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Autor: Knus, M.-A. / Ojanguren, M. / Parimala, R.
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Positive-definite quadratic bundles over the plane

M.-A. KNUS, M. OJANGUREN and RAMAN PARIMALA

Introduction

Indecomposable, positive-definite quadratic spaces of ranks 3 and 4 over $\mathbb{R}[x, y]$ have been constructed in [5] and [13]. A natural question to ask is whether there exist indecomposable quadratic spaces of rank > 4 over $\mathbb{R}[x, y]$ and whether the theorem of Krull–Schmidt holds for orthogonal decompositions of positive-definite quadratic spaces over $\mathbb{R}[x, y]$. (cf [9], p. 204.)

In §1 of this paper we prove a Krull–Schmidt theorem for orthogonal sums of positive-definite quadratic spaces over $\mathbb{R}[x, y]$. In view of [8], Thm. 2.1, it is enough to prove a similar theorem for positive-definite quadratic bundles over $\mathbb{P}_{\mathbb{R}}^2$. More generally, we prove that if X is a projective scheme over \mathbb{R} and $X_{\mathbb{C}}$ the complexification of X , then the theorem of Krull–Schmidt holds for positive-definite σ -hermitian (resp. quadratic) bundles over $X_{\mathbb{C}}$ (resp. X). We also deduce that Witt-cancellation holds for positive-definite quadratic spaces over $\mathbb{R}[x, y]$. In §2, we exhibit a class of vector-bundles of rank 3 and 4 over $\mathbb{P}_{\mathbb{C}}^2$, associated to a pair of projective ideals of $\mathbb{H}[x, y]$, and show, using results of §1, that these bundles are stable. (The examples of rank 4 bundles over $\mathbb{P}_{\mathbb{C}}^2$ constructed here are interesting, particularly in view of the fact that in general it is not easy to decide the stability of bundles of rank > 3 .) In §3, we construct an example of a rank 6, indecomposable quadratic space over $\mathbb{R}[x, y]$. The idea of the construction is to patch certain rank 3 and 4 quadratic spaces over $\mathbb{R}[x, y]$.

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§1. Krull–Schmidt theorem for positive-definite bundles over projective schemes

Let X be a projective scheme over \mathbb{R} and let $X_{\mathbb{C}}$ denote the complexification $\mathbb{C} \times_{\mathbb{R}} X$ of X . Let σ be the involution on $X_{\mathbb{C}}$ induced by the complex conjugation on \mathbb{C} and π the projection of $X_{\mathbb{C}}$ onto X . For any vector bundle \mathcal{F}_0 over X we have a natural isomorphism $\rho: \pi^* \mathcal{F}_0 \rightarrow \sigma^* \pi^* \mathcal{F}_0$, since $\pi \circ \sigma = \pi$. For

any vector bundle \mathcal{F} over $X_{\mathbb{C}}$ we denote by \mathcal{F}' the dual bundle and by \mathcal{F}^* the pull-back $\sigma^*\mathcal{F}'$ of \mathcal{F}' through σ . We define a natural isomorphism (cfr. [11]) $\tau:(\sigma^*\mathcal{F})' \rightarrow \mathcal{F}^*$ by

$$(\sigma^*\mathcal{F})' = \mathcal{H}om(\sigma^*\mathcal{F}, \pi^*\mathcal{O}_{X_{\mathbb{C}}}) \xrightarrow{\mathcal{H}om(\sigma^*\mathcal{F}, \rho)} \mathcal{H}om(\sigma^*\mathcal{F}, \sigma^*\pi^*\mathcal{O}_{X_{\mathbb{C}}}) = \sigma^*\mathcal{F}',$$

In [11] a σ -hermitian structure over \mathcal{F} was defined as an isomorphism $\phi:\mathcal{F} \rightarrow \sigma^*\mathcal{F}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \sigma^*\mathcal{F}' \\ (\sigma^*\phi)' \searrow & & \nearrow \tau \\ & (\sigma^*\mathcal{F})' & \end{array}$$

is commutative. It is convenient to give an equivalent definition, using the terminology of [15]. Let \mathfrak{M} be the category of vector bundles over $X_{\mathbb{C}}$. Associating to every \mathcal{F} the bundle \mathcal{F}^* we get a functor ${}^*:\mathfrak{M} \rightarrow \mathfrak{M}$. Let, for any \mathcal{F} , $i_{\mathcal{F}}:\mathcal{F} \rightarrow \mathcal{F}^{**}$ be the isomorphism defined by

$$\mathcal{F}^{**} = \sigma^*(\mathcal{F}^*)' \xrightarrow{\tau_{\mathcal{F}}^{-1}} (\sigma^*\mathcal{F}^*)' \longrightarrow (\mathcal{F}')' \longrightarrow \mathcal{F}.$$

It is easily checked that i is a natural transformation $id \xrightarrow{\sim} {}^*i$ satisfying $i_{\mathcal{F}}^*i_{\mathcal{F}^*} = id_{\mathcal{F}^{**}}$. Hence * is a duality functor in the sense of [15]. We identify each bundle \mathcal{F} with \mathcal{F}^{**} and each morphism ϕ of bundles with ϕ^{**} . For $\varepsilon = \pm 1$, we define an ε -hermitian structure on \mathcal{F} as an isomorphism $\phi:\mathcal{F} \xrightarrow{\sim} \mathcal{F}^*$ such that $\phi^* = \varepsilon\phi$. A 1-hermitian structure on \mathcal{F} turns out to be the same as a σ -hermitian structure in the sense defined above and in [11] or [8]. If x is a real closed point of $X_{\mathbb{C}}$, i.e. a closed point such that $\sigma(x) = x$, the fibre \mathcal{F}_x at x of a σ -hermitian bundle \mathcal{F} carries a non-degenerate hermitian form. We say that \mathcal{F} is positive definite if the fibre at every real closed point is positive definite. Since the signature of a hermitian form is locally constant, if $X_{\mathbb{R}}$ is connected, \mathcal{F} is positive definite if and only if the induced form on the fibre of some real closed point of $X_{\mathbb{C}}$ is positive definite.

