the proof of 7.13) that there exists

$$u' \in H^3(X; W(X, Y)) \quad \text{such that} \quad u' \cap \alpha = s'.$$

Since the cohomology class u' takes values in $W(X, Y)$, it follows that the element $\varepsilon(\langle u', \lambda\alpha \rangle)$, with ε as in 7.11, lies in $[\pi_2 X, \pi_2 Y] \circ S\gamma$. From the relation 7.14 (with

X there replaced by $X \vee Y$) we get

$$[\pi_2 X, \pi_2 Y] \circ S\gamma \subset [\pi_2 X, \pi_3(Y^{(2)})] \oplus [\pi_2 Y, [\pi_2 X, \pi_2 Y]].$$

Therefore $r + \varepsilon(\langle u', \lambda\alpha \rangle)$ lies in that subgroup, and so we may write

$$r + \varepsilon(\langle u', \lambda\alpha \rangle) = \sigma(r'') + \sigma(r')$$

for some $r'' \in \pi^2(X) \otimes \pi_3(Y^{(2)})$ and $r' \in \pi_2(Y) \otimes W(X, Y)$. Again by assumption (a) there are elements

$$v' \in H^3(Y; W(X, Y)) \quad \text{such that} \quad v' \cap \alpha = r' + t'$$
$$v'' \in H^3(X; \pi_3(Y^{(2)}) \quad \text{such that} \quad v'' \cap \alpha = r''.$$

Now we define, with the obvious identifications,

$$u = -u' - v' - v'' \in H^3(X \vee Y; \pi_3(X \vee Y)),$$

and consider the corresponding map $f = f_u : X \vee Y \to X \vee Y$. By 7.11(b) it is a homotopy equivalence, since $W(X, Y) \subset \pi_3((X \vee Y)^{(2)})$. From 7.11(a) and the equations above we get (observe that $2\varepsilon(\langle\ ,\ \rangle) = 0$):

$$\begin{aligned} f \circ \alpha &= \alpha + \sigma(u \cap \alpha) + \varepsilon(\langle u, \lambda\alpha \rangle) \\ &= \alpha + (-s - \sigma(r') - t - \sigma(r'')) + \varepsilon(\langle -u' - v' - v'', \lambda\alpha \rangle) \\ &= \alpha - (r + s + t) + \varepsilon(\langle v', \lambda\alpha \rangle) + \varepsilon(\langle v'', \lambda\alpha \rangle) \\ &= \alpha_X + \alpha_Y + \varepsilon(\langle v', \lambda\alpha \rangle) + \varepsilon(\langle v'', \lambda\alpha \rangle). \end{aligned}$$

The cohomology class v' lies in the subgroup $H^3(Y; W(X, Y))$ of $H^3(X \vee Y; \pi_3(X \vee Y))$, and, by assumption (b), the homology class $\lambda\alpha$ lies in the subgroup $H_3(X; \mathbf{Z}_2)$ of $H_3(X \vee Y; \mathbf{Z}_2)$. Both facts imply $\langle v', \lambda\alpha \rangle = 0$. The cohomology class v'' takes values in $\pi_3(Y)$, therefore $\varepsilon(\langle v'', \lambda\alpha \rangle) \in \pi_4(Y)$. Thus the last equation says $f \circ \alpha \in \pi_4(X) \oplus \pi_4(Y)$.

8. Calculation of the exotic class

By the Wu formula, the Stiefel–Whitney classes of a Poincaré duality space P are determined by the action of the Steenrod algebra on $H^*(P; \mathbf{Z}_2)$. It is not known whether or not there exists a "Wu formula" which describes the exotic characteristic classes of P by certain (higher, twisted) cohomology operations in P. So we have to look for other methods.

Let P be a Poincaré duality space of the form $P = SA \cup_\alpha e^n$, with A a connected CW complex of dimension $\leq n-3$, and $\alpha \in \pi_{n-1}(SA)$. Let $\beta : \widetilde{KG}(SA) \to \{A, S^0\}$ be the canonical bijection, and let

$$\gamma_m : \pi_r(SA) \to \{S^{r-1}, A^m\} \quad (r, m \geq 2 \text{ and } A^m = A \wedge \cdots \wedge A)$$

be the stable Hopf invariant (see below). The following is one of the main results of [14]:

8.1 THEOREM. *The restriction $\eta = \nu_P \mid SA \in \widetilde{KG}(SA)$ of the Spivak fibration of P onto SA is uniquely characterized by the following equation in $\{S^{n-2}, A\}$:*

$$\{a\} + \sum_{m \geq 2} (\{id_A\} \wedge \beta(\eta) \wedge \cdots \wedge \beta(\eta)) \circ \gamma_m(\alpha) = 0.$$

Thus if we know the attaching map of the top cell and its Hopf invariants, we may calculate the Spivak fibration $\nu_P \mid SA$ and hence its exotic characteristic classes.

