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Positive-definite quadratic bundles over the plane

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Introduction

Indecomposable, positive-definite quadratic spaces of ranks 3 and 4 over $\mathbb{R}[x, y]$ have been constructed in [5] and [13]. A natural question to ask is whether there exist indecomposable quadratic spaces of rank >4 over $\mathbb{R}[x, y]$ and whether the theorem of Krull–Schmidt holds for orthogonal decompositions of positive-definite quadratic spaces over $\mathbb{R}[x, y]$. (cf [9], p. 204.)

In §1 of this paper we prove a Krull–Schmidt theorem for orthogonal sums of positive-definite quadratic spaces over $\mathbb{R}[x, y]$. In view of [8], Thm. 2.1, it is enough to prove a similar theorem for positive-definite quadratic bundles over $\mathbb{P}^2_{\mathbb{R}}$. More generally, we prove that if X is a projective scheme over $\mathbb{R}$ and $X_{\mathbb{C}}$ the complexification of X, then the theorem of Krull–Schmidt holds for positive-definite σ-hermitian (resp. quadratic) bundles over $X_{\mathbb{C}}$ (resp. X). We also deduce that Witt-cancellation holds for positive-definite quadratic spaces over $\mathbb{R}[x, y]$. In §2, we exhibit a class of vector-bundles of rank 3 and 4 over $\mathbb{P}^2_{\mathbb{C}}$, associated to a pair of projective ideals of $\mathbb{H}[x, y]$, and show, using results of §1, that these bundles are stable. (The examples of rank 4 bundles over $\mathbb{P}^2_{\mathbb{C}}$ constructed here are interesting, particularly in view of the fact that in general it is not easy to decide the stability of bundles of rank >3.) In §3, we construct an example of a rank 6, indecomposable quadratic space over $\mathbb{R}[x, y]$. The idea of the construction is to patch certain rank 3 and 4 quadratic spaces over $\mathbb{R}[x, y]$.

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§1. Krull–Schmidt theorem for positive-definite bundles over projective schemes

Let X be a projective scheme over $\mathbb{R}$ and let $X_{\mathbb{C}}$ denote the complexification Spec $\mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} X$ of X. Let σ be the involution on $X_{\mathbb{C}}$ induced by the complex conjugation on $\mathbb{C}$ and π the projection of $X_{\mathbb{C}}$ onto X. For any vector bundle $\mathcal{F}_0$ over X we have a natural isomorphism $\rho: \pi^*\mathcal{F}_0 \to \sigma^*\pi^*\mathcal{F}_0$, since $\pi \circ \sigma = \pi$. For any vector bundle $\mathcal{F}$ over $X_{\mathbb{C}}$ we denote by $\mathcal{F}'$ the dual bundle and by $\mathcal{F}^*$ the pull-back $\sigma^*\mathcal{F}'$ of $\mathcal{F}'$ through σ. We define a natural isomorphism (cfr. [11]) $\tau:(\sigma^*\mathcal{F})' \to \mathcal{F}^*$ by

$$(\sigma^*\mathcal{F})' = \mathcal{H}om\,(\sigma^*\mathcal{F}, \pi^*\mathcal{O}_{X_{\mathbb{C}}}) \xrightarrow{\mathcal{H}om\,(\sigma^*\mathcal{F},\,\rho)} \mathcal{H}om\,(\sigma^*\mathcal{F}, \sigma^*\pi^*\mathcal{O}_{X_{\mathbb{C}}}) = \sigma^*\mathcal{F}',$$

In [11] a σ-hermitian structure over $\mathcal{F}$ was defined as an isomorphism $\phi:\mathcal{F} \to \sigma^*\mathcal{F}'$ such that the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[rr] \arrow[dr, "(\sigma^*\phi)'"'] & & \sigma^*\mathcal{F}' \\ & (\sigma^*\mathcal{F})' \arrow[ur, "\tau"'] & \end{tikzcd}$$

is commutative. It is convenient to give an equivalent definition, using the terminology of [15]. Let $\mathfrak{M}$ be the category of vector bundles over $X_{\mathbb{C}}$. Associating to every $\mathcal{F}$ the bundle $\mathcal{F}^*$ we get a functor $^*:\mathfrak{M} \to \mathfrak{M}$. Let, for any $\mathcal{F}$, $i_{\mathcal{F}}:\mathcal{F} \to \mathcal{F}^{**}$ be the isomorphism defined by

$$\mathcal{F}^{**} = \sigma^*(\mathcal{F}^*)' \xrightarrow[\tau_{\mathcal{F}^*}^{-1}]{} (\sigma^*\mathcal{F}^*)' \longrightarrow (\mathcal{F}')' \longrightarrow \mathcal{F}.$$

It is easily checked that i is a natural transformation $id \xrightarrow{\sim} {}^{**}$ satisfying $i_{\mathcal{F}}^* i_{\mathcal{F}^*} = id_{\mathcal{F}^*}$. Hence * is a duality functor in the sense of [15]. We identify each bundle $\mathcal{F}$ with $\mathcal{F}^{**}$ and each morphism ϕ of bundles with ϕ^{**}. For $\varepsilon = \pm 1$, we define an ε-hermitian structure on $\mathcal{F}$ as an isomorphism $\phi:\mathcal{F} \xrightarrow{\sim} \mathcal{F}^*$ such that $\phi^* = \varepsilon\phi$. A 1-hermitian structure on $\mathcal{F}$ turns out to be the same as a σ-hermitian structure in the sense defined above and in [11] or [8]. If x is a real closed point of $X_{\mathbb{C}}$, i.e. a closed point such that $\sigma(x) = x$, the fibre $\mathcal{F}_x$ at x of a σ-hermitian bundle $\mathcal{F}$ carries a non-degenerate hermitian form. We say that $\mathcal{F}$ is positive definite if the fibre at every real closed point is positive definite. Since the signature of a hermitian form is locally constant, if $X_{\mathbb{R}}$ is connected, $\mathcal{F}$ is positive definite if and only if the induced form on the fibre of some real closed point of $X_{\mathbb{C}}$ is positive definite.

