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Invariant differential operators in hyperbolic space

H. M. REIMANN

1. Introduction

The conformal mappings in real higher dimensional space $\mathbf{R}^n$, $n \geq 3$, are the proper Möbiustransformations. The group $GM(n)$ of Möbiustransformations acts on $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ and there is a subgroup isomorphic to $GM(n-1)$ which stabilizes the unit ball B. It is the action of this group $GM(n-1)$ and the induced action on functions on the hyperbolic space B that will be studied.

The differentiation process leads from functions to vectorfields and tensorfields of higher order. There is a natural setting which reduces the analysis of at least the symmetric tensors with vanishing traces to the study of functions on a bigger space X. Whereas the hyperbolic space B is isomorphic to $O_{\pm}(1, n)/O(n)$ this space X is isomorphic to the quotient space $O_{\pm}(1, n)/O(n-1)$. Geometrically it can be described as the cosphere bundle of the hyperbolic space B. The action of the Möbius group $GM(n-1)$ on X essentially is the action of $GM(n-1)$ on the cotangent space of B.

The approach described here, whereby certain tensorfields on B are interpreted as functions on X, is inspired by a similar construction for the sphere (see Levine [4]). The purpose of that construction was the characterization of invariant systems of singular differential operators on the sphere. In both cases the conformal structure seems to be essential.

The space $C(X)$ of functions on X can be split into a direct sum of subspaces

$$C(X) = \bigoplus_{k=0}^{\infty} E^k$$

The functions in E^k have an interpretation as tensorfields of symmetric tensors with vanishing traces. Their analysis is in a certain sense complementary to the analysis of differential forms, which in the tensor language is a theory of antisymmetric tensors. Certain striking analogies are apparent. The invariant operators S_k and S_k^* defined in Section 5, Theorem 7, are generalizations of the operators grad and div. They play a role similar to the operators d and d^* for differential forms (see Theorems 7 and 9). In particular, S_k maps E^k into E^{k+1}

and S_k^* maps E^k into E^{k-1}. It is shown that the operators S_1 and S_2^* coincide with certain operators studied by Ahlfors [1] (see Theorem 8).

There exists an invariant differential operator D_Z on X which is of first order. As a consequence, the space of solutions of $D_Z f = 0$ is an algebra. The functions $f \in E^1$ which satisfy $D_Z f = 0$ are exactly those which correspond to vectorfields v in the Lie algebra of the Möbius group (Theorem 6).

The algebra of invariant differential operators on X is not commutative. It is generated by 1, the first order differential operator D_Z and a further differential operator $D_{|Y|^2}$ of second order (Theorems 1 and 2). The operator $D_{|Y|^2}$ is basically the Laplace-operator on the sphere $O(n)/O(n-1)$. The spaces E^k appear as eigenspaces of $D_{|Y|^2}$. The Laplace-Casimir operator Δ_X on X preserves the eigenspaces (Theorem 9).

2. The Möbius group and its Lie algebra

The Möbius group $GM(n)$ is the transformation group of $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ which is generated by reflections in the spheres and hyperplanes of $\mathbf{R}^n$. The group is isomorphic to $O_{\pm}(1, n+1)$, the subgroup of $O(1, n+1)$ which preserves the positive cone:

$$\left\{y \in \mathbf{R}^{n+2} : \langle y, y\rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2 > 0,\ y_0 > 0\right\}$$

(see Mostow [5]). The isomorphism is constructed in the following way: The group $O(1, n+1)$ leaves invariant the quadratic form $\langle y, y\rangle = y_0^2 - \sum_{i=1}^{n+1} y_i^2$ and in particular the cone $\{y \in \mathbf{R}^{n+2} : \langle y, y\rangle = 0\}$. If inhomogeneous coordinates $\eta_i = y_i/y_0$ are introduced, the group becomes a transformation group of the sphere $\Sigma = \{\eta \in \mathbf{R}^{n+1} : |\eta| = 1\}$ and the elements g and $-g$ give rise to the same transformation. Stereographic projection from the point $\varepsilon_n = (0, \dots, 0, 1)$ onto the plane $\eta_{n+1} = 0$ then leads to the realization of $O_{\pm}(1, n+1)$ as a transformation group of $\hat{\mathbf{R}}^n$. The subgroup of the Möbius group $GM(n)$, which stabilizes the unit ball $B \subset \mathbf{R}^n$ is isomorphic to the Möbius-group $GM(n-1)$ of one lower dimension. This group which acts on B will again be denoted by $GM(n-1)$. Observe that under the above isomorphism this is exactly the subgroup $O_{\pm}(1, n)$ of $O_{\pm}(1, n+1)$ which stabilizes the lower half space in $\mathbf{R}^{n+2}$. The elements in matrix notation have the special form

$$g = \begin{pmatrix} & & 0 \\ & g_{ij} & \vdots \\ & & 0 \\ 0 & \dots\ 0 & 1 \end{pmatrix} \qquad i, j = 0, 1, \dots, n \quad g_{00} > 0$$

Our main concern is with this group $G = GM(n-1)$, $n \geq 3$, which is the group of conformal and anti-conformal mappings of the unit ball $B \subset \mathbf{R}^n$ onto itself. Referring to the isomorphism $GM(n-1) \cong O_{\pm}(1, n)$ we will speak about the geometric realization of the group, if we consider it as a transformation group of B. The algebraic realization then refers to the group as a matrix group.

The unit ball B has the structure of a symmetric space (the hyperbolic space) $B = G/K$ with the invariant metric $ds^2 = \rho^2 |dx|^2$, $\rho(x) = (1-|x|^2)^{-1}$. The stabilizer K of the origin is the orthogonal group. We start with an explicit description of the action of $G = GM(n-1)$ on $B \subset \mathbf{R}^n$.

The stereographic projection of the sphere $\Sigma = \{\eta \in R^{n+1} : |\eta| = 1\}$ onto the plane $\eta_{n+1} = 0$ is given by the formula

$$x_i = \frac{\eta_i}{1-\eta_{n+1}} \qquad i = 1, \ldots, n$$

and the inverse mapping is

$$\eta_i = \frac{2x_i}{1+|x|^2} \qquad i = 1, \ldots, n$$

$$\eta_{n+1} = \frac{|x|^2-1}{|x|^2+1}$$

Let $g = (g_{ij})$ be an element in $O_{\pm}(1, n)$ and consider $O_{\pm}(1, n)$ as the subgroup of $O_{\pm}(1, n+1)$ which stabilizes the unit vector $e_{n+1} = (0, \ldots, 0, 1) \in \mathbf{R}^{n+2}$. The image of the half line $y = t(e_0 - e_{n+1})$ $t > 0$ is the half line

$$t(ge_0 - ge_{n+1}) = t(g_{00}, \ldots, g_{n0}, -1)$$

which in turn is mapped onto the point

$$\eta = \frac{1}{g_{00}}(g_{10}, \ldots, g_{n0}, -1)$$

Under stereographic projection this point projects onto

$$x = \frac{1}{1+g_{00}}(g_{10}, \ldots, g_{n0}) \in B \tag{2.1}$$

If g is in the subgroup $O(n)$ of $O_{\pm}(1, n)$, then $g_{00} = 1$ and the corresponding point on the ball B is the center $x=0$. This establishes the isomorphism

$$B \cong O_{\pm}(1, n)/O(n)$$

The group $O_{\pm}(1, n)$ acts on the quotient space by left translation. The Möbiustransformation corresponding to the element $g \in O_{\pm}(1, n)$ will be denoted by τ_g. It is a conformal mapping if $g \in SO_{\pm}(1, n)$

$$SO_{\pm}(1, n) = \{g \in O_{\pm}(1, n) : \det g > 0\},$$

otherwise it is an anti-conformal mapping.

Consider the one parameter subgroup

$$a_t = \exp t\begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix} = \begin{pmatrix} \mathrm{Ch}\, t & & & & \mathrm{Sh}\, t \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \mathrm{Sh}\, t & & & & \mathrm{Ch}\, t \end{pmatrix} \tag{2.2}$$

in $O_{\pm}(1, n)$. The curve $x_t = \tau_{a_t}(0)$ in the ball B is given by

$$x_t = \frac{\mathrm{Sh}\, t}{1+\mathrm{Ch}\, t} e_n \qquad e_n = (0, \ldots, 0, 1) \in \mathbf{R}^n$$

The tangent vector to the curve at the origin is the vector

$$\left.\frac{dx_t}{dt}\right|_{t=0} = -e_n/2$$

The element τ_g, $g \in O_{\pm}(1, n)$, maps this curve onto the curve $z_t = \tau_g \tau_{a_t}(0)$

$$(z_t)_i = \frac{g_{i0}\,\mathrm{Ch}\, t + g_{in}\,\mathrm{Sh}\, t}{1+g_{00}\,\mathrm{Ch}\, t + g_{0n}\,\mathrm{Sh}\, t} \qquad i = 1, \ldots, n$$

whose tangent vector at $\tau_g(0)$ is given by

$$\left.\frac{dz_t}{dt}\right|_{t=0} = \frac{-g_{0n}}{(1+g_{00})^2}(g_{10}, \ldots, g_{n0}) + \frac{1}{1+g_{00}}(g_{in}, \ldots, g_{nn})$$

The tangent vector $\varepsilon_n = (0, \ldots, 0, 1)$ at the origin is therefore mapped onto the tangent vector ξ at $x=(1/(1+g_{00}))(g_{10},\ldots,g_{n0})$ with coordinates

$$\xi_i=\frac{2g_{0n}g_{i0}}{(1+g_{00})^2}-\frac{2g_{in}}{1+g_{00}}\qquad i=1,\ldots,n \tag{2.3}$$

