

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 57 (1982)

Artikel: Invariant differential operators in hyperbolic spaces.
Autor: Reimann, H.M.
DOI: <https://doi.org/10.5169/seals-43894>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Invariant differential operators in hyperbolic space

H. M. REIMANN

1. Introduction

The conformal mappings in real higher dimensional space \mathbf{R}^n , $n \geq 3$, are the proper Möbiustransformations. The group $GM(n)$ of Möbiustransformations acts on $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ and there is a subgroup isomorphic to $GM(n-1)$ which stabilizes the unit ball B . It is the action of this group $GM(n-1)$ and the induced action on functions on the hyperbolic space B that will be studied.

The differentiation process leads from functions to vectorfields and tensorfields of higher order. There is a natural setting which reduces the analysis of at least the symmetric tensors with vanishing traces to the study of functions on a bigger space X . Whereas the hyperbolic space B is isomorphic to $O_{\pm}(1, n)/O(n)$ this space X is isomorphic to the quotient space $O_{\pm}(1, n)/O(n-1)$. Geometrically it can be described as the cosphere bundle of the hyperbolic space B . The action of the Möbius group $GM(n-1)$ on X essentially is the action of $GM(n-1)$ on the cotangent space of B .

The approach described here, whereby certain tensorfields on B are interpreted as functions on X , is inspired by a similar construction for the sphere (see Levine [4]). The purpose of that construction was the characterization of invariant systems of singular differential operators on the sphere. In both cases the conformal structure seems to be essential.

The space $C(X)$ of functions on X can be split into a direct sum of subspaces

$$C(X) = \bigoplus_{k=0}^{\infty} E^k$$

The functions in E^k have an interpretation as tensorfields of symmetric tensors with vanishing traces. Their analysis is in a certain sense complementary to the analysis of differential forms, which in the tensor language is a theory of antisymmetric tensors. Certain striking analogies are apparent. The invariant operators S_k and S_k^* defined in Section 5, Theorem 7, are generalizations of the operators grad and div. They play a role similar to the operators d and d^* for differential forms (see Theorems 7 and 9). In particular, S_k maps E^k into E^{k+1}

and S_k^* maps E^k into E^{k-1} . It is shown that the operators S_1 and S_2^* coincide with certain operators studied by Ahlfors [1] (see Theorem 8).

There exists an invariant differential operator D_Z on X which is of first order. As a consequence, the space of solutions of $D_Z f = 0$ is an algebra. The functions $f \in E^1$ which satisfy $D_Z f = 0$ are exactly those which correspond to vectorfields v in the Lie algebra of the Möbius group (Theorem 6).

The algebra of invariant differential operators on X is not commutative. It is generated by 1, the first order differential operator D_Z and a further differential operator $D_{|Y|^2}$ of second order (Theorems 1 and 2). The operator $D_{|Y|^2}$ is basically the Laplace-operator on the sphere $O(n)/O(n-1)$. The spaces E^k appear as eigenspaces of $D_{|Y|^2}$. The Laplace-Casimir operator Δ_X on X preserves the eigenspaces (Theorem 9).

2. The Möbius group and its Lie algebra

The Möbius group $GM(n)$ is the transformation group of $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ which is generated by reflections in the spheres and hyperplanes of \mathbf{R}^n . The group is isomorphic to $O_{\pm}(1, n+1)$, the subgroup of $O(1, n+1)$ which preserves the positive cone:

$$\left\{ y \in \mathbf{R}^{n+2} : \langle y, y \rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2 > 0, y_0 > 0 \right\}$$

(see Mostow [5]). The isomorphism is constructed in the following way: The group $O(1, n+1)$ leaves invariant the quadratic form $\langle y, y \rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2$ and in particular the cone $\{y \in \mathbf{R}^{n+2} : \langle y, y \rangle = 0\}$. If inhomogeneous coordinates $\eta_i = y_i/y_0$ are introduced, the group becomes a transformation group of the sphere $\Sigma = \{\eta \in \mathbf{R}^{n+1} : |\eta| = 1\}$ and the elements g and $-g$ give rise to the same transformation. Stereographic projection from the point $\varepsilon_n = (0, \dots, 0, 1)$ onto the plane $\eta_{n+1} = 0$ then leads to the realization of $O_{\pm}(1, n+1)$ as a transformation group of $\hat{\mathbf{R}}^n$. The subgroup of the Möbius group $GM(n)$, which stabilizes the unit ball $B \subset \mathbf{R}^n$ is isomorphic to the Möbius-group $GM(n-1)$ of one lower dimension. This group which acts on B will again be denoted by $GM(n-1)$. Observe that under the above isomorphism this is exactly the subgroup $O_{\pm}(1, n)$ of $O_{\pm}(1, n+1)$ which stabilizes the lower half space in \mathbf{R}^{n+2} . The elements in matrix notation have the special form

$$g = \begin{pmatrix} & & & 0 \\ & g_{ij} & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad i, j = 0, 1, \dots, n \quad g_{00} > 0$$

Our main concern is with this group $G = GM(n-1)$, $n \geq 3$, which is the group of conformal and anti-conformal mappings of the unit ball $B \subset \mathbf{R}^n$ onto itself. Referring to the isomorphism $GM(n-1) \cong O_{\pm}(1, n)$ we will speak about the geometric realization of the group, if we consider it as a transformation group of B . The algebraic realization then refers to the group as a matrix group.

The unit ball B has the structure of a symmetric space (the hyperbolic space) $B = G/K$ with the invariant metric $ds^2 = \rho^2 |dx|^2$, $\rho(x) = (1 - |x|^2)^{-1}$. The stabilizer K of the origin is the orthogonal group. We start with an explicit description of the action of $G = GM(n-1)$ on $B \subset \mathbf{R}^n$.

The stereographic projection of the sphere $\Sigma = \{\eta \in \mathbf{R}^{n+1} : |\eta| = 1\}$ onto the plane $\eta_{n+1} = 0$ is given by the formula

$$x_i = \frac{\eta_i}{1 - \eta_{n+1}} \quad i = 1, \dots, n$$

and the inverse mapping is

$$\eta_i = \frac{2x_i}{1 + |x|^2} \quad i = 1, \dots, n$$

$$\eta_{n+1} = \frac{|x|^2 - 1}{|x|^2 + 1}$$

Let $g = (g_{ij})$ be an element in $O_{\pm}(1, n)$ and consider $O_{\pm}(1, n)$ as the subgroup of $O_{\pm}(1, n+1)$ which stabilizes the unit vector $e_{n+1} = (0, \dots, 0, 1) \in \mathbf{R}^{n+2}$. The image of the half line $y = t(e_0 - e_{n+1})$ $t > 0$ is the half line

$$t(ge_0 - ge_{n+1}) = t(g_{00}, \dots, g_{n0}, -1)$$

which in turn is mapped onto the point

$$\eta = \frac{1}{g_{00}} (g_{10}, \dots, g_{n0}, -1)$$

Under stereographic projection this point projects onto

$$x = \frac{1}{1 + g_{00}} (g_{10}, \dots, g_{n0}) \in B \tag{2.1}$$

If g is in the subgroup $O(n)$ of $O_{\pm}(1, n)$, then $g_{00} = 1$ and the corresponding point

tangent vector ξ at $x = (1/(1 + g_{00}))(g_{10}, \dots, g_{n0})$ with coordinates

$$\xi_i = \frac{2g_{0n}g_{i0}}{(1 + g_{00})^2} - \frac{2g_{in}}{1 + g_{00}} \quad i = 1, \dots, n \quad (2.3)$$

The invariance of the quadratic form $\langle y, y \rangle$ implies

$$\begin{aligned} 1 &= g_{00}^2 - \sum_{i=1}^n g_{i0}^2 \\ -1 &= g_{0k}^2 - \sum_{i=1}^n g_{ik}^2 \quad k = 1, \dots, n \\ 0 &= g_{00}g_{0k} - \sum_{i=1}^n g_{ik}g_{i0} \end{aligned} \quad (2.4)$$

and it follows that

$$\begin{aligned} |x|^2 &= (1 + g_{00})^{-2} \sum_{i=1}^n g_{i0}^2 = \frac{g_{00} - 1}{g_{00} + 1} \\ \frac{2}{1 + g_{00}} &= 1 - |x|^2 \end{aligned} \quad (2.5)$$

if $x = \tau_g(0)$. The length $|\xi| = (\sum_{i=1}^n \xi_i^2)^{1/2}$ of the tangent vector ξ can now easily be calculated to be $1 - |x|^2$

$$\begin{aligned} \frac{1}{(1 - |x|^2)^2} |\xi|^2 &= 4^{-1} (1 + g_{00})^2 |\xi|^2 \\ &= (1 + g_{00})^{-2} g_{0n}^2 (g_{00}^2 - 1) - (1 + g_{00})^{-1} 2g_{0n}^2 g_{00} + g_{0n}^2 + 1 \\ &= (1 + g_{00})^{-1} g_{0n}^2 (g_{00} - 1 - 2g_{00}) + g_{0n}^2 + 1 = 1 \\ |\xi| &= 1 - |x|^2 \end{aligned} \quad (2.6)$$

This proves the invariance of the metric

$$ds^2 = \rho^2 |dx|^2 \quad \rho = (1 - |x|^2)^{-1}$$

and the conformality (or anti-conformality) of the transformations τ_g .