We assume, from now on, that X has at least one real closed point.

For any bundle \mathcal{F} we denote by $H(\mathcal{F})$ the hyperbolic bundle associated to \mathcal{F} . This is the bundle $\mathcal{F} \oplus \mathcal{F}^*$ with the hermitian structure defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

LEMMA 1.1. *Let \mathcal{N} be an indecomposable vector bundle over $X_{\mathbb{C}}$ such that $\mathcal{N} \cong \mathcal{N}^*$. Then \mathcal{N} carries a σ -hermitian structure.*

Proof. By Proposition 2.5 of [15], \mathcal{N} carries a (1)- or a (-1)-hermitian form. If $\phi: \mathcal{N} \rightarrow \mathcal{N}^*$ is (-1)-hermitian, $i\phi$ is hermitian.

THEOREM 1.2. *Let (\mathcal{E}, ϕ) be a positive-definite σ -hermitian bundle over $X_{\mathbb{C}}$. Then, there is a unique orthogonal decomposition*

$$(\mathcal{E}, \phi) \xrightarrow{i} \bigoplus_i (\mathcal{E}_i, \phi_i),$$

where \mathcal{E}_i are the isotypical components of the vector bundle \mathcal{E} (i.e. $\mathcal{E} \xrightarrow{\sim} \bigoplus \mathcal{N}_i$, where \mathcal{N}_i are indecomposable and for $i \neq j$, $\mathcal{N}_i \not\sim \mathcal{N}_j$). Each \mathcal{E}_i carries a positive-definite σ -hermitian structure which is unique up to isometry.

Proof. Since X is a projective scheme, the category \mathfrak{M} with the duality functor * defined above satisfies the assumptions (i)–(iii) of [15], page 272. Hence, by Theorem 3.2 of [15],

$$(\mathcal{E}, \phi) \cong (\mathcal{E}_1, \phi_1) \perp \cdots \perp (\mathcal{E}_n, \phi_n),$$

where each \mathcal{E}_i is a direct sum of vector bundles isomorphic to a fixed indecomposable \mathcal{N}_i or to its “dual” \mathcal{N}_i^* . By Theorem 3.3 of [15], if $\mathcal{N}_i \not\cong \mathcal{N}_i^*$, \mathcal{E}_i contains a hyperbolic orthogonal summand. Since, by assumption, \mathcal{E} is positive definite, this cannot happen and hence each \mathcal{E}_i is isotypical. Since the orthogonal decomposition written above is unique, it suffices to prove the uniqueness for an isotypical vector bundle.

Let \mathcal{E} be an isotypical vector bundle of type \mathcal{N} and let $\mathcal{E} \xrightarrow{\sim} \bigoplus \mathcal{N}$. We show that if \mathcal{E} carries a positive-definite σ -hermitian structure, then it is unique. Since \mathcal{N} is indecomposable, the ring $E = \text{End } \mathcal{N}$ is a local finite-dimensional \mathbb{C} -algebra. Let $\bar{E} = E/\text{rad } E$. Then \bar{E} is a finite-dimensional division algebra over \mathbb{C} and hence $\bar{E} \xrightarrow{\sim} \mathbb{C}$. One reduces the study of σ -hermitian structures on \mathcal{E} to the study of hermitian-forms over a certain vector space \bar{M} over \bar{E} defined as follows (see [15], 2.2, 2.4). Let $\phi: \mathcal{E} \rightarrow \mathcal{E}^*$ be a σ -hermitian structure on \mathcal{E} . Then, $\bigoplus_i \mathcal{N} \xrightarrow{\sim} \bigoplus_i \mathcal{N}^*$ and by the Krull–Schmidt theorem the vector bundles \mathcal{N} and \mathcal{N}^* are isomorphic. Hence, by Lemma 1.1, there exists an isomorphism $\phi_0: \mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$ which defines a σ -hermitian structure on \mathcal{N} . In what follows, we shall fix this σ -hermitian structure ϕ_0 on \mathcal{N} . The isomorphism ϕ_0 induces an involution τ on $E = \text{End } \mathcal{N}$ defined as

$$\tau f = f^0 = \phi_0^{-1} \circ f^* \circ \phi_0.$$

The map $f \rightarrow f^0$ satisfies $(fg)^0 = g^0 f^0$, $(f^0)^0 = f$ and for $\lambda \in \mathbb{C}$, $(\lambda f)^0 = \bar{\lambda} f^0$, $\bar{\lambda}$ denoting the complex conjugate of λ . This involution passes down to an involution on

$\bar{E} = E/\text{rad } E = \mathbb{C}$ which is just the complex conjugation on \mathbb{C} . Let $M = \text{Hom}(\mathcal{N}, \mathcal{E})$. Then M is a right E -module and the isomorphism $\phi: \mathcal{E} \rightarrow \mathcal{E}^*$ induces an isomorphism $\phi_1: M \rightarrow \text{Hom}_E(M, E)$ which is semilinear with respect to the involution τ . The map ϕ_1 is in fact defined as $\phi_1(f)(g) = \phi_0^{-1} \circ f^* \circ g$ for $f, g \in M$. It is easily verified that ϕ_1 defines a hermitian form on the E -module M with respect to the involution τ on E . Going modulo the radical of E , we obtain on $\bar{M} = M/(\text{rad } E)M$ a hermitian form over \mathbb{C} .