We assume, from now on, that X has at least one real closed point.

For any bundle $\mathcal{F}$ we denote by $H(\mathcal{F})$ the hyperbolic bundle associated to $\mathcal{F}$. This is the bundle $\mathcal{F} \oplus \mathcal{F}^*$ with the hermitian structure defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

LEMMA 1.1. *Let $\mathcal{N}$ be an indecomposable vector bundle over $X_{\mathbb{C}}$ such that $\mathcal{N} \cong \mathcal{N}^*$. Then $\mathcal{N}$ carries a σ-hermitian structure.*

Proof. By Proposition 2.5 of [15], $\mathcal{N}$ carries a (1)- or a (−1)-hermitian form. If $\phi:\mathcal{N}\to\mathcal{N}^*$ is (−1)-hermitian, $i\phi$ is hermitian.

THEOREM 1.2. *Let $(\mathscr{E},\phi)$ be a positive-definite σ-hermitian bundle over $X_{\mathbb{C}}$. Then, there is a unique orthogonal decomposition*

$$(\mathscr{E},\phi)\xrightarrow{\sim}\perp_i(\mathscr{E}_i,\phi_i),$$

where $\mathscr{E}_i$ are the isotypical components of the vector bundle $\mathscr{E}$ (i.e. $\mathscr{E}_i\xrightarrow{\sim}\oplus\mathcal{N}_i$, where $\mathcal{N}_i$ are indecomposable and for $i\neq j$, $\mathcal{N}_i\not\xrightarrow{\sim}\mathcal{N}_j$). Each $\mathscr{E}_i$ carries a positive-definite σ-hermitian structure which is unique up to isometry.

Proof. Since X is a projective scheme, the category $\mathfrak{M}$ with the duality functor * defined above satisfies the assumptions (i)–(iii) of [15], page 272. Hence, by Theorem 3.2 of [15],

$$(\mathscr{E},\phi)\cong(\mathscr{E}_1,\phi_1)\perp\cdots\perp(\mathscr{E}_n,\phi_n),$$

where each $\mathscr{E}_i$ is a direct sum of vector bundles isomorphic to a fixed indecomposable $\mathcal{N}_i$ or to its "dual" $\mathcal{N}_i^*$. By Theorem 3.3 of [15], if $\mathcal{N}_i\not\cong\mathcal{N}_i^*$, $\mathscr{E}_i$ contains a hyperbolic orthogonal summand. Since, by assumption, $\mathscr{E}$ is positive definite, this cannot happen and hence each $\mathscr{E}_i$ is isotypical. Since the orthogonal decomposition written above is unique, it suffices to prove the uniqueness for an isotypical vector bundle.

Let $\mathscr{E}$ be an isotypical vector bundle of type $\mathcal{N}$ and let $\mathscr{E}\xrightarrow{\sim}\oplus\mathcal{N}$. We show that if $\mathscr{E}$ carries a positive-definite σ-hermitian structure, then it is unique. Since $\mathcal{N}$ is indecomposable, the ring $E=\operatorname{End}\mathcal{N}$ is a local finite-dimensional $\mathbb{C}$-algebra. Let $\bar{E}=E/\operatorname{rad}E$. Then $\bar{E}$ is a finite-dimensional division algebra over $\mathbb{C}$ and hence $\bar{E}\xrightarrow{\sim}\mathbb{C}$. One reduces the study of σ-hermitian structures on $\mathscr{E}$ to the study of hermitian-forms over a certain vector space $\bar{M}$ over $\bar{E}$ defined as follows (see [15], 2.2, 2.4). Let $\phi:\mathscr{E}\to\mathscr{E}^*$ be a σ-hermitian structure on $\mathscr{E}$. Then, $\oplus_r\mathcal{N}\xrightarrow{\sim}\oplus_r\mathcal{N}^*$ and by the Krull–Schmidt theorem the vector bundles $\mathcal{N}$ and $\mathcal{N}^*$ are isomorphic. Hence, by Lemma 1.1, there exists an isomorphism $\phi_0:\mathcal{N}\xrightarrow{\sim}\mathcal{N}^*$ which defines a σ-hermitian structure on $\mathcal{N}$. In what follows, we shall fix this σ-hermitian structure ϕ_0 on $\mathcal{N}$. The isomorphism ϕ_0 induces an involution τ on $E=\operatorname{End}\mathcal{N}$ defined as

$$\tau f=f^0=\phi_0^{-1}\circ f^*\circ\phi_0.$$

The map $f\to f^0$ satisfies $(fg)^0=g^0f^0$, $(f^0)^0=f$ and for $\lambda\in\mathbb{C}$, $(\lambda f)^0=\bar{\lambda}f^0$, $\bar{\lambda}$ denoting the complex conjugate of λ. This involution passes down to an involution on

$\bar{E} = E/\text{rad}\, E = \mathbb{C}$ which is just the complex conjugation on $\mathbb{C}$. Let $M = \text{Hom}(\mathcal{N}, \mathscr{E})$. Then M is a right E-module and the isomorphism $\phi : \mathscr{E} \to \mathscr{E}^*$ induces an isomorphism $\phi_1 : M \to \text{Hom}_E(M, E)$ which is semilinear with respect to the involution τ. The map ϕ_1 is in fact defined as $\phi_1(f)(g) = \phi_0^{-1} \circ f^* \circ g$ for $f, g \in M$. It is easily verified that ϕ_1 defines a hermitian form on the E-module M with respect to the involution τ on E. Going modulo the radical of E, we obtain on $\bar{M} = M/(\text{rad}\, E)M$ a hermitian form over $\mathbb{C}$.