Next we define the subgroup M of the Möbius group $\tilde{G} = GM(n-1)$ as the stabilizer of both the origin and the tangent vector ε_n at the origin in B . M is a

subgroup of K . In the algebraic picture this is the orthogonal group

$$O(n-1) = \left\{ g \in O_{\pm}(1, n) : g = \begin{pmatrix} 1 & & \\ & * & \\ & & 1 \end{pmatrix} \right\} \cong M \quad (2.7)$$

The cosets are parametrized by the geometric parameters $x = \tau_g(0)$ and $\xi = d\tau_g(0)\varepsilon_n$. We call the pair (x, ξ) the coordinates for the coset $gO(n-1)$. The equations (2.1) and (2.3) express these coordinates by the matrix elements g_{ij} of g . Geometrically, the quotient space G/M can be realized as the cosphere bundle X of B . Since $|\xi| = 1 - |x|^2$, the group $GM(n-1)$ acts on

$$X = \{(x, \xi) \in B \times \mathbf{R}^n : |\xi| = 1 - |x|^2\} \quad (2.8)$$

and the action is seen to be transitive. It can be described by the formula

$$(x, \xi) \rightarrow (\tau_g x, d\tau_g(x)\xi) \quad (2.9)$$

where $d\tau_g(x)$ is the cotangent mapping which maps the cotangent space at x onto the cotangent space at $z = \tau_g x$.

We now turn to a description of the Lie algebra \mathfrak{g} of $O_{\pm}(1, n)$. Let $E_{ij} \in GL(n+1)$ denote the matrix with element 1 at the place i, j and zero otherwise. A basis for the Lie algebra of $O_{\pm}(1, n)$ is given by the matrices

$$X_{0j} = E_{0j} + E_{j0} \quad j = 1, \dots, n$$

and (2.10)

$$X_{ij} = E_{ij} - E_{ji} \quad 1 \leq i < j \leq n$$

We set

$$X_i = X_{0i} \quad i = 1, \dots, n-1$$

$$Z = X_{0n} \quad (2.11)$$

$$Y_i = X_{in} \quad i = 1, \dots, n-1$$

The stabilizer $O(n)$ of $e_0 \in \mathbf{R}^{n+1}$ is a maximal compact subgroup in $O_{\pm}(1, n)$ and $O_{\pm}(1, n)/O(n) \cong B$ is a symmetric space of rank one. In the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the subalgebra \mathfrak{k} has the vectorspace basis $\{X_{ij} : 1 \leq i < j \leq n\}$ and \mathfrak{p} is the

linear subspace with basis $\{X_{0j} : j = 1, \dots, n\}$. The commutator relations

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \quad (2.12)$$

$$[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p} \quad (2.13)$$

hold. A maximal abelian subalgebra in \mathfrak{p} is given by $\mathfrak{a} = \mathbf{R}Z$, it is one dimensional. If the corresponding subgroup is denoted by A , then the subgroup $O(n-1) \cong M$ defined above (2.7) is the centralizer of A in $O(n) \cong K$. Its Lie algebra \mathfrak{m} has the basis $\{X_{ij} : 1 \leq i < j \leq n-1\}$

The commutator relations are as follows

$$\begin{aligned} [\mathfrak{m}, Z] &= 0 \\ [X_i, Z] &= Y_i \quad [X_i, X_{ij}] = X_j \quad 1 \leq i < j \leq n-1 \\ [Y_i, Z] &= X_i \quad [Y_i, X_{ij}] = Y_j \quad 1 \leq i < j \leq n-1 \\ [X_i, X_j] &= X_{ij} \quad [Y_i, Y_j] = -X_{ij} \quad 1 \leq i < j \leq n-1 \\ [X_i, Y_j] &= \delta_{ij}Z \quad i, j = 1, \dots, n-1 \end{aligned} \quad (2.14)$$

In particular it should be noted that if \mathfrak{q} is the linear subspace with basis $\{X_1, \dots, X_{n-1}, Z, Y_1, \dots, Y_{n-1}\}$ then

$$[\mathfrak{q}, \mathfrak{m}] \subset \mathfrak{q} \quad (2.15)$$

which shows that G/M is a reductive coset space (see Section 3). $\{X_i - Y_i : i = 1, \dots, n-1\}$ is a basis of the α -root space \mathfrak{n} of the pair $(\mathfrak{g}, \mathfrak{a})$:

$$[tZ, X_i - Y_i] = t(X_i - Y_i), \quad \alpha(tZ) = t$$

whereas $\bar{\mathfrak{n}}$ is given by $\{X_i + Y_i : i = 1, \dots, n-1\}$.

The Weyl group $W = O'(n-1)/O(n-1)$ where $O'(n-1)$ and $O(n-1)$ are the normalizer and centralizer of A in $O(n) = K$ consists of two elements only. They are represented by the identity and the matrix

$$w = \begin{pmatrix} 1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \\ & & & & & -1 \end{pmatrix} \quad (2.16)$$

The mapping ω , which maps the cosets $gO(n-1)$ onto the cosets $gwO(n-1)$ can geometrically be described by the formula

$$\omega(x, \xi) = (x, -\xi) \quad (2.17)$$

This mapping is not a Möbiustransformation on X .

Geometrically, the Lie algebra of $G = GM(n-1)$ is given by the vectorfields on B which generate the one parameter subgroups τ_{g_i} of G . The vectorfields are determined by the equation

$$v(x) = \left. \frac{d}{dt} \tau_{g_i}(x) \right|_{t=0}$$

Conversely, the one parameter subgroup τ_{g_i} is obtained from the vectorfield v by solving the differential equation

$$\frac{dz}{dt} = v(z)$$

with initial condition $z(0) = x$. The one parameter subgroup is then given by $\tau_{g_i}(x) = z(t)$.

In a first step the vectorfields on \mathbf{R}^n are determined, which are the infinitesimal generators of the one parameter subgroups of the group $GM(n)$ acting on $\hat{\mathbf{R}}^n$. The vectorfields in the Lie algebra of $GM(n-1)$ are then singled out by the condition

$$(v(x), x) = 0 \quad \text{for } |x| = 1 \quad (2.18)$$

The vectorfield v has to be tangent to the boundary of $B \subset \mathbf{R}^n$. The vectorfields in the Lie algebra of $GM(n)$ are

$$v(x) = a + Bx + \lambda x + c|x|^2 - 2x(c, x) \quad (2.19)$$

with a, c constant vectors in \mathbf{R}^n , B a constant matrix with $B' = -B$ and $\lambda \in \mathbf{R}$. The vectorfields Bx account for the rotations in \mathbf{R}^n (the subgroup M with respect to $GM(n)$), the constant vectors a for the translations (the subgroup N) and λx for the dilations (the subgroup A). The remaining vectorfields $c|x|^2 - 2x(c, x)$ generate the one parameter subgroups τ_{g_i} conjugate to the translations (the subgroup \bar{N}):

$$s \circ \tau_{g_i} \circ s(x) = x + ct$$

where s is the reflection in the unit sphere. The vectorfields in the Lie algebra of $GM(n-1)$ can easily be singled out by condition (2.18). The restrictions are $\lambda = 0$ and

$$(a, x) - (c, x) = 0 \quad \text{for} \quad |x| = 1$$

The Lie algebra of $GM(n-1)$ is therefore described by the vectorfields

$$v(x) = Bx + c(1 + |x|^2) - 2x(c, x) \quad (2.20)$$

The vectorfields Bx now correspond to the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and the remaining vectorfields to the complementary subspace $\mathfrak{p} \subset \mathfrak{g}$.

3. Invariant differential operators

The group $O_{\pm}(1, n)$ is not connected. The connected component of the identity is the subgroup $SO_{\pm}(1, n)$. The spaces $O_{\pm}(1, n)/O(n-1)$ and $SO_{\pm}(1, n)/SO(n-1)$ are isomorphic coset spaces with in the first instance the group $O_{\pm}(1, n)$, in the second the group $SO_{\pm}(1, n)$ acting by left translations.

DEFINITION (Nomizu [6]). Let G be a connected Lie group with Lie algebra \mathfrak{g} and denote the adjoint representation of G on \mathfrak{g} by $\text{Ad}(g)$. Assume that M is a closed subgroup with Lie algebra \mathfrak{m} . The coset space G/M is reductive, if there exists a subspace \mathfrak{q} of \mathfrak{g} , complementary to \mathfrak{m} , such that $\text{Ad}(m)\mathfrak{q} \subset \mathfrak{q}$ for all $m \in M$.

Upon taking $G = SO_{\pm}(1, n)$ and $M = SO(n-1)$ one finds that the subspace \mathfrak{q} with basis $\{X_1, \dots, X_{n-1}, Z, Y_1, \dots, Y_{n-1}\}$ is complementary to the Lie algebra \mathfrak{m} of M and that furthermore $[\mathfrak{m}, \mathfrak{q}] \subset \mathfrak{q}$ (see (2.11) and (2.15)). Since M is connected, this implies $\text{Ad}(m)\mathfrak{q} \subset \mathfrak{q}$ for all $m \in M$. The coset space $X = SO_{\pm}(1, n)/SO(n-1)$ (with $SO_{\pm}(1, n)$ acting on it by left translation) is therefore reductive.

By definition, the differential operator D on G/M is invariant (with respect to left translations $\tau^g f(x) = f(\tau_{g^{-1}}x)$) if $D\tau^g f = \tau^g Df$ for all $f \in C_c(G/M)$ and for all $g \in G$. The algebra of invariant differential operators is denoted by $\underline{D}(G/M)$. It can be determined on the base of a theorem of Helgason [3]. For this purpose let $I(\mathfrak{q})$ denote the polynomials in the symmetric algebra $S(\mathfrak{q})$ over \mathfrak{q} , which are invariant under $\text{Ad}(m)$ for all $m \in M$. The polynomials in $S(\mathfrak{q})$ are polynomials in the variables Z_1, \dots, Z_k where $\{Z_1, \dots, Z_k\}$ is a basis in M .

