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## The module of a 2-component link

J. LEVINE

The most prominent algebraic invariant of a link  $L$  in 3-space is the fundamental group  $\Pi$  of the complement. One might try to extract “abelian” invariants from  $\Pi$ . The most obvious candidate:  $\Pi/\Pi'$ , where  $\Pi'$  is the commutator subgroup of  $\Pi$ , is not very useful since, by Alexander duality, this is just the free abelian group with rank the multiplicity (i.e. number of components) of  $L$ . A reasonable next candidate is  $A(L) = \Pi'/\Pi''$ , considered as a module over  $\Pi/\Pi'$ . If  $L$  is oriented, a canonical basis of  $\Pi/\Pi'$  is defined by the meridians of  $L$ . Thus  $A(L)$  has a well-defined structure as module over  $\Lambda_\mu = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_\mu, t_\mu^{-1}]$  ( $\mu =$  multiplicity of  $L$ ). We refer to this as the *module of  $L$* . An alternative description can be given by considering the universal abelian covering  $\tilde{X}$  of the complement  $X$  of  $L$ . The group of covering translations of  $\tilde{X}$  is canonically identified with  $\Pi/\Pi'$  and then  $H_1(\tilde{X}) \approx A(L)$ , as a  $\Pi/\Pi'$ -module.

A closely related invariant of  $L$  is what is sometimes called the *Alexander module of  $L$* ,  $\tilde{A}(L)$ . This is classically defined as the  $\Lambda_\mu$ -module presented by the Jacobian matrix of any presentation of  $\Pi$ . Equivalently  $\tilde{A}(L) \approx H_1(\tilde{X}, \tilde{*})$ , where  $\tilde{*}$  is the inverse image of a base-point  $*$  of  $X$ . Thus we have an exact sequence:  $0 \rightarrow A(L) \rightarrow \tilde{A}(L) \rightarrow M \rightarrow 0$ , where  $M$  is the “augmentation ideal” of  $\Lambda_\mu$  generated by  $t_1 - 1, \dots, t_\mu - 1$ .

A classical collection of invariants considered by Fox [F] is the sequence of elementary ideals, or Fitting invariants,  $\tilde{E}_i(L)$ ,  $i \geq 0$ .  $\tilde{E}_i(L)$  is defined to be the ideal of  $\Lambda_\mu$  generated by the  $(n - i)$ -order minors of a presentation matrix of  $\tilde{A}(L)$  obtained from  $n$  generators. One also considers the greatest common divisor  $\tilde{\Delta}_i(L)$  of  $\tilde{E}_i(L)$  – note that  $\tilde{E}_{i+1}(L) \supseteq \tilde{E}_i(L)$ , and so  $\tilde{\Delta}_{i+1}(L) \mid \tilde{\Delta}_i(L)$ . Furthermore  $\tilde{E}_0(L) = 0 = \tilde{\Delta}_0(L)$ :  $\tilde{\Delta}_1(L)$  is the *Alexander polynomial* of  $L$ . One can define  $E_i(L)$  and  $\Delta_i(L)$  from  $A(L)$  in the same way; then  $\Delta_i(L) = \tilde{\Delta}_{i+1}(L)$ , but  $E_i(L) \neq \tilde{E}_{i+1}(L)$ , in general. If  $\mu = 1$ , then  $E_i(L) = \tilde{E}_{i+1}(L)$ , in fact,  $\tilde{A}(L) = A(L) \oplus \Lambda_1$ , and  $E_0(L)$  is principal and non-zero.

See [C], [F], [H], [H1], [L], [M] for details and more information.

The torsion submodule  $tA$  of  $A = A(L)$  carries a sesqui-linear Hermitian

pairing  $\langle , \rangle$  with values in  $S(\Lambda) = Q(\Lambda)/\Lambda$  ( $Q(\Lambda)$  is the quotient field of  $\Lambda$ ), referred to as the *Blanchfield pairing* (see [B], [L1]). If  $\beta: \bar{A} \rightarrow \text{Hom}_\Lambda(A, S(\Lambda))$  is the adjoint of  $\langle , \rangle$ , ( $\bar{A}$  is the conjugate of  $A$ , defined by changing the action of  $\Lambda$  on  $A$  via the anti-automorphism  $f(x, y) \rightarrow f(x^{-1}, y^{-1})$ ) then Kernel  $\beta$  is referred to as the *null-space* of  $\langle , \rangle$  and cokernel  $\beta$  as the *conull-space*. If  $\mu = 1$ , the pairing is non-singular. See [B], [H] for more information.

The problem of giving a purely algebraic characterization of  $A(L)$ , with the Blanchfield pairing, has been solved in the case  $\mu = 1$  (see [L1]). Bailey [By] has given a characterization of  $A(L)$  in terms of the presentation matrix, when  $\mu = 2$ . The present paper is devoted to a further examination of  $A(L)$  when  $\mu = 2$ ; in particular the identification of some of its algebraic properties and a characterization of certain natural “parts” of  $A(L)$ .

We write  $\Lambda = \Lambda_2 = \mathbb{Z} [x, x^{-1}, y, y^{-1}]$ , and use the notation  $G = \pi/\pi'$ ,  $A = A(L)$ ,  $B = H_2(\tilde{X})$  – note that  $H_i(\tilde{X}) = 0$ , for  $i > 2$ . We begin by presenting the main results.

A.  $r = \text{rank } A = \text{rank } B \leq 1$ .  $B$  is a free  $\Lambda$ -module. If  $l$  is the linking number of the link components, then  $r = 1$  implies  $l = 0$ .  $A \otimes \mathbb{Z} = \mathbb{Z}/l$ .

B. If  $l \neq 0$ , then  $A$  has projective dimension one, (we will say  $A$  is *one-dimensional*), the Blanchfield pairing is non-degenerate (i.e. null-space = 0) and the conull-space  $\approx \Lambda/I_l$ , where  $I_l$  is the ideal generated by

$$(x - 1)(y - 1) \quad \text{and} \quad \frac{(xy)^l - 1}{xy - 1}.$$

C. If  $l = 0$ , we define *longitudinal elements*  $\xi_x, \xi_y \in A$  by lifting into  $\tilde{X}$  “longitudinal” circles parallel to the  $x$  and  $y$  components of  $L$  which link neither component ( $\xi_x, \xi_y$  are, therefore, determined up to multiplication by elements of  $\Pi/\Pi'$ ).  $\xi_x$  (resp  $\xi_y$ ) generates the submodule of elements invariant under  $x$  (resp.  $y$ ). The annihilator ideal of  $\xi_x$  (resp.  $\xi_y$ ) is generated by  $x - 1$  (resp.  $y - 1$ ) and one more element  $\mu(y)$  (resp.  $\lambda(x)$ ). Thus  $\mu(y)$  (resp.  $\lambda(x)$ ) is well-defined up to unit multiple in  $\mathbb{Z} [y, y^{-1}]$  (resp.  $\mathbb{Z}[x, x^{-1}]$ );  $\lambda(x), \mu(y)$  will be called the *longitudinal orders* of  $L$  and depend only on  $A$ .

D. If  $l = 0$  and  $r = 0$ , then  $\lambda(x) = 0 = \mu(y)$  and  $A$  is one-dimensional and contains an element  $\alpha$  such that  $(y - 1) \alpha = \xi_x$  and  $(x - 1) \alpha = \xi_y$ . Thus the annihilator ideal of  $\alpha$  is generated by  $(x - 1)(y - 1)$ . The null-space of  $\langle , \rangle$  is generated by  $\alpha$ , while the conull-space  $\approx \Lambda/(x - 1)(y - 1)$ . In fact,  $A/(\alpha)$  is one-dimensional and the pairing on  $A/(\alpha)$  induced by the Blanchfield pairing is non-singular.

E. If  $r = 1$ , then, we may choose  $\lambda(1) = 1 = \mu(1)$  and, in fact,  $\lambda(x) \mid \Delta(x)$  and  $\mu(y) \mid \Delta(y)$ , where  $\Delta(x), \Delta(y)$  are the Alexander polynomials of the individual components of  $L$  considered as knots.

Furthermore,  $tA \otimes Z = 0$  and  $fA = A/tA$  is isomorphic to an ideal  $I$  of  $\Lambda$ .  $I$  may be uniquely specified by demanding that its greatest common divisor be 1; in that case,  $I + M = \Lambda$ . Another ideal  $J \subseteq I$  can be defined from  $L$ ;  $J$  is generated by  $(x-1)(y-1)I$  and an element  $\sigma(x, y) \in I$ , which is well-defined modulo  $(x-1)(y-1)I$ . Then  $\sigma(x, y) \equiv \lambda(x^{-1}) + \mu(y^{-1}) - 1 \pmod{(x-1)(y-1)}$  and so  $\sigma(x, y)$  defines a slightly sharper invariant of  $L$  than the pair  $(\lambda(x), \mu(y))$ , since  $I/(x-1)(y-1)I \rightarrow \Lambda/(x-1)(y-1)$  has kernel

$$\frac{I \cap (x-1)(y-1)\Lambda}{(x-1)(y-1)I}.$$

F. If  $r = 1$ , the null-space of  $\langle \ , \ \rangle$  is the “pseudo-null” submodule  $P(\bar{A})$  of  $\bar{A}$  (i.e. the set of all elements whose annihilator ideal has greatest common divisor 1 see [Bo]).  $P(A)$  contains the submodule  $P_0$  generated by  $\xi_x, \xi_y$  which coincides with the submodule generated by  $\xi = \xi_x + \xi_y$ , whose annihilator ideal is generated by  $\sigma(x, y)$  and  $(x-1)(y-1)$ .  $P_0$  is the submodule of elements annihilated by  $(x-1)(y-1)$ .  $P(\bar{A})/\bar{P}_0 \approx e^1(I)$  – we use the notation  $e^i(R) = \text{Ext}_\Lambda^i(R, \Lambda)$  for any  $\Lambda$ -module  $R$ . In fact,  $P(\bar{A}) \approx e^1(J)$ . The conull-space  $C$  is isomorphic to the kernel of a homomorphism  $e^2(I) \rightarrow \Lambda/\bar{J}$ , whose cokernel is isomorphic to  $e^2(tA)$ .  $A$  and  $tA$  have projective dimension  $\leq 2$ .

