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The module of a 2-component link

J. LEVINE

The most prominent algebraic invariant of a link L in 3-space is the fundamental group Π of the complement. One might try to extract “abelian” invariants from Π . The most obvious candidate: Π/Π' , where Π' is the commutator subgroup of Π , is not very useful since, by Alexander duality, this is just the free abelian group with rank the multiplicity (i.e. number of components) of L . A reasonable next candidate is $A(L) = \Pi'/\Pi''$, considered as a module over Π/Π' . If L is oriented, a canonical basis of Π/Π' is defined by the meridians of L . Thus $A(L)$ has a well-defined structure as module over $\Lambda_\mu = \mathbb{Z}[t_1, t_1^{-1}, \dots, t_\mu, t_\mu^{-1}]$ (μ = multiplicity of L). We refer to this as the *module of L* . An alternative description can be given by considering the universal abelian covering \tilde{X} of the complement X of L . The group of covering translations of \tilde{X} is canonically identified with Π/Π' and then $H_1(\tilde{X}) \approx A(L)$, as a Π/Π' -module.

A closely related invariant of L is what is sometimes called the *Alexander module of L* , $\tilde{A}(L)$. This is classically defined as the Λ_μ -module presented by the Jacobian matrix of any presentation of Π . Equivalently $\tilde{A}(L) \approx H_1(\tilde{X}, \tilde{*})$, where $\tilde{*}$ is the inverse image of a base-point $*$ of X . Thus we have an exact sequence: $0 \rightarrow A(L) \rightarrow \tilde{A}(L) \rightarrow M \rightarrow 0$, where M is the “augmentation ideal” of Λ_μ generated by $t_1 - 1, \dots, t_\mu - 1$.

A classical collection of invariants considered by Fox [F] is the sequence of elementary ideals, or Fitting invariants, $\tilde{E}_i(L)$, $i \geq 0$. $\tilde{E}_i(L)$ is defined to be the ideal of Λ_μ generated by the $(n-i)$ -order minors of a presentation matrix of $\tilde{A}(L)$ obtained from n generators. One also considers the greatest common divisor $\tilde{\Delta}_i(L)$ of $\tilde{E}_i(L)$ – note that $\tilde{E}_{i+1}(L) \supseteq \tilde{E}_i(L)$, and so $\tilde{\Delta}_{i+1}(L) \mid \tilde{\Delta}_i(L)$. Furthermore $\tilde{E}_0(L) = 0 = \tilde{\Delta}_0(L)$: $\tilde{\Delta}_1(L)$ is the *Alexander polynomial* of L . One can define $E_i(L)$ and $\Delta_i(L)$ from $A(L)$ in the same way; then $\Delta_i(L) = \tilde{\Delta}_{i+1}(L)$, but $E_i(L) \neq \tilde{E}_{i+1}(L)$, in general. If $\mu = 1$, then $E_i(L) = \tilde{E}_{i+1}(L)$, in fact, $\tilde{A}(L) = A(L) \oplus \Lambda_1$, and $E_0(L)$ is principal and non-zero.

See [C], [F], [H], [H1], [L], [M] for details and more information.

The torsion submodule tA of $A = A(L)$ carries a sesqui-linear Hermitian

pairing $\langle \ , \ \rangle$ with values in $S(\Lambda) = Q(\Lambda)/\Lambda$ ($Q(\Lambda)$ is the quotient field of Λ), referred to as the *Blanchfield pairing* (see [B], [L1]). If $\beta: \bar{A} \rightarrow \text{Hom}_\Lambda(A, S(\Lambda))$ is the adjoint of $\langle \ , \ \rangle$, (\bar{A} is the conjugate of A , defined by changing the action of Λ on A via the anti-automorphism $f(x, y) \rightarrow f(x^{-1}, y^{-1})$) then Kernel β is referred to as the *null-space* of $\langle \ , \ \rangle$ and cokernel β as the *conull-space*. If $\mu = 1$, the pairing is non-singular. See [B], [H] for more information.

The problem of giving a purely algebraic characterization of $A(L)$, with the Blanchfield pairing, has been solved in the case $\mu = 1$ (see [L1]). Bailey [By] has given a characterization of $A(L)$ in terms of the presentation matrix, when $\mu = 2$. The present paper is devoted to a further examination of $A(L)$ when $\mu = 2$; in particular the identification of some of its algebraic properties and a characterization of certain natural "parts" of $A(L)$.

We write $\Lambda = \Lambda_2 = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, and use the notation $G = \pi/\pi'$, $A = A(L)$, $B = H_2(\tilde{X})$ —note that $H_i(\tilde{X}) = 0$, for $i > 2$. We begin by presenting the main results.