The symmetrization mapping λ associates with every polynomial $Q \in S(\mathfrak{q})$ a differential operator on the group G . Symmetrization is a linear mapping, which maps the elements $Y_1 Y_2 \cdots Y_p \in S(\mathfrak{q})$ (where the Y_j are elements in the subspace \mathfrak{q} of \mathfrak{g} , $j = 1, \dots, p$) onto the differential operator

$$\lambda(Y_1 Y_2 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma} Y_{\sigma(1)} \cdot Y_{\sigma(2)} \cdots Y_{\sigma(p)}$$

In this sum σ runs over the symmetric group on p letters. In particular, $\lambda(Y)$ is the differential operator defined by the Lie algebra element $Y \in \mathfrak{g}$

THEOREM (Helgason). *Let G/M be a reductive coset space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$, $\text{Ad}(m)\mathfrak{q} \subset \mathfrak{q}$ for all $m \in M$. Then there exists a linear bijection of $I(\mathfrak{q})$ onto $D(G/M)$. It associates to the polynomial $Q(Z_1, \dots, Z_k) \in I(\mathfrak{q})$ the differential operator D_Q which can be determined by one of the equivalent methods:*

$$(1) D_Q f(x) = Q\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_k}\right) f \circ \pi\left(g \exp \sum_{i=1}^k t_i Z_i\right) \Big|_{t=0} \tag{3.1}$$

where π is the canonical projection of G onto G/M , $\pi(g) = x$.

$$(2) \lambda(Q)(f \circ \pi) = D_Q f \circ \pi \tag{3.2}$$

This formula defines $D_Q f$, since $\lambda(Q)(f \circ \pi)$ is constant on each coset gM if $f \in C_c^\infty(G/M)$.

THEOREM 1. *Let $G = \text{SO}_\pm(1, n)$, $M = \text{SO}(n-1)$ and $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$ with the specified basis $\{X_1, \dots, X_{n-1}, Z, Y_1, \dots, Y_{n-1}\}$ for \mathfrak{q} (see Section 2). Then the algebra $I(\mathfrak{q})$ of $\text{Ad}(M)$ invariant polynomials is generated by the polynomials*

$$1, \quad Z, \quad |X|^2 = \sum_{i=1}^{n-1} X_i^2, \quad (X, Y) = \sum_{i=1}^{n-1} X_i Y_i, \quad |Y|^2 = \sum_{i=1}^{n-1} Y_i^2.$$

We calculate the action of $\text{Ad}(m)$. If $X \in \mathfrak{m}$, $Y \in \mathfrak{q}$ then

$$\text{Ad}(\exp tX) Y = e^{t \text{ad } X} Y = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad } X)^n Y$$

Set $X = X_{ij} \in \mathfrak{m}$ and $Y = Z_i$, which stands for X_i or $Y_i \in \mathfrak{q}$. Then

$$(\text{ad } X_{ij})Z_i = [X_{ij}, Z_i] = -Z_j$$

$$(\text{ad } X_{ij})Z_j = Z_i, \quad (\text{ad } X_{ij})Z_k = 0 \quad k \neq i, j$$

$$\begin{aligned} \text{Ad}(\exp tX_{ij})Z_i &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} Z_i - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} Z_j \\ &= Z_i \cos t - Z_j \sin t \end{aligned}$$

$$\text{Ad}(\exp tX_{ij})Z_j = Z_i \sin t + Z_j \cos t$$

It follows that

$$\text{Ad}(m)X_k = \sum_{h=1}^{n-1} m_{hk} X_h$$

for $m = \exp tX_{ij} \in SO(n-1) \subset GL(n-1)$ with $m = (m_{hk})$. This equation therefore holds for all $m \in SO(n-1)$. Furthermore, if $X = \sum_{k=1}^{n-1} x_k X_k$, then $\text{Ad}(m)X = \sum_{h=1}^{n-1} x'_h X_h$ with $x' = mx$. Similarly, if $Y = \sum_{k=1}^{n-1} y_k Y_k$ then $\text{Ad}(m)Y = \sum_{h=1}^{n-1} y'_h Y_h$ with $y' = my$. Finally, since $\text{Ad}(m)zZ = zZ$ ($z \in \mathbf{R}$), the action of $\text{Ad}(m)$ on the polynomials $P(x, y, z)$ in the variables $x, y \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$ is given by

$$\text{Ad}(m)P(x, y, z) = P(mx, my, z)$$

Assume now that the polynomial Q is invariant under the action of $\text{Ad}(M)$. It can then be written as a finite sum

$$Q(x, y, z) = \sum_k z^k Q_k(x, y)$$

with invariant polynomials $Q_k(x, y)$. It is well known (see e.g. Weyl [7] p. 31 ff.) that the invariant polynomials in the variables x, y under the action

$$(x, y) \rightarrow (mx, my) \quad m \in SO(n-1)$$

are generated by the polynomials 1 , $|x|^2 = \sum_{i=1}^{n-1} x_i^2$, $(x, y) = \sum_{i=1}^{n-1} x_i y_i$ and $|y|^2 = \sum_{i=1}^{n-1} y_i^2$. This proves the theorem.

The invariant operators 1 , D_Z , $D_{|x|^2}$, $D_{(x, Y)}$ and $D_{|y|^2}$ generate the whole

algebra $\underline{D}(G/M)$. This follows from the fact that

$$D_{P_1 P_2} = D_{P_1} \cdot D_{P_2} + D$$

where the order of the invariant differential operator D is less than the sum of the degrees of the polynomials P_1 and P_2 (see Helgason [3] p. 269). In the present situation there is however more that can be said:

THEOREM 2. *The differential operators satisfy the following commutator relations:*

$$[D_Z, D_{|X|^2}] = -2D_{(X, Y)} \quad (3.3)$$

$$[D_Z, D_{|Y|^2}] = -2D_{(X, Y)} \quad (3.4)$$

$$[D_Z, D_{(X, Y)}] = -D_{|X|^2} - D_{|Y|^2} \quad (3.5)$$

Consequently, $\underline{D}(G/M)$ is generated by 1, D_Z and $D_{|Y|^2}$ (or by 1, D_Z and $D_{|X|^2}$).

The proof relies on the symmetrization mapping λ . The differential operator $D_{Z|Y|^2}$ is obtained from the differential operator on G which is given by

$$\lambda(Z|Y|^2) = \frac{1}{3!} \sum_{i=1}^{n-1} 2(Y_i \cdot Y_i \cdot Z + Y_i \cdot Z \cdot Y_i + Z \cdot Y_i \cdot Y_i)$$

The commutator relations for the Lie algebra (2.14) then imply

$$\lambda(Z|Y|^2) = \sum_{i=1}^{n-1} Y_i \cdot Y_i \cdot Z - \frac{1}{2} \sum_{i=1}^{n-1} (X_i \cdot Y_i + Y_i \cdot X_i) + \frac{n-1}{6} Z$$

$$\lambda(Z|Y|^2) = \sum_{i=1}^{n-1} Z \cdot Y_i \cdot Y_i + \frac{1}{2} \sum_{i=1}^{n-1} (X_i \cdot Y_i + Y_i \cdot X_i) + \frac{n-1}{6} Z$$

It follows that

$$D_{|Y|^2} D_Z - D_{(X, Y)} + \frac{n-1}{6} D_Z = D_Z D_{|Y|^2} + D_{(X, Y)} + \frac{n-1}{6} D_Z$$

which proves the first equality. The second is proved in the same way and the

third is a consequence of the following equations:

$$\begin{aligned} \lambda\left(\sum_{i=1}^{n-1} X_i Y_i Z\right) &= \frac{1}{3} \sum_{i=1}^{n-1} (X_i \cdot Y_i \cdot Z + Y_i \cdot X_i \cdot Z + X_i \cdot Z \cdot Y_i + Y_i \cdot Z \cdot X_i \\ &\quad + Z \cdot X_i \cdot Y_i + Z \cdot Y_i \cdot X_i) \\ &= \frac{1}{2} \sum_{i=1}^{n-1} (X_i \cdot Y_i + Y_i \cdot X_i) \cdot Z - \frac{1}{2} \sum_{i=1}^{n-1} (X_i \cdot X_i + Y_i \cdot Y_i) \\ &= \frac{1}{2} \sum_{i=1}^{n-1} Z \cdot (X_i \cdot Y_i + Y_i \cdot X_i) + \frac{1}{2} \sum_{i=1}^{n-1} (X_i \cdot X_i + Y_i \cdot Y_i) \end{aligned}$$

$$D_{(X, Y)} D_Z - \frac{1}{2} D_{|X|^2} - \frac{1}{2} D_{|Y|^2} = D_Z D_{(X, Y)} + \frac{1}{2} D_{|X|^2} + \frac{1}{2} D_{|Y|^2}$$

The Killing form on the Lie algebra of $SO(1, n)$ is given by

$$B(X, X) = 2(n-1) \left\{ \sum_{i=1}^n x_i^2 - \sum_{1 \leq i < j \leq n} x_{ij}^2 \right\}$$

$$X = \sum_{i=1}^n x_i X_i + \sum_{1 \leq i < j \leq n} x_{ij} X_{ij}$$

(see the definitions (2.10) and (2.11) in section 2). The Killing form is invariant under $\text{Ad}(g)$ for all $g \in SO(1, n)$ and in particular for $g \in SO(n)$ or $SO(n-1)$. The Casimir operator restricted to $B \cong SO_{\pm}(1, n)/SO(n)$ is

$$\Delta_K = D_{|X|^2} + D_{Z^2} \tag{3.6}$$

and restricted to $X \cong SO_{\pm}(1, n)/SO(n-1)$ it is

$$\Delta_M = D_{|X|^2} + D_{Z^2} - D_{|Y|^2} \tag{3.7}$$

It follows that the operators Δ_K and Δ_M , considered as operators in $\underline{D}(G/M)$ commute. In fact, Δ_M commutes with every differential operator in $\underline{D}(G/M)$.