G. *Realization:* Let  $\lambda(x), \mu(y)$  be polynomials and  $I$  an ideal of  $\Lambda$  satisfying: (i)  $\lambda(1) = 1 = \mu(1)$ ; (ii) greatest common divisor of  $I$  is 1 and (iii)  $\lambda(x^{-1}) + \mu(y^{-1}) - 1 \in I$ . Then there exists a 2-component link whose module  $A$  has longitudinal orders  $\lambda(x), \mu(y)$  and  $fA \approx I$ . Note (i), (ii) and (iii) are necessary conditions (see (C) and (E)).

We refer the reader to work of Hillman [H], [H1], [H2] and Sato [S] for related and overlapping results.

§1

We begin by considering the Cartan–LeRay spectral sequence of the covering  $\tilde{X} \rightarrow X$ .  $E_{pq}^2 = H_p(G; H_q(\tilde{X})) = 0$  for  $p > 2$  or  $q > 2$  and so  $E_{pq}^3 = E_{pq}^\infty$ . Straightforward examination obtains an exact sequence:  $H_2(X) \xrightarrow{\phi} H_2(G) \rightarrow A \otimes Z \rightarrow 0$  where  $\phi$  is induced by the map  $X \rightarrow K(G, 1)$  corresponding to the covering  $\tilde{X}$ . Now  $H_2(X) = H_2(G) = Z$  and  $\phi =$  multiplication by  $l$ ; thus  $A \otimes Z$  is infinite cyclic, if  $l = 0$ , and cyclic of order  $l$ , if  $l \neq 0$ . Now a standard Nakayama lemma argument allows us to construct  $\Delta \in \Lambda$  such that  $\Delta A = 0$  and  $\Delta(1, 1) = l^k$ , for some integer  $k > 0$ : if  $\{\alpha_i\}$  generate  $A$ , then we may write  $l\alpha_i = \sum \lambda_{ij}\alpha_j$ , where  $\lambda_{ij} \in M$ , and, thus,  $\Delta = \det(l\delta_{ij} - \lambda_{ij})$  annihilates  $A$ . This shows that  $A$  is a torsion module if  $l \neq 0$ .

That rank  $A =$  rank  $B$  follows from consideration of the Euler characteristic:

rank  $B - \text{rank } A = \chi_\Lambda(\tilde{X}) = \chi(X) = 0$ . ( $\chi_\Lambda$  is the Euler characteristic using rank as a  $\Lambda$ -module.) To see that rank  $B \leq 1$ , choose a finite 2-dimensional cellular structure on  $X$  (actually a compact-deformation retract of  $X$ ) and let  $C_*, \tilde{C}_*$  denote the corresponding chain complexes of  $X$  and  $\tilde{X}$ . If  $D_{ij}$  and  $d_{ij}$  are matrix representatives, with respect to the cell basis, of the boundary maps  $C_2(\tilde{X}) \rightarrow C_1(\tilde{X})$  and  $C_2(X) \rightarrow C_1(X)$ , then  $d_{ij} = D_{ij}(1, 1)$ . Now rank  $B = \text{null}_\Lambda(D_{ij}) \leq \text{null}_\mathbb{Z}(D_{ij}(1, 1)) = \text{rank } H_2(X) = 1$ . Note that this argument shows rank  $H_2(\tilde{X}) \leq \mu - 1$  for a  $\mu$ -component link.

§2

We now define the Blanchfield pairing  $\langle \ , \ \rangle$  on  $tA$  with values in  $S(\Lambda)$ .

Let  $K$  be a triangulation of  $X$  and  $K'$  the dual triangulation – let  $\tilde{K}$  and  $\tilde{K}'$  be the induced triangulations of  $\tilde{X}$ . If  $\alpha, \beta \in tA$ , choose representative cycles  $z$  of  $\alpha$  in  $\tilde{K}$  and  $w$  of  $\beta$  in  $\tilde{K}'$ . If  $\lambda\alpha = 0, \lambda \in \Lambda$ , choose a chain  $c$  in  $\tilde{K}$  such that  $\partial c = \lambda z$ .

Now define  $\langle \alpha, \beta \rangle = \frac{c \cdot w}{\lambda} \text{ mod } \Lambda$ . Standard arguments (see [L1]) show this is well-defined. Furthermore  $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ , using the usual symmetry properties of intersection. An alternative definition of the adjoint  $\beta$  of  $\langle \ , \ \rangle$  is obtained by composing the maps:

$$\begin{aligned} \overline{tH_1(\tilde{X})} \subseteq H_1(\tilde{X}) &\xrightarrow{j^*} \overline{H_1(\tilde{X}, \partial\tilde{X})} \stackrel{D}{\cong} H^2(\tilde{X}; \Lambda) \\ &\xrightarrow{\rho} e^1(H_1(\tilde{X})) \longrightarrow e^1(tH_1(X)) \approx \text{Hom}_\Lambda(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \tag{1}$$

$D$  is the Reidemeister–Milnor duality isomorphism ([M]) and  $\rho$  is a “universal coefficient” homomorphism defined on  $Dj_* \overline{tH_1(\tilde{X})}$  which will be explained below. We are now taking  $X$  to be a compact manifold, the complement of an open tubular neighborhood of  $L$ .

It is not hard to equate this definition with the following reformulation;

$$\begin{aligned} \overline{H_1(\tilde{X})} &\xleftarrow{\partial_*} \overline{H_2(\tilde{X}; S(\Lambda))} \longrightarrow \overline{H_2(\tilde{X}, \partial\tilde{X}; S(\Lambda))} \stackrel{D}{\cong} H^1(\tilde{X}; S(\Lambda)) \\ &\xrightarrow{\bar{\rho}} \text{Hom}_\Lambda(H_1(\tilde{X}), S(\Lambda)) \longleftarrow \text{Hom}_\Lambda(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \tag{2}$$

where  $\bar{\rho}$  is the standard Kronecker map on cohomology, and  $\partial_*$  is the Bockstein from the coefficient sequence  $0 \rightarrow \Lambda \rightarrow Q(\Lambda) \rightarrow S(\Lambda) \rightarrow 0$ . Note that  $\text{Image } \partial_* = \overline{tH_1(\tilde{X})}$  and so any element  $\alpha$  of  $tH_1(\tilde{X})$ , can be pulled back to  $\alpha' \in \overline{H_2(\tilde{X}; S(\Lambda))}$ . Any two pull-backs  $\alpha', \alpha''$  differ by the image of an element of  $H_2(\tilde{X}; Q(\Lambda))$ .

Using naturality of the maps of (2) with respect to the homomorphism  $Q(\Lambda) \rightarrow S(\Lambda)$ , we see that  $\alpha' - \alpha''$  passes to an element of  $\text{Hom}_\Lambda (tH_1(\tilde{X}), S(\Lambda))$  which comes from  $\text{Hom}_\Lambda (tH_1(\tilde{X}), Q(\Lambda)) = 0$ . Thus the composition defined by (2) is well-defined on  $tH_1(\tilde{X})$ . This reformulation is seen to be equivalent to our first definition using the definition of  $D$  via the intersection pairing.

§3

To understand the maps  $\rho, \bar{\rho}$  used in our definitions of the Blanchfield pairing we need a “universal coefficient” consideration of the relation between homology and cohomology. Recall the universal coefficient spectral sequence (see [Mc]): Given a free left chain complex  $C_*$  over a ring  $\Lambda$  and a left module  $N$ , there exists a spectral sequence “converging” to  $H^*(C; N)$ , with  $E^2$ -terms given by  $E^2_{pq} = \text{Ext}^q_\Lambda (H_p(C), N)$ , and differential  $d_r$  in  $E^r$  of degree  $(1 - r, r)$ . There is a filtration

$$H^m(C; N) = J_{m,0} \supseteq J_{m-1,1} \supseteq \dots \supseteq J_{1,m-1} \supseteq J_{0,m}$$

where  $J_{pq}/J_{p-1,q+1} \approx E^\infty_{pq}$ . To define  $\bar{\rho}$ , we simply consider  $H^m(C; N) = J_{m,0} \rightarrow E^\infty_{m,0} \subseteq E^2_{m,0} = \text{Hom}_\Lambda (H_m(C), N)$ . To define  $\rho$  (on  $\text{Ker } \bar{\rho}$ ), we take  $\text{Ker } \bar{\rho} = J_{m-1,1} \rightarrow E^\infty_{m-1,1} \subseteq E^2_{m-1,1} = \text{Ext}^1_\Lambda (H_{m-1}(C), N)$ . Looking back at (1), we see that  $\rho$  is well-defined on elements coming from  $tH_1(\tilde{X})$ , since  $\bar{\rho}$  is obviously zero on any torsion element when  $N = \Lambda$  (and  $\Lambda$  is a domain).

We will consider the universal coefficient spectral sequences for  $C = C^*(\tilde{X})$  and  $C = C^*(\tilde{X}, \partial\tilde{X})$ , with  $N = \Lambda$ . In each case the spectral sequence can be reduced to one or more exact sequences. This reduction is straightforward and we omit the details. The exact sequences obtained are the following:

$$0 \rightarrow H^1(\tilde{X}; \Lambda) \xrightarrow{\bar{\rho}} A^* \rightarrow Z \rightarrow J_{11} \xrightarrow{\rho} e^1(A) \rightarrow 0 \tag{3}$$

$$0 \rightarrow J_{11} \rightarrow H^2(\tilde{X}; \Lambda) \xrightarrow{\bar{\rho}} B^* \rightarrow e^2(A) \rightarrow 0 \tag{4}$$

$$e^3(A) \approx e^1(B) \tag{5}$$

$$0 \rightarrow e^1(A_0) \rightarrow H^2(\tilde{X}, \partial\tilde{X}; \Lambda) \rightarrow B^*_0 \rightarrow e^2(A_0) \rightarrow H^3(\tilde{X}, \partial\tilde{X}) \rightarrow e^1(B_0) \rightarrow e^3(A_0) \rightarrow 0 \tag{6}$$

$$A^*_0 \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda) \tag{7}$$

where we use the notation  $A = H_1(\tilde{X})$ ,  $B = H_2(\tilde{X})$ , (as before)  $A_0 = H_1(\tilde{X}, \partial\tilde{X})$ ,  $B_0 = H_2(\tilde{X}, \partial\tilde{X})$ ,  $e^i = \text{Ext}^i_\Lambda ( , \Lambda)$  and  $*$  =  $e^0 = \text{Hom}_\Lambda ( , \Lambda)$ .