To apply this result to our case, we'll need some facts on Hopf invariants. First recall its definition, as given in [14]. Choose $k > r$ and consider the inclusions $i : SA \to SA \vee S^k$ and $j : S^k \to SA \vee S^k$. Given $\alpha \in \pi_r(SA)$, the element $[i \circ \alpha, j] \in \pi_{r+k-1}(SA \vee S^k)$ may be uniquely written as

$$[i \circ \alpha, j] = [i, j] \circ S^k\alpha + [i, [i, j]] \circ \gamma_2^k(\alpha) + [i, [i, [i, j]]] \circ \gamma_3^k(\alpha) + \cdots \tag{8.2}$$

using the Hilton–Milnor formula. Then $\gamma_m(\alpha)$ is defined to be the stable class of $\gamma_m^k(\alpha) \in \pi_{r+k-1}(S^k A^m)$. This defines homomorphisms γ_m as above. If A is a suspension, then $\gamma_m(\alpha)$ is up to some sign just the stable class of the Hopf invariants $\lambda_m(\alpha)$ in [3].

Here is a first application of 8.1:

8.3 LEMMA. *Let $\alpha = [\iota^2, \iota^3] + m(\iota^2 \circ \gamma \circ S\gamma) + n(\iota^3 \circ S\gamma)$ in $\pi_4(S^2 \vee S^3)$, where $m, n \in \mathbf{Z}_2$. Then the Poincaré duality space $P = (S^2 \vee S^3) \cup_\alpha e^5$ has exotic characteristic class $e = m(n+1)(\iota^2 \otimes 1)$ in $H_2(P) \otimes \mathbf{Z}_2$.*

Proof. This is the case $A = S^1 \vee S^2$ in 8.1. The Hopf invariants of α are easy to compute, either directly by the defining equation 8.2, or with [3]. It results that $\gamma_m(\alpha) = 0$ if $m \geq 3$ and

$$\gamma_2(\alpha) = \{\iota^1 \wedge \iota^2\} - \{\iota^2 \wedge \iota^1\} + m\{\iota^1 \wedge \iota^1\} \circ \{\gamma\} \quad \text{in} \quad \{S^3, A \wedge A\}.$$

Thus 8.1 reduces to $\{\alpha\}+(\{id_A\}\wedge\beta(\eta))\circ\gamma_2(\alpha)=0$ in $\{S^3, A\}$. Using the formulas for α and $\gamma_2(\alpha)$, and applying the retraction $A\to S^2$, gives the following equation in $\{S^2, S^0\}$:

$$m\{S^{-2}\gamma\circ S^{-1}\gamma\}+\beta(\eta)\mid S^2+m(\beta(\eta)\mid S^1)\circ\{S^{-1}\gamma\}=0.$$

From $\lambda\alpha=n\iota^3\in H_3(S^2\vee S^3;\mathbf{Z}_2)$, see 6.2, and from 6.4 and 5.1 it follows that $\beta(\eta)\mid S^1=n\in\{S^1, S^0\}=\mathbf{Z}_2$. Therefore the last equation gives $\beta(\eta)\mid S^2=m+mn$ in $\{S^2, S^0\}=\mathbf{Z}_2$, hence $\eta\mid S^3=m+mn$ in $\widetilde{KG}(S^3)=\mathbf{Z}_2$, and this proves 8.3.

For our second application of 8.1 we need some preparations. In the following, $n\geq 2$ is a fixed integer, and we set $A=S^1\cup_n e^2$, thus $SA=X(n)=S^2\cup_n e^3$, and

$$A\vee A=(S_1^1\cup e_1^2)\vee(S_2^1\cup e_2^2),\qquad X=SA\vee SA=(S_1^2\cup e_1^3)\vee(S_2^2\cup e_2^3).$$

Let $i_1, i_2: A\to A\vee A$ and $j_1, j_2: SA\to X$ be the inclusions. We first note the following:

8.4 PROPOSITION. *There exists a map $u_0: S^3\to A\wedge A$ representing the homology class of the cycle $\iota^1\wedge e^2+e^2\wedge\iota^1$. This map is a duality map in the sense of Spanier–Whitehead duality, and therefore induces an isomorphism $D_0:\{A, S^0\}\to\{S^3, A\}$ by $D_0(\alpha)=(\{id\}\wedge\alpha)\circ\{u_0\}$. Finally, this isomorphism maps $\{S^{-2}\gamma\circ S^{-1}\gamma\circ\hat{e}^2\}$ onto $\{\iota^1\circ S^{-1}\gamma\circ\gamma\}$.*

Proof. Since $A\wedge A$ is simply connected, u_0 exists. Since all slant products $\backslash u_0:\bar{H}^i(A)\to\bar{H}_{n-i}(A)$ are isomorphisms, it is a duality map. By homology arguments, the map $(id_A\wedge\hat{e}^2)\circ u_0: S^3\to A\wedge S^2=S^2A=S^3\cup e^4$ is the inclusion; this gives the final statement in 8.4.

Next, consider the following bijections (see 5.1 and 6.1):

$$H^2(SA;\mathbf{Z}_2)\oplus H^3(SA;\mathbf{Z}_2)\xleftarrow{w+e}\widetilde{KG}(SA)\xrightarrow{\beta}\{A, S^0\}\xrightarrow{D_0}\{S^3, A\}\cong\pi_4(SA).$$

From 8.4 it follows that $\iota^2\circ\gamma\circ S\gamma\in\pi_4(SA)$ corresponds to $\tilde{e}^3$ under these bijections. So we get from 6.1 and 6.2:

8.5 PROPOSITION. *If n is even, there exists a unique element $\delta_0\in\pi_4(SA)$ such that, under the bijections above, the elements $0, -\delta_0, \iota^2\circ\gamma\circ S\gamma$ and $\iota^2\circ\gamma\circ S\gamma-\delta_0$ in $\pi_4(SA)$ correspond to the elements $0, \tilde{\iota}^2, \tilde{e}^3$ and $\tilde{\iota}^2+\tilde{e}^3$ in $H^2(SA;\mathbf{Z}_2)\oplus H^3(SA;\mathbf{Z}_2)$, respectively. This element satisfies the equations $\hat{e}^3\circ\delta_0=S\gamma$, $\lambda(\delta_0)=e^3$ and $\tilde{e}^3\cap\delta_0=(n/2)\iota^2$.*

The last equation is just the assertion in the proof of 6.3. Observe that, by 6.1, we have the following two cases. If $n \not\equiv 2 \bmod 4$, then $2\,\delta_0 = 0$, the sequence in 6.1 splits, and β is an isomorphism of groups. If $n \equiv 2 \bmod 4$, then $2\,\delta_0 = \iota^2 \circ \gamma \circ S\gamma$, the sequence in 6.1 does not split, and β is a bijection of sets, but no homomorphism.

8.6 PROPOSITION. *Let* $[j_1, j_2]: S(A \wedge A) \to X$ *be the Whitehead product, and define* $\tau_0 = [j_1, j_2] \circ Su_0 \in \pi_4(X)$. *Then* $\bar{e}_1^3 \cap \tau_0 = -\iota_2^2$ *and* $\bar{e}_2^3 \cap \tau_0 = \iota_1^2$.

This follows from 7.8 and the homological properties of u_0 in 8.4 (the sign results from [4], page 191, (2.12)). Observe that δ_0 and τ_0 depend on the choice of the element u_0 in 8.4. However, we have:

8.7 PROPOSITION. *The elements* τ_0 *and* $\tau_0 + j_1 \circ \delta_0$ *in* $\pi_4(X)$ *are unique up to homotopy equivalences* $f: X \to X$ *with* $f \mid X^{(2)} = id$.

Proof. For τ_0 this follows from 7.13, using 8.6 and $S\tau_0 = 0$. Observe that τ_0 and δ_0 only depend on Su_0, and, by the exact sequence 7.7, Su_0 is unique up to $S(\iota^1 \wedge \iota^1) \circ S\gamma \in \pi_4(S(A \wedge A))$. If we replace Su_0 by $Su_0 + S(\iota^1 \wedge \iota^1) \circ S\gamma$, then τ_0 is replaced by $\tau_0' = \tau_0 + [\iota_1^2, \iota_2^2] \circ S\gamma$. The isomorphism D_0 in 8.4 must be replaced by $D_0'(\alpha) = D_0(\alpha) + (\{\iota^1\} \wedge \alpha \mid \iota^1) \circ \{\gamma\}$, and therefore δ_0 and $\delta_0' = \delta_0 + \iota^2 \circ \gamma \circ S\gamma$. Now let $f: X \to X$ be the map induced by $e_1^3 \to -(\iota^2 \circ \gamma)$ and $e_2^3 \to [\iota_1^2, \iota_2^2]$, see 7.12. Then it follows from 7.11, 7.14, 8.5 and 8.6 that f is a homotopy equivalence such that $f \mid X^{(2)} = id$ and $f \circ (\tau_0 + j_1 \circ \delta_0) = \tau_0' + j_1 \circ \delta_0'$.

8.8 PROPOSITION. *The Hopf invariants of* $\tau_0 \in \pi_4(X)$ *are given by the following formulas:*

(a) $\gamma_2(\tau_0) = (\{i_1 \wedge i_2\} - \{i_2 \wedge i_1\}) \circ \{u_0\}$

(b) $\gamma_3(\tau_0) = 0$ *if* $n \not\equiv 2 \bmod 4$; *otherwise*

$$\gamma_3(\tau_0) = \frac{n}{2}(\{\iota_2^1 \wedge \iota_1^1 \wedge \iota_2^1\} + \{\iota_1^1 \wedge \iota_1^1 \wedge \iota_2^1\} + \{\iota_1^1 \wedge \iota_2^1 \wedge \iota_1^1\}).$$

Proof. By the defining equation 8.2 we have to calculate the following Whitehead product

$$[i \circ \tau_0, j] = [i \circ [j_1, j_2] \circ Su_0, j] = [[i, i], j] \circ S^k(i_1 \wedge i_2) \circ S^k u_0$$

where now $i: S(A \wedge A) \vee S^k$ and $j: S^k \to S(A \wedge A) \vee S^k$. The idea is to express it by Whitehead products as in 8.2, using the Jacobi identity and commutativity. The problem is that $A = S^1 \cup_n e^2$ is not a suspension, so it is not obvious that the reduced diagonal $d: A \to A \wedge A$ is null homotopic. But this is what is needed in the proof of the Jacobi identity. Now it is not difficult to show that d is null homotopic if $n \not\equiv 2 \bmod 4$, while if $n \equiv 2 \bmod 4$ it is homotopic to the composite of $\hat{e}^2: A \to S^2$ and $(n/2)(\iota^1 \wedge \iota^1): S^2 \to A \wedge A$. If with this information we copy the usual proof of the Jacobi identity (see e.g. [3], especially the Witt identity on page 192 of [3]) we get a generalization of it which together with 8.2 implies 8.8.