In the next section it will be shown that the operators in $\underline{D}(G/M)$ are invariant under the whole group $O_{\pm}(1, n)$ and not only under the subgroup $SO_{\pm}(1, n)$.

4. The calculations for some operators

In this section the geometric versions of the operators D_Z , $D_{|Y|^2}$ and $D_{(X, Y)}$ will be calculated. This means that the operators will be expressed as differential

operators in the variables (x, ξ) . Recall that

$$x_i = (1 + g_{00})^{-1} g_{i0} \quad (2.1)$$

and

$$\begin{aligned} \xi_i &= 2g_{0n}g_{i0}(1 + g_{00})^{-2} - 2g_{in}(1 + g_{00})^{-1} \\ &= 2(g_{0n}x_i - g_{in})(1 + g_{00})^{-1} \end{aligned} \quad (2.3)$$

$i = 1, \dots, n$ are the coordinates for the coset $gO(n-1)$. The matrices (g_{ij}) representing g satisfy the relations (2.4) and in particular

$$2(1 + g_{00})^{-1} = 1 - |x|^2 = |\xi|^2 \quad (2.5) \quad (2.6)$$

and

$$\begin{aligned} (x | \xi) &= \sum_{i=1}^n x_i \xi_i = \frac{2}{1 + g_{00}} \left(g_{0n} |x|^2 - (1 + g_{00})^{-1} \sum_{i=1}^n g_{i0} g_{in} \right) \\ &= 2g_{0n}(1 + g_{00})^{-1} (|x|^2 - g_{00}(1 + g_{00})^{-1}) \\ &= -\frac{1}{2}g_{0n}(1 - |x|^2)^2 = -2g_{0n}(1 + g_{00})^{-2} \end{aligned} \quad (4.1)$$

Let $a_t = \exp tZ$ denote the one parameter subgroup of $O_{\pm}(1, n)$ defined by Z . In order to calculate $D_Z f$ at the point (x, ξ) (coordinates of the coset $gO(n-1)$), the definition of Lie derivatives is used:

$$D_Z f(x, \xi) = \left. \frac{d}{dt} f(x_t, \xi_t) \right|_{t=0} \quad (4.2)$$

where (x_t, ξ_t) are the coordinates of the coset $ga_t O(n-1)$:

$$(x_t)_i = (g_{i0} \operatorname{Ch} t + g_{in} \operatorname{Sh} t)(1 + g_{00} \operatorname{Ch} t + g_{0n} \operatorname{Sh} t)^{-1} \quad (4.3)$$

$$\begin{aligned} (\xi_t)_i &= 2(g_{00} \operatorname{Sh} t + g_{0n} \operatorname{Ch} t)(g_{i0} \operatorname{Ch} t + g_{in} \operatorname{Sh} t)(1 + g_{00} \operatorname{Ch} t + g_{0n} \operatorname{Sh} t)^{-2} \\ &\quad - 2(g_{i0} \operatorname{Sh} t + g_{in} \operatorname{Ch} t)(1 + g_{00} \operatorname{Ch} t + g_{0n} \operatorname{Sh} t)^{-1} \end{aligned} \quad (4.4)$$

It follows that

$$\left. \frac{d(x_t)_i}{dt} \right|_{t=0} = g_{in}(1 + g_{00})^{-1} - g_{i0}g_{0n}(1 + g_{00})^{-2} = -\frac{1}{2}\xi_i \quad (4.5)$$

and after some calculations

$$\begin{aligned} \left. \frac{d(\xi_t)_i}{dt} \right|_{t=0} &= -2g_{0n}(1+g_{00})^{-1}\xi_i - 2(1+g_{00})^{-1}x_i \\ &= (1-|x|^2)^{-1}(2(x|\xi)\xi_i - |\xi|^2x_i) \end{aligned} \quad (4.6)$$

The operator D_Z can be expressed by the formula

$$D_Z f(x, \xi) = \sum_{i=1}^n f_{x_i} \left. \frac{dx_t}{dt} \right|_{t=0} + \sum_{i=1}^n f_{\xi_i} \left. \frac{d\xi_t}{dt} \right|_{t=0} \quad (4.7)$$

THEOREM 3.

$$D_Z f(x, \xi) = -\frac{1}{2} \sum_{i=1}^n f_{x_i} \xi_i + (1-|x|^2)^{-1} \sum_{i=1}^n f_{\xi_i} (2(x|\xi)\xi_i - |\xi|^2x_i) \quad (4.8)$$

This operator is invariant under the group $GM(n-1)$ of Möbiustransformations on X . Under the mapping $\omega(x, \xi) = (x, -\xi)$ it transforms into the operator $-D_Z$.

The group $GM(n-1)$ has two components. By construction, the operator D_Z is invariant under proper Möbius transformations. It suffices to prove its invariance for a single transformation $\tau_g, g \notin SO_{\pm}(1, n)$. Such a transformation is

$$\begin{aligned} y_1 &= -x_1 & \eta_1 &= -\xi_1 \\ y_k &= x_k & \eta_k &= \xi_k & k &= 2, \dots, n \end{aligned} \quad (4.9)$$

The transformed operator is

$$\begin{aligned} D_Z^g f(y, \eta) &= \frac{1}{2} \sum_{i,j=1}^n \left(f_{y_i} \frac{\partial y_j}{\partial x_i} + f_{\eta_i} \frac{\partial \eta_j}{\partial x_i} \right) \xi_i \\ &\quad + (1-|x|^2)^{-1} \sum_{i,j=1}^n \left(f_{y_i} \frac{\partial y_j}{\partial \xi_i} + f_{\eta_i} \frac{\partial \eta_j}{\partial \xi_i} \right) (2(x|\xi)\xi_i - |\xi|^2x_i) \\ &= -\frac{1}{2} \sum_{i=1}^n f_{y_i} \eta_i + (1-|y|^2)^{-1} \sum_{i=1}^n f_{\eta_i} (2(y|\eta)\eta_i - |\eta|^2y_i) \end{aligned}$$

It coincides with D_Z . The same calculation shows that the mapping ω (see (2.17)) transforms D_Z into the operator $-D_Z$.

A remark about the derivatives $f_{x_i}, f_{\xi_i}, i = 1, \dots, n$ is appropriate. The function f is defined on

$$X = \{(x, \xi) \in \mathbf{R}^{2n} : |\xi|^2 = 1 - |x|^2\}$$

In order that the derivatives with respect to x and ξ have some meaning, the domain of definition for f first has to be extended into a neighbourhood of X in \mathbf{R}^{2n} . The resulting operator D_Z is however known to depend only on the values of f on X . It is independent of the particular extension of f .

The calculation of the remaining operators $D_{|Y|^2}$ and $D_{(X, Y)}$ is based on the theorem of Helgason (section 3). For fixed g with coordinates (x, ξ) and for a given function $f \in C_c(G/M)$ consider the function

$$\tilde{f}(s, t) = f \circ \pi \left(g \exp \sum_{i=1}^{n-1} (s_i X_i + t_i Y_i) \right) \quad (4.10)$$

π is the canonical projection and $(x(s, t), \xi(s, t))$ are the coordinates of $\pi(g \exp \sum_{i=1}^{n-1} (s_i X_i + t_i Y_i))$. Take as an example the operator $D_{(X, Y)}$. We then have

$$D_{(X, Y)} f(x, \xi) = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial s_i \partial t_i} \tilde{f}(s, t) \Big|_{s=t=0} \quad (4.11)$$

The chain rule for the second derivative $\tilde{f}_{s_i t_j}$ gives

$$\begin{aligned} \tilde{f}_{s_i t_j} = & \sum_{m, l=1}^n f_{x_l x_m} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial t_j} + \sum_{m, l=1}^n f_{x_l \xi_m} \frac{\partial x_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} \\ & + \sum_{m, l=1}^n f_{\xi_l x_m} \frac{\partial \xi_l}{\partial s_j} \frac{\partial x_m}{\partial t_j} + \sum_{m, l=1}^n f_{x_l \xi_m} \frac{\partial \xi_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} + \sum_{l=1}^n f_{x_l} \frac{\partial^2 x_l}{\partial s_j \partial t_j} + \sum_{l=1}^n f_{\xi_l} \frac{\partial^2 \xi_l}{\partial s_j \partial t_j} \end{aligned} \quad (4.12)$$

The partial derivatives of f with respect to x and ξ have the same interpretation as above. In addition, the calculations will show that the derivatives of the coordinate functions at $s = t = 0$ are functions on the group. However the resulting operator maps functions on X into functions on X . It can be expressed in the variables x and ξ .