The invariance of the quadratic form $\langle y, y\rangle$ implies

$$\begin{aligned}1&=g_{00}^2-\sum_{i=1}^n g_{i0}^2\\ -1&=g_{0k}^2-\sum_{i=1}^n g_{ik}^2\qquad k=1,\ldots,n\\ 0&=g_{00}g_{0k}-\sum_{i=1}^n g_{ik}g_{i0}\end{aligned} \tag{2.4}$$

and it follows that

$$\begin{aligned}|x|^2&=(1+g_{00})^{-2}\sum_{i=1}^n g_{i0}^2=\frac{g_{00}-1}{g_{00}+1}\\ \frac{2}{1+g_{00}}&=1-|x|^2\end{aligned} \tag{2.5}$$

if $x=\tau_g(0)$. The length $|\xi|=(\sum_{i=1}^n \xi_i^2)^{1/2}$ of the tangent vector ξ can now easily be calculated to be $1-|x|^2$

$$\begin{aligned}\frac{1}{(1-|x|^2)^2}|\xi|^2&=4^{-1}(1+g_{00})^2|\xi|^2\\ &=(1+g_{00})^{-2}g_{0n}^2(g_{00}^2-1)-(1+g_{00})^{-1}2g_{0n}^2g_{00}+g_{0n}^2+1\\ &=(1+g_{00})^{-1}g_{0n}^2(g_{00}-1-2g_{00})+g_{0n}^2+1=1\end{aligned}$$

$$|\xi|=1-|x|^2 \tag{2.6}$$

This proves the invariance of the metric

$$ds^2=\rho^2|dx|^2\qquad \rho=(1-|x|^2)^{-1}$$

and the conformality (or anti-conformality) of the transformations τ_g.

Next we define the subgroup M of the Möbius group $G=GM(n-1)$ as the stabilizer of both the origin and the tangent vector ε_n at the origin in B. M is a subgroup of K. In the algebraic picture this is the orthogonal group

$$O(n-1)=\left\{g\in O_{\pm}(1,n):g=\begin{pmatrix}1 & & \\ & (\ *\ ) & \\ & & 1\end{pmatrix}\right\}\cong M \tag{2.7}$$

The cosets are parametrized by the geometric parameters $x=\tau_g(0)$ and $\xi=d\tau_g(0)\varepsilon_n$. We call the pair (x,ξ) the coordinates for the coset $gO(n-1)$. The equations (2.1) and (2.3) express these coordinates by the matrix elements g_{ij} of g. Geometrically, the quotient space G/M can be realized as the cosphere bundle X of B. Since $|\xi|=1-|x|^2$, the group $GM(n-1)$ acts on

$$X=\{(x,\xi)\in B\times\mathbf{R}^n:|\xi|=1-|x|^2\} \tag{2.8}$$

and the action is seen to be transitive. It can be described by the formula

$$(x,\xi)\to(\tau_g x, d\tau_g(x)\xi) \tag{2.9}$$

where $d\tau_g(x)$ is the cotangent mapping which maps the cotangent space at x onto the cotangent space at $z=\tau_g x$.

We now turn to a description of the Lie algebra $\mathfrak{g}$ of $O_{\pm}(1,n)$. Let $E_{ij}\in GL(n+1)$ denote the matrix with element 1 at the place i,j and zero otherwise. A basis for the Lie algebra of $O_{\pm}(1,n)$ is given by the matrices

$$X_{0j}=E_{0j}+E_{j0} \qquad j=1,\dots,n$$

and (2.10)

$$X_{ij}=E_{ij}-E_{ji} \qquad 1\le i<j\le n$$

We set

$$X_i=X_{0i} \qquad i=1,\dots,n-1$$

$$Z=X_{0n} \tag{2.11}$$

$$Y_i=X_{in} \qquad i=1,\dots,n-1$$

The stabilizer $O(n)$ of $e_0\in\mathbf{R}^{n+1}$ is a maximal compact subgroup in $O_{\pm}(1,n)$ and $O_{\pm}(1,n)/O(n)\cong B$ is a symmetric space of rank one. In the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the subalgebra $\mathfrak{k}$ has the vectorspace basis $\{X_{ij}:1\le i<j\le n\}$ and $\mathfrak{p}$ is the

linear subspace with basis $\{X_{0j} : j = 1, \ldots, n\}$. The commutator relations

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \tag{2.12}$$

$$[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p} \tag{2.13}$$

hold. A maximal abelian subalgebra in $\mathfrak{p}$ is given by $\mathfrak{a} = \mathbf{R}Z$, it is one dimensional. If the corresponding subgroup is denoted by A, then the subgroup $O(n-1) \cong M$ defined above (2.7) is the centralizer of A in $O(n) \cong K$. Its Lie algebra $\mathfrak{m}$ has the basis $\{X_{ij} : 1 \le i < j \le n-1\}$

The commutator relations are as follows

$$\begin{array}{llll}
[\mathfrak{m}, Z] = 0 & & & \\
[X_i, Z] = Y_i & [X_i, X_{ij}] = X_j & 1 \le i < j \le n-1 & \\
[Y_i, Z] = X_i & [Y_i, X_{ij}] = Y_j & 1 \le i < j \le n-1 & (2.14) \\
[X_i, X_j] = X_{ij} & [Y_i, Y_j] = -X_{ij} & 1 \le i < j \le n-1 & \\
[X_i, Y_j] = \delta_{ij} Z & & i, j = 1, \ldots, n-1 &
\end{array}$$

In particular it should be noted that if $\mathfrak{q}$ is the linear subspace with basis $\{X_1, \ldots, X_{n-1}, Z, Y_1, \ldots, Y_{n-1}\}$ then

$$[\mathfrak{q}, \mathfrak{m}] \subset \mathfrak{q} \tag{2.15}$$

which shows that G/M is a reductive coset space (see Section 3). $\{X_i - Y_i : i = 1, \ldots, n-1\}$ is a basis of the α-root space $\mathfrak{n}$ of the pair $(\mathfrak{g}, \mathfrak{a})$:

$$[tZ, X_i - Y_i] = t(X_i - Y_i), \qquad \alpha(tZ) = t$$

whereas $\bar{\mathfrak{n}}$ is given by $\{X_i + Y_i : i = 1, \ldots, n-1\}$.

The Weyl group $W = O'(n-1)/O(n-1)$ where $O'(n-1)$ and $O(n-1)$ are the normalizer and centralizer of A in $O(n) = K$ consists of two elements only. They are represented by the identity and the matrix

$$w = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{2.16}$$

The mapping ω, which maps the cosets $gO(n-1)$ onto the cosets $gwO(n-1)$ can geometrically be described by the formula

$$\omega(x, \xi) = (x, -\xi) \tag{2.17}$$

This mapping is not a Möbiustransformation on X.

Geometrically, the Lie algebra of $G = GM(n-1)$ is given by the vectorfields on B which generate the one parameter subgroups τ_{g_t} of G. The vectorfields are determined by the equation

$$v(x) = \frac{d}{dt} \tau_{g_t}(x)\bigg|_{t=0}$$

Conversely, the one parameter subgroup τ_{g_t} is obtained from the vectorfield v by solving the differential equation

$$\frac{dz}{dt} = v(z)$$

with initial condition $z(0) = x$. The one parameter subgroup is then given by $\tau_{g_t}(x) = z(t)$.

In a first step the vectorfields on $\mathbf{R}^n$ are determined, which are the infinitesimal generators of the one parameter subgroups of the group $GM(n)$ acting on $\hat{\mathbf{R}}^n$. The vectorfields in the Lie algebra of $GM(n-1)$ are then singled out by the condition

$$(v(x), x) = 0 \quad \text{for} \quad |x| = 1 \tag{2.18}$$

The vectorfield v has to be tangent to the boundary of $B \subset \mathbf{R}^n$. The vectorfields in the Lie algebra of $GM(n)$ are

$$v(x) = a + Bx + \lambda x + c\,|x|^2 - 2x(c, x) \tag{2.19}$$

with a, c constant vectors in $\mathbf{R}^n$, B a constant matrix with $B' = -B$ and $\lambda \in \mathbf{R}$. The vectorfields Bx account for the rotations in $\mathbf{R}^n$ (the subgroup M with respect to $GM(n)$), the constant vectors a for the translations (the subgroup N) and λx for the dilations (the subgroup A). The remaining vectorfields $c\,|x|^2 - 2x(c, x)$ generate the one parameter subgroups τ_{g_t} conjugate to the translations (the subgroup $\bar{N}$):

$$s \circ \tau_{g_t} \circ s(x) = x + ct$$

where s is the reflection in the unit sphere. The vectorfields in the Lie algebra of $GM(n-1)$ can easily be singled out by condition (2.18). The restrictions are $\lambda=0$ and

$$(a, x)-(c, x)=0 \quad \text{for} \quad |x|=1$$

The Lie algebra of $GM(n-1)$ is therefore described by the vectorfields

$$v(x)=Bx+c(1+|x|^2)-2x(c, x) \tag{2.20}$$

The vectorfields Bx now correspond to the subalgebra $\mathfrak{k}\subset\mathfrak{g}$ and the remaining vectorfields to the complementary subspace $\mathfrak{p}\subset\mathfrak{g}$.

3. Invariant differential operators

The group $O_{\pm}(1, n)$ is not connected. The connected component of the identity is the subgroup $SO_{\pm}(1, n)$. The spaces $O_{\pm}(1, n)/O(n-1)$ and $SO_{\pm}(1, n)/SO(n-1)$ are isomorphic coset spaces with in the first instance the group $O_{\pm}(1, n)$, in the second the group $SO_{\pm}(1, n)$ acting by left translations.

DEFINITION (Nomizu [6]). Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and denote the adjoint representation of G on $\mathfrak{g}$ by Ad (g). Assume that M is a closed subgroup with Lie algebra $\mathfrak{m}$. The coset space G/M is reductive, if there exists a subspace $\mathfrak{q}$ of $\mathfrak{g}$, complementary to $\mathfrak{m}$, such that Ad (m) $\mathfrak{q}\subset\mathfrak{q}$ for all $m\in M$.