Now we are ready for our second application of Theorem 8.1:

8.9 LEMMA. *With the notations above, let* $P = X \cup_\alpha e^5$ *with*

$$\alpha = \tau_0 + j_1(a\,\delta_0 + r(\iota^2 \circ \gamma \circ S\gamma)) + s(j_2 \circ \iota^2 \circ \gamma \circ S\gamma)$$

in $\pi_4(X)$, *where* $r, s \in \mathbf{Z}_2$ *and* $a = 0$ *or* $a = 1$ (*of course* $a = r = s = 0$ *if* n *is odd*). *Then* P *is a Poincaré duality space; its oriented homotopy type does not depend on the choice of* u_0 *in* 8.4; *and its invariants are given by* $w = a\bar{\iota}_2^2 \in H^2(P; \mathbf{Z}_2)$ *and* $e = s(a-1)\bar{e}_1^3 + r\bar{e}_2^3 \in H^3(P; \mathbf{Z}_2)$, *or, equivalently,* $e = r(\iota_1^2 \otimes 1) + s(a-1)(\iota_2^2 \otimes 1) \in H_2(P) \otimes \mathbf{Z}_2$.

Proof. P satisfies Poincaré duality by 8.5 and 8.6. From 8.7 it follows that the oriented homotopy type of P does not depend on the choice of u_0 in 8.4. From $S\tau_0 = 0$ and 8.5 we have $\lambda(\alpha) = ae_1^3$, therefore $w = a\bar{\iota}_2^2$ by 6.4 and 8.6.

It remains to calculate the exotic class of P. Let $\eta = \nu_P \mid X$ and $\beta = \beta(\eta) \in \{A \vee A, S^0\}$. By connectivity arguments, Theorem 8.1 reduces to

$$\{\alpha\} + (\{id\} \wedge \beta) \circ \gamma_2(\alpha) + (\{id\} \wedge \beta \wedge \beta) \circ \gamma_3(\alpha) = 0.$$

From (b) in 8.8 and from $\gamma_3(j_1 \circ \delta_0) = \{i_1 \wedge i_1 \wedge i_1\} \circ \gamma_3(\delta_0)$, which is true by naturality properties of the Hopf invariants, it follows that $\gamma_3(\alpha)$ is a sum of elements of the form $\iota_i^1 \wedge \iota_j^1 \wedge \iota_k^1$, where $j = 1$ or $k = 1$. Since $\langle w, \iota_1^2 \rangle = 0$, the fibration η is trivial over $S_1^2 \subset X$ (recall 5.1), so $\beta \circ \iota_1^1 = 0$. Both facts imply that the third summand above is zero, hence

$$\{\alpha\} + (\{id\} \wedge \beta) \circ \gamma_2(\alpha) = 0. \tag{8.10}$$

From the definition of α and 8.8 (a) we get (since $\gamma_2(\gamma) = 1$):

$$\begin{aligned}
\{\alpha\} &= a\{i_1 \circ S^{-1}\delta_0\} + r\{\iota_1^1 \circ S^{-1}\gamma \circ \gamma\} + s\{\iota_2^1 \circ S^{-1}\gamma \circ \gamma\} \\
\gamma_2(\alpha) &= (\{i_1 \wedge i_2\} - \{i_2 \wedge i_1\}) \circ \{u_0\} + a\{i_1 \wedge i_1\} \circ \gamma_2(\delta_0) \\
&\quad + r\{\iota_1^1 \wedge \iota_1^1\} \circ \{\gamma\} + s\{\iota_2^1 \wedge \iota_2^1\} \circ \{\gamma\}.
\end{aligned}$$

Defining $\beta_k = \beta \circ i_k \in \{A, S^0\}$ for $k = 1, 2$, it follows that (recall $\beta \circ \iota_1^1 = 0$)

$$(\{id\} \wedge \beta) \circ \gamma_2(\alpha) = (\{i_1\} \wedge \beta_2 - \{i_2\} \wedge \beta_1) \circ \{u_0\} + a(\{i_1\} \wedge \beta_1) \circ \gamma_2(\delta_0) \\ + s\{\iota_2^1 \wedge \beta_2 \circ \iota^1\} \circ \{\gamma\}.$$

Therefore 8.10 is the following equation in $\{S^3, A \vee A\}$:

$$a\{i_1 \circ S^{-1}\delta_0\} + r\{\iota_1^1 \circ S^{-1}\gamma \circ \gamma\} + s\{\iota_2^1 \circ S^{-1}\gamma \circ \gamma\} = \\ (\{i_1\} \wedge \beta_2 - \{i_2\} \wedge \beta_1) \circ \{u_0\} + a(\{i_1\} \wedge \beta_1) \circ \gamma_2(\delta_0) + s\{\iota_2^1 \wedge \beta_2 \circ \iota^1\} \circ \{\gamma\}.$$

Applying the retractions $r_1, r_2 : A \vee A \to A$ gives two equations in $\{S^3, A\}$, which may be written as follows (recall D_0 in 8.4):

$$D_0(\beta_2) = -a\{S^{-1}\delta_0\} - a(\{i_1\} \wedge \beta_1) \circ \gamma_2(\delta_0) + \{\iota^1 \circ S^{-1}\gamma \circ \gamma\} \tag{8.11}$$

$$D_0(\beta_1) = s\{\iota^1 \circ S^{-1}\gamma \circ \gamma\} + s\{\iota^1 \wedge \beta_2 \circ \iota^1\} \circ \{\gamma\}. \tag{8.12}$$

Suppose $a = 0$. Then 8.11 and 8.4 imply $\beta_2 = r\{S^{-2}\gamma \circ S^{-1}\gamma \circ \hat{e}^2\}$. Furthermore, $\langle w, \iota_2^2 \rangle = 0$, hence η is trivial on S_2^2 and so $\beta_2 \circ \iota^1 = 0$. Therefore 8.12 gives $\beta_1 = s\{S^{-2}\gamma \circ S^{-1}\gamma \circ \hat{e}^2\}$. Both facts imply $e = s\tilde{e}_1^3 + r\tilde{e}_2^3$, as stated.