We also note the exact homology sequence:

$$0 \rightarrow B \rightarrow B_0 \rightarrow H_1(\partial\tilde{X}) \rightarrow A \rightarrow A_0 \rightarrow H_0(\partial\tilde{X}) \rightarrow H_0(\tilde{X}) \rightarrow 0. \tag{8}$$

It is easy to see that  $H_*(\partial\tilde{X})$  depends only on the linking number  $l$  and is given as follows:

$$H_0(\partial\tilde{X}) = \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1) \tag{9}$$

$$H_1(\partial\tilde{X}) = \begin{cases} 0 & l \neq 0 \\ \Lambda/(x-1) \oplus \Lambda/(y-1) & l = 0 \end{cases} \tag{10}$$

In (10), when  $l = 0$ , generators are given by the two longitudes, lifted into  $\tilde{X}$ .

**§4**

In the case  $r = 0$ , it follows from (8) that  $\text{rank } A_0 = \text{rank } B_0 = 0$  also. Thus  $A^* = B^* = A_0^* = B_0^* = 0$ . From (3) and (7) we conclude  $B_0 \approx H^1(\tilde{H}; \Lambda) = 0$  and  $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda) = 0$ . From (4) and (5), we conclude  $e^2(A) = 0 = e^3(A)$  and so  $A$  is one-dimensional (note  $e^q = 0$  for  $q > 3$ , since  $\Lambda$  has homological dimension 3).

The Blanchfield pairing  $\beta: \bar{A} \rightarrow \text{Hom}_\Lambda(A, S(\Lambda)) \approx e^1(A)$  can be written as the composition (according to (1)):

$$\bar{A} \rightarrow \bar{A}_0 \approx H^2(X; \Lambda) = J_{11} \rightarrow e^1(A).$$

If  $P$  denotes the null-space of  $\beta$ , and  $C$  the conull-space, we can deduce from (3) and (8) an exact sequence:

$$0 \rightarrow \overline{H_1(\partial\tilde{X})} \rightarrow P \rightarrow Z \rightarrow \bar{K} \rightarrow C \rightarrow 0 \tag{11}$$

where  $K = \text{Kernel } \{H_0(\partial\tilde{X}) \rightarrow H_0(\tilde{X}) \approx Z\}$  - from (8).

In order to analyze the map  $Z \rightarrow \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}$ , we first recall that the edge homomorphism

$$\text{Ext}_\Lambda^q(H_0(C), N) = E_{0q}^2 \twoheadrightarrow E_{0q}^\infty = J_{0q} \subseteq J_{q0} = H^q(C; N)$$

is equivalent to the homomorphism induced by a chain map  $C_* \rightarrow F_*$ , where  $F_*$  is a free resolution of  $H_0(C)$ , which induces the identity map on  $H_0(C) = H_0(F)$ . In case  $\Lambda = \mathbb{Z}\pi$  and  $C_* = C_*(\tilde{X})$ , where  $\tilde{X}$  is a regular  $\pi$ -covering of  $X$ , this coincides

with the homomorphism  $\text{Ext}_\Lambda^q(Z, N) = H^q(\pi; N) \rightarrow H^q(\tilde{X}, N)$  induced by the classifying map  $X \rightarrow B\pi$  of the covering  $\tilde{X} \rightarrow X$ . Now our map  $Z \rightarrow \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}$  is the composition

$$Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\tilde{X}; \Lambda) \approx \overline{H_1(\tilde{X}, \partial\tilde{X})} \xrightarrow{\partial^*} \overline{H_0(\partial\tilde{X})},$$

where  $\varepsilon'$  is the edge homomorphism of the universal coefficient spectral sequence of  $H^*(\tilde{X}; \Lambda)$ , which coincides with the composition  $Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\partial\tilde{X}; \Lambda) \approx \overline{H_0(\partial\tilde{X})}$ , where  $\varepsilon'$  is the edge homomorphism of the universal coefficient spectral sequence of  $H^*(\partial\tilde{X}; \Lambda)$ . Now the map  $\partial X \rightarrow BG$ , which classifies the covering  $\partial\tilde{X} \rightarrow \partial X$ , is an  $l$ -fold covering on each component of  $\partial X$  ( $\partial X$  is the disjoint union of two tori and  $BG$  a single torus). Therefore the induced map  $H^2(G; \Lambda) \rightarrow H^2(\partial\tilde{X}; \Lambda) \approx \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1)$  maps a generator onto  $(\phi_l(y), \phi_l(x))$ , where  $\phi_l(x) = \frac{x^l-1}{x-1}$ . If  $l \neq 0$ , this is a monomorphism, and, since  $H_1(\partial\tilde{X}) = 0$  (see (10)), we conclude  $P = 0$ . Furthermore we now see that  $\text{Cok}\{Z \rightarrow \bar{K} \subseteq H_0(\partial\tilde{X})\}$  has a presentation  $\{\alpha, \beta: (x-1)\alpha = 0 = (y-1)\beta, \phi_l(y)\alpha = \phi_l(x)\beta\}$ , and it, therefore, follows from (11) that  $C$  corresponds to the submodule of elements  $\lambda\alpha + \mu\beta$  ( $\lambda, \mu \in \Lambda$ ) satisfying:

$$\lambda(1, 1) + \mu(1, 1) = 0.$$

It is not hard to see that  $C$  will, therefore, be generated by  $\gamma = \alpha - \beta$ , subject to the relations

$$(x-1)(y-1)\gamma = 0 = (\phi_l(y) + \phi_l(x) - l)\gamma.$$

To complete the proof of (B) it suffices to check that:

$$\phi_l(xy) \equiv \phi_l(x) + \phi_l(y) - l \pmod{(x-1)(y-1)}.$$

But this follows from the easy fact that, for any  $f(x, y) \in \Lambda$ :

$$f(x, y) \equiv f(x, 1) + f(1, y) - f(1, 1) \pmod{(x-1)(y-1)}.$$

### §5

The longitudinal elements  $\xi_x, \xi_y$  of (C) are the generators of the image  $H_1(\partial\tilde{X}) \rightarrow A$  in (8). According to (10)  $(x-1)\xi_x = 0 = (y-1)\xi_y$ . If  $r = 0$ , then  $B_0 = 0$

and, from (10), we see that  $x - 1 (y - 1)$  generates the annihilator of  $\xi_x(\xi_y)$ . Note that our computation of  $Z \rightarrow \bar{K}$ , in the preceding paragraph, shows that it is zero, when  $l = 0$ , and, therefore, (11) contains the short exact sequence:  $0 \rightarrow H_1(\partial\tilde{X}) \rightarrow P \rightarrow Z \rightarrow 0$ .

If we can show that  $P \approx \Lambda/(x - 1)(y - 1)$  (with generator  $\alpha$ ), then it follows that we may choose  $\xi_x = (y - 1)\alpha$ ,  $\xi_y = (x - 1)\alpha$  as longitudinal elements, i.e. they are images, under  $H_1(\partial\tilde{X}) \rightarrow N$ , of generators of the respective summands (see (10)). Since we have already proved  $C \approx \Lambda/(x - 1)(y - 1)$ , the remaining assertions of (D) follows from the Hermitian property of the Blanchfield pairing together with:

**LEMMA.** *Let  $A$  be a one-dimensional torsion  $\Lambda$ -module equipped with a sesquilinear Hermitian pairing  $\langle , \rangle$  with null-space  $K$  and conull-space  $C$ . Then  $K \approx e^1(\bar{C})$  and, if  $A' = A/K$ , the induced pairing on  $A'$  is non-degenerate with conull-space  $\approx e^2(\bar{C})$ . If  $e^3(C) = 0$ , then  $A'$  is one-dimensional.*

*Proof of Lemma:*

Denote the adjoint of  $\langle , \rangle$  by  $\phi: A \rightarrow e^1(\bar{A})$ ; we have, by hypothesis an exact sequence:  $0 \rightarrow K \rightarrow A \xrightarrow{\phi} e^1(\bar{A}) \rightarrow C \rightarrow 0$ . The transpose of  $\phi: A \rightarrow e^1 e^1 A \xrightarrow{e^2 \phi} e^1 \bar{A}$  coincides with  $\phi$  (this is what Hermitian means), where  $A \rightarrow e^1 e^1 A$  is a standard "double dual" map. Since  $A$  is one-dimensional this double dual map is an isomorphism. Now consider the diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 0 & \rightarrow & K & \rightarrow & A & \rightarrow & A' \rightarrow 0 \\
 & & & & \searrow \phi & & \downarrow \\
 & & & & & & e^1 \bar{A} \\
 & & & & & & \downarrow \\
 & & & & & & C \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

From this we derive the diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 0 & \rightarrow & e^1(C) & \rightarrow & e^1 e^1(\bar{A}) & \rightarrow & e^1(A') \rightarrow e^2(C) \rightarrow e^2 e^1(\bar{A}) = 0 \\
 & & & & \parallel & \searrow e^1 \phi & \downarrow \\
 & & & & \bar{A} & & e^1(A)
 \end{array}$$

as well as the isomorphism  $e^i(A') \approx e^{i+1}(C)$ ,  $i \geq 2$ . We immediately see that  $\bar{K} \approx e^1(C)$ , the cokernel of the map  $\bar{A}' \rightarrow e^1(A')$ , induced by  $e^1\phi = \bar{\phi}$ , is  $e^2(C)$ , and that  $A'$  is one-dimensional if  $e^3(C) = 0$ .

