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## Positive-definite quadratic bundles over the plane

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### Introduction

Indecomposable, positive-definite quadratic spaces of ranks 3 and 4 over  $\mathbb{R}[x, y]$  have been constructed in [5] and [13]. A natural question to ask is whether there exist indecomposable quadratic spaces of rank  $>4$  over  $\mathbb{R}[x, y]$  and whether the theorem of Krull–Schmidt holds for orthogonal decompositions of positive-definite quadratic spaces over  $\mathbb{R}[x, y]$ . (cf [9], p. 204.)

In §1 of this paper we prove a Krull–Schmidt theorem for orthogonal sums of positive-definite quadratic spaces over  $\mathbb{R}[x, y]$ . In view of [8], Thm. 2.1, it is enough to prove a similar theorem for positive-definite quadratic bundles over  $\mathbb{P}_{\mathbb{R}}^2$ . More generally, we prove that if  $X$  is a projective scheme over  $\mathbb{R}$  and  $X_{\mathbb{C}}$  the complexification of  $X$ , then the theorem of Krull–Schmidt holds for positive-definite  $\sigma$ -hermitian (resp. quadratic) bundles over  $X_{\mathbb{C}}$  (resp.  $X$ ). We also deduce that Witt-cancellation holds for positive-definite quadratic spaces over  $\mathbb{R}[x, y]$ . In §2, we exhibit a class of vector-bundles of rank 3 and 4 over  $\mathbb{P}_{\mathbb{C}}^2$ , associated to a pair of projective ideals of  $\mathbb{H}[x, y]$ , and show, using results of §1, that these bundles are stable. (The examples of rank 4 bundles over  $\mathbb{P}_{\mathbb{C}}^2$  constructed here are interesting, particularly in view of the fact that in general it is not easy to decide the stability of bundles of rank  $>3$ .) In §3, we construct an example of a rank 6, indecomposable quadratic space over  $\mathbb{R}[x, y]$ . The idea of the construction is to patch certain rank 3 and 4 quadratic spaces over  $\mathbb{R}[x, y]$ .

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### §1. Krull–Schmidt theorem for positive-definite bundles over projective schemes

Let  $X$  be a projective scheme over  $\mathbb{R}$  and let  $X_{\mathbb{C}}$  denote the complexification  $\text{Spec } \mathbb{C} \times_{\text{Spec } \mathbb{R}} X$  of  $X$ . Let  $\sigma$  be the involution on  $X_{\mathbb{C}}$  induced by the complex conjugation on  $\mathbb{C}$  and  $\pi$  the projection of  $X_{\mathbb{C}}$  onto  $X$ . For any vector bundle  $\mathcal{F}_0$  over  $X$  we have a natural isomorphism  $\rho: \pi^* \mathcal{F}_0 \rightarrow \sigma^* \pi^* \mathcal{F}_0$ , since  $\pi \circ \sigma = \pi$ . For

any vector bundle  $\mathcal{F}$  over  $X_{\mathbb{C}}$  we denote by  $\mathcal{F}'$  the dual bundle and by  $\mathcal{F}^*$  the pull-back  $\sigma^*\mathcal{F}'$  of  $\mathcal{F}'$  through  $\sigma$ . We define a natural isomorphism (cfr. [11])  $\tau: (\sigma^*\mathcal{F})' \rightarrow \mathcal{F}^*$  by

$$(\sigma^*\mathcal{F})' = \mathcal{H}om(\sigma^*\mathcal{F}, \pi^*\mathcal{O}_{X_{\mathbb{C}}}) \xrightarrow{\mathcal{H}om(\sigma^*\mathcal{F}, \rho)} \mathcal{H}om(\sigma^*\mathcal{F}, \sigma^*\pi^*\mathcal{O}_{X_{\mathbb{C}}}) = \sigma^*\mathcal{F}',$$

In [11] a  $\sigma$ -hermitian structure over  $\mathcal{F}$  was defined as an isomorphism  $\phi: \mathcal{F} \rightarrow \sigma^*\mathcal{F}'$  such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad} & \sigma^*\mathcal{F}' \\ (\sigma^*\phi)' \searrow & & \nearrow \tau \\ & & (\sigma^*\mathcal{F})' \end{array}$$

is commutative. It is convenient to give an equivalent definition, using the terminology of [15]. Let  $\mathcal{M}$  be the category of vector bundles over  $X_{\mathbb{C}}$ . Associating to every  $\mathcal{F}$  the bundle  $\mathcal{F}^*$  we get a functor  $*$ :  $\mathcal{M} \rightarrow \mathcal{M}$ . Let, for any  $\mathcal{F}$ ,  $i_{\mathcal{F}}: \mathcal{F} \rightarrow \mathcal{F}^{**}$  be the isomorphism defined by

$$\mathcal{F}^{**} = \sigma^*(\mathcal{F}^*)' \xrightarrow{\tau_{\mathcal{F}}^{-1}} (\sigma^*\mathcal{F}^*)' \longrightarrow (\mathcal{F}')' \longrightarrow \mathcal{F}.$$

It is easily checked that  $i$  is a natural transformation  $id \xrightarrow{**}$  satisfying  $i_{\mathcal{F}}^*i_{\mathcal{F}^*} = id_{\mathcal{F}^*}$ . Hence  $*$  is a duality functor in the sense of [15]. We identify each bundle  $\mathcal{F}$  with  $\mathcal{F}^{**}$  and each morphism  $\phi$  of bundles with  $\phi^{**}$ . For  $\varepsilon = \pm 1$ , we define an  $\varepsilon$ -hermitian structure on  $\mathcal{F}$  as an isomorphism  $\phi: \mathcal{F} \xrightarrow{\sim} \mathcal{F}^*$  such that  $\phi^* = \varepsilon\phi$ . A 1-hermitian structure on  $\mathcal{F}$  turns out to be the same as a  $\sigma$ -hermitian structure in the sense defined above and in [11] or [8]. If  $x$  is a real closed point of  $X_{\mathbb{C}}$ , i.e. a closed point such that  $\sigma(x) = x$ , the fibre  $\mathcal{F}_x$  at  $x$  of a  $\sigma$ -hermitian bundle  $\mathcal{F}$  carries a non-degenerate hermitian form. We say that  $\mathcal{F}$  is positive definite if the fibre at every real closed point is positive definite. Since the signature of a hermitian form is locally constant, if  $X_{\mathbb{R}}$  is connected,  $\mathcal{F}$  is positive definite if and only if the induced form on the fibre of some real closed point of  $X_{\mathbb{C}}$  is positive definite.