Two σ -hermitian structures on \mathcal{E} are isometric if and only if the corresponding hermitian forms on \bar{M} are isometric ([15], 2.2). If the form on \mathcal{E} is positive-definite, then the form on \bar{M} is either positive or negative-definite. In fact, if \bar{M} represents zero, then \bar{M} contains a hyperbolic summand and so does \mathcal{E} by [15], Prop. 2.4. If ϕ and ϕ' are two positive definite forms on \mathcal{E} , the corresponding forms on \bar{M} are either both positive-definite or both negative-definite: otherwise the form corresponding to $\phi \perp \phi'$ on $\mathcal{E} \perp \mathcal{E}$ would be isotropic. Since, up to isometry, there is a unique positive or negative-definite hermitian form on \bar{M} , it follows that there is a unique positive definite σ -hermitian structure over \mathcal{E} . This proves Theorem 1.2.

COROLLARY 1.3. *A vector bundle over $X_{\mathbb{C}}$ carries at the most one positive-definite σ -hermitian structure.*

COROLLARY 1.4 (Krull–Schmidt theorem). *Any σ -hermitian positive-definite bundle (\mathcal{E}, ϕ) over $X_{\mathbb{C}}$ has a decomposition*

$$(\mathcal{E}, \phi) = \bigoplus (\mathcal{N}_i, \nu_i)$$

into indecomposable σ -hermitian bundles. The summands (\mathcal{N}_i, ν_i) are unique up to isometries and permutations.

COROLLARY 1.5. *The Krull–Schmidt theorem holds for positive-definite σ -hermitian spaces over $\mathbb{C}[x, y]$.*

Proof. By (3.1) of [8] any positive-definite σ -hermitian space over $\mathbb{C}[x, y]$ has, up to isometry, a unique extension to $\mathbb{P}_{\mathbb{C}}^2$. Hence the assertion follows from 1.4.

The following theorem and corollaries give the corresponding results for positive-definite quadratic bundles.

THEOREM 1.6. *Let (\mathcal{E}, ϕ) be a positive-definite quadratic bundle over X . Then, there is a unique orthogonal decomposition*

$$(\mathcal{E}, \phi) = \bigoplus_i (\mathcal{E}_i, \phi_i)$$

where \mathcal{E}_i are the isotypical components of the vector bundle \mathcal{E} . The components \mathcal{E}_i carry a positive-definite quadratic structure, unique up to isometry.

A proof on the same lines as of Theorem 1.2 can be given. Let now \mathfrak{M} be the category of real vector bundles over X and, for any such bundle \mathcal{E} let $\mathcal{E}^* = \mathcal{E}' = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ be the dual of \mathcal{E} . By Theorem 3.2 of [15] one reduces immediately to the case of an isotypical bundle $\mathcal{E} \xrightarrow{\sim} \bigoplus \mathcal{N}$, \mathcal{N} indecomposable. Since $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^*$, we have $\mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$ and since $\text{End } \mathcal{N}$ is local, \mathcal{N} carries either a quadratic or a symplectic structure $\phi_0: \mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$. Then ϕ_0 gives rise to an involution τ of $E = \text{End } \mathcal{N}$, which passes down to an involution of $\bar{E} = E/\text{rad } E$. It is clear that $\bar{E} \xrightarrow{\sim} \mathbb{R}, \mathbb{C}$, or \mathbb{H} . If $\bar{E} = \mathbb{R}$, the involution is trivial. If $\bar{E} = \mathbb{C}$, the involution must be complex conjugation. And if $\bar{E} \xrightarrow{\sim} \mathbb{H}$, the involution on \mathbb{H} is either trivial or is a conjugate of the canonical involution. The isometry classes of quadratic structures on \mathcal{E} correspond to isometry classes of positive-definite or negative-definite forms on $\bar{M} = M/(\text{rad } E)M$, where $M = \text{Hom}(\mathcal{N}, \mathcal{E})$. The existence of orthogonal bases for hermitian forms shows that there is unique positive- or negative-definite τ -hermitian form on \bar{M} . It follows that there is a unique positive-definite quadratic structure over \mathcal{E} .

COROLLARY 1.7. *A vector bundle over X carries at the most one positive-definite quadratic structure.*

COROLLARY 1.8. *The Krull-Schmidt theorem holds for positive-definite quadratic bundles over X .*

COROLLARY 1.9. *The Krull-Schmidt theorem holds for positive-definite quadratic spaces over $\mathbb{R}[x, y]$.*

§2. Some stable bundles of rank 3 and 4 associated to projective ideals of $\mathbb{H}[x, y]$

We recall that a bundle \mathcal{E} over $\mathbb{P}'_{\mathbb{C}}$ is said to be *stable* if, for every coherent subsheaf $\mathcal{F} \neq 0$ of \mathcal{E} such that \mathcal{E}/\mathcal{F} is torsionfree we have $c_1(\mathcal{F})/\text{rank } \mathcal{F} < c_1(\mathcal{E})/\text{rank } \mathcal{E}$. In [8] to each non-free projective ideal P of $\mathbb{H}[x, y]$ was associated a rank 2 stable bundle $\mathcal{E}(P)$ with a positive-definite σ -hermitian structure. We recall the construction of these bundles, which in [8] were called \mathfrak{P} -bundles. Let $\phi: \mathbb{C} \otimes \mathbb{H} \rightarrow M_2(\mathbb{C})$ be the isomorphism given by

$$\phi(s \otimes (u + vj)) = s \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} u, v \in \mathbb{C}.$$