A. $r = \text{rank } A = \text{rank } B \leq 1$. B is a free Λ -module. If l is the linking number of the link components, then $r = 1$ implies $l = 0$. $A \otimes \mathbb{Z} = \mathbb{Z}/l$.

B. If $l \neq 0$, then A has projective dimension one, (we will say A is *one-dimensional*), the Blanchfield pairing is non-degenerate (i.e. null-space = 0) and the conull-space $\approx \Lambda/I_l$, where I_l is the ideal generated by

$$(x-1)(y-1) \quad \text{and} \quad \frac{(xy)^l - 1}{xy - 1}.$$

C. If $l = 0$, we define *longitudinal elements* $\xi_x, \xi_y \in A$ by lifting into \tilde{X} "longitudinal" circles parallel to the x and y components of L which link neither component (ξ_x, ξ_y are, therefore, determined up to multiplication by elements of Π/Π'). ξ_x (resp ξ_y) generates the submodule of elements invariant under x (resp. y). The annihilator ideal of ξ_x (resp. ξ_y) is generated by $x-1$ (resp. $y-1$) and one more element $\mu(y)$ (resp. $\lambda(x)$). Thus $\mu(y)$ (resp. $\lambda(x)$) is well-defined up to unit multiple in $\mathbb{Z}[y, y^{-1}]$ (resp. $\mathbb{Z}[x, x^{-1}]$); $\lambda(x), \mu(y)$ will be called the *longitudinal orders* of L and depend only on A .

D. If $l = 0$ and $r = 0$, then $\lambda(x) = 0 = \mu(y)$ and A is one-dimensional and contains an element α such that $(y-1)\alpha = \xi_x$ and $(x-1)\alpha = \xi_y$. Thus the annihilator ideal of α is generated by $(x-1)(y-1)$. The null-space of $\langle \ , \ \rangle$ is generated by α , while the conull-space $\approx \Lambda/(x-1)(y-1)$. In fact, $A/(\alpha)$ is one-dimensional and the pairing on $A/(\alpha)$ induced by the Blanchfield pairing is non-singular.

E. If $r = 1$, then, we may choose $\lambda(1) = 1 = \mu(1)$ and, in fact, $\lambda(x) \mid \Delta(x)$ and $\mu(y) \mid \Delta(y)$, where $\Delta(x), \Delta(y)$ are the Alexander polynomials of the individual components of L considered as knots.

Furthermore, $tA \otimes Z = 0$ and $fA = A/tA$ is isomorphic to an ideal I of Λ . I may be uniquely specified by demanding that its greatest common divisor be 1; in that case, $I + M = \Lambda$. Another ideal $J \subseteq I$ can be defined from L ; J is generated by $(x-1)(y-1)I$ and an element $\sigma(x, y) \in I$, which is well-defined modulo $(x-1)(y-1)I$. Then $\sigma(x, y) \equiv \lambda(x^{-1}) + \mu(y^{-1}) - 1 \pmod{(x-1)(y-1)}$ and so $\sigma(x, y)$ defines a slightly sharper invariant of L than the pair $(\lambda(x), \mu(y))$, since $I/(x-1)(y-1)I \rightarrow \Lambda/(x-1)(y-1)$ has kernel

$$\frac{I \cap (x-1)(y-1)\Lambda}{(x-1)(y-1)I}.$$

F. If $r = 1$, the null-space of $\langle \ , \ \rangle$ is the “pseudo-null” submodule $P(\bar{A})$ of \bar{A} (i.e. the set of all elements whose annihilator ideal has greatest common divisor 1 see [Bo]). $P(A)$ contains the submodule P_0 generated by ξ_x, ξ_y which coincides with the submodule generated by $\xi = \xi_x + \xi_y$, whose annihilator ideal is generated by $\sigma(x, y)$ and $(x-1)(y-1)$. P_0 is the submodule of elements annihilated by $(x-1)(y-1)$. $P(\bar{A})/\bar{P}_0 \approx e^1(I)$ – we use the notation $e^i(R) = \text{Ext}_\Lambda^i(R, \Lambda)$ for any Λ -module R . In fact, $P(\bar{A}) \approx e^1(J)$. The conull-space C is isomorphic to the kernel of a homomorphism $e^2(I) \rightarrow \Lambda/\bar{J}$, whose cokernel is isomorphic to $e^2(tA)$. A and tA have projective dimension ≤ 2 .

G. *Realization*: Let $\lambda(x), \mu(y)$ be polynomials and I an ideal of Λ satisfying: (i) $\lambda(1) = 1 = \mu(1)$; (ii) greatest common divisor of I is 1 and (iii) $\lambda(x^{-1}) + \mu(y^{-1}) - 1 \in I$. Then there exists a 2-component link whose module A has longitudinal orders $\lambda(x), \mu(y)$ and $fA \approx I$. Note (i), (ii) and (iii) are necessary conditions (see (C) and (E)).