Finally, let $a = 1$. Then $\langle w, \iota_2^2 \rangle = 1$, so $\eta \neq 0$ on S_2^2 and $\beta_2 \circ \iota^1 = \{S^{-2}\gamma\}$. Therefore 8.12 gives $D_0(\beta_1) = 0$, hence $\beta_1 = 0$, and therefore $\langle e, e_1^3 \rangle = 0$, as stated. Furthermore, 8.11 reduces to $D_0(\beta_2) = -\delta_0 + r(\iota^2 \circ \gamma \circ S\gamma)$, where we have identified $\{S^3, A\}$ and $\pi_4(SA)$. From this and from 8.5 we get that $\eta \mid S_2^2 \cup e_2^3$ has Stiefel–Whitney class $\tilde{\iota}_2^2$ and exotic class $r\tilde{e}_2^3$, and 8.9 is proved.

9. The Poincaré duality spaces and the proof of the classification theorem

We first describe models for Poincaré duality spaces which generate the semigroup OHP^5. These models are divided into five classes.

Class I. It only contains the sphere S^5 with system of invariants $I(S^5) = 0$.

Class II. It contains the unique Poincaré duality space P such that $H_2(P) \neq 0$ is a finite cyclic group.

Proof of existence: Take $P = (S^2 \cup_2 e^3) \cup e^5$ with the 5-cell attached by $\delta_0 \in \pi_4(S^2 \cup_2 e^3)$, see 8.5.

Proof of uniqueness. By 3.1 we have $H_2(P) = \mathbf{Z}_2$, hence $P = (S^2 \cup_2 e^3) \cup_\alpha e^5$ for some $\alpha \in \pi_4(S^2 \cup_2 e^3)$ such that $\tilde{e}^3 \cap \alpha = \iota^2$. By 6.1, this element is unique up to $\iota^2\circ\gamma\circ S\gamma$. Since by 7.11 there exists $f:X\simeq X$ such that $f\circ\alpha=\alpha+\iota^2\circ\gamma\circ S\gamma$, we see that P is uniquely determined. Following [1], we write $P=X_{-1}$. The invariants are

$$T_{-1}=I(X_{-1})=(\mathbf{Z}_2(x), b, w, 0) \quad \text{with} \quad b(x, x)=\langle w, x\rangle=\tfrac{1}{2}$$

and $(i, j, k)=(1, 0, 0)$; for we must have $b\neq 0$, and w and e are then determined by 2.1.

Class III. It contains the Poincaré duality spaces P such that $H_2(P)=\mathbf{Z}$. There are precisely three such spaces:

$$M_\infty=(S^2\vee S^3)\cup e^5 \text{ with } e^5 \text{ attached by } [\iota^2, \iota^3]$$
$$X_\infty=(S^2\vee S^3)\cup e^5 \text{ with } e^5 \text{ attached by } [\iota^2, \iota^3]+\iota^3\circ S\gamma$$
$$M'_\infty=(S^2\vee S^3)\cup e^5 \text{ with } e^5 \text{ attached by } [\iota^2, \iota^3]+\iota^2\circ\gamma\circ S\gamma.$$

The invariants are as follows (use 6.4, 6.2 for w and 8.3 for e):

$$S_\infty=I(M_\infty)=(\mathbf{Z}, 0, 0, 0) \text{ and } i=j=k=0$$
$$T_\infty=I(X_\infty)=(\mathbf{Z}, 0, w, 0) \text{ with } w\neq 0 \text{ and } i=\infty,\ j=k=0$$
$$S'_\infty=I(M'_\infty)=(\mathbf{Z}, 0, 0, e) \text{ with } e\neq 0 \text{ and } i=k=0,\ j=\infty.$$

These spaces are wellknown. M_∞ is simply $S^2\times S^3$. By [9], X_∞ is the total space of the non trivial S^3-bundle over S^2, and M'_∞ is the total space of the non trivial S^2-fibration over S^3 [5]. This gives other proofs that $e\neq 0$ for M'_∞. (Since $M'_\infty\to S^3$ is not stably equivalent to some bundle, its total space is not a manifold (Theorem 4 of [18]), and therefore has non zero exotic class. Compare also page 32 in [10].) There is one further candidate with $H_2=\mathbf{Z}$, namely $(S^2\vee S^3)\cup e^5$ with the 5-cell attached by $[\iota^2, \iota^3]+\iota^2\circ\gamma\circ S\gamma+\iota^3\circ S\gamma$. But the homotopy equivalence $\iota^2\mapsto\iota^3$, $\iota^3\mapsto\iota^3+\iota^2\circ\gamma$ of $S^2\vee S^3$ shows that it coincides with X_∞.