Let $H = \mathbb{H}[x, y]$ and $C = \mathbb{C}[x, y]$. For any projective ideal P of H , $C \otimes P$ is an $M_2(C)$ -module via ϕ . Hence, there is a ϕ -semilinear isomorphism $\Psi_P : C \otimes P \xrightarrow{\sim} M_2(C)$. We shall call such a map a *splitting* of P . By Galois cohomology, we associate to the splitting Ψ_P the cocycle

$$\alpha_P = \sigma \Psi_P(\sigma \otimes 1) \Psi_P^{-1}(1) \in GL_2(C)$$

where σ is the complex conjugation on \mathbb{C} and the transported action $\phi(\sigma \otimes 1)\phi^{-1}$ on $M_2(C)$. The map Ψ_P can be chosen such that α_P is positive-definite hermitian of determinant one. Such a splitting is called a *normalized splitting*. Hence, α_P defines a σ -hermitian structure on $\mathbb{A}_{\mathbb{C}}^2$. This structure can be uniquely extended to $\mathbb{P}_{\mathbb{C}}^2$ ([8]) and the extension is the complex bundle $\mathcal{E}(P)$. Notice that by (1.2) $\mathcal{E}(P)$ carries a unique positive-definite σ -hermitian structure. Let now P and Q be two projective ideals in H . The reduced norm Nr introduced in [6] defines a quadratic form on the $\mathbb{R}[x, y]$ -module of rank 4 $\text{Hom}_H(P, Q)$. If $\Psi_P : C \otimes P \xrightarrow{\sim} M_2(C)$ and $\Psi_Q : C \otimes Q \xrightarrow{\sim} M_2(C)$ are normalized splittings of P and Q , then, for any $f \in \text{Hom}_H(P, Q)$, $\text{Nr}(f) = \det \Psi_Q(1 \otimes f) \Psi_P^{-1}(1)$. This quadratic space is indecomposable if P and Q are non-free and not isomorphic. If $P \simeq Q$ and P is non-free, then this space decomposes as $\langle 1 \rangle \perp \bar{q}$, where \bar{q} is the orthogonal complement of the submodule $\mathbb{R}[x, y]$ of $\text{End}_H(P)$ for the reduced norm on the algebra $\text{End}_H(P)$. It is shown in [6] that \bar{q} is indecomposable. These indecomposable quadratic spaces of ranks 3 and 4 extend uniquely to indecomposable quadratic bundles over $\mathbb{P}_{\mathbb{R}}^2$, denoted respectively by $\mathcal{F}(P, Q)$ and $\mathcal{F}(P)$. Let $\pi : \mathbb{P}_{\mathbb{C}}^2 \rightarrow \mathbb{P}_{\mathbb{R}}^2$ be the projection and let $\pi^* \mathcal{F}(P, Q) = \mathcal{G}(P, Q)$ and $\pi^* \mathcal{F}(P) = \mathcal{G}(P)$. We shall show that these bundles are stable.

THEOREM 2.1. *The bundle $\mathcal{G}(P, Q)$ is isomorphic to $\mathcal{E}(P) \otimes \mathcal{E}(Q)$.*

COROLLARY 2.2. *We have $c_2(\mathcal{G}(P, Q)) = 2(c_2(\mathcal{E}(P)) + c_2(\mathcal{E}(Q)))$ and $c_2(\mathcal{G}(P)) = 4c_2(\mathcal{E}(P))$.*

Proof. For 2-bundles \mathcal{E} and \mathcal{F} on $\mathbb{P}_{\mathbb{C}}^2$, if $c_1(\mathcal{E}) = c_1(\mathcal{F}) = 0$, then $c_2(\mathcal{E} \otimes \mathcal{F})$ is given by $2(c_2(\mathcal{E}) + c_2(\mathcal{F}))$.

Theorem (2.1) is a consequence of the following results. The first one is implicitly contained in [7], (1.12).

LEMMA 2.3. *Let P be a projective ideal of H , Ψ_P a normalized splitting of P and $\alpha_P \in GL_2(C)$ the corresponding cocycle. Then there is a basis e_1, e_2 of P as a*

C -module such that the matrix of the σ -hermitian form a_P on P defined by

$$\begin{aligned} a_P(e_i, e_i) &= (\Psi_P(e_i), \Psi_P(e_i)) \quad i = 1, 2 \\ a_P(e_1, e_2) &= (\Psi_P(e_1), \Psi_P(e_2)) - i(\Psi_P(e_1), \Psi_P(ie_2)) \end{aligned}$$

where, for $u, v \in M_2(C)$, $(u, v) = \frac{1}{2}(\det(u + v) - \det u - \det v)$, is α_P .

Let $\alpha_P = \alpha + i\beta$ with $\alpha, \beta \in M_2(\mathbb{R}[x, y])$. Then the symmetric matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ represents the reduced norm on P with respect to the basis $e_1, e_2, e_3 = ie_1, e_4 = ie_2$ of P over $\mathbb{R}[x, y]$.

The next lemma is an immediate consequence of (2.3) and of the definition of the reduced norm on $\text{Hom}_H(P, Q)$ by means of the splittings Ψ_P and Ψ_Q .

LEMMA 2.4. *Let $f \in \text{Hom}_H(P, Q)$ and a_P, a_Q the hermitian forms given in (2.1). Then, for any $u, v \in P$*

$$a_Q(f(u), f(v)) = \text{Nr}(f)a_P(u, v).$$

The module $P' = \text{Hom}_C(P, C)$ is a projective right H -module (with the action $(f\lambda)(x) = f(\lambda x)$, $\lambda \in H$). We now compute its cocycle.