§6

From now on we will assume  $r = 1$ , since all the statements for  $r = 0$  have been proved. We first point out that  $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda)$ , by duality, and, by (7), we then conclude  $B \approx A_0^*$ , which is free – over a unique factorization domain,  $R^*$  is free for any module  $R$  of rank  $\leq 1$ .

We examine the longitudinal elements. We can define  $\xi_x, \xi_y \in A$ , when  $l = 0$ , by choosing translates of the components  $K_x, K_y$  of  $L$  into  $X$  which have 0 linking number with their associated components – since  $l = 0$  these translates lift into  $\tilde{X}$  defining  $\xi_x, \xi_y$  up to multiplication by a unit of  $\Lambda$ . Clearly  $\xi_x, \xi_y$  generate  $\text{Image}\{H_1(\partial\tilde{X}) \rightarrow H_1(\tilde{X})\}$ , and we have  $(x - 1)\xi_x = 0 = (y - 1)\xi_y$  (this distinguishes  $\xi_x$  from  $\xi_y$ ). We now show the existence of  $\lambda(x), \mu(y)$ , as in (C).

Consider the infinite cyclic covering  $X_x$  of  $X$  defined by the homomorphism  $\Pi \rightarrow G \rightarrow Z$ , which sends  $x \rightarrow 1$  and  $y \rightarrow 0$ . Thus  $\tilde{X}$  is an infinite cyclic covering of  $X_x$ , and in fact,  $C_*(X_x) \approx C_*(\tilde{X}) / (y - 1)C_*(\tilde{X})$ . We obtain, by tensoring  $C_*(\tilde{X})$  with the short exact sequence:

$$0 \rightarrow \Lambda \xrightarrow{y-1} \Lambda \rightarrow \Lambda / (y - 1) \rightarrow 0$$

the following exact homology sequence:

$$\begin{aligned} 0 \rightarrow H_2(\tilde{X}) \xrightarrow{y-1} H_2(\tilde{X}) \rightarrow H_2(X_x) \rightarrow H_1(\tilde{X}) \xrightarrow{y-1} H_1(\tilde{X}) \\ \rightarrow H_1(X_x) \rightarrow H_0(\tilde{X}) \xrightarrow{y-1} H_0(\tilde{X}) \end{aligned} \tag{12}$$

Now  $X_x$  is closely related to the infinite cyclic covering  $Y_x$  of the complement of  $K_x$ . In fact  $\overline{Y_x - X_x}$  is the union of translates, by powers of  $x$ , of the solid torus formed by lifting a tubular neighborhood of  $K_y$  into  $Y_x$ . Thus  $H_i(Y_x, X_x) \approx \Lambda / (y - 1)$ , if  $i = 2, 3$ , and zero otherwise. By considering the exact sequence of the pair  $(Y_x, X_x)$  and the facts that  $H_i(Y_x) = 0$  if  $i \geq 2$ , we see easily that  $H_2(X_x) \approx \Lambda / (y - 1)$  and obtain an exact sequence:

$$0 \rightarrow \Lambda / (y - 1) \rightarrow H_1(X_x) \rightarrow H_1(Y_x) \rightarrow 0. \tag{13}$$

The sequence (12) can now be put in the simpler form:

$$0 \rightarrow \Lambda/(y-1) \rightarrow \Lambda/(y-1) \rightarrow A \xrightarrow{y-1} A \rightarrow H_1(X_x) \rightarrow Z \rightarrow 0 \tag{12'}$$

since  $H_2(\tilde{X}) = B \approx \Lambda$ . The image of a generator, under the injection  $\Lambda/(y-1) \rightarrow \Lambda/(y-1)$  is represented by a non-zero polynomial  $\lambda(x)$ . Since a generator  $\hat{\xi}_y$  of  $H_2(X_x) \approx \Lambda/(y-1)$  is represented by the boundary torus of a tubular neighborhood of  $K_y$  (lifted into  $X_x$ ), it is straightforward to check, from the definition of the boundary homomorphism  $H_2(X_x) \rightarrow H_1(\tilde{X}) = A$ , that  $\hat{\xi}_y \rightarrow \xi_y \in A$ . It follows immediately that  $\lambda(x)$  and  $y-1$  generate the annihilator ideal of  $\xi_y$ . A similar argument establishes the existence of  $\mu(y)$ .

Note from (12') that  $\xi_y$  generates the submodule of elements invariant under  $y$ . Thus  $\lambda(x)$  is defined, purely algebraically, up to unit multiple, by the property of being a generator, together with  $y-1$ , of the annihilator ideal of this submodule – similarly for  $\mu(y)$ .

We now show  $\lambda(x) \mid \Delta(x)$ , where  $\Delta(x)$  is the Alexander polynomial of  $K_x$  – this will imply  $\lambda(1) = \pm 1$ . Let  $T$  be the torsion sub-module of  $A$ . We first derive from (12') and (13) an exact sequence:

$$0 \rightarrow R \rightarrow T \xrightarrow{y-1} T \rightarrow S \rightarrow 0 \tag{14}$$

where  $R = \Lambda/(\lambda(x), y-1)$ ,  $S \subseteq H_1(Y_x)$  is the image of  $T$  under  $A \rightarrow H_1(X_x) \rightarrow H_1(Y_x)$ . The only point not immediately obvious is:  $\text{Ker}\{T \rightarrow S\} \subseteq (y-1)T$ . Suppose  $\alpha \in T$  and  $\alpha \rightarrow 0$  in  $S$ . If  $\alpha \rightarrow 0$  in  $H_1(X_x)$ , then  $\alpha = (y-1)\beta$  for some  $\beta \in A$ , by exactness of (12'). But then  $\alpha \in T$  implies  $\beta \in T$ . To see  $\alpha \rightarrow 0$  in  $H_1(X_x)$  it suffices by (13) to show  $f(x)\alpha \rightarrow 0$  for any non-zero  $f(x)$ . But, since  $\alpha \in T$ ,  $f(x, y)\alpha = 0$  for some non-zero  $f(x, y)$ . If we write  $f(x, y) = f(x) + (y-1)g(x, y)$ , then  $0 = f(x)\alpha + (y-1)g(x, y)\alpha$ . Since  $(y-1)A \rightarrow 0$  in  $H_1(X_x)$ , so does  $f(x)\alpha$ . If  $f(x) = 0$ , then, by (12'),  $\lambda(x)g(x, y)\alpha = 0$ . But this would be impossible if we had chosen  $f(x, y)$  with the smallest number of  $y-1$  factors.

Now recall that  $\Delta(x) = \Delta(H_1(Y_x))$ , where  $\Delta(A)$ , for any  $\Lambda_x$ -module  $A$  ( $\Lambda_x = z[x, x^{-1}] \approx \Lambda/(y-1)$ ) is the greatest common divisor of the *order ideal* of  $A$  (see [L]). We also recall the following property of  $\Delta(A)$ : if  $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$  is a short exact sequence of  $\Lambda_x$ -modules, then  $\Delta(A) = \Delta(A') \Delta(A'')$  (see [L] for a proof). Thus, for example,  $\Delta(S) \mid \Delta(x)$  and, so, it suffices to prove that  $\lambda(x)$  ( $= \Delta(R)$ ,  $R$  considered as a  $\Lambda_x$ -module) divides  $\Delta(S)$ . Define

$$T_i = \frac{\text{Ker } \phi^{i+1}}{\text{Ker } \phi^i} \quad \text{and} \quad T^i = \frac{\phi^i T}{\phi^{i+1} T},$$

where  $\phi: T \rightarrow T$  is multiplication by  $y - 1$ . These are  $\Lambda_x$ -modules and we have a family of short exact sequences:  $0 \rightarrow T_{i+1} \rightarrow T_i \rightarrow T^i \rightarrow T^{i+1} \rightarrow 0$ , for  $i \geq 0$  (see [L2]). From (14) we see that  $T_0 \approx R$  and  $T^0 \approx S$ . From the above-mentioned multiplicative property of  $\Delta$  we have  $\Delta(T_{i+1})\Delta(T^i) = \Delta(T_i)\Delta(T^{i+1})$  for  $i \geq 0$ . Therefore, we see that  $\Delta(T_{i+1}) \mid \Delta(T^{i+1})$  would imply  $\Delta(T_i) \mid \Delta(T^i)$  - note that these are all non-zero, since  $\Delta(T_{i+1}) \mid \Delta(T_i)$ ,  $\Delta(T^{i+1}) \mid \Delta(T^i)$  and  $\Delta(T_0), \Delta(T^0)$  are non-zero. Thus it suffices to show  $\Delta(T_i) \mid \Delta(T^i)$  for some value of  $i$ . But  $T_i = 0$ , for large enough  $i$ , since  $\{\text{Ker } \phi_i\}$  is an increasing sequence of submodules in a finitely-generated module over a Noetherian ring. This completes the proof.

Of course, by a similar argument, we can show  $\mu(y) \mid \Delta(y)$ .

We can now show that  $P_0$ , the submodule of  $A$  generated by  $\xi_x$  and  $\xi_y$ , is the submodule of elements annihilated by  $(x - 1)(y - 1)$ . Suppose  $(x - 1)(y - 1)\alpha = 0$ ; then  $(y - 1)\alpha = f\xi_x$  for some  $f \in \Lambda$ . So  $\mu(y)(y - 1)\alpha = 0$  which means  $\mu(y)\alpha = g\xi_y$ . Since  $\mu(1) = 1$ , we may write  $\mu(y) = 1 + (y - 1)\mu'(y)$  and so  $\alpha + (y - 1)\mu'(y)\alpha = g\xi_y$  or  $\alpha + \mu'(y)f\xi_x = g\xi_y$ . Thus  $\alpha \in P_0$ .