We assume, from now on, that  $X$  has at least one real closed point.

For any bundle  $\mathcal{F}$  we denote by  $H(\mathcal{F})$  the hyperbolic bundle associated to  $\mathcal{F}$ . This is the bundle  $\mathcal{F} \oplus \mathcal{F}^*$  with the hermitian structure defined by the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**LEMMA 1.1.** *Let  $\mathcal{N}$  be an indecomposable vector bundle over  $X_{\mathbb{C}}$  such that  $\mathcal{N} \cong \mathcal{N}^*$ . Then  $\mathcal{N}$  carries a  $\sigma$ -hermitian structure.*

*Proof.* By Proposition 2.5 of [15],  $\mathcal{N}$  carries a (1)- or a (-1)-hermitian form. If  $\phi: \mathcal{N} \rightarrow \mathcal{N}^*$  is (-1)-hermitian,  $i\phi$  is hermitian.

**THEOREM 1.2.** *Let  $(\mathcal{E}, \phi)$  be a positive-definite  $\sigma$ -hermitian bundle over  $X_{\mathbb{C}}$ . Then, there is a unique orthogonal decomposition*

$$(\mathcal{E}, \phi) \xrightarrow{\sim} \perp_i (\mathcal{E}_i, \phi_i),$$

where  $\mathcal{E}_i$  are the isotypical components of the vector bundle  $\mathcal{E}$  (i.e.  $\mathcal{E}_i \xrightarrow{\sim} \bigoplus \mathcal{N}_i$ , where  $\mathcal{N}_i$  are indecomposable and for  $i \neq j$ ,  $\mathcal{N}_i \not\cong \mathcal{N}_j$ ). Each  $\mathcal{E}_i$  carries a positive-definite  $\sigma$ -hermitian structure which is unique up to isometry.

*Proof.* Since  $X$  is a projective scheme, the category  $\mathfrak{M}$  with the duality functor  $*$  defined above satisfies the assumptions (i)–(iii) of [15], page 272. Hence, by Theorem 3.2 of [15],

$$(\mathcal{E}, \phi) \cong (\mathcal{E}_1, \phi_1) \perp \cdots \perp (\mathcal{E}_n, \phi_n),$$

where each  $\mathcal{E}_i$  is a direct sum of vector bundles isomorphic to a fixed indecomposable  $\mathcal{N}_i$  or to its “dual”  $\mathcal{N}_i^*$ . By Theorem 3.3 of [15], if  $\mathcal{N}_i \not\cong \mathcal{N}_i^*$ ,  $\mathcal{E}_i$  contains a hyperbolic orthogonal summand. Since, by assumption,  $\mathcal{E}$  is positive definite, this cannot happen and hence each  $\mathcal{E}_i$  is isotypical. Since the orthogonal decomposition written above is unique, it suffices to prove the uniqueness for an isotypical vector bundle.

Let  $\mathcal{E}$  be an isotypical vector bundle of type  $\mathcal{N}$  and let  $\mathcal{E} \xrightarrow{\sim} \bigoplus \mathcal{N}$ . We show that if  $\mathcal{E}$  carries a positive-definite  $\sigma$ -hermitian structure, then it is unique. Since  $\mathcal{N}$  is indecomposable, the ring  $E = \text{End } \mathcal{N}$  is a local finite-dimensional  $\mathbb{C}$ -algebra. Let  $\bar{E} = E/\text{rad } E$ . Then  $\bar{E}$  is a finite-dimensional division algebra over  $\mathbb{C}$  and hence  $\bar{E} \xrightarrow{\sim} \mathbb{C}$ . One reduces the study of  $\sigma$ -hermitian structures on  $\mathcal{E}$  to the study of hermitian-forms over a certain vector space  $\bar{M}$  over  $\bar{E}$  defined as follows (see [15], 2.2, 2.4). Let  $\phi: \mathcal{E} \rightarrow \mathcal{E}^*$  be a  $\sigma$ -hermitian structure on  $\mathcal{E}$ . Then,  $\bigoplus_r \mathcal{N} \xrightarrow{\sim} \bigoplus_r \mathcal{N}^*$  and by the Krull–Schmidt theorem the vector bundles  $\mathcal{N}$  and  $\mathcal{N}^*$  are isomorphic. Hence, by Lemma 1.1, there exists an isomorphism  $\phi_0: \mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$  which defines a  $\sigma$ -hermitian structure on  $\mathcal{N}$ . In what follows, we shall fix this  $\sigma$ -hermitian structure  $\phi_0$  on  $\mathcal{N}$ . The isomorphism  $\phi_0$  induces an involution  $\tau$  on  $E = \text{End } \mathcal{N}$  defined as

$$\tau f = f^0 = \phi_0^{-1} \circ f^* \circ \phi_0.$$

The map  $f \rightarrow f^0$  satisfies  $(fg)^0 = g^0 f^0$ ,  $(f^0)^0 = f$  and for  $\lambda \in \mathbb{C}$ ,  $(\lambda f)^0 = \bar{\lambda} f^0$ ,  $\bar{\lambda}$  denoting the complex conjugate of  $\lambda$ . This involution passes down to an involution on

$\bar{E} = E/\text{rad } E = \mathbb{C}$  which is just the complex conjugation on  $\mathbb{C}$ . Let  $M = \text{Hom}(\mathcal{N}, \mathcal{E})$ . Then  $M$  is a right  $E$ -module and the isomorphism  $\phi: \mathcal{E} \rightarrow \mathcal{E}^*$  induces an isomorphism  $\phi_1: M \rightarrow \text{Hom}_E(M, E)$  which is semilinear with respect to the involution  $\tau$ . The map  $\phi_1$  is in fact defined as  $\phi_1(f)(g) = \phi_0^{-1} \circ f^* \circ g$  for  $f, g \in M$ . It is easily verified that  $\phi_1$  defines a hermitian form on the  $E$ -module  $M$  with respect to the involution  $\tau$  on  $E$ . Going modulo the radical of  $E$ , we obtain on  $\bar{M} = M/(\text{rad } E)M$  a hermitian form over  $\mathbb{C}$ .