LEMMA 2.5. *Let Ψ_P be a splitting of P with cocycle α_P . Then, there is a splitting $\Psi_{P'}$ of P' with cocycle $\alpha_{P'} = \alpha_P^{-1}$.*

Proof. Let $T: M_2(C) \xrightarrow{\sim} \text{Hom}_C(M_2(C), C)$ be the isomorphism given by the trace, i.e. $T_a(b) = \text{Tr}(ab)$, $a, b \in M_2(C)$. Let $\hat{P} = \text{Hom}_{\mathbb{R}[x, y]}(P, \mathbb{R}[x, y])$. Then the map $\Psi_{P'} = T^{-1}(\Psi_P)^\wedge$ (where \wedge means dualization with respect to $\mathbb{R}[x, y]$) is a splitting of \hat{P} and one computes that the corresponding cocycle is α_P^{-1} . Let now $t: P' \xrightarrow{\sim} \hat{P}$ be the isomorphism (of H -modules) induced by the trace $\mathbb{C} \rightarrow \mathbb{R}$. Then the map $\Psi_{P'} = \Psi_{P'} \circ (1 \otimes t)$ is a splitting of P' such that $\alpha_{P'} = \alpha_{P'} = \alpha_P^{-1}$.

Let now $a_{P'}$ be the hermitian structure on P' given by

$$a_{P'}(e'_i, e'_j) = \frac{1}{2}(\alpha_{P'})_{j,i} = \frac{1}{2}(\alpha_P^{-1})_{j,i},$$

where e'_i , $i = 1, 2$ is the dual basis of the basis e_i , $i = 1, 2$ given in (2.3). Let S be the σ -hermitian space obtained by extending the reduced norm Nr on $\text{Hom}_H(P, Q)$ to $\mathbb{C} \otimes_{\mathbb{R}} \text{Hom}_H(P, Q)$, i.e. $S(\lambda \otimes f) = \lambda \bar{\lambda} \text{Nr}(f)$.

LEMMA 2.6. *The map $\rho : \mathbb{C} \otimes \text{Hom}_H(P, Q) \xrightarrow{\sim} \text{Hom}_C(P, Q) \xrightarrow{\sim} P' \otimes_C Q$ where the first map is the multiplication and the second is the canonical map, is an isomorphism of σ -hermitian spaces $\rho : S \xrightarrow{\sim} a_{P'} \otimes a_Q$.*

Proof. For any basis $\{e_i\}$ of P , ρ is given by $\rho(\lambda \otimes f) = \sum_i e_i^* \otimes f(\lambda e_i)$. Choosing the basis given in (2.3), we have, using (2.4),

$$\begin{aligned} (a_{P'} \otimes a_Q) \left(\sum_i e'_i \otimes f(\lambda e_i) \right) &= \sum_{i,j} a_{P'}(e'_i, e'_j) a_Q(f(\lambda e_i), f(\lambda e_j)) \\ &= \text{Nr}(f) \lambda \bar{\lambda} \sum_{i,j} a_{P'}(e'_i, e'_j) a_Q(e_i, e_j) = \text{Nr}(f) \lambda \bar{\lambda}. \end{aligned}$$

This shows that ρ is an isometry.

Theorem (2.1) now follows from (2.6) noting that the extension of a positive definite σ -hermitian form from \mathbb{A}_C^2 to \mathbb{P}_C^2 is unique and that $\mathcal{E}(P^*) \cong \mathcal{E}(P)$.

To show that the bundles $\mathcal{G}(P, Q)$ and $\mathcal{G}(P)$ are stable, we begin with

LEMMA 2.7. *Let K be a field of characteristic $\neq 2$ and let (\mathcal{E}, ϕ) be a quadratic bundle of rank 2 over \mathbb{P}_K' . If (\mathcal{E}, ϕ) is anisotropic, (\mathcal{E}, ϕ) is extended from K . If (\mathcal{E}, ϕ) is isotropic, then $(\mathcal{E}, \phi) \xrightarrow{\sim} H(\mathcal{O}(n))$, a hyperbolic space.*

Proof. The first part of the lemma is proved in ([8], 2.4). If (\mathcal{E}, ϕ) is isotropic, then restricted to each affine piece $D(x_i)$, the quadratic form can be given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One then easily checks that $(\mathcal{E}, \phi) \xrightarrow{\sim} H(\mathcal{O}(n))$ for some n .

LEMMA 2.8. *Let K be a field of characteristic $\neq 2$ and let \mathcal{E} be an indecomposable anisotropic quadratic bundle over \mathbb{P}_K' . Then \mathcal{E} has no non-zero section.*

Proof. Evaluating the quadratic form on a global section one gets a global function on \mathbb{P}_K' , hence a constant. This constant must be zero, since the bundle is indecomposable as a quadratic bundle. The section has to be zero since the form is anisotropic.

For any bundle \mathcal{E} over \mathbb{P}_C^2 the “type” of \mathcal{E} is the pair of Chern classes $(c_1(\mathcal{E}), c_2(\mathcal{E}))$.