### §7

We now examine  $fA$  and prove  $fA \otimes_{\Lambda} Z$  is infinite cyclic. (over  $Z$ ) We already know  $A \otimes_{\Lambda} Z$  is infinite cyclic, which implies  $fA \otimes_{\Lambda} Z$  is cyclic. If  $fA \otimes_{\Lambda} Z$  were finite of order  $k > 0$ , then  $fA \otimes_{\Lambda} Z/p = 0$ , for any  $p$  relatively prime to  $k$ . If so, by Nakayama's lemma,  $\Delta \cdot fA = 0$  for some  $\Delta \notin M_p$ , where  $M_p = \ker \{\Lambda \rightarrow Z/p\}$ . But  $fA$  is torsion-free. If we define  $I$  to be the ideal of  $\Lambda$  with greatest common divisor 1 which is isomorphic to  $fA$ , then  $I + M = \Lambda$ . To see this choose  $\lambda \in I$  which generates  $I/MI \approx I \otimes Z \approx Z$  - we will show  $\lambda(1, 1) = \pm 1$ .  $M(I/(\lambda)) = I/(\lambda)$ , which implies, by Nakayama's lemma, that  $\Delta \cdot I/(\lambda) = 0$ , i.e.  $\Delta I \subseteq (\lambda)$ , for some  $\Delta \equiv 1 \pmod M$  - i.e.  $\Delta(1, 1) = \pm 1$ . Since  $I$  has greatest common divisor one,  $\Delta \in (\lambda)$  and so  $\lambda(1, 1) = \pm 1$ . To see that  $tA \otimes_{\Lambda} Z = 0$  (when  $r = 1$ ) consider the short exact sequence  $0 \rightarrow tA \rightarrow A \rightarrow fA \rightarrow 0$  and apply  $\otimes_{\Lambda} Z$  to obtain  $\text{Tor}^1(fA, Z) \rightarrow tA \otimes Z \rightarrow A \otimes Z \rightarrow fA \otimes Z \rightarrow 0$ . Since  $A \otimes Z \approx Z \approx fA \otimes Z$ , it suffices to show  $\text{Tor}^1(fA, Z) = \text{Tor}^1(I, Z) = 0$ . Now  $\text{Tor}^1(I, Z) = \text{Tor}^2(\Lambda/I, Z)$  which can be considered to be the submodule of *invariant* elements of  $\Lambda/I$  - i.e. of elements  $\alpha$  satisfying  $x\alpha = y\alpha = \alpha$ . But  $\lambda\alpha = 0$ , where  $\lambda \in I$  satisfying  $\lambda(1, 1) = 1$  has been found in the preceding paragraph, and so  $0 = (1 + (x - 1)\lambda' + (y - 1)\lambda'')\alpha = \alpha + \lambda'(x - 1)\alpha + \lambda''(y - 1)\alpha = \alpha$ .

By the results of §7, we may break (8) up into two shorter exact sequences (for

$r = 1$ ):

$$0 \rightarrow B \rightarrow B_0 \rightarrow \Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow 0 \quad (15a)$$

$$0 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow A \rightarrow A_0 \rightarrow \Lambda/(x-1)(y-1) \rightarrow 0 \quad (15b)$$

where  $\rho = \lambda(x) + \mu(y) - 1$  (choosing  $\lambda(1) = 1 = \mu(1)$ ). Note that the quotient of  $\Lambda/(x-1) \oplus \Lambda/(y-1)$  by the submodule generated by  $(\mu(y), 0)$  and  $(0, \lambda(x))$  is isomorphic to  $\Lambda/(\rho, (x-1)(y-1))$ , using the generator  $(1, 1)$ , and the kernel of the epimorphism  $\Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow Z$  is isomorphic to  $\Lambda/(x-1)(y-1)$ , using the generator  $(1, -1)$ . Applying  $\text{Hom}(\ , \Lambda)$  to (15a) yields an exact sequence:

$$0 \rightarrow B_0^* \rightarrow B^* \rightarrow \Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow e^1(B_0) \rightarrow e^1(B).$$

Now  $e^1(B) = 0$ , since  $B$  is free. From (3), we conclude that  $\bar{B}_0 \approx H^1(\tilde{X}; \Lambda)$  is free or isomorphic to  $M$  (the ideal in  $\Lambda$  generated by  $(x-1, y-1)$ ), since  $A^*$  is free. But (15a) is possible only if  $\bar{B}_0 \approx M$ . Thus  $e^1(B_0) \approx Z$ . We, therefore, have the exact sequence:

$$0 \rightarrow B_0^* \rightarrow B^* \rightarrow \Lambda/(x-1)(y-1) \rightarrow 0 \quad (16)$$

We now apply  $\text{Hom}(\ , \Lambda)$  to (15b) and obtain exact sequences:

$$0 \rightarrow A_0^* \rightarrow A^* \rightarrow \Lambda/(x-1)(y-1) \rightarrow e^1(A_0) \rightarrow e^1(A) \rightarrow 0 \quad (17a)$$

$$0 \rightarrow e^2(A_0) \rightarrow e^2(A) \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow e^3(A_0) \rightarrow 0. \quad (17b)$$

Note that

$$e^1(\Lambda/(\rho, (x-1)(y-1))) = 0$$

$$e^2(\Lambda/(\rho, (x-1)(y-1))) \approx \Lambda/(\rho, (x-1)(y-1))$$

and  $e^3(A) = 0$  (by (5), since  $B$  is free).

We now examine the homomorphisms  $tA \rightarrow tA_0$  and  $fA \rightarrow fA_0$ , using (15b). Denoting the kernel and cokernel, respectively, by  $K_1, K_2$  and  $C_1, C_2$ , we can apply the snake lemma, using (15b) to obtain an exact sequence:  $0 \rightarrow K_1 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow K_2 \rightarrow C_1 \rightarrow \Lambda/(x-1)(y-1) \rightarrow C_2 \rightarrow 0$ . Since  $K_2$  is torsion-free and  $C_1$  torsion, we see that  $K_2 = 0$ . For some ideal  $S \supseteq (x-1)(y-1)$ , we have

$$C_1 \approx S/(x-1)(y-1), \quad C_2 \approx \Lambda/S$$

and we have:

$$0 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow tA \rightarrow tA_0 \rightarrow S/(x-1)(y-1) \rightarrow 0 \tag{18a}$$

$$0 \rightarrow fA \rightarrow fA_0 \rightarrow \Lambda/S \rightarrow 0. \tag{18b}$$

**§8**

We now deduce some facts from (3)–(6) and duality:

$$\overline{tA} = e^1(A_0). \tag{19}$$

This follows from (6), since  $e^1(A_0)$  is torsion and  $B_0^*$  is free.

$$0 \rightarrow \overline{B}_0 \rightarrow A^* \rightarrow Z \rightarrow \overline{tA}_0 \rightarrow e^1(A) \rightarrow 0. \tag{20}$$

This is just (3), since  $J_{11} \simeq \overline{tA}_0$  from (4).

$$0 \rightarrow \overline{fA}_0 \rightarrow B^* \rightarrow e^2(A) \rightarrow 0. \tag{21}$$

This follows from (4).

$$0 \rightarrow \overline{fA} \rightarrow B_0^* \rightarrow e^2(A_0) \rightarrow 0, e^3(A_0) \simeq Z/k, \quad (\text{some } k > 0). \tag{22}$$

This follows from (6), since  $H^3(\tilde{X}, \partial\tilde{X}) \simeq Z$  and  $e^3(A_0)$  cannot be isomorphic to  $Z$ , since  $e^2(Z) \neq 0$  but  $e^2e^3 = 0$  over  $\Lambda$  (see [Ba]).

We use the map  $\tilde{X} \rightarrow (\tilde{X}, \partial\tilde{X})$  to map (22)  $\rightarrow$  (21). Using (18b), (16) and (17b), we obtain a commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \overline{fA} & \rightarrow & B_0^* & \rightarrow & e^2(A_0) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \overline{fA}_0 & \rightarrow & B^* & \rightarrow & e^2(A) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \Lambda/S & & \Lambda/(x-1)(y-1) & & \Lambda/(\rho, (x-1)(y-1)) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & e^3(A_0) & \\
 & & & & & \downarrow & \\
 & & & & & 0 & 
 \end{array} \tag{23}$$

From the snake lemma we deduce an exact sequence:

$$0 \rightarrow \Lambda/S \rightarrow \Lambda/(x-1)(y-1) \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow e^3(A_0) \rightarrow 0. \tag{24}$$

From this sequence we may deduce:  $e^3(A_0) = 0$  and  $S = (x-1)(y-1)$ . The first of these follows from the fact that any epimorphism  $\Lambda \rightarrow \mathbb{Z}/k$  is of the form  $f(x, y) \rightarrow af(1, 1)$ , where  $a \in \mathbb{Z}$  is relatively prime to  $k$ , and  $\rho(1, 1) = 1$ . To see the second, let  $\alpha \in \Lambda$  represent the image of 1 under  $\Lambda/S \rightarrow \Lambda/(x-1)(y-1)$ . Then  $(\alpha, (x-1)(y-1)) = (\rho, (x-1)(y-1))$  and so  $\alpha(1, 1) = \pm 1$ . But  $\beta \in S$  if and only if  $(x-1)(y-1) \mid \alpha\beta$  and so  $S \subseteq (x-1)(y-1)$ . Since we already know  $S \supseteq (x-1)(y-1)$ , we have  $S = (x-1)(y-1)$ .

Now define  $J$  to be the ideal of  $\Lambda$ , with greatest common divisor 1, isomorphic to  $fA_0$ . We can rewrite (23) as follows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \rightarrow & I & \xrightarrow{\tau'} & \Lambda & \rightarrow & e^2(\bar{A}_0) & \rightarrow 0 \\
 & \downarrow \tau & & \downarrow & & \downarrow & \\
 0 \rightarrow & J & \xrightarrow{\tau''} & \Lambda & \rightarrow & e^2(\bar{A}) & \rightarrow 0 \tag{25} \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \Lambda/(x-1)(y-1) & \rightarrow & \Lambda/(x-1)(y-1) & \rightarrow & \Lambda/(\bar{\rho}, (x-1)(y-1)) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & 
 \end{array}$$

The maps indicated by  $\tau, \tau', \tau''$  and  $\tau_0$  are all multiplication by elements of  $Q(\Lambda)$  – we also use  $\tau, \tau', \tau'', \tau_0$  to denote these elements. Obviously  $\tau_0 \in \Lambda$  and is a unit multiple of  $(x-1)(y-1)$  and, since  $I$  and  $J$  have greatest common divisor one,  $\tau'$  and  $\tau''$  are also in  $\Lambda$ . Now  $e^2(\bar{A}_0)$  and  $e^2(\bar{A})$  are pseudo-null since they are grade  $\geq 2$  (see [Ba]) and grade  $\geq 2$  means pseudo-null (see [R]). Therefore  $\tau'$  and  $\tau''$  must be units of  $\Lambda$ ; so  $\tau \in \Lambda$  and is a unit multiple of  $\tau_0$  or  $(x-1)(y-1)$ .