Two σ-hermitian structures on $\mathscr{E}$ are isometric if and only if the corresponding hermitian forms on $\bar{M}$ are isometric ([15], 2.2). If the form on $\mathscr{E}$ is positive-definite, then the form on $\bar{M}$ is either positive or negative-definite. In fact, if $\bar{M}$ represents zero, then $\bar{M}$ contains a hyperbolic summand and so does $\mathscr{E}$ by [15], Prop. 2.4. If ϕ and ϕ' are two positive definite forms on $\mathscr{E}$, the corresponding forms on $\bar{M}$ are either both positive-definite or both negative-definite: otherwise the form corresponding to $\phi \perp \phi'$ on $\mathscr{E} \perp \mathscr{E}$ would be isotropic. Since, up to isometry, there is a unique positive or negative-definite hermitian form on $\bar{M}$, it follows that there is a unique positive definite σ-hermitian structure over $\mathscr{E}$. This proves Theorem 1.2.

COROLLARY 1.3. *A vector bundle over $X_{\mathbb{C}}$ carries at the most one positive-definite σ-hermitian structure.*

COROLLARY 1.4 (Krull–Schmidt theorem). *Any σ-hermitian positive-definite bundle $(\mathscr{E}, \phi)$ over $X_{\mathbb{C}}$ has a decomposition*

$$(\mathscr{E}, \phi) = \perp (\mathcal{N}_i, \nu_i)$$

into indecomposable σ-hermitian bundles. The summands $(\mathcal{N}_i, \nu_i)$ are unique up to isometries and permutations.

COROLLARY 1.5. *The Krull-Schmidt theorem holds for positive-definite σ-hermitian spaces over $\mathbb{C}[x, y]$.*

Proof. By (3.1) of [8] any positive-definite σ-hermitian space over $\mathbb{C}[x, y]$ has, up to isometry, a unique extension to $\mathbb{P}^2_{\mathbb{C}}$. Hence the assertion follows from 1.4.

The following theorem and corollaries give the corresponding results for positive-definite quadratic bundles.

THEOREM 1.6. *Let $(\mathscr{E}, \phi)$ be a positive-definite quadratic bundle over X. Then, there is a unique orthogonal decomposition*

$$(\mathscr{E}, \phi) = \perp_i (\mathscr{E}_i, \phi_i)$$

where $\mathscr{E}_i$ are the isotypical components of the vector bundle $\mathscr{E}$. The components $\mathscr{E}_i$ carry a positive-definite quadratic structure, unique up to isometry.

A proof on the same lines as of Theorem 1.2 can be given. Let now $\mathfrak{M}$ be the category of real vector bundles over X and, for any such bundle $\mathscr{E}$ let $\mathscr{E}^* = \mathscr{E}' = \mathscr{H}om(\mathscr{E}, \mathcal{O}_X)$ be the dual of $\mathscr{E}$. By Theorem 3.2 of [15] one reduces immediately to the case of an isotypical bundle $\mathscr{E} \xrightarrow{\sim} \oplus \mathscr{N}$, $\mathscr{N}$ indecomposable. Since $\mathscr{E} \xrightarrow{\sim} \mathscr{E}^*$, we have $\mathscr{N} \xrightarrow{\sim} \mathscr{N}^*$ and since $\operatorname{End} \mathscr{N}$ is local, $\mathscr{N}$ carries either a quadratic or a symplectic structure $\phi_0 : \mathscr{N} \xrightarrow{\sim} \mathscr{N}^*$. Then ϕ_0 gives rise to an involution τ of $E = \operatorname{End} \mathscr{N}$, which passes down to an involution of $\bar{E} = E/\operatorname{rad} E$. It is clear that $\bar{E} \xrightarrow{\sim} \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. If $\bar{E} = \mathbb{R}$, the involution is trivial. If $\bar{E} = \mathbb{C}$, the involution must be complex conjugation. And if $\bar{E} \xrightarrow{\sim} \mathbb{H}$, the involution on $\mathbb{H}$ is either trivial or is a conjugate of the canonical involution. The isometry classes of quadratic structures on $\mathscr{E}$ correspond to isometry classes of positive-definite or negative-definite forms on $\bar{M} = M/(\operatorname{rad} E)M$, where $M = \operatorname{Hom}(\mathscr{N}, \mathscr{E})$. The existence of orthogonal bases for hermitian forms shows that there is unique positive- or negative-definite τ-hermitian form on $\bar{M}$. It follows that there is a unique positive-definite quadratic structure over $\mathscr{E}$.

COROLLARY 1.7. *A vector bundle over X carries at the most one positive-definite quadratic structure.*

COROLLARY 1.8. *The Krull–Schmidt theorem holds for positive-definite quadratic bundles over X.*

COROLLARY 1.9. *The Krull–Schmidt theorem holds for positive-definite quadratic spaces over $\mathbb{R}[x, y]$.*

§2. Some stable bundles of rank 3 and 4 associated to projective ideals of $\mathbb{H}[x, y]$

We recall that a bundle $\mathscr{E}$ over $\mathbb{P}^r_{\mathbb{C}}$ is said to be *stable* if, for every coherent subsheaf $\mathscr{F} \neq 0$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is torsionfree we have $c_1(\mathscr{F})/\operatorname{rank} \mathscr{F} < c_1(\mathscr{E})/\operatorname{rank} \mathscr{E}$. In [8] to each non-free projective ideal P of $\mathbb{H}[x, y]$ was associated a rank 2 stable bundle $\mathscr{E}(P)$ with a positive-definite σ-hermitian structure. We recall the construction of these bundles, which in [8] were called $\mathfrak{P}$-bundles. Let $\phi : \mathbb{C} \otimes \mathbb{H} \to M_2(\mathbb{C})$ be the isomorphism given by

$$\phi(s \otimes (u + vj)) = s\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} u, v \in \mathbb{C}.$$