THEOREM 2.9. *The bundles $\mathcal{G}(P)$ are stable rank 3 bundles of type $(0, 8n)$, where $c_2(\mathcal{E}(P)) = 2n$, $\mathcal{E}(P)$ denoting the \mathfrak{P} -bundle associated to a non-free projective ideal P of $\mathbb{H}[x, y]$. The bundles $\mathcal{E}(P, Q)$ are stable rank 4 of type $(0, 4(m+n))$ if P and Q are non-isomorphic, non-free, $\mathcal{E}(P)$ of type $(0, 2n)$ and $\mathcal{E}(Q)$ of type $(0, 2m)$.*

Proof. Since $\mathcal{E}(P)$ supports a quadratic form it follows that $c_1(\mathcal{E}(P)) = 0$. If we consider global sections, we have $H^0(\mathbb{P}_{\mathbb{C}}^2, \mathcal{G}(P)) \xrightarrow{\sim} \mathbb{C} \otimes H^0(P_{\mathbb{R}}, \mathcal{F}(P)) = 0$ by Lemma 2.7, since $\mathcal{F}(P)$ supports an anisotropic indecomposable quadratic form. Further, being a quadratic bundle, $\mathcal{G}(P) \xrightarrow{\sim} \mathcal{G}(P)'$. Hence $\mathcal{G}(P)$ is stable by [12], 1.2.6.

We shall now show that $\mathcal{G}(P, Q)$ is stable for P, Q non-isomorphic, non-free. We show that for every subsheaf \mathcal{F} of $\mathcal{G} = \mathcal{G}(P, Q)$ with the quotient $\mathcal{G}(P, Q)/\mathcal{F}$ torsion free, $c_1(\mathcal{F})/\text{rank } \mathcal{F} < c_1(\mathcal{G})/\text{rank } \mathcal{G}$. Since $\mathbb{P}_{\mathbb{C}}^2$ is regular of dimension 2, such a sheaf is locally free. Hence it suffices to show that for any locally free subsheaf \mathcal{F} of \mathcal{G} , $c_1(\mathcal{F}) < 0$. If \mathcal{F} is a line bundle with $c_1(\mathcal{F}) = n$, necessarily $n < 0$ since, otherwise, \mathcal{F} and hence \mathcal{G} would have a non-zero global section. If \mathcal{F} is of rank 3 we have a surjection $\mathcal{G}' \rightarrow \mathcal{F}' \rightarrow 0$ whose kernel is a line bundle \mathcal{L} . Since $\mathcal{G}' \xrightarrow{\sim} \mathcal{G}$ also does not admit of global sections, it follows that $c_1(\mathcal{L}) < 0$. Hence $c_1(\mathcal{F}') > 0$ so that $c_1(\mathcal{F}) = -c_1(\mathcal{F}') < 0$. Let \mathcal{F} be of rank 2. The bundle \mathcal{G} restricted to a real line L of $\mathbb{P}_{\mathbb{C}}^2$ is trivial, since \mathcal{G} supports an anisotropic quadratic form ([16], Prop. 5). The restriction of \mathcal{F} to L is isomorphic to $\mathcal{O}(n) \oplus \mathcal{O}(m)$. Since $\mathcal{F}|_L$ is a subsheaf of $\mathcal{G}|_L \xrightarrow{\sim} \mathcal{O}|_L \oplus \mathcal{O}|_L$, we have $c_1(\mathcal{F}) = n + m \leq 0$. Suppose that $c_1(\mathcal{F}) = 0$. Then \mathcal{F} is a rank 2 bundle with no global sections and with $c_1(\mathcal{F}) = 0$. Hence \mathcal{F} is a stable bundle ([12], 1.2.5). The quadratic structure on $\mathcal{F}(P, Q)$ extends to a positive-definite σ -hermitian structure, denoted by ϕ , on $\mathcal{G}(P, Q)$. The restriction of ϕ to \mathcal{F} induces a map $\mathcal{F} \rightarrow \sigma^* \mathcal{F}^* = \mathcal{F}^*$. This map cannot be zero since \mathcal{F} is anisotropic (positive-definite). By the corollary to Lemma 1.2.8 of [12], ϕ is an isomorphism and $(\mathcal{F}, \phi | \mathcal{F})$ splits off as an orthogonal summand of (\mathcal{G}, ϕ) . Then, $\mathcal{G} \xrightarrow{\sim} \mathcal{F} \perp \mathcal{F}_1$. The bundle \mathcal{G} supports a quadratic form, namely the extension of the quadratic structure on $\mathcal{F}(P, Q)$. The bundle \mathcal{F} cannot support a quadratic structure, since, otherwise, $\mathcal{F} \xrightarrow{\sim} H(\mathcal{O}(n))$ by Lemma 2.7 contradicting the stability of \mathcal{F} . Thus, by the uniqueness of the quadratic structure on \mathcal{G} , it follows that $\mathcal{G} \xrightarrow{\sim} H(\mathcal{F})$ and hence $\mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}' \xrightarrow{\sim} \mathcal{F}$. In fact, by the uniqueness of the positive-definite structure (see (1.6)) $(\mathcal{F}_1, \phi | \mathcal{F}_1) \xrightarrow{\sim} (\mathcal{F}, \phi | \mathcal{F})$ and $(\mathcal{G}, \phi) \xrightarrow{\sim} (\mathcal{F}, \phi | \mathcal{F}) \perp (\mathcal{F}_1, \phi | \mathcal{F}_1)$. Since \mathcal{F} is a rank 2 stable bundle with a positive-definite σ -hermitian structure, it follows by [8] that \mathcal{F} is a \mathfrak{P} -bundle, i.e. $\mathcal{F} \xrightarrow{\sim} \mathcal{E}(P_0)$, where P_0 is some non-free projective ideal of $\mathbb{H}[x, y]$. By [8], Prop. 3.2, $\mathcal{G} \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0) \xrightarrow{\sim} \pi^* \pi_* \mathcal{E}(P_0) = \pi^*(\mathcal{F}(\mathbb{H}[x, y]), P_0)$. Since $\text{End}(\mathcal{E}(P_0) \oplus \mathcal{E}(P_0)) \xrightarrow{\sim} M_2(\mathbb{C})$, the isomorphism classes of vector-bundles on $\mathbb{P}_{\mathbb{R}}^2$ with $\pi^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$ are classified by $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C}))$ for an action on $GL_2(\mathbb{C})$ which is the restriction of an action on $M_2(\mathbb{C})$. Since $\pi^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$ is \mathbb{C} -linear, $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C} \subset M_2(\mathbb{C}) = \text{End}(\mathcal{E}(P_0) \oplus \mathcal{E}(P_0))$ by conjugation, and hence the action on $M_2(\mathbb{C})$ is of the form $\alpha \rightarrow u\bar{\alpha}u^{-1}$ for some fixed $u \in GL_2(\mathbb{C})$. It is easily checked that in this case $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C})) = 0$. Hence, there is a unique descent for $\mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$, i.e. $\mathcal{F}(P, Q) \xrightarrow{\sim} \mathcal{F}(\mathbb{H}[x, y], P_0)$. By the