Two  $\sigma$ -hermitian structures on  $\mathcal{E}$  are isometric if and only if the corresponding hermitian forms on  $\bar{M}$  are isometric ([15], 2.2). If the form on  $\mathcal{E}$  is positive-definite, then the form on  $\bar{M}$  is either positive or negative-definite. In fact, if  $\bar{M}$  represents zero, then  $\bar{M}$  contains a hyperbolic summand and so does  $\mathcal{E}$  by [15], Prop. 2.4. If  $\phi$  and  $\phi'$  are two positive definite forms on  $\mathcal{E}$ , the corresponding forms on  $\bar{M}$  are either both positive-definite or both negative-definite: otherwise the form corresponding to  $\phi \perp \phi'$  on  $\mathcal{E} \perp \mathcal{E}$  would be isotropic. Since, up to isometry, there is a unique positive or negative-definite hermitian form on  $\bar{M}$ , it follows that there is a unique positive definite  $\sigma$ -hermitian structure over  $\mathcal{E}$ . This proves Theorem 1.2.

**COROLLARY 1.3.** *A vector bundle over  $X_{\mathbb{C}}$  carries at the most one positive-definite  $\sigma$ -hermitian structure.*

**COROLLARY 1.4** (Krull-Schmidt theorem). *Any  $\sigma$ -hermitian positive-definite bundle  $(\mathcal{E}, \phi)$  over  $X_{\mathbb{C}}$  has a decomposition*

$$(\mathcal{E}, \phi) = \perp (\mathcal{N}_i, \nu_i)$$

*into indecomposable  $\sigma$ -hermitian bundles. The summands  $(\mathcal{N}_i, \nu_i)$  are unique up to isometries and permutations.*

**COROLLARY 1.5.** *The Krull-Schmidt theorem holds for positive-definite  $\sigma$ -hermitian spaces over  $\mathbb{C}[x, y]$ .*

*Proof.* By (3.1) of [8] any positive-definite  $\sigma$ -hermitian space over  $\mathbb{C}[x, y]$  has, up to isometry, a unique extension to  $\mathbb{P}_{\mathbb{C}}^2$ . Hence the assertion follows from 1.4.

The following theorem and corollaries give the corresponding results for positive-definite quadratic bundles.

**THEOREM 1.6.** *Let  $(\mathcal{E}, \phi)$  be a positive-definite quadratic bundle over  $X$ . Then, there is a unique orthogonal decomposition*

$$(\mathcal{E}, \phi) = \perp_i (\mathcal{E}_i, \phi_i)$$

where  $\mathcal{E}_i$  are the isotypical components of the vector bundle  $\mathcal{E}$ . The components  $\mathcal{E}_i$  carry a positive-definite quadratic structure, unique up to isometry.

A proof on the same lines as of Theorem 1.2 can be given. Let now  $\mathfrak{M}$  be the category of real vector bundles over  $X$  and, for any such bundle  $\mathcal{E}$  let  $\mathcal{E}^* = \mathcal{E}' = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$  be the dual of  $\mathcal{E}$ . By Theorem 3.2 of [15] one reduces immediately to the case of an isotypical bundle  $\mathcal{E} \xrightarrow{\sim} \bigoplus \mathcal{N}$ ,  $\mathcal{N}$  indecomposable. Since  $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^*$ , we have  $\mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$  and since  $\text{End } \mathcal{N}$  is local,  $\mathcal{N}$  carries either a quadratic or a symplectic structure  $\phi_0: \mathcal{N} \xrightarrow{\sim} \mathcal{N}^*$ . Then  $\phi_0$  gives rise to an involution  $\tau$  of  $E = \text{End } \mathcal{N}$ , which passes down to an involution of  $\bar{E} = E/\text{rad } E$ . It is clear that  $\bar{E} \xrightarrow{\sim} \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ . If  $\bar{E} = \mathbb{R}$ , the involution is trivial. If  $\bar{E} = \mathbb{C}$ , the involution must be complex conjugation. And if  $\bar{E} \xrightarrow{\sim} \mathbb{H}$ , the involution on  $\mathbb{H}$  is either trivial or is a conjugate of the canonical involution. The isometry classes of quadratic structures on  $\mathcal{E}$  correspond to isometry classes of positive-definite or negative-definite forms on  $\bar{M} = M/(\text{rad } E)M$ , where  $M = \text{Hom}(\mathcal{N}, \mathcal{E})$ . The existence of orthogonal bases for hermitian forms shows that there is unique positive- or negative-definite  $\tau$ -hermitian form on  $\bar{M}$ . It follows that there is a unique positive-definite quadratic structure over  $\mathcal{E}$ .

**COROLLARY 1.7.** *A vector bundle over  $X$  carries at the most one positive-definite quadratic structure.*

**COROLLARY 1.8.** *The Krull–Schmidt theorem holds for positive-definite quadratic bundles over  $X$ .*

**COROLLARY 1.9.** *The Krull–Schmidt theorem holds for positive-definite quadratic spaces over  $\mathbb{R}[x, y]$ .*

## §2. Some stable bundles of rank 3 and 4 associated to projective ideals of $\mathbb{H}[x, y]$

We recall that a bundle  $\mathcal{E}$  over  $\mathbb{P}_{\mathbb{C}}^r$  is said to be *stable* if, for every coherent subsheaf  $\mathcal{F} \neq 0$  of  $\mathcal{E}$  such that  $\mathcal{E}/\mathcal{F}$  is torsionfree we have  $c_1(\mathcal{F})/\text{rank } \mathcal{F} < c_1(\mathcal{E})/\text{rank } \mathcal{E}$ . In [8] to each non-free projective ideal  $P$  of  $\mathbb{H}[x, y]$  was associated a rank 2 stable bundle  $\mathcal{E}(P)$  with a positive-definite  $\sigma$ -hermitian structure. We recall the construction of these bundles, which in [8] were called  $\mathfrak{B}$ -bundles. Let  $\phi: \mathbb{C} \otimes \mathbb{H} \rightarrow M_2(\mathbb{C})$  be the isomorphism given by

$$\phi(s \otimes (u + vj)) = s \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} u, v \in \mathbb{C}.$$