Upon taking $G=SO_{\pm}(1, n)$ and $M=SO(n-1)$ one finds that the subspace $\mathfrak{q}$ with basis $\{X_1, \ldots, X_{n-1}, Z, Y_1, \ldots, Y_{n-1}\}$ is complementary to the Lie algebra $\mathfrak{m}$ of M and that furthermore $[\mathfrak{m}, \mathfrak{q}]\subset\mathfrak{q}$ (see (2.11) and (2.15)). Since M is connected, this implies Ad (m) $\mathfrak{q}\subset\mathfrak{q}$ for all $m\in M$. The coset space $X=SO_{\pm}(1, n)/SO(n-1)$ (with $SO_{\pm}(1, n)$ acting on it by left translation) is therefore reductive.

By definition, the differential operator D on G/M is invariant (with respect to left translations $\tau^g f(x)=f(\tau_{g^{-1}}x)$) if $D\tau^g f=\tau^g Df$ for all $f\in C_c(G/M)$ and for all $g\in G$. The algebra of invariant differential operators is denoted by $\underset{\sim}{D}(G/M)$. It can be determined on the base of a theorem of Helgason [3]. For this purpose let $I(\mathfrak{q})$ denote the polynomials in the symmetric algebra $S(\mathfrak{q})$ over $\mathfrak{q}$, which are invariant under Ad (m) for all $m\in M$. The polynomials in $S(\mathfrak{q})$ are polynomials in the variables $Z_1, \ldots, Z_k$ where $\{Z_1, \ldots, Z_k\}$ is a basis in M.

The symmetrization mapping λ associates with every polynomial $Q \in S(\mathfrak{q})$ a differential operator on the group G. Symmetrization is a linear mapping, which maps the elements $Y_1 Y_2 \cdots Y_p \in S(\mathfrak{q})$ (where the Y_j are elements in the subspace $\mathfrak{q}$ of $\mathfrak{g}$, $j = 1, \ldots, p$) onto the differential operator

$$\lambda(Y_1 Y_2 \cdots Y_p) = \frac{1}{p!} \sum_{\sigma} Y_{\sigma(1)} \cdot Y_{\sigma(2)} \cdot \cdots \cdot Y_{\sigma(p)}$$

In this sum σ runs over the symmetric group on p letters. In particular, $\lambda(Y)$ is the differential operator defined by the Lie algebra element $Y \in \mathfrak{g}$

THEOREM (Helgason). *Let G/M be a reductive coset space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$, $\mathrm{Ad}\,(m)\mathfrak{q} \subset \mathfrak{q}$ for all $m \in M$. Then there exists a linear bijection of $I(\mathfrak{q})$ onto $D(G/M)$. It associates to the polynomial $Q(Z_1, \ldots, Z_k) \in I(\mathfrak{q})$ the differential operator D_Q which can be determined by one of the equivalent methods*:

$$(1)\ D_Q f(x) = Q\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}\right) f \circ \pi\left(g \exp \sum_{i=1}^{k} t_i Z_i\right)\bigg|_{t=0} \tag{3.1}$$

where π is the canonical projection of G onto G/M, $\pi(g) = x$.

$$(2)\ \lambda(Q)(f \circ \pi) = D_Q f \circ \pi \tag{3.2}$$

This formula defines $D_Q f$, since $\lambda(Q)(f \circ \pi)$ is constant on each coset gM if $f \in C_c^\infty(G/M)$.

THEOREM 1. *Let $G = SO_\pm(1, n)$, $M = SO(n-1)$ and $\mathfrak{g} = \mathfrak{m} + \mathfrak{q}$ with the specified basis $\{X_1, \ldots, X_{n-1}, Z, Y_1, \ldots, Y_{n-1}\}$ for $\mathfrak{q}$ (see Section 2). Then the algebra $I(\mathfrak{q})$ of $\mathrm{Ad}\,(M)$ invariant polynomials is generated by the polynomials*

$$1,\quad Z,\quad |X|^2 = \sum_{i=1}^{n-1} X_i^2, \qquad (X, Y) = \sum_{i=1}^{n-1} X_i Y_i, \qquad |Y|^2 = \sum_{i=1}^{n-1} Y_i^2.$$

We calculate the action of $\mathrm{Ad}\,(m)$. If $X \in \mathfrak{m}$, $Y \in \mathfrak{q}$ then

$$\mathrm{Ad}\,(\exp tX)\,Y = e^{t\,\mathrm{ad}\,X}\,Y = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathrm{ad}\,X)^n Y$$

Set $X = X_{ij} \in \mathfrak{m}$ and $Y = Z_i$, which stands for X_i or $Y_i \in \mathfrak{q}$. Then

$$(\operatorname{ad} X_{ij})Z_i = [X_{ij}, Z_i] = -Z_j$$

$$(\operatorname{ad} X_{ij})Z_j = Z_i, \qquad (\operatorname{ad} X_{ij})Z_k = 0 \qquad k \neq i, j$$

$$\operatorname{Ad}(\exp tX_{ij})Z_i = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} Z_i - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} Z_j$$
$$= Z_i \cos t - Z_j \sin t$$
$$\operatorname{Ad}(\exp tX_{ij})Z_j = Z_i \sin t + Z_j \cos t$$

It follows that

$$\operatorname{Ad}(m)X_k = \sum_{h=1}^{n-1} m_{hk} X$$

for $m = \exp tX_{ij} \in SO(n-1) \subset GL(n-1)$ with $m = (m_{hk})$. This equation therefore holds for all $m \in SO(n-1)$. Furthermore, if $X = \sum_{k=1}^{n-1} x_k X_k$, then $\operatorname{Ad}(m)X = \sum_{h=1}^{n-1} x'_h X_h$ with $x' = mx$. Similarly, if $Y = \sum_{k=1}^{n-1} y_k Y_k$ then $\operatorname{Ad}(m)Y = \sum_{h=1}^{n-1} y'_h Y_h$ with $y' = my$. Finally, since $\operatorname{Ad}(m)zZ = zZ$ $(z \in \mathbf{R})$, the action of $\operatorname{Ad}(m)$ on the polynomials $P(x, y, z)$ in the variables $x, y \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$ is given by

$$\operatorname{Ad}(m)P(x, y, z) = P(mx, my, z)$$

Assume now that the polynomial Q is invariant under the action of $\operatorname{Ad}(M)$. It can then be written as a finite sum

$$Q(x, y, z) = \sum_k z^k Q_k(x, y)$$

with invariant polynomials $Q_k(x, y)$. It is well known (see e.g. Weyl [7] p. 31 ff.) that the invariant polynomials in the variables x, y under the action

$$(x, y) \to (mx, my) \qquad m \in SO(n-1)$$

are generated by the polynomials $1, |x|^2 = \sum_{i=1}^{n-1} x_i^2, (x, y) = \sum_{i=1}^{n-1} x_i y_i$ and $|y|^2 = \sum_{i=1}^{n-1} y_i^2$. This proves the theorem.

The invariant operators $1, D_Z, D_{|X|^2}, D_{(X,Y)}$ and $D_{|Y|^2}$ generate the whole algebra $\underset{\sim}{D}(G/M)$. This follows from the fact that

$$D_{P_1P_2} = D_{P_1} \cdot D_{P_2} + D$$

where the order of the invariant differential operator D is less than the sum of the degrees of the polynomials P_1 and P_2 (see Helgason [3] p. 269). In the present situation there is however more that can be said:

THEOREM 2. *The differential operators satisfy the following commutator relations*:

$$[D_Z, D_{|X|^2}] = -2D_{(X,Y)} \tag{3.3}$$

$$[D_Z, D_{|Y|^2}] = -2D_{(X,Y)} \tag{3.4}$$

$$[D_Z, D_{(X,Y)}] = -D_{|X|^2} - D_{|Y|^2} \tag{3.5}$$

Consequently, $\underset{\sim}{D}(G/M)$ is generated by $1, D_Z$ *and* $D_{|Y|^2}$ *(or by* $1, D_Z$ *and* $D_{|X|^2}$*).*

The proof relies on the symmetrization mapping λ. The differential operator $D_{Z|Y|^2}$ is obtained from the differential operator on G which is given by

$$\lambda(Z\,|Y|^2) = \frac{1}{3!}\sum_{i=1}^{n-1} 2(Y_i \cdot Y_i \cdot Z + Y_i \cdot Z \cdot Y_i + Z \cdot Y_i \cdot Y_i)$$

The commutator relations for the Lie algebra (2.14) then imply

$$\lambda(Z\,|Y|^2) = \sum_{i=1}^{n-1} Y_i \cdot Y_i \cdot Z - \frac{1}{2}\sum_{i=1}^{n-1}(X_i \cdot Y_i + Y_i \cdot X_i) + \frac{n-1}{6} Z$$

$$\lambda(Z\,|Y|^2) = \sum_{i=1}^{n-1} Z \cdot Y_i \cdot Y_i + \frac{1}{2}\sum_{i=1}^{n-1}(X_i \cdot Y_i + Y_i \cdot X_i) + \frac{n-1}{6} Z$$