The first derivatives of the coordinate functions

Let e_1, \dots, e_{n-1} be the canonical basis in the parameter spaces \mathbf{R}^{n-1} for the s

and t variables. If $h \in \mathbf{R}$ then

$$x_m(he_j, 0) = \frac{g_{m0} \operatorname{Ch} h + g_{mj} \operatorname{Sh} h}{1 + g_{00} \operatorname{Ch} h + g_{0j} \operatorname{Sh} h}$$

$$x_m(0, he_j) = \frac{g_{m0}}{1 + g_{00}}$$

$$\xi_m(he_j, 0) = \frac{2(g_{m0} \operatorname{Ch} h + g_{mj} \operatorname{Sh} h)g_{0n}}{(1 + g_{00} \operatorname{Ch} h + g_{0j} \operatorname{Sh} h)^2} - \frac{2g_{mn}}{1 + g_{00} \operatorname{Ch} h + g_{0j} \operatorname{Sh} h}$$

$$\xi_m(0, he_j) = \frac{2g_{m0}(g_{0j} \sin h + g_{0n} \cos h)}{(1 + g_{00})^2} - \frac{2(g_{mj} \sin h + g_{mn} \cos h)}{1 + g_{00}}$$

The partial derivatives at $(s, t) = (0, 0)$ are

$$\frac{\partial x_m}{\partial s_j} = \frac{d}{dh} x_m(he_j, 0) \Big|_{h=0} = \frac{g_{mj}}{1 + g_{00}} - \frac{g_{m0}g_{0j}}{(1 + g_{00})^2}$$

$$\frac{\partial x_m}{\partial t_j} = 0$$

$$\frac{\partial \xi_m}{\partial s_j} = -2 \frac{2g_{m0}g_{0n}g_{0j}}{(1 + g_{00})^3} + \frac{2g_{mj}g_{0n}}{(1 + g_{00})^2} + \frac{2g_{mn}g_{0j}}{(1 + g_{00})^2}$$

$$\frac{\partial \xi_m}{\partial t_j} = \frac{2g_{m0}g_{0j}}{(1 + g_{00})^2} - \frac{2g_{mj}}{1 + g_{00}} = -2 \frac{\partial x_m}{\partial s_j}$$

The following expressions are needed for the differential operators:

$$\begin{aligned} \sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial s_j} &= -\frac{1}{2} \sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} = \frac{1}{4} \sum_{j=1}^{n-1} \frac{\partial \xi_l}{\partial t_j} \frac{\partial \xi_m}{\partial t_j} \\ &= -\frac{1}{4} \xi_l \xi_m + \frac{1}{4} \delta_{lm} |\xi|^2 \end{aligned} \quad (4.13)$$

$$\sum_{j=1}^{n-1} \frac{\partial \xi_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} = -\frac{(x | \xi)}{1 - |x|^2} \left(2\xi_l \xi_m - \delta_{lm} |\xi|^2 - \frac{x_m \xi_l}{(x | \xi)} |\xi|^2 \right) \quad (4.14)$$

As an example, the calculation of formula (4.13) is given:

$$\begin{aligned} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial s_j} &= \left(\frac{g_{mj}}{1+g_{00}} - \frac{g_{m0}g_{0j}}{(1+g_{00})^2} \right) \left(\frac{g_{lj}}{1+g_{00}} - \frac{g_{l0}g_{0j}}{(1+g_{00})^2} \right) \\ &= (1+g_{00})^{-2} g_{mj} g_{lj} - (1+g_{00})^{-3} (g_{l0} g_{mj} g_{0j} + g_{m0} g_{lj} g_{0j}) \\ &\quad + (1+g_{00})^{-4} g_{l0} g_{m0} g_{0j}^2 \\ \sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial s_j} &= (1+g_{00})^{-2} (\delta_{lm} + g_{l0} g_{m0} - g_{ln} g_{mn}) \\ &\quad - (1+g_{00})^{-3} (g_{l0} (g_{00} g_{m0} - g_{0n} g_{mn}) + g_{m0} (g_{00} g_{l0} - g_{0n} g_{ln})) \\ &\quad + (1+g_{00})^{-4} g_{l0} g_{m0} (g_{00}^2 - 1 - g_{0n}^2) \end{aligned}$$

The expression $\frac{1}{4} \xi_l \xi_m$ has the value

$$(1+g_{00})^{-4} g_{0n}^2 g_{l0} g_{m0} - g_{0n} (1+g_{00})^{-3} (g_{l0} g_{mn} + g_{m0} g_{ln}) + (1+g_{00})^{-2} g_{ln} g_{mn}$$

Therefore

$$\sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial s_j} = -\frac{1}{4} \xi_l \xi_m + \delta_{lm} (1+g_{00})^{-2} = \frac{1}{4} (-\xi_l \xi_m + \delta_{lm} |\xi|^2)$$

(All partial derivatives are taken at $s = t = 0$.)

The second derivatives of the coordinate functions

The second derivatives are calculated according to the formulas

$$\left. \frac{\partial^2 x}{\partial s_j \partial t_j} \right|_{s=t=0} = \lim_{h \rightarrow 0} h^{-2} (x(he_j, he_j) - x(he_j, 0) - x(0, he_j) + x(0, 0))$$

$$\left. \frac{\partial^2 \xi}{\partial t_j^2} \right|_{s=t=0} = \lim_{h \rightarrow 0} h^{-2} (\xi(0, he_j) + \xi(0, -he_j) - 2\xi(0, 0))$$

Up to third order terms

$$x_m(he_j, he_j) \approx \frac{1}{N} (g_{m0}(1+h^2/2) + g_{mj}h - g_{mn}h^2/2)$$

$$\begin{aligned} \xi_m(he_j, he_j) &\approx \frac{2}{N} (g_{00}h^2/2 + g_{0j}h + g_{0n}(1-h^2/2)) x_m(he_j, he_j) \\ &\quad - \frac{2}{N} (g_{m0}h^2/2 + g_{mj}h + g_{mn}(1-h^2/2)) \end{aligned}$$

with

$$N = 1 + g_{00}(1 + h^2/2) + g_{0j}h - g_{0n}h^2/2$$

The resulting expressions (at $s = t = 0$) are

$$\frac{\partial^2 x_m}{\partial s_j \partial t_j} = \frac{1}{4} \xi_m$$

$$\frac{\partial^2 x_m}{\partial t_j^2} = 0$$

$$\frac{\partial^2 \xi_m}{\partial s_j \partial t_j} = 2g_{m0}(g_{0n}^2 - 2g_{0j}^2)(1 + g_{00})^{-3} + (-2g_{mn}g_{0n} - g_{m0} + 4g_{0j}g_{mj})(1 + g_{00})^{-2}$$

$$\frac{\partial^2 \xi_m}{\partial t_j^2} = -\xi_m$$

As above this leads to the required equations

$$\sum_{j=1}^{n-1} \frac{\partial^2 x_m}{\partial s_j \partial t_j} = \frac{n-1}{4} \xi_m \quad (4.15)$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 x_m}{\partial t_j^2} = 0 \quad (4.16)$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 \xi_m}{\partial s_j \partial t_j} = -(n+1) \frac{(x | \xi)}{1 - |x|^2} \xi_m - \frac{n-5}{2} (1 - |x|^2) x_m \quad (4.17)$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 \xi_m}{\partial t_j^2} = -(n-1) \xi_m \quad (4.18)$$

THEOREM 4. *The operator $D_{|Y|^2}$ on $X \cong O_{\pm}(1, n)/O(n-1)$ is given by*

$$D_{|Y|^2} f = - \sum_{l, m=1}^n f_{\xi_l \xi_m} (\xi_l \xi_m - \delta_{lm} |\xi|^2) - (n-1) \sum_{m=1}^n f_{\xi_m} \xi_m \quad (4.19)$$

It is invariant under the Möbius group $GM(n-1)$ and under the mapping $\omega(x, \xi) = (x, -\xi)$. At the same time, $D_{|Y|^2}$ is the Laplace operator on the sphere $\{\xi \in \mathbf{R}^n : |\xi| = 1\}$.

Consider the stabilizer $K \cong O(n)$ of the sphere

$$\Sigma = \{(x, \xi) \in X : x = 0, |\xi| = 1\}$$

The Lie algebra elements Y_1, \dots, Y_{n-1} (see (2.11)) are in the Lie algebra \mathfrak{k} of $O(n)$. The invariant differential operator $D_{|Y|^2}$ is therefore a differential operator on the subgroup $O(n)$. Furthermore, it is the restriction of the Casimir operator $\sum_{1 \leq i < j \leq n} X_{ij}^2$ of \mathfrak{k} onto the quotient space $\Sigma \cong O(n)/O(n-1)$. This operator is the Laplace operator on the sphere.

According to the preceding formulas (4.13)–(4.18), the operator $D_{|Y|^2}$ on X has the explicit form given in the theorem. In particular it is seen to be independent of the x coordinate (apart from the restriction $|\xi| = 1 - |x|^2$).

The invariance of the operator $D_{|Y|^2}$ under the whole Möbius group $GM(n-1)$ and under the mapping ω can be established with the same method which was used in connection with the operator D_Z .