Now choose an element  $\sigma \in J$  which maps onto a generator of  $\Lambda/(x-1)(y-1)$ . Therefore  $J = (x-1)(y-1)I + (\sigma)$ . From the left-most vertical row of (25) we see that  $f\sigma \in (x-1)(y-1)I$  if and only if  $(x-1)(y-1) \mid f$ . If  $f = (x-1)(y-1)$ , this says  $\sigma \in I$ , and so  $J \subseteq I$ . If  $\sigma'$  is another element such that  $J = (x-1)(y-1)I + (\sigma')$ , then a straight-forward computation shows  $\sigma' = \alpha\sigma \text{ mod } (x-1)(y-1) I$ , where  $\alpha = u \text{ mod } (x-1)(y-1)$  for some unit  $u$  of  $\Lambda$ . Since  $\sigma \in I$ , we have  $\sigma' \equiv u\sigma \text{ mod } (x-1)(y-1) I$  and so  $\sigma$  is well-defined up to unit multiple, mod  $(x-1)(y-1) I$ . Finally, it follows from (25) that  $(\sigma, (x-1)(y-1)) = (\bar{\rho}, (x-1)(y-1))$ , which implies  $\sigma \equiv u'\bar{\rho} \text{ mod } (x-1)(y-1)$  for some unit  $u'$  of  $\Lambda$ .

§9

We now determine the null-space  $N$  and co-null space  $C$  of the Blanchfield pairing. Its adjoint  $\overline{tA} \rightarrow e^1(tA)$ , whose kernel and cokernel are  $N$  and  $C$ , can be described as the composition:

$$\overline{tA} \rightarrow e^1(A_0) \rightarrow e^1(tA_0) \rightarrow e^1(tA) \tag{26}$$

where the first homomorphism is the isomorphism of (19) and the others are induced by inclusion  $tA_0 \subseteq A_0$  and  $A \rightarrow A_0$ . By its Hermitian property this coincides with the composition;

$$\overline{tA} \rightarrow \overline{tA_0} \rightarrow e^1(A) \rightarrow e^1(tA). \tag{27}$$

The middle map comes from (20).

In (26), the last map is also an isomorphism – this follows from (18a), since  $S = (x - 1)(y - 1)$  and  $e^0(\Lambda/(\rho, (x - 1)(y - 1))) = e^1(\Lambda/(\rho, (x - 1)(y - 1))) = 0$ . Thus  $N$  and  $C$  are isomorphic to the kernel and cokernel, respectively, of  $e^1(A_0) \rightarrow e^1(tA_0)$ . From the short exact sequence  $0 \rightarrow tA_0 \rightarrow A_0 \rightarrow fA_0 \rightarrow 0$ , we conclude  $N \approx e^1(fA_0) = e^1(J)$ . Since  $e^1(tA) \approx \text{Hom}_\Lambda(tA, S(\Lambda))$  is pseudo-null free, and  $e^1(J) \approx e^2(\Lambda/J)$  is pseudo-null, it follows that  $N$  is the pseudo-null submodule of  $\overline{tA}$ .

We show that the map  $\overline{tA_0} \rightarrow e^1(A)$  in (27) is an isomorphism. Referring to (20) we have already seen that  $A^* \rightarrow Z$  is non-trivial, since  $B_0 \approx M$  and  $A^*$  is free. It remains to show that  $tA_0$  cannot contain a submodule isomorphic to  $Z/k$ , unless  $k = 0$  or  $1$ . But we have seen  $e^3(A_0) = 0$  and an inclusion  $Z/k \rightarrow A_0$  would induce an epimorphism  $e^3(A_0) \rightarrow e^3(Z/k) \approx Z/k$  (if  $k > 0$ ).

From the short exact sequence  $0 \rightarrow tA \rightarrow A \rightarrow fA \rightarrow 0$  we deduce an exact sequence.

$$0 \rightarrow e^1(I) \rightarrow e^1(A) \rightarrow e^1(tA) \rightarrow e^2(I) \rightarrow e^2(A) \rightarrow e^2(tA) \rightarrow 0 \tag{28}$$

since  $fA \approx I$  and  $e^3(I) \approx e^4(\Lambda/I) = 0$ . From (28) and (18a), we can deduce exact sequences:

$$0 \rightarrow \Lambda/(\bar{\rho}, (x - 1)(y - 1)) \rightarrow N \rightarrow e^1(I) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow C \rightarrow e^2(I) \rightarrow e^2(A) \rightarrow e^2(tA) \rightarrow 0.$$

Recall  $S = (x - 1)(y - 1)$ . Since  $e^2(A) \approx \Lambda/\bar{J}$  from (25), we have completed the proof of (F).

§10

To prove (G) we will use the following special case of a result of Bailey [By]

**THEOREM.** *Let  $(\lambda_{ij})$  be an  $(n \times n)$ -matrix over  $\Lambda$  satisfying (i)  $\lambda_{11} = 0$ ; (ii)  $\lambda_{ij} = \bar{\lambda}_{ji}$  if  $n \geq i, j > 1$  (iii)  $\lambda_{1j} = (x^{-1} - 1)(y^{-1} - 1)\bar{\lambda}_{j1}$  for  $1 \leq j \leq n$  (iv)  $\lambda_{ij}(1, 1) = \pm \delta_{ij}$  if  $n \geq i, j > 1$ . Then there exists a link, with  $l = 0$ , whose module  $A$  has presentation*

$$\left\{ \alpha_1, \dots, \alpha_n : \sum_{j=1}^n \lambda_{ij} \alpha_j = 0, i = 1, \dots, n \right\}.$$

Since a proof this theorem has not appeared in a journal, we present one in the Appendix. We point out that our proof is very different from Bailey's. Also see [N]. ( $(\lambda_{ij})$  is referred to as a *presentation matrix* of  $A$ .) We prove two lemmas.

**LEMMA 1.** *Let  $A$  be a link module with a presentation matrix  $(\lambda_{ij})$  satisfying (i)–(iv). Suppose  $(\sigma_i)$  is an  $(n \times 1)$ -row vector, whose entries are relatively prime, such that  $\sum_{i=1}^n \sigma_i \lambda_{ij} = 0$  for  $j = 1, \dots, n$ . Then  $\sigma_1(s, 1)$  and  $\sigma_1(1, y)$  are the longitudinal orders of  $A$ .*

The next lemma deals with a more general situation.

**LEMMA 2.** *Let  $(\lambda_{ij})$  be an  $(n \times n)$  matrix over a domain  $\Lambda$ , a presentation matrix of a module  $A$  of rank one. Let  $M$  be the  $(n - 1) \times (n - 1)$ -matrix  $(\lambda_{ij}), 2 \leq i, j \leq n$  and suppose  $\Delta = \det M \neq 0$ . Let  $(\mu_{ij}) = \Delta \cdot M^{-1} (2 \leq i, j \leq n)$ , the cofactor matrix of  $M$  and set  $\rho_i = \sum_{j=2}^n \mu_{ij} \lambda_{j1}$ . Then  $fM$  is isomorphic to the ideal of  $\Lambda$  generated by  $(\Delta, \rho_2, \dots, \rho_n)$ .*

*Proof of Lemma 1.* Let  $0 \rightarrow W \rightarrow F_0 \xrightarrow{d} F_1 \rightarrow A \rightarrow 0$  be the resolution defined by  $(\lambda_{ij})$ , i.e.  $F_0$  and  $F_1$  are free modules of rank  $n$  with bases  $\{\alpha_i\}, \{\beta_i\}$  with  $d(\beta_i) = \sum_j \lambda_{ij} \alpha_j$ . Since  $\text{rank } A = 1$  and projective dimension  $A \leq 2$ ,  $W$  is free of rank one. If a generator of  $W \subset F_0$  is  $\sum_i \sigma'_i \alpha_i$ , then  $\sigma'_i = u \sigma_i$ , for some unit  $u$  in  $\Lambda$ . Let  $\Lambda_x = \Lambda/(y - 1)$ : then  $\text{Tor}_1^\Lambda(A, \Lambda_x)$  is the submodule of elements of  $A$  annihilated by  $y - 1$  (using exact sequence  $0 \rightarrow \Lambda \xrightarrow{y-1} \Lambda \rightarrow \Lambda_x \rightarrow 0$ ) which, by (C), is isomorphic to  $\Lambda_x/(\lambda(x))$ . Using the resolution of  $A$  given above  $\text{Tor}_1^\Lambda(A, \Lambda_x)$  is the homology of the chain complex:

$$W \otimes_{\Lambda} \Lambda_x \xrightarrow{d'} F_0 \otimes_{\Lambda} \Lambda_x \xrightarrow{d''} F_1 \otimes_{\Lambda} \Lambda_x$$

where the modules are free over  $\Lambda_x$  and  $d', d''$  are represented by the matrices:  $(\sigma_i(x, 1))$  and  $(\lambda_{ij}(x, 1))$ , respectively. Since  $\lambda_{1j}(x, 1) = 0$ , for all  $j$ , and  $\lambda_{ij}(1, 1) = \pm \delta_{ij}$  for  $i, j \geq 2$ , it follows easily that kernel  $d''$  is the free submodule of  $F_0 \otimes_{\Lambda} \Lambda_x$  generated by  $\beta_1 \otimes 1$ . Thus, since  $\text{Image } d' \subseteq \text{kernel } d''$ ,  $\sigma_i(x, 1) = 0$  for  $i > 1$ , and  $\text{Tor}_1^{\Lambda}(A, \Lambda_x) \approx \Lambda / (\sigma_1(x, 1))$ .