Let  $H = \mathbb{H}[x, y]$  and  $C = \mathbb{C}[x, y]$ . For any projective ideal  $P$  of  $H$ ,  $C \otimes P$  is an  $M_2(C)$ -module via  $\phi$ . Hence, there is a  $\phi$ -semilinear isomorphism  $\Psi_P : C \otimes P \xrightarrow{\sim} M_2(C)$ . We shall call such a map a *splitting* of  $P$ . By Galois cohomology, we associate to the splitting  $\Psi_P$  the cocycle

$$\alpha_P = \sigma \Psi_P(\sigma \otimes 1) \Psi_P^{-1}(1) \in GL_2(C)$$

where  $\sigma$  is the complex conjugation on  $\mathbb{C}$  and the transported action  $\phi(\sigma \otimes 1)\phi^{-1}$  on  $M_2(C)$ . The map  $\Psi_P$  can be chosen such that  $\alpha_P$  is positive-definite hermitian of determinant one. Such a splitting is called a *normalized splitting*. Hence,  $\alpha_P$  defines a  $\sigma$ -hermitian structure on  $\mathbb{A}_{\mathbb{C}}^2$ . This structure can be uniquely extended to  $\mathbb{P}_{\mathbb{C}}^2$  ([8]) and the extension is the complex bundle  $\mathcal{E}(P)$ . Notice that by (1.2)  $\mathcal{E}(P)$  carries a unique positive-definite  $\sigma$ -hermitian structure. Let now  $P$  and  $Q$  be two projective ideals in  $H$ . The reduced norm  $Nr$  introduced in [6] defines a quadratic form on the  $\mathbb{R}[x, y]$ -module of rank 4  $\text{Hom}_H(P, Q)$ . If  $\Psi_P : C \otimes P \xrightarrow{\sim} M_2(C)$  and  $\Psi_Q : C \otimes Q \xrightarrow{\sim} M_2(C)$  are normalized splittings of  $P$  and  $Q$ , then, for any  $f \in \text{Hom}_H(P, Q)$ ,  $Nr(f) = \det \Psi_Q(1 \otimes f) \Psi_P^{-1}(1)$ . This quadratic space is indecomposable if  $P$  and  $Q$  are non-free and not isomorphic. If  $P \simeq Q$  and  $P$  is non-free, then this space decomposes as  $\langle 1 \rangle \perp \bar{q}$ , where  $\bar{q}$  is the orthogonal complement of the submodule  $\mathbb{R}[x, y]$  of  $\text{End}_H(P)$  for the reduced norm on the algebra  $\text{End}_H(P)$ . It is shown in [6] that  $\bar{q}$  is indecomposable. These indecomposable quadratic spaces of ranks 3 and 4 extend uniquely to indecomposable quadratic bundles over  $\mathbb{P}_{\mathbb{R}}^2$ , denoted respectively by  $\mathcal{F}(P, Q)$  and  $\mathcal{F}(P)$ . Let  $\pi : \mathbb{P}_{\mathbb{C}}^2 \rightarrow \mathbb{P}_{\mathbb{R}}^2$  be the projection and let  $\pi^* \mathcal{F}(P, Q) = \mathcal{G}(P, Q)$  and  $\pi^* \mathcal{F}(P) = \mathcal{G}(P)$ . We shall show that these bundles are stable.

**THEOREM 2.1.** *The bundle  $\mathcal{G}(P, Q)$  is isomorphic to  $\mathcal{E}(P) \otimes \mathcal{E}(Q)$ .*

**COROLLARY 2.2.** *We have  $c_2(\mathcal{G}(P, Q)) = 2(c_2(\mathcal{E}(P)) + c_2(\mathcal{E}(Q)))$  and  $c_2(\mathcal{G}(P)) = 4c_2(\mathcal{E}(P))$ .*

*Proof.* For 2-bundles  $\mathcal{E}$  and  $\mathcal{F}$  on  $\mathbb{P}_{\mathbb{C}}^2$ , if  $c_1(\mathcal{E}) = c_1(\mathcal{F}) = 0$ , then  $c_2(\mathcal{E} \otimes \mathcal{F})$  is given by  $2(c_2(\mathcal{E}) + c_2(\mathcal{F}))$ .

Theorem (2.1) is a consequence of the following results. The first one is implicitly contained in [7], (1.12).

**LEMMA 2.3.** *Let  $P$  be a projective ideal of  $H$ ,  $\Psi_P$  a normalized splitting of  $P$  and  $\alpha_P \in GL_2(C)$  the corresponding cocycle. Then there is a basis  $e_1, e_2$  of  $P$  as a*

$C$ -module such that the matrix of the  $\sigma$ -hermitian form  $a_P$  on  $P$  defined by

$$a_P(e_i, e_i) = (\Psi_P(e_i), \Psi_P(e_i)) \quad i = 1, 2$$

$$a_P(e_1, e_2) = (\Psi_P(e_1), \Psi_P(e_2)) - i(\Psi_P(e_1), \Psi_P(ie_2))$$

where, for  $u, v \in M_2(C)$ ,  $(u, v) = \frac{1}{2}(\det(u + v) - \det u - \det v)$ , is  $\alpha_P$ .

Let  $\alpha_P = \alpha + i\beta$  with  $\alpha, \beta \in M_2(\mathbb{R}[x, y])$ . Then the symmetric matrix  $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$  represents the reduced norm on  $P$  with respect to the basis  $e_1, e_2, e_3 = ie_1, e_4 = ie_2$  of  $P$  over  $\mathbb{R}[x, y]$ .

The next lemma is an immediate consequence of (2.3) and of the definition of the reduced norm on  $\text{Hom}_H(P, Q)$  by means of the splittings  $\Psi_P$  and  $\Psi_Q$ .