COROLLARY. *All differential operators on X which are invariant under the group of special Möbius transformations $SM(n-1) \cong SO_{\pm}(1, n)$ are invariant under the whole group $GM(n-1)$. The operators $D_{|Y|^2}$ and $D_{|X|^2}$ are also invariant under the mapping ω , yet ω transforms D_Z and $D_{(X, Y)}$ into $-D_Z$ and $-D_{(X, Y)}$ respectively.*

THEOREM 5. *The operator $D_{(X, Y)}$ is given by*

$$\begin{aligned} D_{(X, Y)}f = & \frac{1}{2} \sum_{l, m=1}^n f_{x_l \xi_m} (\xi_l \xi_m - \delta_{lm} |\xi|^2) + \frac{n-1}{4} \sum_{m=1}^n f_{x_m} \xi_m \\ & - \sum_{l, m=1}^n f_{\xi_l \xi_m} \left[\frac{(x | \xi)}{1 - |x|^2} (2 \xi_l \xi_m - \delta_{lm} |\xi|^2) - (1 - |x|^2) x_l \xi_m \right] \\ & - \sum_{m=1}^n f_{\xi_m} \left[(n+1) \frac{(x | \xi)}{1 - |x|^2} \xi_m + \frac{n-5}{2} (1 - |x|^2) x_m \right] \end{aligned} \tag{4.20}$$

5. Spherical harmonics and the operators S_k and S_k^*

A spherical harmonic of degree k on the sphere $\Sigma = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$ is the restriction of a harmonic polynomial in \mathbf{R}^n which is homogeneous of degree k . The space of spherical harmonics of degree k will be denoted by H^k . Alternatively, it can be described as the eigenspace with eigenvalue $-k(k+n-2)$ of the Laplace operator Δ_{Σ} on the sphere. The system of spherical harmonics is

complete in $L^2(\Sigma)$. It gives a decomposition of this space as a direct orthogonal Hilbert sum

$$L^2(\Sigma) = \bigoplus_{k=0}^{\infty} H^k$$

DEFINITION. A spherical harmonic of degree k on $X \cong O_{\pm}(1, n)/O(n-1)$ is an eigenfunction of the operator $D_{|Y|^2}$ with eigenvalue $-k(k+n-2)$.

$$E^k(X) = \{f \in C^{\infty}(X) : D_{|Y|^2}f = -k(k+n-2)f\} \quad (5.1)$$

If a function $f \in C^{\infty}(X)$ is an eigenfunction of the operator $D_{|Y|^2}$, then for every fixed x

$$-\sum_{l,m=1}^n f_{\xi_l \xi_m} (\xi_l \xi_m - \delta_{lm} |\xi|^2) - (n-1) \sum_{m=1}^n f_{\xi_m} \xi_m = \lambda f$$

But the left hand side is the spherical Laplace operator Δ_{Σ} applied to $f(x, \xi)$ with x fixed. Therefore the eigenvalue λ is of the form $-k(k+n-2)$ for some non negative integer k . If $\{h_{k_1}, \dots, h_{k_d}\}$, $d = d(k)$, is an orthogonal basis in H^k , then

$$f(x, \xi) = \sum_{j=1}^d c_{k_j}(x) h_{k_j}(\xi)$$

with coefficients c_{k_j} $j = 1, \dots, d$ which will depend (smoothly) on x . Conversely, any such function is in E^k .

From the completeness property of the system of spherical harmonics on Σ we conclude that any function $f \in C^{\infty}(X)$ has an expansion of the form

$$f(x, \xi) = \sum_{k=0}^{\infty} \sum_{j=1}^d c_{k_j}(x) h_{k_j}(\xi) \quad (5.2)$$

which converges for every fixed x in $L^2(\Sigma)$.

A harmonic polynomial p of degree k in (\mathbf{R}^n) defines a symmetric tensor t of order k with vanishing traces

$$p(\xi) = \sum_{i_1, \dots, i_k=1}^n t_{i_1 \dots i_k} \xi_{i_1} \dots \xi_{i_k}$$

$$t_{i_1 \dots i_k} = t_{i_{\sigma(1)} \dots i_{\sigma(k)}} \text{ for any permutation } \sigma \text{ on the indices} \quad (5.3)$$

$$\sum_{i=1}^n t_{i i i_3 \dots i_k} = 0$$

Conversely, to any such tensor the formula associates a harmonic polynomial p which is homogeneous of degree k . The functions $f \in E^k$ therefore can be viewed as tensorfields of order k on the hyperbolic space $B = O_{\pm}(1, n)/O(n)$:

$$E^k = \{f \in C^{\infty}(X) : f(x, \xi) = (1 - |x|^2)^{-2k} \sum_{i_1, \dots, i_k} t_{i_1, \dots, i_k}(x) \xi_{i_1} \cdots \xi_{i_k}\} \quad (5.4)$$

In this representation $t(x) = t_{i_1, \dots, i_k}(x)$ is a tensorfield of symmetric tensors with vanishing trace. The factor $(1 - |x|^2)^{-2k}$ is a normalizing factor.

The type of the tensorfield t is given by its transformation behaviour under Möbius transformations. Recall that the action of $GM(n-1)$ on X is defined by

$$(x, \xi) \rightarrow (\tau_g x, d\tau_g \xi) \quad (2.9)$$

The action on $C(X)$ then becomes

$$f^{g^{-1}}(x, \xi) = f(\tau_g x, d\tau_g \xi) \quad (5.5)$$

First consider the special case of a vectorfield

$$f(x, \xi) = (1 - |x|^2)^{-2} \sum_{i=1}^n v_i(x) \xi_i \in E^1$$

$$f^{g^{-1}}(x, \xi) = (1 - |\tau_g x|^2)^{-2} \sum_{i=1}^n v_i(\tau_g x) (d\tau_g \xi)_i$$

We set $y = \tau_g x$. Since $ds^2 = (1 - |x|^2)^{-2} |dx|^2$ is an invariant metric, the Jacobian determinant of the matrix

$$G(x) = \left(\frac{\partial y_i}{\partial x_k}(x) \right)$$

representing the tangent mapping $d\tau_g(x)$ is given by

$$\det G(x) = \pm (1 - |y|^2)^n (1 - |x|^2)^{-n}$$

The conformality (or anti-conformality) implies that $((1 - |x|^2)/(1 - |y|^2))G(x)$ is an orthogonal matrix. In particular

$$G^{-1}(x) = (1 - |x|^2)^2 (1 - |y|^2)^{-2} G'(x) \quad (5.6)$$

(G^t is the transposed matrix). It then follows that

$$\begin{aligned} f^{g^{-1}}(x, \xi) &= (1 - |x|^2)^{-2} \sum_{k=1}^n \sum_{i=1}^n v_i(y) \frac{\partial y_i}{\partial x_k}(x) \xi_k (1 - |x|^2)^2 (1 - |y|^2)^{-2} \\ &= (1 - |x|^2)^{-2} \sum_{k=1}^n v_k^{g^{-1}}(x) \xi_k \end{aligned}$$

with

$$v^{g^{-1}}(x) = G^{-1}(x)v(\tau_g x) \quad (5.7)$$

Next assume that $f \in E^k$,

$$f(x, \xi) = (1 - |x|^2)^{-2k} \sum_{i_1, \dots, i_k} t_{i_1 \dots i_k}(x) \xi_{i_1} \cdots \xi_{i_k}$$

Then the same calculations show that

$$f^{g^{-1}}(x, \xi) = (1 - |x|^2)^{-2k} \sum_{i_1, \dots, i_k} t_{i_1 \dots i_k}^{g^{-1}}(x) \xi_{i_1} \cdots \xi_{i_k} \quad (5.8)$$

with

$$t_{i_1 \dots i_k}^{g^{-1}}(x) = \sum_{j_1, \dots, j_k} a_{i_1 j_1} \cdots a_{i_k j_k} t_{j_1 \dots j_k}(\tau_g x)$$

where the a_{kj} are the components of the matrix $G^{-1}(x)$. The transformation behaviour of the tensors is influenced by the choice of the normalizing factor $(1 - |x|^2)^{-2k}$. To illustrate this set

$$f(x, \xi) = (1 - |x|^2)^{-2} \sum_{i, k} \varphi_{ik}(x) \xi_i \xi_k \quad (5.9)$$

Here, $\Phi(x) = (\varphi_{ik}(x))$ is a symmetric matrix with vanishing trace. The same calculations as above then show that

$$f^{g^{-1}}(x, \xi) = (1 - |x|^2)^{-2} \sum_{i, k} \varphi_{ik}^{g^{-1}}(x) \xi_i \xi_k$$

where the transformed matrix is given by

$$\Phi^{g^{-1}}(x) = G^{-1}(x)\Phi(\tau_g x)G(x) \quad (5.10)$$

This transformation behaviour differs from the preceding by a factor $(\det G(x))^{2/n}$.

THEOREM 6. *A function $f(x, \xi) = (1 - |x|^2)^{-2} \sum_{i=1}^n v_i(x)\xi_i \in E^1$ satisfies $D_Z f = 0$ if and only if v is a vectorfield in the Lie algebra of $GM(n-1)$.*