It follows that

$$D_{|Y|^2}D_Z - D_{(X,Y)} + \frac{n-1}{6} D_Z = D_Z D_{|Y|^2} + D_{(X,Y)} + \frac{n-1}{6} D_Z$$

which proves the first equality. The second is proved in the same way and the third is a consequence of the following equations:

$$\lambda\left(\sum_{i=1}^{n-1} X_iY_iZ\right)=\tfrac{1}{3}\sum_{i=1}^{n-1}(X_i\cdot Y_i\cdot Z+Y_i\cdot X_i\cdot Z+X_i\cdot Z\cdot Y_i+Y_i\cdot Z\cdot X_i$$
$$+Z\cdot X_i\cdot Y_i+Z\cdot Y_i\cdot X_i)$$
$$=\tfrac{1}{2}\sum_{i=1}^{n-1}(X_i\cdot Y_i+Y_i\cdot X_i)\cdot Z-\tfrac{1}{2}\sum_{i=1}^{n-1}(X_i\cdot X_i+Y_i\cdot Y_i)$$
$$=\tfrac{1}{2}\sum_{i=1}^{n-1}Z\cdot(X_i\cdot Y_i+Y_i\cdot X_i)+\tfrac{1}{2}\sum_{i=1}^{n-1}(X_i\cdot X_i+Y_i\cdot Y_i)$$

$$D_{(X,Y)}D_Z-\tfrac{1}{2}D_{|X|^2}-\tfrac{1}{2}D_{|Y|^2}=D_ZD_{(X,Y)}+\tfrac{1}{2}D_{|X|^2}+\tfrac{1}{2}D_{|Y|^2}$$

The Killing form on the Lie algebra of $SO(1, n)$ is given by

$$B(X,X)=2(n-1)\left\{\sum_{i=1}^{n}x_i^2-\sum_{1\le i<j\le n}x_{ij}^2\right\}$$

$$X=\sum_{i=1}^{n}x_iX_i+\sum_{1\le i<j\le n}x_{ij}X_{ij}$$

(see the definitions (2.10) and (2.11) in section 2). The Killing form is invariant under $\operatorname{Ad}(g)$ for all $g\in SO(1,n)$ and in particular for $g\in SO(n)$ or $SO(n-1)$. The Casimir operator restricted to $B\cong SO_{\pm}(1,n)/SO(n)$ is

$$\Delta_K=D_{|X|^2}+D_{Z^2} \tag{3.6}$$

and restricted to $X\cong SO_{\pm}(1,n)/SO(n-1)$ it is

$$\Delta_M=D_{|X|^2}+D_{Z^2}-D_{|Y|^2} \tag{3.7}$$

It follows that the operators Δ_K and Δ_M, considered as operators in $\underset{\sim}{D}(G/M)$ commute. In fact, Δ_M commutes with every differential operator in $\underset{\sim}{D}(G/M)$.

In the next section it will be shown that the operators in $\underset{\sim}{D}(G/M)$ are invariant under the whole group $O_{\pm}(1,n)$ and not only under the subgroup $SO_{\pm}(1,n)$.

4. The calculations for some operators

In this section the geometric versions of the operators D_Z, $D_{|Y|^2}$ and $D_{(X,Y)}$ will be calculated. This means that the operators will be expressed as differential operators in the variables (x, ξ). Recall that

$$x_i = (1+g_{00})^{-1} g_{i0} \tag{2.1}$$

and

$$\begin{aligned} \xi_i &= 2g_{0n}g_{i0}(1+g_{00})^{-2} - 2g_{in}(1+g_{00})^{-1} \\ &= 2(g_{0n}x_i - g_{in})(1+g_{00})^{-1} \end{aligned} \tag{2.3}$$

$i = 1, \ldots, n$ are the coordinates for the coset $gO(n-1)$. The matrices (g_{ij}) representing g satisfy the relations (2.4) and in particular

$$2(1+g_{00})^{-1} = 1 - |x|^2 = |\xi|^2 \qquad (2.5) \quad (2.6)$$

and

$$\begin{aligned} (x \mid \xi) &= \sum_{i=1}^{n} x_i\xi_i = \frac{2}{1+g_{00}}\left(g_{0n}\,|x|^2 - (1+g_{00})^{-1}\sum_{i=1}^{n} g_{i0}g_{in}\right) \\ &= 2g_{0n}(1+g_{00})^{-1}(|x|^2 - g_{00}(1+g_{00})^{-1}) \\ &= -\tfrac{1}{2}g_{0n}(1-|x|^2)^2 = -2g_{0n}(1+g_{00})^{-2} \end{aligned} \tag{4.1}$$

Let $a_t = \exp tZ$ denote the one parameter subgroup of $O_{\pm}(1, n)$ defined by Z. In order to calculate $D_Z f$ at the point (x, ξ) (coordinates of the coset $gO(n-1)$), the definition of Lie derivatives is used:

$$D_Z f(x, \xi) = \frac{d}{dt} f(x_t, \xi_t)\bigg|_{t=0} \tag{4.2}$$

where (x_t, ξ_t) are the coordinates of the coset $ga_tO(n-1)$:

$$(x_t)_i = (g_{i0}\,\mathrm{Ch}\,t + g_{in}\,\mathrm{Sh}\,t)(1 + g_{00}\,\mathrm{Ch}\,t + g_{0n}\,\mathrm{Sh}\,t)^{-1} \tag{4.3}$$

$$\begin{aligned} (\xi_t)_i &= 2(g_{00}\,\mathrm{Sh}\,t + g_{0n}\,\mathrm{Ch}\,t)(g_{i0}\,\mathrm{Ch}\,t + g_{in}\,\mathrm{Sh}\,t)(1 + g_{00}\,\mathrm{Ch}\,t + g_{0n}\,\mathrm{Sh}\,t)^{-2} \\ &\quad -2(g_{i0}\,\mathrm{Sh}\,t + g_{in}\,\mathrm{Ch}\,t)(1 + g_{00}\,\mathrm{Ch}\,t + g_{0n}\,\mathrm{Sh}\,t)^{-1} \end{aligned} \tag{4.4}$$

It follows that

$$\frac{d(x_t)_i}{dt}\bigg|_{t=0} = g_{in}(1+g_{00})^{-1} - g_{i0}g_{0n}(1+g_{00})^{-2} = -\tfrac{1}{2}\xi_i \tag{4.5}$$

and after some calculations

$$\left.\frac{d(\xi_t)_i}{dt}\right|_{t=0} = -2g_{0n}(1+g_{00})^{-1}\xi_i - 2(1+g_{00})^{-1}x_i$$
$$= (1-|x|^2)^{-1}(2(x \mid \xi)\xi_i - |\xi|^2 x_i) \tag{4.6}$$

The operator D_Z can be expressed by the formula

$$D_Z f(x,\xi) = \sum_{i=1}^n f_{x_i} \left.\frac{(dx_t)_i}{dt}\right|_{t=0} + \sum_{i=1}^n f_{\xi_i} \left.\frac{(d\xi_t)_i}{dt}\right|_{t=0} \tag{4.7}$$

THEOREM 3.

$$D_Z f(x,\xi) = -\tfrac{1}{2}\sum_{i=1}^n f_{x_i}\xi_i + (1-|x|^2)^{-1}\sum_{i=1}^n f_{\xi_i}(2(x \mid \xi)\xi_i - |\xi|^2 x_i) \tag{4.8}$$

This operator is invariant under the group $GM(n-1)$ of Möbiustransformations on X. Under the mapping $\omega(x,\xi)=(x,-\xi)$ it transforms into the operator $-D_Z$.

The group $GM(n-1)$ has two components. By construction, the operator D_Z is invariant under proper Möbius transformations. It suffices to prove its invariance for a single transformation τ_g, $g \notin SO_\pm(1,n)$. Such a transformation is

$$y_1 = -x_1 \qquad \eta_1 = -\xi_1$$
$$y_k = x_k \qquad \eta_k = \xi_k \qquad k = 2, \ldots, n \tag{4.9}$$

The transformed operator is

$$D_Z^g f(y,\eta) = \frac{1}{2}\sum_{i,j=1}^n \left(f_{y_j}\frac{\partial y_j}{\partial x_i} + f_{\eta_j}\frac{\partial \eta_j}{\partial x_i}\right)\xi_i$$
$$+(1-|x|^2)^{-1}\sum_{i,j=1}^n \left(f_{y_j}\frac{\partial y_j}{\partial \xi_i} + f_{\eta_j}\frac{\partial n_j}{\partial \xi_i}\right)(2(x \mid \xi)\xi_i - |\xi|^2 x_i)$$
$$= -\frac{1}{2}\sum_{i=1}^n f_{y_i}\eta_i + (1-|y|^2)^{-1}\sum_{i=1}^n f_{\eta_i}(2(y \mid \eta)\eta_i - |\eta|^2 y_i)$$

It coincides with D_Z. The same calculation shows that the mapping ω (see (2.17)) transforms D_Z into the operator $-D_Z$.

A remark about the derivatives f_{x_i}, f_{ξ_i} $i = 1, \ldots, n$ is appropriate. The function f is defined on

$$X = \{(x, \xi) \in \mathbf{R}^{2n} : |\xi|^2 = 1 - |x|^2\}$$

In order that the derivatives with respect to x and ξ have some meaning, the domain of definition for f first has to be extended into a neighbourhood of X in $\mathbf{R}^{2n}$. The resulting operator D_Z is however known to depend only on the values of f on X. It is independent of the particular extension of f.

The calculation of the remaining operators $D_{|Y|^2}$ and $D_{(X, Y)}$ is based on the theorem of Helgason (section 3). For fixed g with coordinates (x, ξ) and for a given function $f \in C_c(G/M)$ consider the function

$$\tilde{f}(s, t) = f \circ \pi\left(g \exp \sum_{i=1}^{n-1} (s_i X_i + t_i Y_i)\right) \tag{4.10}$$

π is the canonical projection and $(x(s, t), \xi(s, t))$ are the coordinates of $\pi(g \exp \sum_{i=1}^{n-1} (s_i X_i + t_i Y_i))$. Take as an example the operator $D_{(X, Y)}$. We then have

$$D_{(X, Y)} f(x, \xi) = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial s_i \, \partial t_i} \tilde{f}(s, t) \bigg|_{s=t=0} \tag{4.11}$$

The chain rule for the second derivative $\tilde{f}_{s_j t_j}$ gives

$$\begin{aligned} \tilde{f}_{s_j t_j} = &\sum_{m, l=1}^{n} f_{x_l x_m} \frac{\partial x_l}{\partial s_j} \frac{\partial x_m}{\partial t_j} + \sum_{m, l=1}^{n} f_{x_l \xi_m} \frac{\partial x_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} \\ &+ \sum_{m, l=1}^{n} f_{\xi_l x_m} \frac{\partial \xi_l}{\partial s_j} \frac{\partial x_m}{\partial t_j} + \sum_{m, l=1}^{n} f_{x_l \xi_m} \frac{\partial \xi_l}{\partial s_j} \frac{\partial \xi_m}{\partial t_j} + \sum_{l=1}^{n} f_{x_l} \frac{\partial^2 x_l}{\partial s_j \, \partial t_j} + \sum_{l=1}^{n} f_{\xi_l} \frac{\partial^2 \xi_l}{\partial s_j \, \partial t_j} \end{aligned} \tag{4.12}$$

The partial derivatives of f with respect to x and ξ have the same interpretation as above. In addition, the calculations will show that the derivatives of the coordinate functions at $s = t = 0$ are functions on the group. However the resulting operator maps functions on X into functions on X. It can be expressed in the variables x and ξ.