LEMMA 2.4. Let  $f \in \text{Hom}_H(P, Q)$  and  $a_P, a_Q$  the hermitian forms given in (2.1). Then, for any  $u, v \in P$

$$a_Q(f(u), f(v)) = \text{Nr}(f) a_P(u, v).$$

The module  $P' = \text{Hom}_C(P, C)$  is a projective right  $H$ -module (with the action  $(f\lambda)(x) = f(\lambda x)$ ,  $\lambda \in H$ ). We now compute its cocycle.

LEMMA 2.5. Let  $\Psi_P$  be a splitting of  $P$  with cocycle  $\alpha_P$ . Then, there is a splitting  $\Psi_{P'}$  of  $P'$  with cocycle  $\alpha_{P'} = \alpha_P^{-1}$ .

*Proof.* Let  $T: M_2(C) \xrightarrow{\sim} \text{Hom}_C(M_2(C), C)$  be the isomorphism given by the trace, i.e.  $T_a(b) = \text{Tr}(ab)$ ,  $a, b \in M_2(C)$ . Let  $\hat{P} = \text{Hom}_{\mathbb{R}[x, y]}(P, \mathbb{R}[x, y])$ . Then the map  $\Psi_{\hat{P}} = T^{-1}(\Psi_P)^\wedge$  (where  $\hat{\phantom{x}}$  means dualization with respect to  $\mathbb{R}[x, y]$ ) is a splitting of  $\hat{P}$  and one computes that the corresponding cocycle is  $\alpha_P^{-1}$ . Let now  $t: P' \xrightarrow{\sim} \hat{P}$  be the isomorphism (of  $H$ -modules) induced by the trace  $\mathbb{C} \rightarrow \mathbb{R}$ . Then the map  $\Psi_{P'} = \Psi_{\hat{P}} \circ (1 \otimes t)$  is a splitting of  $P'$  such that  $\alpha_{P'} = \alpha_{\hat{P}} = \alpha_P^{-1}$ .

Let now  $a_{P'}$  be the hermitian structure on  $P'$  given by

$$a_{P'}(e'_i, e'_j) = \frac{1}{2}(\alpha_{P'})_{j,i} = \frac{1}{2}(\alpha_P^{-1})_{j,i}$$

where  $e'_i, i = 1, 2$  is the dual basis of the basis  $e_i, i = 1, 2$  given in (2.3). Let  $S$  be the  $\sigma$ -hermitian space obtained by extending the reduced norm  $\text{Nr}$  on  $\text{Hom}_H(P, Q)$  to  $\mathbb{C} \otimes_{\mathbb{R}} \text{Hom}_H(P, Q)$ , i.e.  $S(\lambda \otimes f) = \lambda \bar{\lambda} \text{Nr}(f)$ .

LEMMA 2.6. *The map  $\rho: \mathbb{C} \otimes \text{Hom}_H(P, Q) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(P, Q) \xrightarrow{\sim} P' \otimes_{\mathbb{C}} Q$  where the first map is the multiplication and the second is the canonical map, is an isomorphism of  $\sigma$ -hermitian spaces  $\rho: S \xrightarrow{\sim} a_{P'} \otimes a_Q$ .*

*Proof.* For any basis  $\{e_i\}$  of  $P$ ,  $\rho$  is given by  $\rho(\lambda \otimes f) = \sum_i e_i^* \otimes f(\lambda e_i)$ . Choosing the basis given in (2.3), we have, using (2.4),

$$\begin{aligned} (a_{P'} \otimes a_Q) \left( \sum_i e_i^* \otimes f(\lambda e_i) \right) &= \sum_{i,j} a_{P'}(e_i^*, e_j^*) a_Q(f(\lambda e_i), f(\lambda e_j)) \\ &= \text{Nr}(f) \lambda \bar{\lambda} \sum_{i,j} a_{P'}(e_i^*, e_j^*) a_Q(e_i, e_j) = \text{Nr}(f) \lambda \bar{\lambda}. \end{aligned}$$

This shows that  $\rho$  is an isometry.

Theorem (2.1) now follows from (2.6) noting that the extension of a positive definite  $\sigma$ -hermitian form from  $\mathbb{A}_{\mathbb{C}}^2$  to  $\mathbb{P}_{\mathbb{C}}^2$  is unique and that  $\mathcal{E}(P^*) \cong \mathcal{E}(P)$ .

To show that the bundles  $\mathcal{G}(P, Q)$  and  $\mathcal{G}(P)$  are stable, we begin with

LEMMA 2.7. *Let  $K$  be a field of characteristic  $\neq 2$  and let  $(\mathcal{E}, \phi)$  be a quadratic bundle of rank 2 over  $\mathbb{P}'_K$ . If  $(\mathcal{E}, \phi)$  is anisotropic,  $(\mathcal{E}, \phi)$  is extended from  $K$ . If  $(\mathcal{E}, \phi)$  is isotropic, then  $(\mathcal{E}, \phi) \xrightarrow{\sim} H(\mathcal{O}(n))$ , a hyperbolic space.*

*Proof.* The first part of the lemma is proved in ([8], 2.4). If  $(\mathcal{E}, \phi)$  is isotropic, then restricted to each affine piece  $D(x_i)$ , the quadratic form can be given by the matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . One then easily checks that  $(\mathcal{E}, \phi) \xrightarrow{\sim} H(\mathcal{O}(n))$  for some  $n$ .

LEMMA 2.8. *Let  $K$  be a field of characteristic  $\neq 2$  and let  $\mathcal{E}$  be an indecomposable anisotropic quadratic bundle over  $\mathbb{P}'_K$ . Then  $\mathcal{E}$  has no non-zero section.*

*Proof.* Evaluating the quadratic form on a global section one gets a global function on  $\mathbb{P}'_K$ , hence a constant. This constant must be zero, since the bundle is indecomposable as a quadratic bundle. The section has to be zero since the form is anisotropic.

For any bundle  $\mathcal{E}$  over  $\mathbb{P}_{\mathbb{C}}^2$  the ‘‘type’’ of  $\mathcal{E}$  is the pair of Chern classes  $(c_1(\mathcal{E}), c_2(\mathcal{E}))$ .