Class IV. It contains all Poincaré duality spaces P such that $H_2(P)=\mathbf{Z}_n\oplus\mathbf{Z}_n$ for some $n\geq 2$ and such that $w=0$. We may assume that $P=X\cup_\alpha e^5$ with $X=(S_1^2\cup_n e_1^3)\vee(S_2^2\cup_n e_2^3)$ and $b(\iota_1^2, \iota_2^2)=1/n$, $b(\iota_i^2, \iota_i^2)=0$ for $i=1, 2$ (compare the arguments following 9.1 below). Furthermore, if n is even and $e\neq 0$, we may assume that $e=\iota_1^2\otimes 1$. It follows that $\bigcap\alpha=\bigcap\tau_0$, where τ_0 is from 8.6, and (since $\lambda\alpha=0$ by 6.4)

$$\{\alpha\}=\{\tau_0+r(\iota_1^2\circ\gamma\circ S\gamma)+s(\iota_2^2\circ\gamma\circ S\gamma)\}\in\pi_4^s(X)$$

for some $r, s\in\mathbf{Z}_2$. Therefore, by 7.13, there exists a homotopy equivalence of X

sending α to this sum, and so we may assume

$$\alpha = \tau_0 + r(\iota_1^2 \circ \gamma \circ S\gamma) + s(\iota_2^2 \circ \gamma \circ S\gamma) \in \pi_4(X).$$

From 8.9 it follows that $r = s = 0$ if $e = 0$, and $r = 1$, $s = 0$ if $e \neq 0$. Thus class IV contains precisely the following spaces (recall 8.7):

$$M_n = (S_1^2 \cup_n e_1^3) \vee (S_2^2 \cup_n e_2^3) \cup e^5 \text{ with } e^5 \text{ attached by } \tau_0$$
$$M'_n = (S_1^2 \cup_n e_1^3) \vee (S_2^2 \cup_n e_2^3) \cup e^5 \text{ with } e^5 \text{ attached by } \tau_0 + \iota_1^2 \circ \gamma \circ S\gamma.$$

The spaces M'_n are only defined if n is even. The invariants are as follows:

$$S_n = I(M_n) = (\mathbf{Z}_n(x) \oplus \mathbf{Z}_n(y), b, 0, 0)$$
$$S'_n = I(M'_n) = (\mathbf{Z}_n(x) \oplus \mathbf{Z}_n(y), b, 0, x \otimes 1)$$

where in both cases $b(x, y) = 1/n$ and $b(x, x) = b(y, y) = 0$. Thus $i = j = k = 0$ for M_n and $i = k = 0$, $j = t$ for M'_n, where $n = 2^t s$ with s odd.

Class V. This class consists of all spaces P such that $H_2(P) = \mathbf{Z}_n \oplus \mathbf{Z}_n$ for some even integer $n \geq 2$, and such that $w \neq 0$. Again we may assume that $P = X \cup_\alpha e^5$ with X as above, and $b(\iota_1^2, \iota_2^2) = 1/n$, $b(\iota_1^2, \iota_1^2) = 0$ and $b(\iota_2^2, \iota_2^2) = \frac{1}{2}$. This gives the following formulas (recall 6.4):

$$\tilde{e}_1^3 \cap \alpha = \frac{n}{2} \iota_1^2 - \iota_2^2, \qquad \tilde{e}_2^3 \cap \alpha = \iota_1^2, \qquad w = \tilde{\iota}_2^2, \qquad \lambda\alpha = e_1^3.$$

From 8.5 and 8.6 it therefore follows that $\bigcap \alpha = \bigcap (\tau_0 + j_1 \circ \delta_0)$ and $\lambda\alpha = \lambda(\tau_0 + j_1 \circ \delta_0)$. Then, by 7.15, we may assume that

$$\alpha = \tau_0 + j_1 \circ \delta_0 + r(\iota_1^2 \circ \gamma \circ S\gamma) + s(\iota_2^2 \circ \gamma \circ S\gamma) + t[\iota_1^2, \iota_2^2] \circ S\gamma$$

for some $r, s, t \in \mathbf{Z}_2$. The homotopy equivalence f in the proof of 8.7 adds the term $\iota_1^2 \circ \gamma \circ S\gamma + [\iota_1^2, \iota_2^2] \circ S\gamma$ to the right hand side, so we may assume $t = 0$. Let $g: X \simeq X$ be the map induced by $e_1^3 \to \iota_2^2 \circ \gamma$ and $e_2^3 \to (n/2)(\iota_2^2 \circ \gamma)$, see 7.12. Then, by 7.11, 8.5 and 8.6, g adds the summand $\iota_2^2 \circ \gamma \circ S\gamma$, and hence we may also assume that $s = 0$. Finally, the coefficient r is determined by the exotic class: from 8.9 we have $r = 0$ if and only if $e = 0$. Thus class V contains for each even integer $n \geq 2$ precisely the following two spaces (recall 8.7):

$$X_n = (S_1^2 \cup_n e_1^3) \vee (S_2^2 \cup_n e_2^3) \cup e^5 \text{ with } e^5 \text{ attached by } \tau_0 + j_1 \circ \delta_0$$
$$X'_n = (S_1^2 \cup_n e_1^3) \vee (S_2^2 \cup_n e_2^3) \cup e^5 \text{ with } e^5 \text{ attached by } \tau_0 + j_1 \circ (\delta_0 + \iota^2 \circ \gamma \circ S\gamma)$$

The systems of invariants are

$$T_n = I(X_n) = (\mathbf{Z}_n(x) \oplus \mathbf{Z}_n(y), b, w, 0)$$
$$T'_n = I(X'_n) = (\mathbf{Z}_n(x) \oplus \mathbf{Z}_n(y), b, w, x \otimes 1)$$

where in both cases $b(x, y) = 1/n$, $b(x, x) = 0$ by $b(y, y) = \frac{1}{2}$. Thus if $n = 2^t s$ with s odd, we get $(i, j, k) = (t, 0, 0)$ in the first and $(i, j, k) = (t, t, 1)$ in the second case.