The first derivatives of the coordinate functions

Let $e_1, \ldots, e_{n-1}$ be the canonical basis in the parameter spaces $\mathbf{R}^{n-1}$ for the s

and t variables. If $h \in \mathbf{R}$ then

$$x_m(he_j, 0) = \frac{g_{m0}\,\mathrm{Ch}\,h + g_{mj}\,\mathrm{Sh}\,h}{1 + g_{00}\,\mathrm{Ch}\,h + g_{0j}\,\mathrm{Sh}\,h}$$

$$x_m(0, he_j) = \frac{g_{m0}}{1 + g_{00}}$$

$$\xi_m(he_j, 0) = \frac{2(g_{m0}\,\mathrm{Ch}\,h + g_{mj}\,\mathrm{Sh}\,h)g_{0n}}{(1 + g_{00}\,\mathrm{Ch}\,h + g_{0j}\,\mathrm{Sh}\,h)^2} - \frac{2g_{mn}}{1 + g_{00}\,\mathrm{Ch}\,h + g_{0j}\,\mathrm{Sh}\,h}$$

$$\xi_m(0, he_j) = \frac{2g_{m0}(g_{0j}\sin h + g_{0n}\cos h)}{(1 + g_{00})^2} - \frac{2(g_{mj}\sin h + g_{mn}\cos h)}{1 + g_{00}}$$

The partial derivatives at $(s, t) = (0, 0)$ are

$$\frac{\partial x_m}{\partial s_j} = \frac{d}{dh}\, x_m(he_j, 0)\bigg|_{h=0} = \frac{g_{mj}}{1 + g_{00}} - \frac{g_{m0}g_{0j}}{(1 + g_{00})^2}$$

$$\frac{\partial x_m}{\partial t_j} = 0$$

$$\frac{\partial \xi_m}{\partial s_j} = -2\frac{2g_{m0}g_{0n}g_{0j}}{(1 + g_{00})^3} + \frac{2g_{mj}g_{0n}}{(1 + g_{00})^2} + \frac{2g_{mn}g_{0j}}{(1 + g_{00})^2}$$

$$\frac{\partial \xi_m}{\partial t_j} = \frac{2g_{m0}g_{0j}}{(1 + g_{00})^2} - \frac{2g_{mj}}{1 + g_{00}} = -2\frac{\partial x_m}{\partial s_j}$$

The following expressions are needed for the differential operators:

$$\sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j}\frac{\partial x_m}{\partial s_j} = -\frac{1}{2}\sum_{j=1}^{n-1} \frac{\partial x_l}{\partial s_j}\frac{\partial \xi_m}{\partial t_j} = \frac{1}{4}\sum_{j=1}^{n-1} \frac{\partial \xi_l}{\partial t_j}\frac{\partial \xi_m}{\partial t_j}$$
$$= -\tfrac{1}{4}\xi_l\xi_m + \tfrac{1}{4}\delta_{lm}|\xi|^2 \tag{4.13}$$

$$\sum_{j=1}^{n-1} \frac{\partial \xi_l}{\partial s_j}\frac{\partial \xi_m}{\partial t_j} = -\frac{(x \mid \xi)}{1 - |x|^2}\left(2\xi_l\xi_m - \delta_{lm}\,|\xi|^2 - \frac{x_m\xi_l}{(x \mid \xi)}\,|\xi|^2\right) \tag{4.14}$$

As an example, the calculation of formula (4.13) is given:

$$\frac{\partial x_l}{\partial s_j}\frac{\partial x_m}{\partial s_j}=\left(\frac{g_{mj}}{1+g_{00}}-\frac{g_{m0}g_{0j}}{(1+g_{00})^2}\right)\left(\frac{g_{lj}}{1+g_{00}}-\frac{g_{l0}g_{0j}}{(1+g_{00})^2}\right)$$

$$=(1+g_{00})^{-2}g_{mj}g_{lj}-(1+g_{00})^{-3}(g_{l0}g_{mj}g_{0j}+g_{m0}g_{lj}g_{0j})$$

$$+(1+g_{00})^{-4}g_{l0}g_{m0}g_{0j}^2$$

$$\sum_{j=1}^{n-1}\frac{\partial x_l}{\partial s_j}\frac{\partial x_m}{\partial s_j}=(1+g_{00})^{-2}(\delta_{lm}+g_{l0}g_{m0}-g_{ln}g_{mn})$$

$$-(1+g_{00})^{-3}(g_{l0}(g_{00}g_{m0}-g_{0n}g_{mn})+g_{m0}(g_{00}g_{l0}-g_{0n}g_{ln}))$$

$$+(1+g_{00})^{-4}g_{l0}g_{m0}(g_{00}^2-1-g_{0n}^2)$$

The expression $\frac{1}{4}\xi_l\xi_m$ has the value

$$(1+g_{00})^{-4}g_{0n}^2g_{l0}g_{m0}-g_{0n}(1+g_{00})^{-3}(g_{l0}g_{mn}+g_{m0}g_{ln})+(1+g_{00})^{-2}g_{ln}g_{mn}$$

Therefore

$$\sum_{j=1}^{n-1}\frac{\partial x_l}{\partial s_j}\frac{\partial x_m}{\partial s_j}=-\tfrac{1}{4}\xi_l\xi_m+\delta_{lm}(1+g_{00})^{-2}=\tfrac{1}{4}(-\xi_l\xi_m+\delta_{lm}|\xi|^2)$$

(All partial derivatives are taken at $s=t=0$.)

The second derivatives of the coordinate functions

The second derivatives are calculated according to the formulas

$$\left.\frac{\partial^2 x}{\partial s_j\,\partial t_j}\right|_{s=t=0}=\lim_{h\to 0}h^{-2}(x(he_j,he_j)-x(he_j,0)-x(0,he_j)+x(0,0))$$

$$\left.\frac{\partial^2 \xi}{\partial t_j^2}\right|_{s=t=0}=\lim_{h\to 0}h^{-2}(\xi(0,he_j)+\xi(0,-he_j)-2\xi(0,0))$$

Up to third order terms

$$x_m(he_j,he_j)\simeq\frac{1}{N}(g_{m0}(1+h^2/2)+g_{mj}h-g_{mn}h^2/2)$$

$$\xi_m(he_j,he_j)\simeq\frac{2}{N}(g_{00}h^2/2+g_{0j}h+g_{0n}(1-h^2/2))x_m(he_j,he_j)$$

$$-\frac{2}{N}(g_{m0}h^2/2+g_{mj}h+g_{mn}(1-h^2/2))$$

with

$$N = 1 + g_{00}(1 + h^2/2) + g_{0j}h - g_{0n}h^2/2$$

The resulting expressions (at $s = t = 0$) are

$$\frac{\partial^2 x_m}{\partial s_j\, \partial t_j} = \tfrac{1}{4}\xi_m$$

$$\frac{\partial^2 x_m}{\partial t_j^2} = 0$$

$$\frac{\partial^2 \xi_m}{\partial s_j\, \partial t_j} = 2g_{m0}(g_{0n}^2 - 2g_{0j}^2)(1 + g_{00})^{-3} + (-2g_{mn}g_{0n} - g_{m0} + 4g_{0j}g_{mj})(1 + g_{00})^{-2}$$

$$\frac{\partial^2 \xi_m}{\partial t_j^2} = -\xi_m$$

As above this leads to the required equations

$$\sum_{j=1}^{n-1} \frac{\partial^2 x_m}{\partial s_j\, \partial t_j} = \frac{n-1}{4}\, \xi_m \tag{4.15}$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 x_m}{\partial t_j^2} = 0 \tag{4.16}$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 \xi_m}{\partial s_j\, \partial t_j} = -(n+1)\frac{(x \mid \xi)}{1 - |x|^2}\, \xi_m - \frac{n-5}{2}(1 - |x|^2)x_m \tag{4.17}$$

$$\sum_{j=1}^{n-1} \frac{\partial^2 \xi_m}{\partial t_j^2} = -(n-1)\xi_m \tag{4.18}$$

THEOREM 4. *The operator $D_{|Y|^2}$ on $X \cong O_\pm(1, n)/O(n-1)$ is given by*

$$D_{|Y|^2} f = -\sum_{l,\, m=1}^{n} f_{\xi_l \xi_m}(\xi_l \xi_m - \delta_{lm}\, |\xi|^2) - (n-1) \sum_{m=1}^{n} f_{\xi_m} \xi_m \tag{4.19}$$

It is invariant under the Möbius group $GM(n-1)$ and under the mapping $\omega(x, \xi) = (x, -\xi)$. At the same time, $D_{|Y|^2}$ is the Laplace operator on the sphere $\{\xi \in \mathbf{R}^n : |\xi| = 1\}$.