THEOREM 2.9. *The bundles  $\mathcal{G}(P)$  are stable rank 3 bundles of type  $(0, 8n)$ , where  $c_2(\mathcal{E}(P)) = 2n$ ,  $\mathcal{E}(P)$  denoting the  $\mathfrak{B}$ -bundle associated to a non-free projective ideal  $P$  of  $\mathbb{H}[x, y]$ . The bundles  $\mathcal{E}(P, Q)$  are stable rank 4 of type  $(0, 4(m+n))$  if  $P$  and  $Q$  are non-isomorphic, non-free,  $\mathcal{E}(P)$  of type  $(0, 2n)$  and  $\mathcal{E}(Q)$  of type  $(0, 2m)$ .*

*Proof.* Since  $\mathcal{E}(P)$  supports a quadratic form it follows that  $c_1(\mathcal{E}(P)) = 0$ . If we consider global sections, we have  $H^0(\mathbb{P}_{\mathbb{C}}^2, \mathcal{G}(P)) \xrightarrow{\sim} \mathbb{C} \otimes H^0(\mathbb{P}_{\mathbb{R}}^2, \mathcal{F}(P)) = 0$  by Lemma 2.7, since  $\mathcal{F}(P)$  supports an anisotropic indecomposable quadratic form. Further, being a quadratic bundle,  $\mathcal{G}(P) \xrightarrow{\sim} \mathcal{G}(P)'$ . Hence  $\mathcal{G}(P)$  is stable by [12], 1.2.6.

We shall now show that  $\mathcal{G}(P, Q)$  is stable for  $P, Q$  non-isomorphic, non-free. We show that for every subsheaf  $\mathcal{F}$  of  $\mathcal{G} = \mathcal{G}(P, Q)$  with the quotient  $\mathcal{G}(P, Q)/\mathcal{F}$  torsion free,  $c_1(\mathcal{F})/\text{rank } \mathcal{F} < c_1(\mathcal{G})/\text{rank } \mathcal{G}$ . Since  $\mathbb{P}_{\mathbb{C}}^2$  is regular of dimension 2, such a sheaf is locally free. Hence it suffices to show that for any locally free subsheaf  $\mathcal{F}$  of  $\mathcal{G}$ ,  $c_1(\mathcal{F}) < 0$ . If  $\mathcal{F}$  is a line bundle with  $c_1(\mathcal{F}) = n$ , necessarily  $n < 0$  since, otherwise,  $\mathcal{F}$  and hence  $\mathcal{G}$  would have a non-zero global section. If  $\mathcal{F}$  is of rank 3 we have a surjection  $\mathcal{G}' \rightarrow \mathcal{F}' \rightarrow 0$  whose kernel is a line bundle  $\mathcal{L}$ . Since  $\mathcal{G}' \xrightarrow{\sim} \mathcal{G}$  also does not admit of global sections, it follows that  $c_1(\mathcal{L}) < 0$ . Hence  $c_1(\mathcal{F}') > 0$  so that  $c_1(\mathcal{F}) = -c_1(\mathcal{F}') < 0$ . Let  $\mathcal{F}$  be of rank 2. The bundle  $\mathcal{G}$  restricted to a real line  $L$  of  $\mathbb{P}_{\mathbb{C}}^2$  is trivial, since  $\mathcal{G}$  supports an anisotropic quadratic form ([16], Prop. 5). The restriction of  $\mathcal{F}$  to  $L$  is isomorphic to  $\mathcal{O}(n) \oplus \mathcal{O}(m)$ . Since  $\mathcal{F}|_L$  is a subsheaf of  $\mathcal{G}|_L \xrightarrow{\sim} \mathcal{O}|_L$ , we have  $c_1(\mathcal{F}) = n + m \leq 0$ . Suppose that  $c_1(\mathcal{F}) = 0$ . Then  $\mathcal{F}$  is a rank 2 bundle with no global sections and with  $c_1(\mathcal{F}) = 0$ . Hence  $\mathcal{F}$  is a stable bundle ([12], 1.2.5). The quadratic structure on  $\mathcal{F}(P, Q)$  extends to a positive-definite  $\sigma$ -hermitian structure, denoted by  $\phi$ , on  $\mathcal{G}(P, Q)$ . The restriction of  $\phi$  to  $\mathcal{F}$  induces a map  $\mathcal{F} \rightarrow \sigma^* \mathcal{F}^* = \mathcal{F}^*$ . This map cannot be zero since  $\mathcal{F}$  is anisotropic (positive-definite). By the corollary to Lemma 1.2.8 of [12],  $\phi$  is an isomorphism and  $(\mathcal{F}, \phi|_{\mathcal{F}})$  splits off as an orthogonal summand of  $(\mathcal{G}, \phi)$ . Then,  $\mathcal{G} \xrightarrow{\sim} \mathcal{F} \perp \mathcal{F}_1$ . The bundle  $\mathcal{G}$  supports a quadratic form, namely the extension of the quadratic structure on  $\mathcal{F}(P, Q)$ . The bundle  $\mathcal{F}$  cannot support a quadratic structure, since, otherwise,  $\mathcal{F} \xrightarrow{\sim} H(\mathcal{O}(n))$  by Lemma 2.7 contradicting the stability of  $\mathcal{F}$ . Thus, by the uniqueness of the quadratic structure on  $\mathcal{G}$ , it follows that  $\mathcal{G} \xrightarrow{\sim} H(\mathcal{F})$  and hence  $\mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}' \xrightarrow{\sim} \mathcal{F}$ . In fact, by the uniqueness of the positive-definite structure (see (1.6))  $(\mathcal{F}_1, \phi|_{\mathcal{F}_1}) \xrightarrow{\sim} (\mathcal{F}, \phi|_{\mathcal{F}})$  and  $(\mathcal{G}, \phi) \xrightarrow{\sim} (\mathcal{F}, \phi|_{\mathcal{F}}) \perp (\mathcal{F}_1, \phi|_{\mathcal{F}_1})$ . Since  $\mathcal{F}$  is a rank 2 stable bundle with a positive-definite  $\sigma$ -hermitian structure, it follows by [8] that  $\mathcal{F}$  is a  $\mathfrak{B}$ -bundle, i.e.  $\mathcal{F} \xrightarrow{\sim} \mathcal{E}(P_0)$ , where  $P_0$  is some non-free projective ideal of  $\mathbb{H}[x, y]$ . By [8], Prop. 3.2,  $\mathcal{G} \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0) \xrightarrow{\sim} \pi^* \pi_* \mathcal{E}(P_0) = \pi^*(\mathcal{F}(\mathbb{H}[x, y]), P_0)$ . Since  $\text{End}(\mathcal{E}(P_0) \oplus \mathcal{E}(P_0)) \xrightarrow{\sim} M_2(\mathbb{C})$ , the isomorphism classes of vector-bundles on  $\mathbb{P}_{\mathbb{R}}^2$  with  $\pi^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$  are classified by  $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C}))$  for an action on  $GL_2(\mathbb{C})$  which is the restriction of an action on  $M_2(\mathbb{C})$ . Since  $\pi^*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$  is  $\mathbb{C}$ -linear,  $\mathbb{Z}/2\mathbb{Z}$  acts on  $\mathbb{C} \subset M_2(\mathbb{C}) = \text{End}(\mathcal{E}(P_0) \oplus \mathcal{E}(P_0))$  by conjugation, and hence the action on  $M_2(\mathbb{C})$  is of the form  $\alpha \rightarrow u\bar{\alpha}u^{-1}$  for some fixed  $u \in GL_2(\mathbb{C})$ . It is easily checked that in this case  $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C})) = 0$ . Hence, there is a unique descent for  $\mathcal{E}(P_0) \oplus \mathcal{E}(P_0)$ , i.e.  $\mathcal{F}(P, Q) \xrightarrow{\sim} \mathcal{F}(\mathbb{H}[x, y], P_0)$ . By the