uniqueness of the positive-definite quadratic structure on a vector-bundle over $\mathbb{P}_{\mathbb{R}}^2$ [(1.7)], it follows that $\mathcal{F}(P, Q)$ is isomorphic as a quadratic bundle to $\mathcal{F}(\mathbb{H}[x, y], P_0)$. By restricting these bundles to $\mathbb{A}_{\mathbb{R}}^2$ and using ([6], Thm. 4.6), it follows that P or Q is free, a contradiction. The statement in the theorem regarding the second Chern classes of $\mathcal{G}(P)$ and $\mathcal{G}(P, Q)$ was proved in (2.2).

§3. An example of an indecomposable quadratic space of rank 6 over $\mathbb{R}[x, y]$

LEMMA 3.1. *Let R be a local domain in which 2 is invertible and let q_1, q_2 be quadratic spaces over $R[x]$ such that $q_1 \perp q_2$ is anisotropic. If $q_1(v) + q_2(w)$ is a unit of $R[x]$, then $q_1(v)$ or $q_2(w)$ is a unit of $R[x]$.*

Proof. Let K denote the quotient field of R . Since R is local, if bar denotes reduction modulo x , one has $\bar{q}_1 \xrightarrow{\sim} \langle \lambda_1, \dots, \lambda_n \rangle$, $\bar{q}_2 \xrightarrow{\sim} \langle \mu_1, \dots, \mu_m \rangle$, $\lambda_i, \mu_i \in U(R)$. By a theorem of Harder, we have, over $K[x]$, $q_1 \xrightarrow{\sim} \langle \lambda_1, \dots, \lambda_n \rangle$, $q_2 \xrightarrow{\sim} \langle \mu_1, \dots, \mu_m \rangle$. Thus, there exist $\theta_i, \phi_i \in K[x]$ such that $q_1(v) = \sum \lambda_i \theta_i^2$ and $q_2(w) = \sum \mu_i \phi_i^2$. Since the forms q_1 and q_2 are anisotropic over $K[x]$, if $q_1(v) = a_0 + a_1 x + \dots + a_r x^r$, then $q_2(w) = b_0 - b_1 x - \dots - b_r x^r$, and $a_r = \sum \lambda_i c_i^2 = -\sum \lambda_i d_i^2$, where c_i, d_i denote the leading coefficients of θ_i and ϕ_i respectively. Then, $\bar{q}_1 \perp \bar{q}_2$ represents zero over K and hence $q_1 \perp q_2$ represents zero over $K[x]$, contradicting the assumption that $q_1 \perp q_2$ is anisotropic.

The next lemma is a generalization of Proposition 1.1 of [13].

LEMMA 3.2. *Let A be a normal ring in which 2 is invertible. Every quadratic space of rank 2 over $A[X_1, \dots, X_n]$ is extended from A .*

Proof. By [3, 4.15, Remark 4] we may assume that A is local. Let K be the field of fractions of A and M a quadratic space of rank 2 over $A[X]$, X denoting (X_1, \dots, X_n) . If the signed discriminant of M_K is trivial, by [2, Proposition 5.1] M is of the form $H(I)$, where I is a projective ideal of $A[X]$. Since $\text{Pic } A = \text{Pic } A[X]$, M is extended. If the signed discriminant d of M_K is not a square in K , put $L = K[\sqrt{d}]$ and $B = A[\sqrt{d}]$. Then B is the integral closure of A in L hence is a normal semilocal ring. The signed discriminant of M_B is trivial and hence M_B is of the form $H(I)$, where I is a projective ideal of $B[X]$. Since $\text{Pic } B[X] = \text{Pic } B = 0$, $M_B = H(B[X])$. This shows that M is represented by an element of $H^1(\text{Gal}(L/K), O_2(B[X]))$. But $O_2(B[X]) = O_2(B)$ (compare [11], §1) and hence M is in the image of $H^1(\text{Gal}(L/K), O_2(B))$ in $H^1(\text{Gal}(L/K), O_2(B[X]))$. This shows that M is extended from A .