Consider the stabilizer $K \cong O(n)$ of the sphere

$$\sum = \{(x, \xi) \in X : x = 0, |\xi| = 1\}$$

The Lie algebra elements $Y_1, \ldots, Y_{n-1}$ (see (2.11)) are in the Lie algebra $\mathfrak{k}$ of $O(n)$. The invariant differential operator $D_{|Y|^2}$ is therefore a differential operator on the subgroup $O(n)$. Furthermore, it is the restriction of the Casimir operator $\sum_{1 \le i < j \le n} X_{ij}^2$ of $\mathfrak{k}$ onto the quotient space $\Sigma \cong O(n)/O(n-1)$. This operator is the Laplace operator on the sphere.

According to the preceeding formulas (4.13)–(4.18), the operator $D_{|Y|^2}$ on X has the explicit form given in the theorem. In particular it is seen to be independent of the x coordinate (apart from the restriction $|\xi| = 1 - |x|^2$).

The invariance of the operator $D_{|Y|^2}$ under the whole Möbius group $GM(n-1)$ and under the mapping ω can be established with the same method which was used in connection with the operator D_Z.

COROLLARY. *All differential operators on X which are invariant under the group of special Möbius transformations $SM(n-1) \cong SO_{\pm}(1, n)$ are invariant under the whole group $GM(n-1)$. The operators $D_{|Y|^2}$ and $D_{|X|^2}$ are also invariant under the mapping ω, yet ω transforms D_Z and $D_{(X,Y)}$ into $-D_Z$ and $-D_{(X,Y)}$ respectively.*

THEOREM 5. *The operator $D_{(X,Y)}$ is given by*

$$\begin{aligned} D_{(X,Y)}f = \tfrac{1}{2} \sum_{l,m=1}^{n} f_{x_l \xi_m}(\xi_l \xi_m - \delta_{lm} |\xi|^2) + \frac{n-1}{4} \sum_{m=1}^{n} f_{x_m} \xi_m \\ - \sum_{l,m=1}^{n} f_{\xi_l \xi_m} \left[\frac{(x \mid \xi)}{1 - |x|^2} (2\xi_l \xi_m - \delta_{lm} |\xi|^2) - (1 - |x|^2) x_l \xi_m \right] \\ - \sum_{m=1}^{n} f_{\xi_m} \left[(n+1) \frac{(x \mid \xi)}{1 - |x|^2} \xi_m + \frac{n-5}{2} (1 - |x|^2) x_m \right] \end{aligned} \tag{4.20}$$

5. Spherical harmonics and the operators S_k and S_k^*

A spherical harmonic of degree k on the sphere $\Sigma = \{\xi \in \mathbf{R}^n : |\xi| = 1\}$ is the restriction of a harmonic polynomial in $\mathbf{R}^n$ which is homogeneous of degree k. The space of spherical harmonics of degree k will be denoted by H^k. Alternatively, it can be described as the eigenspace with eigenvalue $-k(k+n-2)$ of the Laplace operator Δ_Σ on the sphere. The system of spherical harmonics is complete in $L^2(\Sigma)$. It gives a decomposition of this space as a direct orthogonal Hilbert sum

$$L^2(\Sigma) = \bigoplus_{k=0}^{\infty} H^k$$

DEFINITION. A spherical harmonic of degree k on $X \cong O_{\pm}(1, n)/O(n-1)$ is an eigenfunction of the operator $D_{|Y|^2}$ with eigenvalue $-k(k+n-2)$.

$$E^k(X) = \{f \in C^\infty(X) : D_{|Y|^2} f = -k(k+n-2)f\} \tag{5.1}$$

If a function $f \in C^\infty(X)$ is an eigenfunction of the operator $D_{|Y|^2}$, then for every fixed x

$$-\sum_{l,m=1}^{n} f_{\xi_l\xi_m}(\xi_l\xi_m - \delta_{lm}|\xi|^2) - (n-1)\sum_{m=1}^{n} f_{\xi_m}\xi_m = \lambda f$$

But the left hand side is the spherical Laplace operator Δ_Σ applied to $f(x, \xi)$ with x fixed. Therefore the eigenvalue λ is of the form $-k(k+n-2)$ for some non negative integer k. If $\{h_{k1}, \ldots, h_{kd}\}$, $d = d(k)$, is an orthogonal basis in H^k, then

$$f(x, \xi) = \sum_{j=1}^{d} c_{kj}(x) h_{kj}(\xi)$$

with coefficients c_{kj} $j = 1, \ldots, d$ which will depend (smoothly) on x. Conversely, any such function is in E^k.

From the completeness property of the system of spherical harmonics on Σ we conclude that any function $f \in C^\infty(X)$ has an expansion of the form

$$f(x, \xi) = \sum_{k=0}^{\infty}\sum_{j=1}^{d} c_{kj}(x) h_{kj}(\xi) \tag{5.2}$$

which converges for every fixed x in $L^2(\Sigma)$.

A harmonic polynomial p of degree k in $(\mathbf{R}^n)$ defines a symmetric tensor t of order k with vanishing traces

$$p(\xi) = \sum_{i_1, \ldots, i_k = 1}^{n} t_{i_1 \cdots i_k} \xi_{i_1} \cdots \xi_{i_k}$$

$$t_{i_1 \cdots i_k} = t_{i_{\sigma(1)} \cdots i_{\sigma(k)}} \text{ for any permutation } \sigma \text{ on the indices} \tag{5.3}$$

$$\sum_{i=1}^{n} t_{iii_3 \cdots i_k} = 0$$

Conversely, to any such tensor the formula associates a harmonic polynomial p which is homogeneous of degree k. The functions $f \in E^k$ therefore can be viewed as tensorfields of order k on the hyperbolic space $B = O_{\pm}(1, n)/O(n)$:

$$E^k = \{f \in C^\infty(X): f(x, \xi) = (1-|x|^2)^{-2k} \sum_{i_1, \ldots, i_k} t_{i_1 \cdots i_k}(x)\xi_{i_1} \cdots \xi_{i_k}\} \tag{5.4}$$

In this representation $t(x) = t_{i_1 \cdots i_k}(x)$ is a tensorfield of symmetric tensors with vanishing trace. The factor $(1-|x|^2)^{-2k}$ is a normalizing factor.

The type of the tensorfield t is given by its transformation behaviour under Möbius transformations. Recall that the action of $GM(n-1)$ on X is defined by

$$(x, \xi) \to (\tau_g x, d\tau_g \xi) \tag{2.9}$$

The action on $C(X)$ then becomes

$$f^{g^{-1}}(x, \xi) = f(\tau_g x, d\tau_g \xi) \tag{5.5}$$

First consider the special case of a vectorfield

$$f(x, \xi) = (1-|x|^2)^{-2} \sum_{i=1}^{n} v_i(x)\xi_i \in E^1$$

$$f^{g^{-1}}(x, \xi) = (1-|\tau_g x|^2)^{-2} \sum_{i=1}^{n} v_i(\tau_g x)(d\tau_g \xi)_i$$

We set $y = \tau_g x$. Since $ds^2 = (1-|x|^2)^{-2}|dx|^2$ is an invariant metric, the Jacobian determinant of the matrix

$$G(x) = \left(\frac{\partial y_i}{\partial x_k}(x)\right)$$

representing the tangent mapping $d\tau_g(x)$ is given by

$$\det G(x) = \pm(1-|y|^2)^n(1-|x|^2)^{-n}$$

The conformality (or anti-conformality) implies that $((1-|x|^2)/(1-|y|^2))G(x)$ is an orthogonal matrix. In particular

$$G^{-1}(x) = (1-|x|^2)^2(1-|y|^2)^{-2}G^t(x) \tag{5.6}$$

(G^t is the transposed matrix). It then follows that

$$f^{g^{-1}}(x, \xi) = (1-|x|^2)^{-2} \sum_{k=1}^{n} \sum_{i=1}^{n} v_i(y) \frac{\partial y_i}{\partial x_k}(x)\xi_k(1-|x|^2)^2(1-|y|^2)^{-2}$$

$$= (1-|x|^2)^{-2} \sum_{k=1}^{n} v_k^{g^{-1}}(x)\xi_k$$

with

$$v^{g^{-1}}(x) = G^{-1}(x)v(\tau_g x) \tag{5.7}$$

Next assume that $f \in E^k$,

$$f(x, \xi) = (1-|x|^2)^{-2k} \sum_{i_1, \cdots, i_k} t_{i_1 \cdots i_k}(x)\xi_{i_1} \cdots \xi_{i_k}$$

Then the same calculations show that

$$f^{g^{-1}}(x, \xi) = (1-|x|^2)^{-2k} \sum_{i_1, \cdots, i_k} t^{g^{-1}}_{i_1 \cdots i_k}(x)\xi_{i_1} \cdots \xi_{i_k} \tag{5.8}$$

with

$$t^{g^{-1}}_{i_1 \cdots i_k}(x) = \sum_{j_1, \ldots, j_k} a_{i_1 j_1} \cdots a_{i_k j_k} t_{j_1 \cdots j_k}(\tau_g x)$$

where the a_{kj} are the components of the matrix $G^{-1}(x)$. The transformation behaviour of the tensors is influenced by the choice of the normalizing factor $(1-|x|^2)^{-2k}$. To illustrate this set

$$f(x, \xi) = (1-|x|^2)^{-2} \sum_{i, k} \varphi_{ik}(x)\xi_i\xi_k \tag{5.9}$$

Here, $\Phi(x) = (\varphi_{ik}(x))$ is a symmetric matrix with vanishing trace. The same calculations as above then show that

$$f^{g^{-1}}(x, \xi) = (1-|x|^2)^{-2} \sum_{i, k} \varphi^{g^{-1}}_{ik}(x)\xi_i\xi_k$$

where the transformed matrix is given by

$$\Phi^{g^{-1}}(x)=G^{-1}(x)\Phi(\tau_g x)G(x) \tag{5.10}$$

This transformation behaviour differs from the preceding by a factor $(\det G(x))^{2/n}$.