uniqueness of the positive-definite quadratic structure on a vector-bundle over  $\mathbb{P}_{\mathbb{R}}^2$  [(1.7)], it follows that  $\mathcal{F}(P, Q)$  is isomorphic as a quadratic bundle to  $\mathcal{F}(\mathbb{H}[x, y], P_0)$ . By restricting these bundles to  $\mathbb{A}_{\mathbb{R}}^2$  and using ([6], Thm. 4.6), it follows that  $P$  or  $Q$  is free, a contradiction. The statement in the theorem regarding the second Chern classes of  $\mathcal{G}(P)$  and  $\mathcal{G}(P, Q)$  was proved in (2.2).

### §3. An example of an indecomposable quadratic space of rank 6 over $\mathbb{R}[x, y]$

LEMMA 3.1. *Let  $R$  be a local domain in which 2 is invertible and let  $q_1, q_2$  be quadratic spaces over  $R[x]$  such that  $q_1 \perp q_2$  is anisotropic. If  $q_1(v) + q_2(w)$  is a unit of  $R[x]$ , then  $q_1(v)$  or  $q_2(w)$  is a unit of  $R[x]$ .*

*Proof.* Let  $K$  denote the quotient field of  $R$ . Since  $R$  is local, if bar denotes reduction modulo  $x$ , one has  $\bar{q}_1 \xrightarrow{\sim} \langle \lambda_1, \dots, \lambda_n \rangle$ ,  $\bar{q}_2 \xrightarrow{\sim} \langle \mu_1, \dots, \mu_m \rangle$ ,  $\lambda_i, \mu_i \in U(R)$ . By a theorem of Harder, we have, over  $K[x]$ ,  $q_1 \xrightarrow{\sim} \langle \lambda_1, \dots, \lambda_n \rangle$ ,  $q_2 \xrightarrow{\sim} \langle \mu_1, \dots, \mu_m \rangle$ . Thus, there exist  $\theta_i, \phi_i \in K[x]$  such that  $q_1(v) = \sum \lambda_i \theta_i^2$  and  $q_2(w) = \sum \mu_i \phi_i^2$ . Since the forms  $q_1$  and  $q_2$  are anisotropic over  $K[x]$ , if  $q_1(v) = a_0 + a_1x + \dots + a_r x^r$ , then  $q_2(w) = b_0 - a_1x - \dots - a_r x^r$ , and  $a_r = \sum \lambda_i c_i^2 = -\sum \mu_i d_i^2$ , where  $c_i, d_i$  denote the leading coefficients of  $\theta_i$  and  $\phi_i$  respectively. Then,  $\bar{q}_1 \perp \bar{q}_2$  represents zero over  $K$  and hence  $q_1 \perp q_2$  represents zero over  $K[x]$ , contradicting the assumption that  $q_1 \perp q_2$  is anisotropic.

The next lemma is a generalization of Proposition 1.1 of [13].

LEMMA 3.2. *Let  $A$  be a normal ring in which 2 is invertible. Every quadratic space of rank 2 over  $A[X_1, \dots, X_n]$  is extended from  $A$ .*

*Proof.* By [3, 4.15, Remark 4] we may assume that  $A$  is local. Let  $K$  be the field of fractions of  $A$  and  $M$  a quadratic space of rank 2 over  $A[X]$ ,  $X$  denoting  $(X_1, \dots, X_n)$ . If the signed discriminant of  $M_K$  is trivial, by [2, Proposition 5.1]  $M$  is of the form  $H(I)$ , where  $I$  is a projective ideal of  $A[X]$ . Since  $\text{Pic } A = \text{Pic } A[X]$ ,  $M$  is extended. If the signed discriminant  $d$  of  $M_K$  is not a square in  $K$ , put  $L = K[\sqrt{d}]$  and  $B = A[\sqrt{d}]$ . Then  $B$  is the integral closure of  $A$  in  $L$  hence is a normal semilocal ring. The signed discriminant of  $M_B$  is trivial and hence  $M_B$  is of the form  $H(I)$ , where  $I$  is a projective ideal of  $B[X]$ . Since  $\text{Pic } B[X] = \text{Pic } B = 0$ ,  $M_B = H(B[X])$ . This shows that  $M$  is represented by an element of  $H^1(\text{Gal}(L/K), O_2(B[X]))$ . But  $O_2(B[X]) = O_2(B)$  (compare [11], §1) and hence  $M$  is in the image of  $H^1(\text{Gal}(L/K), O_2(B))$  in  $H^1(\text{Gal}(L/K), O_2(B[X]))$ . This shows that  $M$  is extended from  $A$ .