The following table shows all spaces together:

P	$I(P)$	$H_2(P)$	w	e
S^5	0	0	0	0
M_∞	S_∞	$\mathbf{Z}$	0	0
X_∞	T_∞	$\mathbf{Z}$	$\neq 0$	0
M'_∞	S'_∞	$\mathbf{Z}$	0	$\neq 0$
X_{-1}	T_{-1}	$\mathbf{Z}_2$	$\neq 0$	0
M_n	S_n	$\mathbf{Z}_n \oplus \mathbf{Z}_n$	0	0
M'_n	S'_n	$\mathbf{Z}_n \oplus \mathbf{Z}_n$	0	$\neq 0$
X_n	T_n	$\mathbf{Z}_n \oplus \mathbf{Z}_n$	$\neq 0$	0
X'_n	T'_n	$\mathbf{Z}_n \oplus \mathbf{Z}_n$	$\neq 0$	$\neq 0$

Now it is easy to prove that the function $OHP^5 \to J$ in Theorem 2.2 is surjective. In fact, from Proposition 3.2 it follows that the systems of invariants in the table above generate the semigroup J. Since all of these are realized by Poincaré duality spaces, it follows that $OHP^5 \to J$ is surjective.

To prove injectivity of $OHP^5 \to J$, we first observe that the discussion of the five classes above has shown the following:

9.1 PROPOSITION. *Given the dates in the last three columns of the table above, there is one and only one Poincaré duality space with these dates* (*namely the space in the left column and in the corresponding row*).

Now let P be an arbitrary simply connected 5-dimensional Poincaré duality space. We are going to prove that P is uniquely determined (up to oriented homotopy type) by its system of invariants $I(P) = (H_2(P), b, w, e)$, or, equivalently, by $H_2(P)$ and its invariants i_P, j_P and k_P.

Following 3.2, we choose a special basis $\{x_i \mid -s \leq i \leq 2t+1\}$ of $H_2(P)$ such that the following holds:

9.2 $|x_i| = \infty$ for $i \leq 0$; if $i_P = \infty$, then x_0 is the w-exceptional element.

9.3 For $j=1,3,\ldots,2t-1$ we have $|x_j|=|x_{j+1}|=k_j=k_{j+1}$, and $b(x_j,x_{j+1})=-b(x_{j+1},x_j)=1/k_j$. All other linking numbers between these elements are zero, except $b(x_2,x_2)=\frac{1}{2}$ in the almost symplectic case.

9.4 x_{2t+1} with $|x_{2t+1}|=k_{2t+1}=2$ is the w-exceptional element in the quasi-symplectic case.

9.5 If $j_P\neq 0$, we denote by j_0 the unique index between $-s$ and $2t$ such that x_{j_0} is the e-exceptional element (Lemma 2.1(b) implies $j_0\neq 2t+1$).

If $H_2(P)$ is free resp. finite, forget $x_1,\ldots,x_{2t+1}$ resp. $x_{-s},\ldots,x_0$; if we don't have the quasi-symplectic case, forget x_{2t+1}; and if $H_2(P)=0$, forget all: then $P\simeq S^5$.

Let X be the wedge of the following spaces:

$$X_i=S_i^2\vee S_i^3\quad (i\leq 0)$$
$$X_j=(S_j^2\cup e_j^3)\vee(S_{j+1}^2\cup e_{j+1}^3)\quad (j=1,3,\ldots,2t-1)$$
$$X_{2t+1}=S_{2t+1}^2\cup e_{2t+1}^3$$

Here, for $1\leq n\leq 2t+1$, the 3-cell e_n^3 is attached by a map of degree k_n. We may assume that $P=P^{(3)}\cup_\beta e^5$ for some $\beta\in\pi_4(P^{(3)})$. There exists a homotopy equivalence $f:X\to P^{(3)}$ such that $f_*(\iota_n^2)=x_n$ for all $-s\leq n\leq 2t+1$, and such that $f^*(D^{-1}x_i)=\iota_i^3$ for $i\leq 0$, where $D:H^3(P)\to H_2(P)$ is Poincaré duality. If $\alpha=f_*^{-1}(\beta)\in\pi_4(X)$, then $X\cup_\alpha e^5$ and P have the same oriented homotopy type. Therefore we may assume that $P=X\cup_\alpha e^5$, that the basis above is the geometric basis, i.e. $x_n=\iota_n^2$ for all n, and, furthermore, that $\tilde{\iota}_i^3\cap\alpha=\iota_i^2$ for $i\leq 0$. From this and from 9.3 and 9.4 it follows that $\bigcap\alpha$ maps $H^3(X_i)$ isomorphically onto $H_2(X_i)$ for $i\leq 0$ or $i=1,3,\ldots,2t+1$. Furthermore, since $w\cap\alpha=\lambda(\alpha)$ by 6.4, we have $\lambda(\alpha)=0$ (if $w=0$) or $\lambda(\alpha)=\iota_0^3$ (if $i_P=\infty$, by 9.2) or $\lambda(\alpha)=e_1^3$ (in the almost-symplectic case, by 9.3) or $\lambda(\alpha)=e_{2t+1}^3$ (in the quasi-symplectic case, by 9.4). In any case, we have all assumptions we need to apply the splitting principle 7.16, and by induction it follows that there exists a homotopy equivalence $f:X\to X$ with $f\mid X^{(2)}=id$ such that $f_*\alpha=\sum\alpha_i$ with $\alpha_i\in\pi_4(X_i)$. Therefore we may assume, without changing the properties of the basis $x_i=\iota_i^2$ above, that