We refer the reader to work of Hillman [H], [H1], [H2] and Sato [S] for related and overlapping results.

§1

We begin by considering the Cartan–LeRay spectral sequence of the covering $\tilde{X} \rightarrow X$. $E_{pq}^2 = H_p(G; H_q(\tilde{X})) = 0$ for $p > 2$ or $q > 2$ and so $E_{pq}^3 = E_{pq}^\infty$. Straightforward examination obtains an exact sequence: $H_2(X) \xrightarrow{\phi} H_2(G) \rightarrow A \otimes Z \rightarrow 0$ where ϕ is induced by the map $X \rightarrow K(G, 1)$ corresponding to the covering \tilde{X} . Now $H_2(X) = H_2(G) = Z$ and $\phi = \text{multiplication by } l$; thus $A \otimes Z$ is infinite cyclic, if $l = 0$, and cyclic of order l , if $l \neq 0$. Now a standard Nakayama lemma argument allows us to construct $\Delta \in \Lambda$ such that $\Delta A = 0$ and $\Delta(1, 1) = l^k$, for some integer $k > 0$: if $\{\alpha_i\}$ generate A , then we may write $l\alpha_i = \sum \lambda_{ij}\alpha_j$, where $\lambda_{ij} \in M$, and, thus, $\Delta = \det(l\delta_{ij} - \lambda_{ij})$ annihilates A . This shows that A is a torsion module if $l \neq 0$.

That $\text{rank } A = \text{rank } B$ follows from consideration of the Euler characteristic:

$\text{rank } B - \text{rank } A = \chi_\Lambda(\tilde{X}) = \chi(X) = 0$. (χ_Λ is the Euler characteristic using rank as a Λ -module.) To see that $\text{rank } B \leq 1$, choose a finite 2-dimensional cellular structure on X (actually a compact-deformation retract of X) and let C_* , \tilde{C}_* denote the corresponding chain complexes of X and \tilde{X} . If D_{ij} and d_{ij} are matrix representatives, with respect to the cell basis, of the boundary maps $C_2(\tilde{X}) \rightarrow C_1(\tilde{X})$ and $C_2(X) \rightarrow C_1(X)$, then $d_{ij} = D_{ij}(1, 1)$. Now $\text{rank } B = \text{null}_\Lambda(D_{ij}) \leq \text{null}_\mathbb{Z}(D_{ij}(1, 1)) = \text{rank } H_2(X) = 1$. Note that this argument shows $\text{rank } H_2(\tilde{X}) \leq \mu - 1$ for a μ -component link.

§2

We now define the Blanchfield pairing $\langle \ , \ \rangle$ on tA with values in $S(\Lambda)$.

Let K be a triangulation of X and K' the dual triangulation – let \tilde{K} and \tilde{K}' be the induced triangulations of \tilde{X} . If $\alpha, \beta \in tA$, choose representative cycles z of α in \tilde{K} and w of β in \tilde{K}' . If $\lambda\alpha = 0$, $\lambda \in \Lambda$, choose a chain c in \tilde{K} such that $\partial c = \lambda z$.

Now define $\langle \alpha, \beta \rangle = \frac{c \cdot w}{\lambda} \bmod \Lambda$. Standard arguments (see [L1]) show this is well-defined. Furthermore $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$, using the usual symmetry properties of intersection. An alternative definition of the adjoint β of $\langle \ , \ \rangle$ is obtained by composing the maps:

$$\begin{aligned} \overline{tH_1(\tilde{X})} \subseteq H_1(\tilde{X}) &\xrightarrow{j^*} \overline{H_1(\tilde{X}, \partial\tilde{X})} \stackrel{D}{\cong} H^2(\tilde{X}; \Lambda) \\ &\xrightarrow{\rho} e^1(H_1(\tilde{X})) \longrightarrow e^1(tH_1(X)) \approx \text{Hom}_\Lambda(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \quad (1)$$

D is the Reidemeister–Milnor duality isomorphism ([M]) and ρ is a “universal coefficient” homomorphism defined on $Dj_* \overline{tH_1(\tilde{X})}$ which will be explained below. We are now taking X to be a compact manifold, the complement of an open tubular neighborhood of L .