A similar argument for  $\mu(y)$  completes the proof.

*Proof of Lemma 2.* Suppose  $(\rho_{ij}), 1 \leq i, j \leq n$ , is any matrix over  $\Lambda$ ; consider the module  $A'$  presented by the product matrix  $(\rho_{ij})(\lambda_{ij})$ —i.e.  $A' = \{\beta_1, \dots, \beta_n : \sum_{j,s} \rho_{is} \lambda_{sj} \beta_j = 0, i = 1, \dots, n\}$ . If  $\{\alpha_i\}$  are the generators of  $A$ , subject to relations  $\sum_j \lambda_{ij} \alpha_j = 0 (i = 1, \dots, n)$ , then  $\beta_i \rightarrow \alpha_i$  defines an epimorphism  $\phi: A' \rightarrow A$ . The kernel of  $\phi$  is generated by  $\{\gamma_i\}$ , where  $\gamma_i = \sum_j \lambda_{ij} \beta_j (i = 1, \dots, n)$ , and the  $\{\gamma_i\}$  are subject to relations  $\sum_j \rho_{ij} \gamma_j = 0 (i = 1, \dots, n)$ . We apply these observations to the matrix  $(\rho_{ij})$  given by

$$\rho_{ij} = \begin{cases} \mu_{ij} & i, j \geq 2 \\ \delta_{ij} & i = 1 \text{ or } j = 1. \end{cases}$$

The matrix  $(\sigma_{ij}) = (\rho_{ij})(\lambda_{ij})$  is given by

$$\sigma_{ij} = \begin{cases} \lambda_{ij} & i = 1 \\ \rho_i & j = 1, \quad i > 1 \\ \Delta \delta_{ij} & i, j \geq 2. \end{cases}$$

Now  $\det(\rho_{ij}) = \Delta \neq 0$ , which implies, since  $(\rho_{ij})$  is a relation matrix for  $\text{Ker } \phi$ , that  $\text{Ker } \phi$  is a torsion module. Thus  $\phi$  induces an isomorphism  $fA \approx fA'$ . To compute  $fA'$ , we define a homomorphism  $\psi: A' \rightarrow \Lambda$  by  $\psi(\beta_1) = -\Delta, \psi(\beta_i) = \rho_i$  for  $i \geq 2$ . This is well-defined since it preserves the relations given by all the rows of  $(\sigma_{ij})$ , except perhaps the first—but, since  $\text{rank } A' = 1$ , the rows of  $(\sigma_{ij})$  are linearly dependent and, therefore, the relations given by the first row must also be preserved (note that rows 2 through  $n$  are linearly independent). Since  $\text{rank } A' = 1, \psi$  induces an isomorphism  $fA' \approx \text{Image } \psi = (\Delta, \rho_2, \dots, \rho_n)$ . This completes the proof of lemma 2.

**§11**

We can now prove the realization theorem (G). Let  $\sigma(x, y) = \lambda(x^{-1}) + \mu(y^{-1}) - 1$  and choose elements  $\tau_1, \dots, \tau_k \in I$  so that  $(\sigma, \tau_1, \dots, \tau_k) = I$ .

Define the  $(n \times n)$ -matrix  $(\lambda_{ij})$ , where  $n = 2k + 1$  as follows:

$$\lambda_{ij} = \begin{cases} 0 & i = 1 = j \\ \bar{\sigma}\tau_{i-1} & 2 \leq i \leq k+1, j = 1 \\ \tau_{i-k-1} & k+1 < i \leq n, j = 1 \\ (x^{-1} - 1)(y^{-1} - 1)\sigma\bar{\tau}_{j-1} & i = 1, 2 \leq j \leq k+1 \\ (x^{-1} - 1)(y^{-1} - 1)\bar{\tau}_{j-k-1} & i = 1, k+1 < j \leq n \\ -\delta_{ij}\sigma\bar{\sigma} & k+1 \geq i \geq 2 \\ \delta_{ij} & n \geq i \geq k+1. \end{cases}$$

This matrix satisfies the conditions of Bailey's theorem and is, therefore, the presentation matrix of a 2-link module  $A$ .

We can define a row-vector  $(\sigma_i)$  satisfying the hypothesis of lemma 1 by setting:

$$\sigma_i = \begin{cases} \bar{\sigma} & i = 1 \\ (x^{-1} - 1)(y^{-1} - 1)\bar{\tau}_{i-1} & 2 \leq i \leq k+1 \\ -(x^{-1} - 1)(y^{-1} - 1)\bar{\sigma}\bar{\tau}_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Since  $\sigma, \tau_1, \dots, \tau_k$  are relatively prime, and  $\sigma(1, 1) = 1$ , the  $\{\sigma_i\}$  are relatively prime. Clearly  $\sigma_1(x, 1) = \lambda(x)$ ,  $\sigma_1(1, y) = \mu(y)$  and so, by lemma 1, these are the longitudinal orders of  $A$ . To show  $fA \approx I$ , we apply lemma 2. For our matrix  $(\lambda_{ij})$ ,  $\Delta = (-\sigma\bar{\sigma})^k$  and  $(\mu_{ij})$  is given by:

$$\mu_{ij} = \begin{cases} (-\sigma\bar{\sigma})^{k-1}\delta_{ij} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\delta_{ij} & k+1 < i \leq n. \end{cases}$$

Then

$$\rho_i = \sum_j \mu_{ij}\lambda_{j1} = \begin{cases} (-\sigma)^{k-1}\bar{\sigma}^k\tau_{i-1} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\tau_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Thus  $fA \approx$  ideal generated by  $\{(-\sigma\bar{\sigma})^k, (-\sigma)^{k-1}\bar{\sigma}^k\tau_i(1 \leq i \leq k), (-\sigma\bar{\sigma})^k\tau_i(1 \leq i \leq k)\}$ .

If we divide out  $\pm\sigma^{k-1}\bar{\sigma}^k$  from these elements, we find  $fA \approx$  ideal generated by  $\{\sigma, \tau_i, \sigma\tau_i\} = \{\sigma, \tau_i\} = I$ .

This completes the proof of (G).

### Appendix

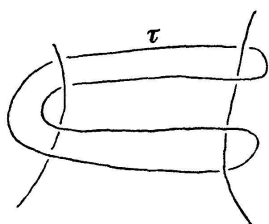
We outline a proof of Bailey's Theorem as stated in §11. The construction of the desired link proceeds, in the spirit of [L], by surgery on the complement of the "unlink", i.e. the link formed by the boundary of two disjoint 2-disks in 3-space.

Let  $X_0$  be the complement of the unlink – then  $H_1(\tilde{X}_0)$  is free of rank one. Choose a generator  $e$  of  $H_1(\tilde{X}_0)$  and let  $\{\sigma_i\}$  ( $2 \leq i \leq n$ ) be disjoint imbedded circles in  $X_0$  which lift to imbedded circles  $\{\tilde{\sigma}_i\}$  in  $\tilde{X}_0$  such that  $\tilde{\sigma}_i$  represents  $\lambda_{i1}e$ . We would also like  $\{\sigma_i\}$ , considered as a link in 3-space, to be the  $(n-1)$ -component unlink. If we give each  $\sigma_i$  the normal framing which winds once around and do surgery on  $S^3$ , using these framed imbedded circles, the result  $\Sigma$ , as in [L], is again diffeomorphic to  $S^3$ . The desired link  $L$  will be the original unlink regarded, now, as a link in  $\Sigma$ .

Let  $Y$  be the complement of the  $\{\sigma_i\}$  in  $X_0$  and  $X$  be the complement of  $L$  in  $\Sigma$ .  $\tilde{Y}$  and  $\tilde{X}$  will be the coverings of  $Y$  and  $X$  inherited from  $\tilde{X}_0$ ;  $\tilde{X}$  is the universal abelian covering of  $X$ . To compute  $H_1(\tilde{Y})$  we examine the homology sequence of  $(\tilde{X}_0, \tilde{Y})$ . From this we conclude that  $H_1(\tilde{Y})$  is generated by elements  $\{e', \varepsilon_2, \dots, \varepsilon_n\}$  where  $e' \rightarrow e$  under the inclusion  $\tilde{Y} \rightarrow \tilde{X}_0$ , and  $\varepsilon_i$  is represented by a small circle which links  $\tilde{\sigma}_i$  simply. There is a single relation  $\sum_{i=2}^n \alpha_i \varepsilon_i = 0$ , where  $\alpha_i = E \cdot \tilde{\sigma}_i$ , the intersection in  $\Lambda$  of a generator  $E$  of  $H_2(\tilde{X}_0)$  with  $\tilde{\sigma}_i$ . Since  $\tilde{\sigma}_i$  represents  $\lambda_{i1}e$ , we have

$$\alpha_i = \bar{\lambda}_{i1}(E \cdot e).$$

Finally, one may calculate  $E \cdot e = (x-1)(y-1)$  by a direct computation:  $E$  is represented by a 2-sphere separating the components of the unlink and  $e$  is represented by the loop  $\tau$  as follows:



So the relation is  $(x-1)(y-1) \sum_{i=2}^n \bar{\lambda}_{i1} \varepsilon_i = 0$ .