Given a pair  $f, g$  of polynomials in  $\mathbb{R}[x, y]$ , let  $\alpha_{f,g}$  (respectively  $\beta_{f,g}$ ) denote the rank 3 (rank 4) quadratic spaces over  $\mathbb{R}[x, y]$  defined as the orthogonal complement of the identity in  $\text{End}(P_{f,g})$  (respectively reduced norm on  $P_{f,g}$ ), where  $P_{f,g}$  is the projective ideal of  $\mathbb{H}[x, y]$  defined as the kernel of the  $\mathbb{H}[x, y]$ -linear map  $\mathbb{H}[x, y]^2 \rightarrow \mathbb{H}[x, y]$  given by  $(1, 0) \rightarrow f + i$ ,  $(0, 1) \rightarrow g + j$  ([8], 1.2). Then  $\alpha = \alpha_{x,y}$  is an indecomposable quadratic space over  $\mathbb{R}[x, y]$ . This space remains indecomposable over  $\mathbb{R}[x]_{(1+x^2)}[y]$ . In fact, if it decomposes as  $\alpha' \perp \alpha''$ , then the ranks of  $\alpha'$  and  $\alpha''$  are 1 or 2 and hence, by Lemma 3.2,  $\alpha$  is extended from  $\mathbb{R}[x]_{(1+x^2)}$ . Since over  $\mathbb{R}[x, 1/1+x^2][y]$ ,  $P_{x,y}$  is free ([7], §5),  $\alpha$  is  $\cong \langle 1, 1, 1 \rangle$  over this ring. Therefore by [3, 4.15, Remark 4],  $\alpha$  is extended from  $\mathbb{R}$ , contrary to the assumption. The form  $\beta = \beta_{x\sqrt{2},y}$  is an indecomposable quadratic space over  $\mathbb{R}[x, y]$  which is isometric to  $\langle 1, 1, 1, 1 \rangle$  over  $\mathbb{R}[x, 1/2+x^2][y]$ . We claim that  $\beta$  remains indecomposable over  $\mathbb{R}[x]_{(2+x^2)}[y]$ . Suppose that  $\beta = \beta' \perp \beta''$  over  $\mathbb{R}[x]_{(2+x^2)}[y]$ . If  $\text{rank } \beta' = \text{rank } \beta'' = 2$  the same argument as above shows that  $\beta$  is extended from  $\mathbb{R}$ , which is absurd. If  $\text{rank } \beta' = 1$ , then  $\beta$  represents a unit over  $\mathbb{R}[x]_{(2+x^2)}[y]$  and therefore, by [6], (3.19)  $P_{x\sqrt{2},y}$  is free over  $\mathbb{H}[x]_{(2+x^2)}[y]$  and, in particular, extended from  $\mathbb{H}$ . Since it is also free over  $\mathbb{H}[x, 1/2+x^2][y]$  ([7], §5), by Quillen's theorem  $P_{x\sqrt{2},y} = \mathbb{H}[x, y]$ , contrary to the assumption.

We define a quadratic space over  $\mathbb{R}[x, y]$  of rank 6 as follows: we consider the covering

$$\text{Spec } \mathbb{R}[x, y] = \text{Spec } \mathbb{R}[x, y][1/1+x^2] \cup \text{Spec } \mathbb{R}[x, y][1/2+x^2].$$

We take the space  $\beta \perp 1 \perp 1$  over  $\text{Spec } \mathbb{R}[x, y][1/1+x^2]$  and the space  $\alpha \perp \alpha$  over  $\text{Spec } \mathbb{R}[x, y][1/2+x^2]$  and some patching isometry  $\phi: \alpha \perp \alpha \xrightarrow{\sim} \beta \perp 1 \perp 1$  over  $\text{Spec } \mathbb{R}[x, y][1/(1+x^2)(2+x^2)]$  (note that both quadratic spaces are equivalent to the identity over this intersection) to get a quadratic space  $\gamma$  of rank 6 over  $\text{Spec } \mathbb{R}[x, y]$ .

We show that  $\gamma$  is indecomposable. Suppose that  $\gamma$  represents a unit of  $\mathbb{R}[x, y]$ . Since  $\gamma \xrightarrow{\sim} \alpha \perp \alpha$  over  $\mathbb{R}[x]_{(1+x^2)}[y]$ , it follows that  $\alpha \perp \alpha$  represents a unit of  $\mathbb{R}[x]_{(1+x^2)}[y]$  and since  $\alpha \perp \alpha$  is anisotropic, by Lemma 3.1,  $\alpha$  represents a unit of  $\mathbb{R}[x]_{(1+x^2)}[y]$  contradicting the indecomposability of  $\alpha$  over  $\mathbb{R}[x]_{(1+x^2)}[y]$ . Since by (3.2) any quadratic space of rank  $\leq 2$  over  $\mathbb{R}[x, y]$  is extended from  $\mathbb{R}$  and hence represents units, we assume now that  $\gamma = \gamma_1 \perp \gamma_2$ , where  $\gamma_1$  and  $\gamma_2$  are indecomposable rank 3 spaces. Over  $\mathbb{R}[x]_{(2+x^2)}[y]$ , we have  $\gamma_1 \perp \gamma_2 \xrightarrow{\sim} \beta \perp 1 \perp 1$ , so that if  $\gamma_1(v) + \gamma_2(w) = 1$ , we have by Lemma 3.1 that  $\gamma_1(v)$  or  $\gamma_2(w)$  is a unit. Suppose that  $\gamma_1(v)$  is a unit. Then  $\gamma_1 \xrightarrow{\sim} \langle \gamma_1(v) \rangle \perp \gamma'_1$  and the orthogonal complement of  $\gamma_1(v) + \gamma_2(w)$  in  $\gamma_1 \perp \gamma_2$  is  $\gamma'_1 \perp \gamma'_2$ , where  $\gamma'_2$  is the orthogonal complement of  $\gamma_1(v) + \gamma_2(w)$  in  $\langle \gamma_1(v) \rangle \perp \gamma_2$ . We therefore have  $\gamma'_1 \perp \gamma'_2 \xrightarrow{\sim} \beta \perp 1$ . Repeating the arguments over again, we get that  $\beta$  is decomposable over  $\mathbb{R}[x]_{(2+x^2)}[y]$ , which is a contradiction.

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