The vectorfields v in the Lie algebra of $GM(n-1)$ are of the form

$$v(x) = Bx + c(1 + |x|^2) - 2x(c, x) \quad (2.20)$$

with $B^t = -B$ and $c \in \mathbf{R}^n$. Direct verification shows that the functions $f \in E^1$ which are associated to these vectorfields satisfy $D_Z f = 0$. Conversely, assume that $f \in E^1$ satisfies

$$\begin{aligned} D_Z f &= -(1 - |x|^2)^{-2} \frac{1}{2} \sum_{i,j} v_{i,j} \xi_i \xi_j - (1 - |x|^2)^{-3} (v, x) \sum_{i,j} \delta_{ij} \xi_i \xi_j \\ &= 0 \end{aligned}$$

for all $(x, \xi) \in X$ ($v_{i,j}$ is the notation for the partial derivative $(\partial v_i / \partial x_j)(x)$). It follows that

$$\begin{aligned} v_{i,j} &= -v_{j,i} & i \neq j \\ v_{i,i} &= -2(1 - |x|^2)^{-1} (v(x), x) & i = 1, \dots, n \end{aligned}$$

and in particular

$$v_{i,i} = v_{j,j}$$

Assume now that i, j and k are different indices. Then the differentiated equations

$$v_{i,jk} + v_{j,ik} = 0$$

$$v_{k,ij} + v_{i,kj} = 0$$

$$v_{j,ki} + v_{k,ji} = 0$$

show that $v_{i,jk} = 0$. Similarly

$$v_{i,ij} = v_{k,kij} = 0$$

and therefore

$$v_{k,ij} = 0 \quad v_{i,kkk} = 0.$$

This shows that all third order derivatives vanish. The vectorfield is therefore given by a second order polynomial

$$v_i(x) = \frac{1}{2} \sum_{k,l} a_{ikl} x_k x_l + \sum_k b_{ik} x_k + c_i \quad i = 1, \dots, n$$

and it can be assumed that

$$a_{ikl} = a_{ilk} = v_{i,kl}$$

A comparison of the coefficients in the equations

$$(1 - |x|^2)v_{i,i} = -2(v, x) \quad i = 1, \dots, n$$

with

$$v_{i,i} = \frac{1}{2} \sum_k (a_{iki} + a_{iik}) x_k + b_{ii}$$

$$(v, x) = \frac{1}{2} \sum_{i,k} a_{ikl} x_i x_k x_l + \sum_{i,k} b_{ik} x_i x_k + \sum_i c_i x_i$$

now results in the equations

$$b_{ii} = 0$$

$$a_{iik} = -2c_k \quad i, k = 1, \dots, n$$

$$b_{ik} = -b_{ki}$$

Since it is already known that

$$a_{ijk} = 0 \quad \text{if } i \neq j \neq k \neq i$$

and

$$a_{kii} = -a_{iki} = 2c_k \quad \text{if } k \neq i$$

it can be concluded that

$$\begin{aligned} v_i(x) &= \sum_{k=1}^n a_{iki} x_k x_i - \frac{1}{2} a_{iii} x_i^2 + \frac{1}{2} \sum_{k \neq i} a_{ikk} x_k^2 + \sum_{k=1}^n b_{ik} x_k + c_i \\ &= -2x_i \sum_{k=1}^n c_k x_k + c_i x_i^2 + \sum_{k \neq i} c_i x_k^2 + \sum_{k=1}^n b_{ik} x_k + c_i \end{aligned}$$

This shows that

$$v(x) = c(1 + |x|^2) - 2x(c, x) + Bx \quad B' = -B$$

It should be noted that the theorem is still true for the dimension $n = 2$, yet for this case the proof has to be modified slightly.

The theorem shows that the operator D_Z applied to vectorfields (i.e. to the spherical harmonics of degree 1 on X) singles out exactly the Lie algebra of the Möbius group $GM(n-1)$.

The space of functions $f \in C^\infty(X)$ satisfying $D_Z f = 0$ is an algebra, since D_Z is a first order differential operator. If $\{v^{(1)}, \dots, v^{(d)}\}$, $d = \frac{1}{2}n(n+1)$, is a basis of the Lie algebra of $GM(n-1)$ and if

$$f_j(x, \xi) = (1 - |x|^2)^{-2} \sum_i v_i^{(j)}(x) \xi_i \in E^1 \quad j = 1, \dots, d$$

then any convergent power series in f_1, \dots, f_d will be a solution of $D_Z f = 0$.

THEOREM 7. *The operator*

$$S_k = D_{(X, Y)} + \left(\frac{1}{2} - \left(\frac{n}{2} + k - 1 \right) \right) D_Z \quad (5.11)$$

maps E^k into E^{k+1} , and the operator

$$S_k^* = D_{(X, Y)} + \left(\frac{1}{2} + \left(\frac{n}{2} + k - 1 \right) \right) D_Z \quad (5.12)$$

maps E^k into E^{k-1} , $k = 1, 2, 3 \dots$

COROLLARY. The operators D_Z and $D_{(X, Y)}$ on E^k take the form

$$D_Z = -(n+2k-2)^{-1}S_k + (n+2k-2)^{-1}S_k^* \quad (5.13)$$

$$2D_{(X, Y)} = (1+(n+2k-2)^{-1})S_k + (1-(n+2k-2)^{-1})S_k^* \quad (5.14)$$

For the proof of the theorem the operator $D_{(X, Y)} + cD_Z$, $c \in \mathbf{R}$, is applied to the function

$$f(x, \xi) = \rho^r \sum_{i_1, \dots, i_k} t_{i_1 \dots i_k}(x) \xi_{i_1} \cdots \xi_{i_k}$$

where t is a symmetric tensor with vanishing traces, $r \in \mathbf{R}$ and $\rho(x) = (1 - |x|^2)^{-1}$. The summation convention will be applied (summation over indices which appear twice). The derivatives of the components of t are denoted by

$$\frac{\partial}{\partial x_m} t_{i_1 \dots i_k} = t_{i_1 \dots i_k, m}$$

and these are no longer the components of a symmetric tensor. The result is as follows:

$$\begin{aligned} D_{(X, Y)}f + cD_Zf &= \frac{k}{2} \rho^{r-2} t_{i_1 \dots i_{k-1} m, m} \xi_{i_1} \cdots \xi_{i_{k-1}} + A \rho^r t_{i_1 \dots i_{k-1} m, l} \xi_{i_1} \cdots \xi_{i_{k-1}} \xi_m \xi_l \\ &\quad + B \rho^{r-1} t_{i_1 \dots i_{k-1} m} \xi_{i_1} \cdots \xi_{i_{k-1} x_m} + C \rho^{r+1} t_{i_1 \dots i_k} \xi_{i_1} \cdots \xi_{i_k}(x | \xi) \end{aligned} \quad (5.15)$$

with

$$A = -\frac{k}{2} - \frac{n-1}{4} + \frac{c}{2}$$

$$B = kr - k(k-1) + k \frac{n-5}{2} + kc$$

$$C = -kr + 2k(k-1) - \frac{n-1}{2} r + k(n+1) + c(r-2k)$$

The first observation is that $C = 0$ if $r = 2k$. This motivates the normalizing factor

$(1-|x|^2)^{-2k}$ occurring in the description (5.4). Having fixed $r=2k$, the operators S_k^* and S_k are now defined by the equations $A=0$ and $B=0$ respectively.

The constant c for the operator S_k is determined by the equations $r=2k$, $B=0$. It follows that

$$c = \frac{1}{2} - \left(\frac{n}{2} + k - 1 \right)$$

$$A = - \left(\frac{n}{2} + k - 1 \right) \quad (5.16)$$

$$S_k f = \frac{k}{2} \rho^{2k-2} t_{i_1 \dots i_k m, m} \xi_{i_1} \dots \xi_{i_k} - \left(\frac{n}{2} + k - 1 \right) \rho^{2k} t_{i_1 \dots i_{k-1} m, l} \xi_{i_1} \dots \xi_{i_{k-1}} \xi_m \xi_l$$

It remains to be shown that $S_k f \in E^{k+1}$. For this purpose set

$$q_{i_1 \dots i_{k+1}} = \frac{1}{k+1} \sum_{j=1}^{k+1} t_{i_1 \dots \hat{i}_j \dots i_{k+1}, j} \quad (5.17)$$

(the symbol \hat{i}_j indicates that the index i_j is omitted). q is a symmetric tensor and

$$t_{i_1 \dots i_{k+1}} \xi_{i_1} \dots \xi_{i_{k+1}} = q_{i_1 \dots i_{k+1}} \xi_{i_1} \dots \xi_{i_{k+1}} \quad (5.18)$$

However in general the traces of q will not vanish:

$$q_{i_1 \dots i_{k-1} j j} = \frac{2}{k+1} t_{i_1 \dots i_{k-1} j, j} \quad (5.19)$$

Consider the symmetric tensor z

$$z_{i_1 \dots i_{k+1}} = \delta_{i_1 i_2} q_{j j i_3 \dots i_{k+1}} + \delta_{i_1 i_3} q_{j i_2 j i_4 \dots i_{k+1}} + \dots + \delta_{i_k i_{k+1}} q_{i_1 \dots i_{k-1} j j} \quad (5.20)$$

Summation gives

$$\delta_{i_1 i_2} q_{j j i_3 \dots i_{k+1}} \xi_{i_1} \dots \xi_{i_{k+1}} = |\xi|^2 q_{i_1 \dots i_{k-1} j j} \xi_{i_1} \dots \xi_{i_{k-1}} \quad (5.21)$$

Since there are $\frac{k(k+1)}{2}$ terms in the definition of z , the equations (5.19), (5.20) and (5.21) show that

$$z_{i_1 \dots i_{k+1}} \xi_{i_1} \dots \xi_{i_{k+1}} = k |\xi|^2 t_{i_1 \dots i_{k-1} j, j} \xi_{i_1} \dots \xi_{i_{k-1}} \quad (5.22)$$

This implies

$$S_k f = \frac{1}{2} \rho^{2k} (-(n+2k-2) q_{i_1 \dots i_{k+1}} + z_{i_1 \dots i_{k+1}}) \xi_{i_1} \dots \xi_{i_{k+1}} \quad (5.23)$$

and it can now be shown that $S_k f$ is defined by a tensor with vanishing traces:

$$\begin{aligned} z_{jji_3 \dots i_{k+1}} &= n q_{jji_3 \dots i_{k+1}} + q_{ji_3ji_4 \dots i_{k+1}} + \dots \\ &+ q_{i_3jji_4 \dots i_{k+1}} + \dots \\ &+ 0 \quad . \\ &= (n+2(k-1)) q_{jji_3 \dots i_{k+1}} \end{aligned}$$

(Observe that e.g. $q_{jji_3 \dots i_{k-1}ii} = 0$ if $k \geq 3$). This completes the proof for the fact that $S_k f \in E^{k+1}$ if $f \in E^k$.