To compute  $H_1(\tilde{X})$  we now examine the homology sequence of  $(\tilde{X}, \tilde{Y})$ . From this we conclude that  $H_1(\tilde{X})$  has generators  $e'', \varepsilon'_2, \dots, \varepsilon'_n$ , the images of  $e', \varepsilon_2, \dots, \varepsilon_n$  under the inclusion  $\tilde{Y} \rightarrow \tilde{X}$ , with the relation:

$$(x-1)(y-1) \sum_{i=2}^n \bar{\lambda}_{i1} \varepsilon'_i = 0$$

and, in addition, new relations

$$(*) \quad \lambda_{i1}e'' + \sum_j \lambda'_{ij}\epsilon'_j = 0, \quad \text{for some } \{\lambda'_{ij}\}.$$

$\lambda_{i1}e' + \sum_j \lambda'_{ij}\epsilon_j \in H_1(\tilde{Y})$  is the class represented by the circle  $\tilde{\sigma}'_i$  obtained by translating  $\tilde{\sigma}_i$  along one of the vector fields of the normal framing of  $\tilde{\sigma}_i$  used in the surgery. That the coefficient of  $e'$  is  $\lambda_{i1}$  follows from the fact that  $\tilde{\sigma}_i$  represents  $\lambda_{i1}e$  in  $H_1(\tilde{X}_0)$ . We show that the correct original choice of  $e'$  results in the following properties:

- (i)  $\lambda'_{ij} = \bar{\lambda}'_{ji}$
- (ii)  $\phi(\lambda'_{ij}) = \delta_{ij}$

where  $\phi: \Lambda \rightarrow Z$  is the usual augmentation  $f(x, y) \rightarrow f(1, 1)$ .

LEMMA. Suppose  $X$  is a compact oriented 3-manifold,  $\tilde{X} \rightarrow X$  a regular covering with  $\tau$  as the group of covering transformations. Let  $T_1, \dots, T_n$  be tori components of  $\partial X$  which lift to  $\tilde{T}_i \subseteq \tilde{X}$  trivially covering  $T_i$ , for each  $i$ . Let  $\alpha_i, \beta_i$  be the canonical generators of  $H_1(\tilde{T}_i)$  represented by meridian and longitude circles. Satisfying  $\alpha_i\alpha_j = 0 = \beta_i\beta_j$  and  $\alpha_i \cdot \beta_j = \delta_{ij}$ . If  $\sum_j \lambda_{ij}i_*(\alpha_j) + \sum_j \mu_{ij}u_*(\beta_j) = 0, i = 1, \dots, m$ , is any set of relations in  $H_1(\tilde{X}), i: \tilde{T}_i \subseteq \tilde{X}$ , then, for any  $i, j$

$$\sum_s \lambda_{is}\bar{\mu}_{js} = \sum_s \mu_{is}\bar{\lambda}_{js},$$

where  $\mu \rightarrow \bar{\mu}$  is the usual conjugation in  $Z\pi$ .

*Proof.* Write  $\sum_j (\lambda_{ij}\alpha_j + \mu_{ij}\beta_j) = \partial_*\theta_i$  for some  $\theta_i \in H_2(\tilde{X}, \tilde{T})$  where  $\partial_*: H_2(\tilde{X}, \tilde{T}) \rightarrow H_1(\tilde{T})$  is the boundary homomorphism. Then, using the property: If  $\alpha \in H_1(\tilde{T}), \theta \in H_2(\tilde{X}, \tilde{T})$ , then  $\partial_*\theta \cdot \alpha = \theta \cdot i_*\alpha$  we conclude that  $\theta_i \cdot i_*(\alpha_j) = -\mu_{ij}; \theta_i \cdot i_*(\beta_j) = \lambda_{ij}$ . Now

$$\begin{aligned} 0 &= \theta_i \cdot (i_*\partial_*\theta_j) = \theta_i \cdot \sum_k (\lambda_{jk}i_*(\alpha_k) + \mu_{jk}i_*(\beta_k)) \\ &= \sum_k (\bar{\lambda}_{jk}\theta_i \cdot i_*(\alpha_k) + \bar{\mu}_{jk}\theta_i \cdot i_*(\beta_k)) \\ &= \sum_k (-\bar{\lambda}_{jk}\mu_{ik} + \bar{\mu}_{jk}\lambda_{ik}). \end{aligned}$$

We have the equality  $\tilde{\sigma}'_i = \lambda_{i1}e' + \sum_j \lambda'_{ij}\epsilon_j$  in  $H_1(\tilde{Y})$ . If we remove a tubular neighborhood of the loop  $\tau$ , representing  $e'$ , from  $Y$  to obtain a new manifold  $W$ , we obtain new equations:  $\tilde{\sigma}'_{0i} = \lambda_{i1}e'_0 + \sum \lambda'_{ij}\epsilon_{0j} + \mu_i C$  in  $H_1(\tilde{W})$  where  $C$  is represented by a meridian of the newly removed tube,  $e'_0$  is represented by a

translate  $\tilde{\tau}'$  of  $\tilde{\tau}$  into  $\tilde{W}$ , and  $\varepsilon_{0j} \rightarrow \varepsilon_j, \sigma'_{0i} \rightarrow \sigma'_i$ . We apply the lemma to these relations and conclude:

$$\lambda'_{ij} - \lambda_{i1}\mu_j = \bar{\lambda}'_{ji} - \bar{\mu}_i\bar{\lambda}_{j1}$$

assuming that  $\{\varepsilon_j\}$  and  $C$  are oriented correctly. We now replace our original choice of  $e'$  by  $e' + \sum_j \mu_j \varepsilon_j$  and check that  $\lambda'_{ij}$  is replaced by  $\lambda'_{ij} - \lambda_{i1}\mu_j$ . Now property (i) is satisfied.

To verify property (ii), we need to add to the above argument the constraint that  $\tau'$  be chosen to have linking number 0 with  $\tau$  in  $S^3$ . If we now project everything to  $W \subseteq S^3$ , the above equations imply:

(a)  $\phi(\mu_i) = l(\tau, -\sigma_i + \phi(\lambda_{i1})\tau')$

(b)  $\phi(\lambda'_{ij}) = l(\sigma_j, \sigma'_i - \phi(\lambda_{i1})\tau)$

where  $l$  denotes linking number in  $S^3$ . Since  $l(\tau, \tau') = 0$  by choice, and  $l(\sigma_i, \sigma'_j) = \delta_{ij}$  by definition of  $\sigma'_j$ , (a) and (b) imply:

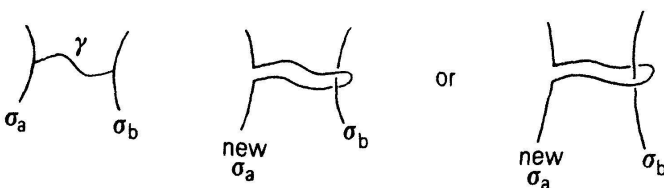
$$\phi(\lambda'_{ij}) = \delta_{ij} + \phi(\lambda_{i1})\phi(\mu_j)$$

or  $\phi(\lambda'_{ij} - \lambda_{i1}\mu_j) = \delta_{ij}$ , as desired.

We finally propose to alter the  $\{\sigma_i\}$  in order to change the  $\{\lambda'_{ij}\}$  to the prescribed  $\{\lambda_{ij}\}$  for  $2 \leq i, j \leq n$ . As a preliminary consideration we show how to make certain elementary changes in the  $\{\lambda'_{ij}\}$ . Choose  $g \in G$ , and  $2 \leq a, b \leq n$ ; we will change  $\sigma_a$  to effect the change:

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm g & i = a, j = b, a \neq b \\ \lambda'_{ij} \pm g^{-1} & i = b, j = a, a \neq b \\ \lambda'_{ij} \pm (g + g^{-1}) & i = j = a = b \\ \lambda'_{ij} & (i, j) \neq (a, b) \text{ or } (b, a). \end{cases}$$

Choose an arc  $\tilde{\gamma}$  in  $\tilde{X}_0$  from  $\tilde{\sigma}_a$  to  $g\tilde{\sigma}_b$  avoiding all lifts of  $\sigma_i, \tau$ , and use  $\gamma$  to form a connected sum of  $\sigma_a$  with a small circle linking  $\sigma_b$ , as in the following picture:

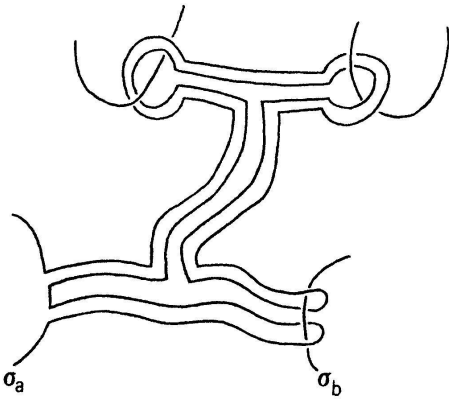


To see that the  $\{\lambda'_{ij}\}$  are changed as claimed, we use the following characterization: given chains  $\theta_i$  in  $\tilde{X}_0$  such that  $\tilde{\sigma}_i - \lambda_{i1}\tilde{\tau} = \partial\theta_i$ , then  $\lambda'_{ij} = \theta_i \cdot \tilde{\sigma}'_j$ . If we now make the obvious change in  $\theta_a$  to accompany our change of  $\sigma_a$ , it is straight forward to verify the new values of  $\{\lambda'_{ij}\}$ . The ambiguity in sign is achieved by the ambiguity in the connected sum, as in the picture.

Note that this construction will destroy the property that  $\{\sigma_i\}$  should form a trivial link in  $S^3$ , as well as property (ii) of  $\{\lambda'_{ij}\}$ . The elementary changes in  $\{\lambda'_{ij}\}$  which would generate an arbitrary change preserving properties (i), (ii) are of the following type: give  $g \in G$  and  $2 \leq a, b \leq n$ :

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm (g - 1) & i = a, j = b, a \neq b \\ \lambda'_{ij} \pm (g^{-1} - 1) & i = b, j = a, a \neq b. \\ \lambda'_{ij} \pm (g + g^{-1} - 2) & i = j = a = b \\ \lambda'_{ij} & (i, j) \neq (a, b) \text{ or } (b, a) \end{cases}$$

But this change is realized by a pair of changes of the original type and, therefore, we will be done if such a pair can be effected without changing the link type of  $\{\sigma_i\}$  in  $S^3$ . To see this it is merely necessary to choose the two arcs from  $\sigma_a$  to  $\sigma_b$  so that, in  $S^3 - \{\sigma_i\}$ , they will be isotopic rel boundary, as suggested by the following picture.



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