Let $H = \mathbb{H}[x, y]$ and $C = \mathbb{C}[x, y]$. For any projective ideal P of H, $C \otimes P$ is an $M_2(C)$-module via ϕ. Hence, there is a ϕ-semilinear isomorphism $\Psi_P : C \otimes P \xrightarrow{\sim} M_2(C)$. We shall call such a map a *splitting* of P. By Galois cohomology, we associate to the splitting Ψ_P the cocycle

$$\alpha_P = \sigma\Psi_P(\sigma \otimes 1)\Psi_P^{-1}(1) \in GL_2(C)$$

where σ is the complex conjugation on $\mathbb{C}$ and the transported action $\phi(\sigma \otimes 1)\phi^{-1}$ on $M_2(C)$. The map Ψ_P can be chosen such that α_P is positive-definite hermitian of determinant one. Such a splitting is called a *normalized splitting*. Hence, α_P defines a σ-hermitian structure on $\mathbb{A}^2_{\mathbb{C}}$. This structure can be uniquely extended to $\mathbb{P}^2_{\mathbb{C}}$ ([8]) and the extension is the complex bundle $\mathscr{E}(P)$. Notice that by (1.2) $\mathscr{E}(P)$ carries a unique positive-definite σ-hermitian structure. Let now P and Q be two projective ideals in H. The reduced norm Nr introduced in [6] defines a quadratic form on the $\mathbb{R}[x, y]$-module of rank 4 $\mathrm{Hom}_H(P, Q)$. If $\Psi_P : \mathbb{C} \otimes P \simeq M_2(C)$ and $\Psi_Q : \mathbb{C} \otimes Q \xrightarrow{\sim} M_2(C)$ are normalized splittings of P and Q, then, for any $f \in \mathrm{Hom}_H(P, Q)$, $\mathrm{Nr}(f) = \det \Psi_Q(1 \otimes f)\Psi_P^{-1}(1)$. This quadratic space is indecomposable if P and Q are non-free and not isomorphic. If $P \simeq Q$ and P is non-free, then this space decomposes as $\langle 1 \rangle \perp \bar{q}$, where $\bar{q}$ is the orthogonal complement of the submodule $\mathbb{R}[x, y]$ of $\mathrm{End}_H(P)$ for the reduced norm on the algebra $\mathrm{End}_H(P)$. It is shown in [6] that $\bar{q}$ is indecomposable. These indecomposable quadratic spaces of ranks 3 and 4 extend uniquely to indecomposable quadratic bundles over $\mathbb{P}^2_{\mathbb{R}}$, denoted respectively by $\mathscr{F}(P, Q)$ and $\mathscr{F}(P)$. Let $\pi : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{R}}$ be the projection and let $\pi^*\mathscr{F}(P, Q) = \mathscr{G}(P, Q)$ and $\pi^*\mathscr{F}(P) = \mathscr{G}(P)$. We shall show that these bundles are stable.

THEOREM 2.1. *The bundle $\mathscr{G}(P, Q)$ is isomorphic to $\mathscr{E}(P) \otimes \mathscr{E}(Q)$.*

COROLLARY 2.2. *We have* $c_2(\mathscr{G}(P, Q) = 2(c_2(\mathscr{E}(P) + c_2(\mathscr{E}(Q))$ *and* $c_2(\mathscr{G}(P)) = 4c_2(\mathscr{E}(P))$.

Proof. For 2-bundles $\mathscr{E}$ and $\mathscr{F}$ on $\mathbb{P}^2_{\mathbb{C}}$, if $c_1(\mathscr{E}) = c_1(\mathscr{F}) = 0$, then $c_2(\mathscr{E} \otimes \mathscr{F})$ is given by $2(c_2(\mathscr{E}) + c_2(\mathscr{F}))$.

Theorem (2.1) is a consequence of the following results. The first one is implicitly contained in [7], (1.12).

LEMMA 2.3. *Let P be a projective ideal of H, Ψ_P a normalized splitting of P and $\alpha_P \in GL_2(C)$ the corresponding cocycle. Then there is a basis e_1, e_2 of P as a*

C-module such that the matrix of the σ-hermitian form a_P on P defined by

$$a_P(e_i, e_i) = (\Psi_P(e_i), \Psi_P(e_i)) \qquad i = 1, 2$$
$$a_P(e_1, e_2) = (\Psi_P(e_1), \Psi_P(e_2)) - i(\Psi_P(e_1), \Psi_P(ie_2))$$

where, for $u, v \in M_2(C)$, $(u, v) = \frac{1}{2}(\det(u+v) - \det u - \det v)$, is α_P.

Let $\alpha_P = \alpha + i\beta$ with $\alpha, \beta \in M_2(\mathbb{R}[x, y])$. Then the symmetric matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ represents the reduced norm on P with respect to the basis $e_1, e_2, e_3 = ie_1, e_4 = ie_2$ of P over $\mathbb{R}[x, y]$.

The next lemma is an immediate consequence of (2.3) and of the definition of the reduced norm on $\mathrm{Hom}_H(P, Q)$ by means of the splittings Ψ_P and Ψ_Q.

LEMMA 2.4. *Let $f \in \mathrm{Hom}_H(P, Q)$ and a_P, a_Q the hermitian forms given in* (2.1). *Then, for any $u, v \in P$*

$$a_Q(f(u), f(v)) = \mathrm{Nr}(f) a_P(u, v).$$

The module $P' = \mathrm{Hom}_C(P, C)$ is a projective right H-module (with the action $(f\lambda)(x) = f(\lambda x)$, $\lambda \in H$). We now compute its cocycle.