THEOREM 6. *A function* $f(x,\xi)=(1-|x|^2)^{-2}\sum_{i=1}^n v_i(x)\xi_i\in E^1$ *satisfies* $D_Z f=0$ *if and only if* v *is a vectorfield in the Lie algebra of* $GM(n-1)$.

The vectorfields v in the Lie algebra of $GM(n-1)$ are of the form

$$v(x)=Bx+c(1+|x|^2)-2x(c,x) \tag{2.20}$$

with $B^t=-B$ and $c\in\mathbf{R}^n$. Direct verification shows that the functions $f\in E^1$ which are associated to these vectorfields satisfy $D_Z f=0$. Conversely, assume that $f\in E^1$ satisfies

$$\begin{aligned}D_Z f&=-(1-|x|^2)^{-2}\tfrac{1}{2}\sum_{i,j}v_{i,j}\xi_i\xi_j-(1-|x|^2)^{-3}(v,x)\sum_{i,j}\delta_{ij}\xi_i\xi_j\\&=0\end{aligned}$$

for all $(x,\xi)\in X$ ($v_{i,j}$ is the notation for the partial derivative $(\partial v_i/\partial x_j)(x)$). It follows that

$$\begin{aligned}&v_{i,j}=-v_{j,i} && i\neq j\\&v_{i,i}=-2(1-|x|^2)^{-1}(v(x),x) && i=1,\dots,n\end{aligned}$$

and in particular

$$v_{i,i}=v_{j,j}$$

Assume now that i, j and k are different indices. Then the differentiated equations

$$v_{i,jk}+v_{j,ik}=0$$

$$v_{k,ij}+v_{i,kj}=0$$

$$v_{j,ki}+v_{k,ji}=0$$

show that $v_{i,jk}=0$. Similarly

$$v_{i,iij}=v_{k,kij}=0$$

and therefore

$$v_{k,iij}=0 \qquad v_{i,kkk}=0.$$

This shows that all third order derivatives vanish. The vectorfield is therefore given by a second order polynomial

$$v_i(x)=\tfrac{1}{2}\sum_{k,l} a_{ikl}x_kx_l+\sum_k b_{ik}x_k+c_i \qquad i=1,\ldots,n$$

and it can be assumed that

$$a_{ikl}=a_{ilk}=v_{i,kl}$$

A comparison of the coefficients in the equations

$$(1-|x|^2)v_{i,i}=-2(v,x) \qquad i=1,\ldots,n$$

with

$$v_{i,i}=\tfrac{1}{2}\sum_k (a_{iki}+a_{iik})x_k+b_{ii}$$

$$(v,x)=\tfrac{1}{2}\sum_{i,k} a_{ikl}x_ix_kx_l+\sum_{i,k} b_{ik}x_ix_k+\sum_i c_ix_i$$

now results in the equations

$$b_{ii}=0$$

$$a_{iik}=-2c_k \qquad i,k=1,\ldots,n$$

$$b_{ik}=-b_{ki}$$

Since it is already known that

$$a_{ijk}=0 \quad \text{if} \quad i\neq j\neq k\neq i$$

and

$$a_{kii} = -a_{iki} = 2c_k \quad \text{if} \quad k \neq i$$

it can be concluded that

$$v_i(x) = \sum_{k=1}^{n} a_{iki}x_k x_i - \tfrac{1}{2}a_{iii}x_i^2 + \tfrac{1}{2}\sum_{k\neq i} a_{ikk}x_k^2 + \sum_{k=1}^{n} b_{ik}x_k + c_i$$

$$= -2x_i \sum_{k=1}^{n} c_k x_k + c_i x_i^2 + \sum_{k\neq i} c_i x_k^2 + \sum_{k=1}^{n} b_{ik}x_k + c_i$$

This shows that

$$v(x) = c(1+|x|^2) - 2x(c, x) + Bx \qquad B^t = -B$$

It should be noted that the theorem is still true for the dimension $n = 2$, yet for this case the proof has to be modified slightly.

The theorem shows that the operator D_Z applied to vectorfields (i.e. to the spherical harmonics of degree 1 on X) singles out exactly the Lie algebra of the Möbius group $GM(n-1)$.

The space of functions $f \in C^\infty(X)$ satisfying $D_Z f = 0$ is an algebra, since D_Z is a first order differential operator. If $\{v^{(1)}, \ldots, v^{(d)}\}$, $d = \frac{1}{2}n(n+1)$, is a basis of the Lie algebra of $GM(n-1)$ and if

$$f_j(x, \xi) = (1-|x|^2)^{-2} \sum_i v_i^{(j)}(x)\xi_i \in E^1 \qquad j = 1, \ldots, d$$

then any convergent power series in $f_1, \ldots, f_d$ will be a solution of $D_Z f = 0$.

THEOREM 7. *The operator*

$$S_k = D_{(X,Y)} + \left(\frac{1}{2} - \left(\frac{n}{2} + k - 1\right)\right) D_Z \tag{5.11}$$

maps E^k into E^{k+1}, and the operator

$$S_k^* = D_{(X,Y)} + \left(\frac{1}{2} + \left(\frac{n}{2} + k - 1\right)\right) D_Z \tag{5.12}$$

maps E^k into E^{k-1}, $k = 1, 2, 3 \ldots$

COROLLARY. *The operators D_Z and $D_{(X,Y)}$ on E^k take the form*

$$D_Z = -(n+2k-2)^{-1}S_k + (n+2k-2)^{-1}S_k^* \tag{5.13}$$

$$2D_{(X,Y)} = (1+(n+2k-2)^{-1})S_k + (1-(n+2k-2)^{-1})S_k^* \tag{5.14}$$

For the proof of the theorem the operator $D_{(X,Y)} + cD_Z$, $c \in \mathbf{R}$, is applied to the function

$$f(x,\xi) = \rho^r \sum_{i_1,\dots,i_k} t_{i_1\dots i_k}(x)\xi_{i_1}\cdots\xi_{i_k}$$

where t is a symmetric tensor with vanishing traces, $r \in \mathbf{R}$ and $\rho(x) = (1-|x|^2)^{-1}$. The summation convention will be applied (summation over indices which appear twice). The derivatives of the components of t are denoted by

$$\frac{\partial}{\partial x_m} t_{i_1\dots i_k} = t_{i_1\dots i_k,m}$$

and these are no longer the components of a symmetric tensor. The result is as follows:

$$\begin{aligned} D_{(X,Y)}f + cD_Zf = {} & \frac{k}{2}\rho^{r-2} t_{i_1\dots i_{k-1}m,m}\xi_{i_1}\cdots\xi_{i_{k-1}} + A\rho^r t_{i_1\dots i_{k-1}m,l}\xi_{i_1}\cdots\xi_{i_{k-1}}\xi_m\xi_l \\ & + B\rho^{r-1} t_{i_1\dots i_{k-1}m}\xi_{i_1}\cdots\xi_{i_{k-1}}x_m + C\rho^{r+1} t_{i_1\dots i_k}\xi_{i_1}\cdots\xi_{i_k}(x \mid \xi) \end{aligned} \tag{5.15}$$

with

$$A = -\frac{k}{2} - \frac{n-1}{4} + \frac{c}{2}$$

$$B = kr - k(k-1) + k\frac{n-5}{2} + kc$$

$$C = -kr + 2k(k-1) - \frac{n-1}{2}r + k(n+1) + c(r-2k)$$

The first observation is that $C = 0$ if $r = 2k$. This motivates the normalizing factor

$(1-|x|^2)^{-2k}$ occurring in the description (5.4). Having fixed $r=2k$, the operators S_k^* and S_k are now defined by the equations $A=0$ and $B=0$ respectively.

The constant c for the operator S_k is determined by the equations $r=2k$, $B=0$. It follows that

$$c=\frac{1}{2}-\left(\frac{n}{2}+k-1\right)$$

$$A=-\left(\frac{n}{2}+k-1\right) \tag{5.16}$$

$$S_kf=\frac{k}{2}\rho^{2k-2}t_{i_1\cdots i_km,m}\xi_{i_1}\cdots\xi_{i_{k-1}}-\left(\frac{n}{2}+k-1\right)\rho^{2k}t_{i_1\cdots i_{k-1}m,l}\xi_{i_1}\cdots\xi_{i_{k-1}}\xi_m\xi_l$$

It remains to be shown that $S_kf\in E^{k+1}$. For this purpose set

$$q_{i_1\cdots i_{k+1}}=\frac{1}{k+1}\sum_{j=1}^{k+1}t_{i_1\cdots\hat{i}_j\cdots i_{k+1},i_j} \tag{5.17}$$

(the symbol $\hat{i}_j$ indicates that the index i_j is omitted). q is a symmetric tensor and

$$t_{i_1\cdots i_{k+1}}\xi_{i_1}\cdots\xi_{i_{k+1}}=q_{i_1\cdots i_{k+1}}\xi_{i_1}\cdots\xi_{i_{k+1}} \tag{5.18}$$

However in general the traces of q will not vanish:

$$q_{i_1\cdots i_{k-1}jj}=\frac{2}{k+1}t_{i_1\cdots i_{k-1}j,j} \tag{5.19}$$

Consider the symmetric tensor z

$$z_{i_1\cdots i_{k+1}}=\delta_{i_1i_2}q_{jji_3\cdots i_{k+1}}+\delta_{i_1i_3}q_{ji_2ji_4\cdots i_{k+1}}+\cdots+\delta_{i_ki_{k+1}}q_{i_1\cdots i_{k-1}jj} \tag{5.20}$$

Summation gives

$$\delta_{i_1i_2}q_{jji_3\cdots i_{k+1}}\xi_{i_1}\cdots\xi_{i_{k+1}}=|\xi|^2q_{i_1\cdots i_{k-1}jj}\xi_{i_1}\cdots\xi_{i_{k-1}} \tag{5.21}$$

Since there are $\frac{k(k+1)}{2}$ terms in the definition of z, the equations (5.19), (5.20) and (5.21) show that

$$z_{i_1 \cdots i_{k+1}} \xi_{i_1} \cdots \xi_{i_{k+1}} = k\,|\xi|^2 t_{i_1 \cdots i_{k-1} j, j} \xi_{i_1} \cdots \xi_{i_{k-1}} \tag{5.22}$$

This implies

$$S_k f = \tfrac{1}{2}\rho^{2k}(-(n+2k-2)q_{i_1 \cdots i_{k+1}} + z_{i_1 \cdots i_{k+1}})\xi_{i_1} \cdots \xi_{i_{k+1}} \tag{5.23}$$

and it can now be shown that $S_k f$ is defined by a tensor with vanishing traces:

$$\begin{aligned} z_{jji_3 \cdots i_{k+1}} &= nq_{jji_3 \cdots i_{k+1}} + q_{ji_3ji_4 \cdots i_{k+1}} + \cdots \\ &\quad + q_{i_3jji_4 \cdots i_{k+1}} + \cdots \\ &\quad + 0 \qquad . \\ &= (n+2(k-1))q_{jji_3 \cdots i_{k+1}} \end{aligned}$$

(Observe that e.g. $q_{jji_3 \cdots i_{k-1} ii} = 0$ if $k \geq 3$). This completes the proof for the fact that $S_k f \in E^{k+1}$ if $f \in E^k$.