$$\alpha=\alpha_{-s}+\cdots+\alpha_0+\alpha_1+\alpha_3+\cdots+\alpha_{2t-1}+\alpha_{2t+1}$$

where $\alpha_i\in\pi_4(X_i)$. This means that P splits as a connected sum

$$P=P_{-s}\#\cdots\# P_0\# P_1\# P_3\#\cdots\# P_{2t-1}\# P_{2t+1}$$

where $P_i = X_i \cup e^5$ with e^5 attached by α_i. Furthermore, the elements of the special basis $x_n = \iota_n^2$ which lie in $H_2(P_i)$ form a special basis of $H_2(P_i)$. Therefore we get from 9.2–9.5 and from Proposition 9.1 the following:

(a) $P_i = M_\infty$ for $i<0$ and $i \neq j_0$; if $i = j_0 < 0$, then $P_i = M'_\infty$.

(b) $P_0 = X_\infty$ if $j_P = \infty$ (then we must have $j_0 \neq 0$). If $j_P < \infty$, then $P_0 = M_\infty$ or $P_0 = M'_\infty$ according as $j_0 \neq 0$ or $j_0 = 0$.

(c) $P_1 = X_{k_1}$ in the almost-symplectic case, if $j_0 \neq 1$, and $P_1 = X'_{k_1}$ in the almost-symplectic case if $j_0 = 1$. If we do not have the almost symplectic case, then $P_1 = M_{k_1}$ or $P_1 = M'_{k_1}$ according as $j_0 \notin \{1, 2\}$ or $j_0 \in \{1, 2\}$.

(d) For $j = 3, 5, \ldots, 2t-1$ we have $P_j = M_{k_j}$ if $j_0 \notin \{j, j+1\}$ and $P_j = M'_{k_j}$ if $j_0 \in \{j, j+1\}$.

(e) $P_{2t+1} = X_{-1}$.

It follows that P is uniquely determined by the special basis above, hence by its invariants, and this completes the proof of the classification theorem.

10. A complete list of the Poincaré duality spaces

We now give a third version of our main theorem:

10.1 CLASSIFICATION THEOREM. *The following is a complete list of (oriented) homotopy types of simply connected 5-dimensional Poincaré duality spaces and their numerical invariants (i, j, k):*

P		$(0,0,0)$
$P \# X_{-1}$		$(1,0,0)$
$P \# X_{2^m}$	for $1 \leq m \leq \infty$	$(m,0,0)$
$P \# M'_{2^n}$	for $1 \leq n \leq \infty$	$(0,n,0)$
$P \# X_{-1} \# M'_{2^n}$	for $1 \leq n \leq \infty$	$(1,n,0)$
$P \# X_{2^m} \# M'_{2^n}$	for $1 \leq m, n \leq \infty$	$(m,n,0)$
$P \# X'_{2^n}$	for $1 \leq n < \infty$	$(n,n,1)$

Here $P = M_\infty \# \cdots \# M_\infty \# M_{k_1} \# \cdots \# M_{k_r}$ (s times M_∞) with prime powers $k_1, \ldots, k_r$ and $r, s \geq 0$ (if $r = s = 0$, then $P = S^5$).

Proof. Replacing the model spaces by their systems of invariants and using 2.2, this reduces to an easy algebraic exercise. Any system of invariants splits as a direct sum $I = I' + I''$, where I' has a free and I'' a finite group. I' is a direct sum of copies of S_∞, T_∞ and T'_∞, and I'' is a direct sum of the other generators in Section

9. In both cases it depends on (i, j, k) which summands can occur. Using this and the obvious isomorphisms ($n \geq 2$ even)

$$
\begin{aligned}
S'_\infty + T'_n &= S'_\infty + T_n \\
S'_\infty + S'_n &= S'_\infty + S_n \\
T_\infty + T'_n &= S_\infty + T'_n \\
T_\infty + T_n &= S_\infty + T_n \\
T_{-1} + T_\infty &= T_{-1} + S_\infty
\end{aligned}
$$

one gets the list above. It is complete since it members have different invariants (i, j, k) or different second homology group.

Remarks. (a) The summands in the list above are all indecomposable in the semigroup OHP^5, except X_2: we have $X_2 = X_{-1} \# X_{-1}$, since obviously $T_2 = T_{-1} + T_{-1}$.

(b) Here is an example for the importance of the linking order: the spaces $X'_n \# M_n$ and $X_n \# M'_n$ ($n \geq 2$ even) have the same second homology group, the same Stiefel–Whitney order and the same exotic order, but they have different linking order.

(c) Our notation differs from that in [1]: the spaces M_{2^n} and X_{2^n} are denoted by M_n and X_n, respectively, in [1].

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Received July 27, 1981/June 25, 1982