LEMMA 2.5. *Let Ψ_P be a splitting of P with cocycle α_P. Then, there is a splitting $\Psi_{P'}$ of P' with cocycle $\alpha_{P'} = \alpha_P^{-1}$.*

Proof. Let $T: M_2(C) \xrightarrow{\sim} \mathrm{Hom}_C(M_2(C), C)$ be the isomorphism given by the trace, i.e. $T_a(b) = \mathrm{Tr}(ab)$, $a, b \in M_2(C)$. Let $P^\wedge = \mathrm{Hom}_{\mathbb{R}[x,y]}(P, \mathbb{R}[x, y])$. Then the map $\Psi_{P^\wedge} = T^{-1}(\Psi_P)^\wedge$ (where $^\wedge$ means dualization with respect to $\mathbb{R}[x, y]$) is a splitting of $P^\wedge$ and one computes that the corresponding cocycle is α_P^{-1}. Let now $t: P' \xrightarrow{\sim} P^\wedge$ be the isomorphism (of H-modules) induced by the trace $\mathbb{C} \to \mathbb{R}$. Then the map $\Psi_{P'} = \Psi_{P^\wedge} \circ (1 \otimes t)$ is a splitting of P' such that $\alpha_{P'} = \alpha_{P^\wedge} = \alpha_P^{-1}$.

Let now $a_{P'}$ be the hermitian structure on P' given by

$$a_{P'}(e_i', e_j') = \tfrac{1}{2}(\alpha_{P'})_{j,i} = \tfrac{1}{2}(\alpha_P^{-1})_{j,i},$$

where e_i', $i = 1, 2$ is the dual basis of the basis e_i, $i = 1, 2$ given in (2.3). Let S be the σ-hermitian space obtained by extending the reduced norm Nr on $\mathrm{Hom}_H(P, Q)$ to $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Hom}_H(P, Q)$, i.e. $S(\lambda \otimes f) = \lambda\bar{\lambda}\, \mathrm{Nr}(f)$.

LEMMA 2.6. *The map* $\rho:\mathbb{C}\otimes \mathrm{Hom}_H(P,Q)\xrightarrow{\sim}\mathrm{Hom}_C(P,Q)\xrightarrow{\sim}P'\otimes_C Q$ *where the first map is the multiplication and the second is the canonical map, is an isomorphism of* σ*-hermitian spaces* $\rho:S\xrightarrow{\sim}a_{P'}\otimes a_Q$.

Proof. For any basis $\{e_i\}$ of P, ρ is given by $\rho(\lambda\otimes f)=\sum_i e_i^*\otimes f(\lambda e_i)$. Choosing the basis given in (2.3), we have, using (2.4),

$$(a_{P'}\otimes a_Q)\Big(\sum_i e'_i\otimes f(\lambda e_i)\Big)=\sum_{i,j} a_{P'}(e'_i,e'_j)a_Q(f(\lambda e_i),f(\lambda e_j))$$

$$=\mathrm{Nr}\,(f)\lambda\bar{\lambda}\sum_{i,j} a_{P'}(e'_i,e'_j)a_Q(e_i,e_j)=\mathrm{Nr}\,(f)\lambda\bar{\lambda}.$$

This shows that ρ is an isometry.

Theorem (2.1) now follows from (2.6) noting that the extension of a positive definite σ-hermitian form from $\mathbb{A}^2_{\mathbb{C}}$ to $\mathbb{P}^2_{\mathbb{C}}$ is unique and that $\mathscr{E}(P^*)\cong\mathscr{E}(P)$.

To show that the bundles $\mathscr{G}(P,Q)$ and $\mathscr{G}(P)$ are stable, we begin with

LEMMA 2.7. *Let* K *be a field of characteristic* $\neq 2$ *and let* $(\mathscr{E},\phi)$ *be a quadratic bundle of rank* 2 *over* $\mathbb{P}^r_K$. *If* $(\mathscr{E},\phi)$ *is anisotropic,* $(\mathscr{E},\phi)$ *is extended from* K. *If* $(\mathscr{E},\phi)$ *is isotropic, then* $(\mathscr{E},\phi)\xrightarrow{\sim}H(\mathscr{O}(n))$, *a hyperbolic space.*

Proof. The first part of the lemma is proved in ([8], 2.4). If $(\mathscr{E},\phi)$ is isotropic, then restricted to each affine piece $D(x_i)$, the quadratic form can be given by the matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. One then easily checks that $(\mathscr{E},\phi)\xrightarrow{\sim}H(\mathscr{O}(n))$ for some n.

LEMMA 2.8. *Let* K *be a field of characteristic* $\neq 2$ *and let* $\mathscr{E}$ *be an indecomposable anisotropic quadratic bundle over* $\mathbb{P}^r_K$. *Then* $\mathscr{E}$ *has no non-zero section.*

Proof. Evaluating the quadratic form on a global section one gets a global function on $\mathbb{P}^r_K$, hence a constant. This constant must be zero, since the bundle is indecomposable as a quadratic bundle. The section has to be zero since the form is anisotropic.

For any bundle $\mathscr{E}$ over $\mathbb{P}^2_{\mathbb{C}}$ the "type" of $\mathscr{E}$ is the pair of Chern classes $(c_1(\mathscr{E}), c_2(\mathscr{E}))$.

THEOREM 2.9. *The bundles* $\mathscr{G}(P)$ *are stable rank* 3 *bundles of type* $(0,8n)$, *where* $c_2(\mathscr{E}(P))=2n$, $\mathscr{E}(P)$ *denoting the* $\mathfrak{P}$*-bundle associated to a non-free projective ideal* P *of* $\mathbb{H}[x,y]$. *The bundles* $\mathscr{E}(P,Q)$ *are stable rank* 4 *of type* $(0,4(m+n))$ *if* P *and* Q *are non-isomorphic, non-free,* $\mathscr{E}(P)$ *of type* $(0,2n)$ *and* $\mathscr{E}(Q)$ *of type* $(0,2m)$.