The constant c for the operator S_k^* is determined by the equations $r = 2k$, $A = 0$. It follows that

$$c = \frac{1}{2} + \left(\frac{n}{2} + k - 1\right)$$

$$S_k^* f = \frac{k}{2}\rho^{2k-2} t_{i_1 \cdots i_{k-1} m, m} \xi_{i_1} \cdots \xi_{i_{k-1}} + k(n+2k-2)\rho^{2k-1} t_{i_1 \cdots i_{k-1} m} \xi_{i_1} \cdots \xi_{i_{k-1}} x_m \tag{5.24}$$

This clearly shows that $S_k^* f \in E^{k-1}$.

The operator S_k^* can be put into a different form:

$$S_k^* f = \frac{k}{2}\rho^{-n} \sum_{i_1, \ldots, i_{k+1}} \sum_{m=1}^{n} \frac{\partial}{\partial x_m}(\rho^{n+2k-2} t_{i_1 \cdots i_{k-1} m})\xi_{i_1} \cdots \xi_{i_{k-1}} \tag{5.25}$$

The case $k = 0$ is special. The functions $f \in E^0$ are identified with the functions on the hyperbolic space B. The operators D_Z and $D_{(X,Y)}$ map E^0 into E^1:

$$D_Z f = -\tfrac{1}{2} \sum_{i=1}^{n} f_{x_i} \xi_i$$

$$D_{(X,Y)} f = \frac{n-1}{4} \sum_{i=1}^{n} f_{x_i} \xi_i$$

and S_0 can be defined by the formula

$$S_0 f = D_{(X,Y)} f - \frac{n+1}{2} D_Z f \tag{5.26}$$

$$S_0 f = \frac{n}{2} \sum_{i=1}^{n} f_{x_i} \xi_i = \frac{n}{2} (1-|x|^2)^{-2} \sum_{i=1}^{n} (1-|x|^2)^2 f_{x_i} \xi_i$$

The operator $S_1^* S_0$ then takes the form

$$S_1^* S_0 f = \frac{1}{2} \frac{n}{2} \rho^{-n} \sum_{m=1}^{n} \frac{\partial}{\partial x_m} (\rho^n (1-|x|^2)^2 f_{x_m})$$

$$= \frac{n}{4} \rho^{-n} \operatorname{div} (\rho^{n-2} \operatorname{grad} f) \tag{5.27}$$

This is (a multiple of) the Laplace operator for the hyperbolic space B.

Following Ahlfors [1] the invariant operator P mapping vectorfields v on B into tensorfields φ is defined by the equation

$$\rho^{-n} (Pv)_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) - \delta_{ij} \frac{1}{n} \sum_{k=1}^{n} v_{k,k} \tag{5.28}$$

The tensors $Pv(x)$ are symmetric and have zero trace. The operator P^* mapping such tensorfields into vectorfields is defined by the formula

$$(P^* \varphi)_i = \rho^{-n-2} \sum_{j=1}^{n} \varphi_{ij,j} \tag{5.29}$$

THEOREM 8. *The operator* S_1 *on* E^1 *coincides with the operator* $-(n/2)\rho^{-n}P$

on vectorfields and S_2^* on E^2 coincides with $P^*\rho^n$ provided the following identifications are made:

(1) *The vectorfield v on B is identified with the function*

$$V(x,\xi)=\sum_{i=1}^{n}(1-|x|^2)^{-2}v_i(x)\xi_i\in E^1$$

(2) *The tensorfield φ on B is identified with the function*

$$\Phi(x,\xi)=\sum_{i,j=1}^{n}(1-|x|^2)^{-2}\varphi_{ij}(x)\xi_i\xi_j\in E^2.$$

In particular it follows that $S_2^*S_1$ is the same operator as $-(n/2)P^*P$.

The operator S_1 is applied to the function $V(x,\xi)\in E^1$:

$$\begin{aligned}S_1V&=\frac{1}{2}\rho^2|\xi|^2\sum_{m=1}^{n}v_{m,m}-\frac{n}{2}\rho^2\sum_{m,l=1}^{n}v_{m,l}\xi_m\xi_l\\&=-\frac{n}{2}(1-|x|^2)^{-2}\sum_{m,l=1}^{n}\left(\frac{1}{2}(v_{m,l}+v_{l,m})-\frac{1}{n}\delta_{lm}\sum_{k=1}^{n}v_{k,k}\right)\xi_l\xi_m\end{aligned}$$

This shows that S_1 corresponds to $-(n/2)\rho^{-n}P$.

Similarly, if S_2^* is applied to Φ, it follows from (5.25) that

$$\begin{aligned}S_2^*\Phi&=S_2^*\sum_{i,j=1}^{n}(1-|x|^2)^{-4}(1-|x|^2)^2\varphi_{ij}\xi_i\xi_j\\&=\rho^{-n}\sum_{i,m=1}^{n}\frac{\partial}{\partial x_m}(\rho^{n+4-2}(1-|x|^2)^2\varphi_{im})\xi_i\\&=\rho^{-n-2}(1-|x|^2)^{-2}\sum_{i,m=1}^{n}\frac{\partial}{\partial x_m}(\rho^n\varphi_{im})\xi_i\end{aligned}$$

This completes the proof of Theorem 8.

Equation (5.7) gives the transformation behaviour of the vectorfields under Möbiustransformations. The transformation of the tensorfields $(\varphi_{ij}(x))$ is described by (5.10). These formulas coincide with formulas (1.5) and (1.7) in [1].

COROLLARY (Ahlfors [2], equation (2.1)). *The solutions of $S_1f=0$, $f(x,\xi)=(1-|x|^2)^{-2}\sum_{i=1}^{n}v_i(x)\xi_i\in E^1$ are of the form*

$$v(x)=a+Bx+\lambda x+c\,|x|^2-2x(c,x),\qquad \lambda\in\mathbf{R},\qquad a,c\in\mathbf{R}^n,\qquad B^t=-B.$$

The solutions of $S_1 f=0, f\in E^1$ describe exactly the Lie algebra of the Möbius group $M(n)$ as a transformation group of $\mathbf{R}^n$ (see equation (2.19)).

THEOREM 9. *For all $f\in E^k$, $k=1,2,\dots$ there is equality*

$$D_{|X|^2+Z^2+|Y|^2}f=-(n+2k-2)^{-1}(S_{k+1}^*S_k f-S_{k-1}S_k^* f) \tag{5.30}$$

COROLLARY. *$\Delta_K=D_{|X|^2+Z^2}$ and $\Delta_X=D_{|X|^2+Z^2-|Y|^2}$ map E^k into E^k, $k=0,1,2,\dots$*

For the proof of the theorem let us calculate the commutator

$$D_{|X|^2+|Y|^2}=[D_{(X,Y)},D_Z]$$

using equations (5.13) and (5.14). Assume that $f\in E^k$, $k\in\mathbb{N}$.

$$\begin{aligned}
&(n+2k-2)[D_{(X,Y)},D_Z]\\
&\quad=S_{k+1}S_k(n+2k)^{-1}\left(\frac{1}{2}-\left(\frac{n}{2}+k\right)-\frac{1}{2}+\left(\frac{n}{2}+k-1\right)\right)\\
&\qquad+S_{k+1}^*S_k(n+2k)^{-1}\left(-\frac{1}{2}-\left(\frac{n}{2}+k\right)+\frac{1}{2}-\left(\frac{n}{2}+k-1\right)\right)\\
&\qquad+S_{k-1}S_k^*(n+2k-4)^{-1}\left(-\frac{1}{2}+\left(\frac{n}{2}+k-2\right)+\frac{1}{2}+\left(\frac{n}{2}+k-1\right)\right)\\
&\qquad+S_{k-1}^*S_k^*(n+2k-4)^{-1}\left(\frac{1}{2}+\left(\frac{n}{2}+k-2\right)-\frac{1}{2}-\left(\frac{n}{2}+k-1\right)\right)
\end{aligned}$$

If the expression

$$\begin{aligned}
&(n+2k-2)D_{Z^2}\\
&\qquad=(n+2k)^{-1}(S_{k+1}S_k-S_{k+1}^*S_k)-(n+2k-4)^{-1}(S_{k-1}S_k^*-S_{k-1}^*S_k^*)
\end{aligned}$$

is added, the formula of the theorem follows:

$$(n+2k-2)(D_{|X|^2}+D_{|Y|^2}+D_{Z^2})=-S_{k+1}^*S_k+S_{k-1}S_k^*$$

The case $k=0$ reduces to the Laplace operator (5.27).

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