Given a pair f, g of polynomials in $\mathbb{R}[x, y]$, let $\alpha_{f,g}$ (respectively $\beta_{f,g}$) denote the rank 3 (rank 4) quadratic spaces over $\mathbb{R}[x, y]$ defined as the orthogonal complement of the identity in $\text{End}(P_{f,g})$ (respectively reduced norm on $P_{f,g}$), where $P_{f,g}$ is the projective ideal of $\mathbb{H}[x, y]$ defined as the kernel of the $\mathbb{H}[x, y]$ -linear map $\mathbb{H}[x, y]^2 \rightarrow \mathbb{H}[x, y]$ given by $(1, 0) \rightarrow f + i, (0, 1) \rightarrow g + j$ ([8], 1.2). Then $\alpha = \alpha_{x,y}$ is an indecomposable quadratic space over $\mathbb{R}[x, y]$. This space remains indecomposable over $\mathbb{R}[x]_{(1+x^2)}[y]$. In fact, if it decomposes as $\alpha' \perp \alpha''$, then the ranks of α' and α'' are 1 or 2 and hence, by Lemma 3.2, α is extended from $\mathbb{R}[x]_{(1+x^2)}$. Since over $\mathbb{R}[x, 1/1+x^2][y]$, $P_{x,y}$ is free ([7], §5), α is $\cong \langle 1, 1, 1 \rangle$ over this ring. Therefore by [3, 4.15, Remark 4], α is extended from \mathbb{R} , contrary to the assumption. The form $\beta = \beta_{x\sqrt{2},y}$ is an indecomposable quadratic space over $\mathbb{R}[x, y]$ which is isometric to $\langle 1, 1, 1, 1 \rangle$ over $\mathbb{R}[x, 1/2+x^2][y]$. We claim that β remains indecomposable over $\mathbb{R}[x]_{(2+x^2)}[y]$. Suppose that $\beta = \beta' \perp \beta''$ over $\mathbb{R}[x]_{(2+x^2)}[y]$. If $\text{rank } \beta' = \text{rank } \beta'' = 2$ the same argument as above shows that β is extended from \mathbb{R} , which is absurd. If $\text{rank } \beta' = 1$, then β represents a unit over $\mathbb{R}[x]_{(2+x^2)}[y]$ and therefore, by [6], (3.19) $P_{x\sqrt{2},y}$ is free over $\mathbb{H}[x]_{(2+x^2)}[y]$ and, in particular, extended from \mathbb{H} . Since it is also free over $\mathbb{H}[x, 1/2+x^2][y]$ ([7], §5), by Quillen's theorem $P_{x\sqrt{2},y} = \mathbb{H}[x, y]$, contrary to the assumption.

We define a quadratic space over $\mathbb{R}[x, y]$ of rank 6 as follows: we consider the covering

$$\text{Spec } \mathbb{R}[x, y] = \text{Spec } \mathbb{R}[x, y][1/1+x^2] \cup \text{Spec } \mathbb{R}[x, y][1/2+x^2].$$

We take the space $\beta \perp 1 \perp 1$ over $\text{Spec } \mathbb{R}[x, y][1/1+x^2]$ and the space $\alpha \perp \alpha$ over $\text{Spec } \mathbb{R}[x, y][1/2+x^2]$ and some patching isometry $\phi: \alpha \perp \alpha \xrightarrow{\sim} \beta \perp 1 \perp 1$ over $\text{Spec } \mathbb{R}[x, y][1/(1+x^2)(2+x^2)]$ (note that both quadratic spaces are equivalent to the identity over this intersection) to get a quadratic space γ of rank 6 over $\text{Spec } \mathbb{R}[x, y]$.

We show that γ is indecomposable. Suppose that γ represents a unit of $\mathbb{R}[x, y]$. Since $\gamma \xrightarrow{\sim} \alpha \perp \alpha$ over $\mathbb{R}[x]_{(1+x^2)}[y]$, it follows that $\alpha \perp \alpha$ represents a unit of $\mathbb{R}[x]_{(1+x^2)}[y]$ and since $\alpha \perp \alpha$ is anisotropic, by Lemma 3.1, α represents a unit of $\mathbb{R}[x]_{(1+x^2)}[y]$ contradicting the indecomposability of α over $\mathbb{R}[x]_{(1+x^2)}[y]$. Since by (3.2) any quadratic space of rank ≤ 2 over $\mathbb{R}[x, y]$ is extended from \mathbb{R} and hence represents units, we assume now that $\gamma = \gamma_1 \perp \gamma_2$, where γ_1 and γ_2 are indecomposable rank 3 spaces. Over $\mathbb{R}[x]_{(2+x^2)}[y]$, we have $\gamma_1 \perp \gamma_2 \xrightarrow{\sim} \beta \perp 1 \perp 1$, so that if $\gamma_1(v) + \gamma_2(w) = 1$, we have by Lemma 3.1 that $\gamma_1(v)$ or $\gamma_2(w)$ is a unit. Suppose that $\gamma_1(v)$ is a unit. Then $\gamma_1 \xrightarrow{\sim} \langle \gamma_1(v) \rangle \perp \gamma'_1$ and the orthogonal complement of $\gamma_1(v) + \gamma_2(w)$ in $\gamma_1 \perp \gamma_2$ is $\gamma'_1 \perp \gamma'_2$, where γ'_2 is the orthogonal complement of $\gamma_1(v) + \gamma_2(w)$ in $\langle \gamma_1(v) \rangle \perp \gamma_2$. We therefore have $\gamma'_1 \perp \gamma'_2 \xrightarrow{\sim} \beta \perp 1$. Repeating the arguments over again, we get that β is decomposable over $\mathbb{R}[x]_{(2+x^2)}[y]$, which is a contradiction.

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*Eidg. Technische Hochschule
Mathematik
CH-8092 Zürich/Switzerland*

*Institut de Mathématiques
Université de Lausanne
CH-1015 Lausanne-Dorigny/Switzerland*

*School of Mathematics
Tata Institute of Fundamental Research
Bombay 400 005/India*

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