Proof. Since $\mathscr{E}(P)$ supports a quadratic form it follows that $c_1(\mathscr{E}(P))=0$. If we consider global sections, we have $H^0(\mathbb{P}^2_{\mathbb{C}}, \mathscr{G}(P)) \xrightarrow{\sim} \mathbb{C}\otimes H^0\ (P^2_{\mathbb{R}}, \mathscr{F}(P))=0$ by Lemma 2.7, since $\mathscr{F}(P)$ supports an anisotropic indecomposable quadratic form. Further, being a quadratic bundle, $\mathscr{G}(P)\xrightarrow{\sim}\mathscr{G}(P)'$. Hence $\mathscr{G}(P)$ is stable by [12], 1.2.6.

We shall now show that $\mathscr{G}(P, Q)$ is stable for P, Q non-isomorphic, non-free. We show that for every subsheaf $\mathscr{F}$ of $\mathscr{G}=\mathscr{G}(P, Q)$ with the quotient $\mathscr{G}(P, Q)/\mathscr{F}$ torsion free, $c_1(\mathscr{F})/\text{rank } \mathscr{F} < c_1(\mathscr{G})/\text{rank } \mathscr{G}$. Since $\mathbb{P}^2_{\mathbb{C}}$ is regular of dimension 2, such a sheaf is locally free. Hence it suffices to show that for any locally free subsheaf $\mathscr{F}$ of $\mathscr{G}$, $c_1(\mathscr{F})<0$. If $\mathscr{F}$ is a line bundle with $c_1(\mathscr{F})=n$, necessarily $n<0$ since, otherwise, $\mathscr{F}$ and hence $\mathscr{G}$ would have a non-zero global section. If $\mathscr{F}$ is of rank 3 we have a surjection $\mathscr{G}'\to\mathscr{F}'\to 0$ whose kernel is a line bundle $\mathscr{L}$. Since $\mathscr{G}'\xrightarrow{\sim}\mathscr{G}$ also does not admit of global sections, it follows that $c_1(\mathscr{L})<0$. Hence $c_1(\mathscr{F}')>0$ so that $c_1(\mathscr{F})=-c_1(\mathscr{F}')<0$. Let $\mathscr{F}$ be of rank 2. The bundle $\mathscr{G}$ restricted to a real line L of $\mathbb{P}^2_{\mathbb{C}}$ is trivial, since $\mathscr{G}$ supports an anisotropic quadratic form ([16], Prop. 5). The restriction of $\mathscr{F}$ to L is isomorphic to $\mathcal{O}(n)\oplus\mathcal{O}(m)$. Since $\mathscr{F}|_L$ is a subsheaf of $\mathscr{G}|_L\xrightarrow{\sim}\oplus\mathcal{O}|_L$, we have $c_1(\mathscr{F})=n+m\leqslant 0$. Suppose that $c_1(\mathscr{F})=0$. Then $\mathscr{F}$ is a rank 2 bundle with no global sections and with $c_1(\mathscr{F})=0$. Hence $\mathscr{F}$ is a stable bundle ([12], 1.2.5). The quadratic structure on $\mathscr{F}(P, Q)$ extends to a positive-definite σ-hermitian structure, denoted by ϕ, on $\mathscr{G}(P, Q)$. The restriction of ϕ to $\mathscr{F}$ induces a map $\mathscr{F}\to\sigma^*\mathscr{F}^*=\mathscr{F}^*$. This map cannot be zero since $\mathscr{F}$ is anisotropic (positive-definite). By the corollary to Lemma 1.2.8 of [12], ϕ is an isomorphism and $(\mathscr{F}, \phi \mid \mathscr{F})$ splits off as an orthogonal summand of $(\mathscr{G}, \phi)$. Then, $\mathscr{G}\xrightarrow{\sim}\mathscr{F}\perp\mathscr{F}_1$. The bundle $\mathscr{G}$ supports a quadratic form, namely the extension of the quadratic structure on $\mathscr{F}(P, Q)$. The bundle $\mathscr{F}$ cannot support a quadratic structure, since, otherwise, $\mathscr{F}\xrightarrow{\sim}H(\mathcal{O}(n))$ by Lemma 2.7 contradicting the stability of $\mathscr{F}$. Thus, by the uniqueness of the quadratic structure on $\mathscr{G}$, it follows that $\mathscr{G}\xrightarrow{\sim}H(\mathscr{F})$ and hence $\mathscr{F}_1\xrightarrow{\sim}\mathscr{F}'\xrightarrow{\sim}\mathscr{F}$. In fact, by the uniqueness of the positive-definite structure (see (1.6)) $(\mathscr{F}_1, \phi\mid\mathscr{F}_1)\xrightarrow{\sim}(\mathscr{F}, \phi\mid\mathscr{F})$ and $(\mathscr{G}, \phi)\xrightarrow{\sim}(\mathscr{F}, \phi\mid\mathscr{F})\perp(\mathscr{F}, \phi\mid\mathscr{F})$. Since $\mathscr{F}$ is a rank 2 stable bundle with a positive-definite σ-hermitian structure, it follows by [8] that $\mathscr{F}$ is a $\mathfrak{P}$-bundle, i.e. $\mathscr{F}\xrightarrow{\sim}\mathscr{E}(P_0)$, where P_0 is some non-free projective ideal of $\mathbb{H}[x, y]$. By [8], Prop. 3.2, $\mathscr{G}\xrightarrow{\sim}\mathscr{E}(P_0)\oplus\mathscr{E}(P_0)\xrightarrow{\sim}\pi^*\pi_*\mathscr{E}(P_0)\simeq\pi^*(\mathscr{F}(\mathbb{H}[x, y]), P_0))$. Since $\text{End}\,(\mathscr{E}(P_0)\oplus\mathscr{E}(P_0))\xrightarrow{\sim}M_2(\mathbb{C})$, the isomorphism classes of vector-bundles on $\mathbb{P}^2_{\mathbb{R}}$ with $\pi^*(\mathscr{E})\xrightarrow{\sim}\mathscr{E}(P_0)\oplus\mathscr{E}(P_0)$ are classified by $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C})$ for an action on $GL_2(\mathbb{C})$ which is the restriction of an action on $M_2(\mathbb{C})$. Since $\pi^*(\mathscr{E})\xrightarrow{\sim}\mathscr{E}(P_0)\oplus\mathscr{E}(P_0)$ is $\mathbb{C}$-linear, $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{C}\subset M_2(\mathbb{C})=\text{End}\,(\mathscr{E}(P_0)+\mathscr{E}(P_0))$ by conjugation, and hence the action on $M_2(\mathbb{C})$ is of the form $\alpha\to u\bar{\alpha}u^{-1}$ for some fixed $u\in GL_2(\mathbb{C})$. It is easily checked that in this case $H^1(\mathbb{Z}/2\mathbb{Z}, GL_2(\mathbb{C}))=0$. Hence, there is a unique descent for $\mathscr{E}(P_0)\oplus\mathscr{E}(P_0)$, i.e. $\mathscr{F}(P, Q)\xrightarrow{\sim}\mathscr{F}(\mathbb{H}[x, y], P_0)$. By the uniqueness of the positive-definite quadratic structure on a vector-bundle over $\mathbb{P}^2_{\mathbb{R}}$ [(1.7)], it follows that $\mathscr{F}(P, Q)$ is isomorphic as a quadratic bundle to $\mathscr{F}(\mathbb{H}[x, y], P_0)$. By restricting these bundles to $\mathbb{A}^2_{\mathbb{R}}$ and using ([6], Thm. 4.6), it follows that P or Q is free, a contradiction. The statement in the theorem regarding the second Chern classes of $\mathscr{G}(P)$ and $\mathscr{G}(P, Q)$ was proved in (2.2).