The constant c for the operator S_k^* is determined by the equations $r = 2k$, $A = 0$. It follows that

$$c = \frac{1}{2} + \left(\frac{n}{2} + k - 1 \right) \quad (5.24)$$

$$S_k^* f = \frac{k}{2} \rho^{2k-2} t_{i_1 \dots i_{k-1} m, m} \xi_{i_1} \dots \xi_{i_{k-1}} + k(n+2k-2) \rho^{2k-1} t_{i_1 \dots i_{k-1} m} \xi_{i_1} \dots \xi_{i_{k-1}} x_m$$

This clearly shows that $S_k^* f \in E^{k-1}$.

The operator S_k^* can be put into a different form:

$$S_k^* f = \frac{k}{2} \rho^{-n} \sum_{i_1, \dots, i_{k+1}} \sum_{m=1}^n \frac{\partial}{\partial x_m} (\rho^{n+2k-2} t_{i_1 \dots i_{k-1} m}) \xi_{i_1} \dots \xi_{i_{k-1}} \quad (5.25)$$

The case $k = 0$ is special. The functions $f \in E^0$ are identified with the functions on

the hyperbolic space B . The operators D_Z and $D_{(X, Y)}$ map E^0 into E^1 :

$$D_Z f = -\frac{1}{2} \sum_{i=1}^n f_{x_i} \xi_i$$

$$D_{(X, Y)} f = \frac{n-1}{4} \sum_{i=1}^n f_{x_i} \xi_i$$

and S_0 can be defined by the formula

$$S_0 f = D_{(X, Y)} f - \frac{n+1}{2} D_Z f \quad (5.26)$$

$$S_0 f = \frac{n}{2} \sum_{i=1}^n f_{x_i} \xi_i = \frac{n}{2} (1-|x|^2)^{-2} \sum_{i=1}^n (1-|x|^2)^2 f_{x_i} \xi_i$$

The operator $S_1^* S_0$ then takes the form

$$\begin{aligned} S_1^* S_0 f &= \frac{1}{2} \frac{n}{2} \rho^{-n} \sum_{m=1}^n \frac{\partial}{\partial x_m} (\rho^n (1-|x|^2)^2 f_{x_m}) \\ &= \frac{n}{4} \rho^{-n} \operatorname{div} (\rho^{n-2} \operatorname{grad} f) \end{aligned} \quad (5.27)$$

This is (a multiple of) the Laplace operator for the hyperbolic space B .

Following Ahlfors [1] the invariant operator P mapping vectorfields v on B into tensorfields φ is defined by the equation

$$\rho^{-n} (Pv)_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) - \delta_{ij} \frac{1}{n} \sum_{k=1}^n v_{k,k} \quad (5.28)$$

The tensors $Pv(x)$ are symmetric and have zero trace. The operator P^* mapping such tensorfields into vectorfields is defined by the formula

$$(P^* \varphi)_i = \rho^{-n-2} \sum_{j=1}^n \varphi_{ij,j} \quad (5.29)$$

THEOREM 8. *The operator S_1 on E^1 coincides with the operator $-(n/2)\rho^{-n}P$*

on vectorfields and S_2^* on E^2 coincides with $P^*\rho^n$ provided the following identifications are made:

(1) The vectorfield v on B is identified with the function

$$V(x, \xi) = \sum_{i=1}^n (1-|x|^2)^{-2} v_i(x) \xi_i \in E^1$$

(2) The tensorfield φ on B is identified with the function

$$\Phi(x, \xi) = \sum_{i,j=1}^n (1-|x|^2)^{-2} \varphi_{ij}(x) \xi_i \xi_j \in E^2.$$

In particular it follows that $S_2^* S_1$ is the same operator as $-(n/2)P^*P$.

The operator S_1 is applied to the function $V(x, \xi) \in E^1$:

$$\begin{aligned} S_1 V &= \frac{1}{2} \rho^2 |\xi|^2 \sum_{m=1}^n v_{m,m} - \frac{n}{2} \rho^2 \sum_{m,l=1}^n v_{m,l} \xi_m \xi_l \\ &= -\frac{n}{2} (1-|x|^2)^{-2} \sum_{m,l=1}^n \left(\frac{1}{2} (v_{m,l} + v_{l,m}) - \frac{1}{n} \delta_{lm} \sum_{k=1}^n v_{k,k} \right) \xi_l \xi_m \end{aligned}$$

This shows that S_1 corresponds to $-(n/2)\rho^{-n}P$.

Similarly, if S_2^* is applied to Φ , it follows from (5.25) that

$$\begin{aligned} S_2^* \Phi &= S_2^* \sum_{i,j=1}^n (1-|x|^2)^{-4} (1-|x|^2)^2 \varphi_{ij} \xi_i \xi_j \\ &= \rho^{-n} \sum_{i,m=1}^n \frac{\partial}{\partial x_m} (\rho^{n+4-2} (1-|x|^2)^2 \varphi_{im}) \xi_i \\ &= \rho^{-n-2} (1-|x|^2)^{-2} \sum_{i,m=1}^n \frac{\partial}{\partial x_m} (\rho^n \varphi_{im}) \xi_i \end{aligned}$$

This completes the proof of Theorem 8.

Equation (5.7) gives the transformation behaviour of the vectorfields under Möbiustransformations. The transformation of the tensorfields $(\varphi_{ij}(x))$ is described by (5.10). These formulas coincide with formulas (1.5) and (1.7) in [1].

COROLLARY (Ahlfors [2], equation (2.1)). *The solutions of $S_1 f = 0$, $f(x, \xi) = (1-|x|^2)^{-2} \sum_{i=1}^n v_i(x) \xi_i \in E^1$ are of the form*

$$v(x) = a + Bx + \lambda x + c|x|^2 - 2x(c, x), \quad \lambda \in \mathbf{R}, \quad a, c \in \mathbf{R}^n, \quad B^t = -B.$$

The solutions of $S_1 f = 0$, $f \in E^1$ describe exactly the Lie algebra of the Möbius group $M(n)$ as a transformation group of \mathbf{R}^n (see equation (2.19)).

THEOREM 9. *For all $f \in E^k$, $k = 1, 2, \dots$ there is equality*

$$D_{|X|^2+Z^2+|Y|^2} f = -(n+2k-2)^{-1} (S_{k+1}^* S_k f - S_{k-1} S_k^* f) \quad (5.30)$$

COROLLARY. $\Delta_K = D_{|X|^2+Z^2}$ and $\Delta_X = D_{|X|^2+Z^2-|Y|^2}$ map E^k into E^k , $k = 0, 1, 2, \dots$

For the proof of the theorem let us calculate the commutator

$$D_{|X|^2+|Y|^2} = [D_{(X,Y)}, D_Z]$$

using equations (5.13) and (5.14). Assume that $f \in E^k$, $k \in \mathbb{N}$.

$$\begin{aligned} & (n+2k-2)[D_{(X,Y)}, D_Z] \\ &= S_{k+1} S_k (n+2k)^{-1} \left(\frac{1}{2} - \left(\frac{n}{2} + k \right) - \frac{1}{2} + \left(\frac{n}{2} + k - 1 \right) \right) \\ & \quad + S_{k+1}^* S_k (n+2k)^{-1} \left(-\frac{1}{2} - \left(\frac{n}{2} + k \right) + \frac{1}{2} - \left(\frac{n}{2} + k - 1 \right) \right) \\ & \quad + S_{k-1} S_k^* (n+2k-4)^{-1} \left(-\frac{1}{2} + \left(\frac{n}{2} + k - 2 \right) + \frac{1}{2} + \left(\frac{n}{2} + k - 1 \right) \right) \\ & \quad + S_{k-1}^* S_k^* (n+2k-4)^{-1} \left(\frac{1}{2} + \left(\frac{n}{2} + k - 2 \right) - \frac{1}{2} - \left(\frac{n}{2} + k - 1 \right) \right) \end{aligned}$$

If the expression

$$\begin{aligned} & (n+2k-2)D_{Z^2} \\ &= (n+2k)^{-1} (S_{k+1} S_k - S_{k+1}^* S_k^*) - (n+2k-4)^{-1} (S_{k-1} S_k^* - S_{k-1}^* S_k^*) \end{aligned}$$

is added, the formula of the theorem follows:

$$(n+2k-2)(D_{|X|^2} + D_{|Y|^2} + D_{Z^2}) = -S_{k+1}^* S_k + S_{k-1} S_k^*$$

The case $k = 0$ reduces to the Laplace operator (5.27).

REFERENCES

- [1] AHLFORS, L. V. *Invariant operators and integral representations in hyperbolic space*, Math. Scand. 36 (1975), 27–43.
- [2] AHLFORS, L. V. *Quasiconformal deformations in several variables*. *Contributions to Analysis*, Academic Press, New York and London, 1974, p. 19–25.
- [3] HELGASON, S. *Differential operators on homogeneous spaces*, Acta Math. 102 (1959), 239–299.
- [4] LEVINE, D. A. *Systems of singular integral operators on spheres*, Trans. AMS 144 (1969), 493–522.
- [5] MOSTOW, G. D. *Quasiconformal mappings in n -space and the rigidity of hyperbolic space forms*, Publications mathématiques IHES 34 (1968), 53–104.
- [6] NOMIZU, K. *Invariant affine connections on homogenous spaces*, Amer. J. Math. 76 (1954), 33–65
- [7] WEYL, H. *The classical groups*, Princeton Univ. Press, Princeton 1939.

Universität Bern
Mathematisches Institut
Sidlerstr. 5
CH-3012 Bern

Received November 4, 1981