§3. An example of an indecomposable quadratic space of rank 6 over $\mathbb{R}[x, y]$

LEMMA 3.1. *Let R be a local domain in which 2 is invertible and let q_1, q_2 be quadratic spaces over $R[x]$ such that $q_1 \perp q_2$ is anisotropic. If $q_1(v)+q_2(w)$ is a unit of $R[x]$, then $q_1(v)$ or $q_2(w)$ is a unit of $R[x]$.*

Proof. Let K denote the quotient field of R. Since R is local, if bar denotes reduction modulo x, one has $\bar{q}_1 \xrightarrow{\sim} \langle\lambda_1, \ldots, \lambda_n\rangle$, $\bar{q}_2 \xrightarrow{\sim} \langle\mu_1, \ldots, \mu_m\rangle$, $\lambda_i, \mu_i \in U(R)$. By a theorem of Harder, we have, over $K[x]$, $q_1 \xrightarrow{\sim} \langle\lambda_1, \ldots, \lambda_n\rangle$, $q_2 \xrightarrow{\sim} \langle\mu_1, \ldots, \mu_m\rangle$. Thus, there exist $\theta_i, \phi_i \in K[x]$ such that $q_1(v) = \sum \lambda_i\theta_i^2$ and $q_2(w) = \sum \mu_i\phi_i^2$. Since the forms q_1 and q_2 are anisotropic over $K[x]$, if $q_1(v) = a_0 + a_1x + \cdots + a_rx^r$, then $q_2(w) = b_0 - a_1x - \cdots - a_rx^r$, and $a_r = \sum \lambda_i c_i^2 = -\sum \lambda_i d_i^2$, where c_i, d_i denote the leading coefficients of θ_i and ϕ_i respectively. Then, $\bar{q}_1 \perp \bar{q}_2$ represents zero over K and hence $q_1 \perp q_2$ represents zero over $K[x]$, contradicting the assumption that $q_1 \perp q_2$ is anisotropic.

The next lemma is a generalization of Proposition 1.1 of [13].

LEMMA 3.2. *Let A be a normal ring in which 2 is invertible. Every quadratic space of rank 2 over $A[X_1, \ldots, X_n]$ is extended from A.*

Proof. By [3, 4.15, Remark 4] we may assume that A is local. Let K be the field of fractions of A and M a quadratic space of rank 2 over $A[X]$, X denoting $(X_1, \ldots, X_n)$. If the signed discriminant of M_K is trivial, by [2, Proposition 5.1] M is of the form $H(I)$, where I is a projective ideal of $A[X]$. Since $\operatorname{Pic} A = \operatorname{Pic} A[X]$, M is extended. If the signed discriminant d of M_K is not a square in K, put $L = K[\sqrt{d}]$ and $B = A[\sqrt{d}]$. Then B is the integral closure of A in L hence is a normal semilocal ring. The signed discriminant of M_B is trivial and hence M_B is of the form $H(I)$, where I is a projective ideal of $B[X]$. Since $\operatorname{Pic} B[X] = \operatorname{Pic} B = 0$, $M_B = H(B[X])$. This shows that M is represented by an element of $H^1(\operatorname{Gal}(L/K), O_2(B[X]))$. But $O_2(B[X]) = O_2(B)$ (compare [11], §1) and hence M is in the image of $H^1(\operatorname{Gal}(L/K), O_2(B))$ in $H^1(\operatorname{Gal}(L/K), O_2(B[X]))$. This shows that M is extended from A.

Given a pair f, g of polynomials in $\mathbb{R}[x, y]$, let $\alpha_{f,g}$ (respectively $\beta_{f,g}$) denote the rank 3 (rank 4) quadratic spaces over $\mathbb{R}[x, y]$ defined as the orthogonal complement of the identity in $\mathrm{End}\,(P_{f,g})$ (respectively reduced norm on $P_{f,g}$), where $P_{f,g}$ is the projective ideal of $\mathbb{H}[x, y]$ defined as the kernel of the $\mathbb{H}[x, y]$-linear map $\mathbb{H}[x, y]^2 \to \mathbb{H}[x, y]$ given by $(1, 0) \to f+i$, $(0, 1) \to g+j$ ([8], 1.2). Then $\alpha = \alpha_{x,y}$ is an indecomposable quadratic space over $\mathbb{R}[x, y]$. This space remains indecomposable over $\mathbb{R}[x]_{(1+x^2)}[y]$. In fact, if it decomposes as $\alpha' \perp \alpha''$, then the ranks of α' and α'' are 1 or 2 and hence, by Lemma 3.2, α is extended from $\mathbb{R}[x]_{(1+x^2)}$. Since over $\mathbb{R}[x, 1/1+x^2][y]$, $P_{x,y}$ is free ([7], §5), α is $\cong \langle 1, 1, 1\rangle$ over this ring. Therefore by [3, 4.15, Remark 4], α is extended from $\mathbb{R}$, contrary to the assumption. The form $\beta = \beta_{x\sqrt{2},y}$ is an indecomposable quadratic space over $\mathbb{R}[x, y]$ which is isometric to $\langle 1, 1, 1, 1\rangle$ over $\mathbb{R}[x, 1/2+x^2][y]$. We claim that β remains indecomposable over $\mathbb{R}[x]_{(2+x^2)}[y]$. Suppose that $\beta = \beta' \perp \beta''$ over $\mathbb{R}[x]_{(2+x^2)}[y]$. If rank $\beta' =$ rank $\beta'' = 2$ the same argument as above shows that β is extended from $\mathbb{R}$, which is absurd. If rank $\beta' = 1$, then β represents a unit over $\mathbb{R}[x]_{(2+x^2)}[y]$ and therefore, by [6], (3.19) $P_{x\sqrt{2},y}$ is free over $\mathbb{H}[x]_{(2+x^2)}[y]$ and, in particular, extended from $\mathbb{H}$. Since it is also free over $\mathbb{H}[x, 1/2+x^2][y]$ ([7], §5), by Quillen's theorem $P_{x\sqrt{2},y} = \mathbb{H}[x, y]$, contrary to the assumption.

We define a quadratic space over $\mathbb{R}[x, y]$ of rank 6 as follows: we consider the covering

$$\operatorname{Spec} \mathbb{R}[x, y] = \operatorname{Spec} \mathbb{R}[x, y][1/1+x^2] \cup \operatorname{Spec} \mathbb{R}[x, y][1/2+x^2].$$

We take the space $\beta \perp 1 \perp 1$ over $\operatorname{Spec} \mathbb{R}[x, y][1/1+x^2]$ and the space $\alpha \perp \alpha$ over $\operatorname{Spec} \mathbb{R}[x, y][1/2+x^2]$ and some patching isometry $\phi : \alpha \perp \alpha \xrightarrow{\sim} \beta \perp 1 \perp 1$ over $\operatorname{Spec} \mathbb{R}[x, y][1/(1+x^2)(2+x^2)]$ (note that both quadratic spaces are equivalent to the identity over this intersection) to get a quadratic space γ of rank 6 over $\operatorname{Spec} \mathbb{R}[x, y]$.

We show that γ is indecomposable. Suppose that γ represents a unit of $\mathbb{R}[x, y]$. Since $\gamma \xrightarrow{\sim} \alpha \perp \alpha$ over $\mathbb{R}[x]_{(1+x^2)}[y]$, it follows that $\alpha \perp \alpha$ represents a unit of $\mathbb{R}[x]_{(1+x^2)}[y]$ and since $\alpha \perp \alpha$ is anisotropic, by Lemma 3.1, α represents a unit of $\mathbb{R}[x]_{(1+x^2)}[y]$ contradicting the indecomposability of α over $\mathbb{R}[x]_{(1+x^2)}[y]$. Since by (3.2) any quadratic space of rank ≤ 2 over $\mathbb{R}[x, y]$ is extended from $\mathbb{R}$ and hence represents units, we assume now that $\gamma = \gamma_1 \perp \gamma_2$, where γ_1 and γ_2 are indecomposable rank 3 spaces. Over $\mathbb{R}[x]_{(2+x^2)}[y]$, we have $\gamma_1 \perp \gamma_2 \xrightarrow{\sim} \beta \perp 1 \perp 1$, so that if $\gamma_1(v) + \gamma_2(w) = 1$, we have by Lemma 3.1 that $\gamma_1(v)$ or $\gamma_2(w)$ is a unit. Suppose that $\gamma_1(v)$ is a unit. Then $\gamma_1 \xrightarrow{\sim} \langle \gamma_1(v)\rangle \perp \gamma_1'$ and the orthogonal complement of $\gamma_1(v) + \gamma_2(w)$ in $\gamma_1 \perp \gamma_2$ is $\gamma_1' \perp \gamma_2'$, where γ_2' is the orthogonal complement of $\gamma_1(v) + \gamma_2(w)$ in $\langle \gamma_1(v)\rangle \perp \gamma_2$. We therefore have $\gamma_1' \perp \gamma_2' \xrightarrow{\sim} \beta \perp 1$. Repeating the arguments over again, we get that β is decomposable over $\mathbb{R}[x]_{(2+x^2)}[y]$, which is a contradiction.

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