It is not hard to equate this definition with the following reformulation;

$$\begin{aligned} \overline{H_1(\tilde{X})} &\xleftarrow{\partial_*} \overline{H_2(\tilde{X}; S(\Lambda))} \longrightarrow \overline{H_2(\tilde{X}, \partial\tilde{X}; S(\Lambda))} \stackrel{D}{\cong} H^1(\tilde{X}; S(\Lambda)) \\ &\xrightarrow{\bar{\rho}} \text{Hom}_\Lambda(H_1(\tilde{X}), S(\Lambda)) \longleftarrow \text{Hom}_\Lambda(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \quad (2)$$

where $\bar{\rho}$ is the standard Kronecker map on cohomology, and ∂_* is the Bockstein from the coefficient sequence $0 \rightarrow \Lambda \rightarrow Q(\Lambda) \rightarrow S(\Lambda) \rightarrow 0$. Note that $\text{Image } \partial_* = \overline{tH_1(\tilde{X})}$ and so any element α of $tH_1(\tilde{X})$, can be pulled back to $\alpha' \in \overline{H_2(X; S(\Lambda))}$. Any two pull-backs α', α'' differ by the image of an element of $H_2(\tilde{X}; Q(\Lambda))$.

Using naturality of the maps of (2) with respect to the homomorphism $Q(\Lambda) \rightarrow S(\Lambda)$, we see that $\alpha' - \alpha''$ passes to an element of $\text{Hom}_\Lambda(tH_1(\tilde{X}), S(\Lambda))$ which comes from $\text{Hom}_\Lambda(tH_1(\tilde{X}), Q(\Lambda)) = 0$. Thus the composition defined by (2) is well-defined on $tH_1(\tilde{X})$. This reformulation is seen to be equivalent to our first definition using the definition of D via the intersection pairing.

§3

To understand the maps $\rho, \bar{\rho}$ used in our definitions of the Blanchfield pairing we need a “universal coefficient” consideration of the relation between homology and cohomology. Recall the universal coefficient spectral sequence (see [Mc]): Given a free left chain complex C_* over a ring Λ and a left module N , there exists a spectral sequence “converging” to $H^*(C; N)$, with E^2 -terms given by $E_{pq}^2 = \text{Ext}_\Lambda^q(H_p(C), N)$, and differential d_r in E^r of degree $(1-r, r)$. There is a filtration

$$H^m(C; N) = J_{m0} \supseteq J_{m-1,1} \supseteq \cdots \supseteq J_{1,m-1} \supseteq J_{0,m}$$

where $J_{pq}/J_{p-1,q+1} \approx E_{pq}^\infty$. To define $\bar{\rho}$, we simply consider $H^m(C; N) = J_{m,0} \rightarrow E_{m0}^\infty \subseteq E_{m0}^2 = \text{Hom}_\Lambda(H_m(C), N)$. To define ρ (on $\text{Ker } \bar{\rho}$), we take $\text{Ker } \bar{\rho} = J_{m-1,1} \rightarrow E_{m-1,1}^\infty \subseteq E_{m-1,1}^2 = \text{Ext}_\Lambda^1(H_{m-1}(C), N)$. Looking back at (1), we see that ρ is well-defined on elements coming from $tH_1(\tilde{X})$, since $\bar{\rho}$ is obviously zero on any torsion element when $N = \Lambda$ (and Λ is a domain).

We will consider the universal coefficient spectral sequences for $C = C^*(\tilde{X})$ and $C = C^*(\tilde{X}, \partial\tilde{X})$, with $N = \Lambda$. In each case the spectral sequence can be reduced to one or more exact sequences. This reduction is straightforward and we omit the details. The exact sequences obtained are the following:

$$0 \rightarrow H^1(\tilde{X}; \Lambda) \xrightarrow{\bar{\rho}} A^* \rightarrow Z \rightarrow J_{11} \xrightarrow{\rho} e^1(A) \rightarrow 0 \quad (3)$$

$$0 \rightarrow J_{11} \rightarrow H^2(\tilde{X}; \Lambda) \xrightarrow{\bar{\rho}} B^* \rightarrow e^2(A) \rightarrow 0 \quad (4)$$

$$e^3(A) \approx e^1(B) \quad (5)$$

$$\begin{aligned} 0 \rightarrow e^1(A_0) \rightarrow H^2(\tilde{X}, \partial\tilde{X}; \Lambda) \rightarrow B_0^* \rightarrow e^2(A_0) \rightarrow H^3(\tilde{X}, \partial\tilde{X}) \\ \rightarrow e^1(B_0) \rightarrow e^3(A_0) \rightarrow 0 \end{aligned} \quad (6)$$

$$A_0^* \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda) \quad (7)$$

where we use the notation $A = H_1(\tilde{X})$, $B = H_2(\tilde{X})$, (as before) $A_0 = H_1(\tilde{X}, \partial\tilde{X})$, $B_0 = H_2(\tilde{X}, \partial\tilde{X})$, $e^i = \text{Ext}_\Lambda^i(\ , \Lambda)$ and $*$ = $e^0 = \text{Hom}_\Lambda(\ , \Lambda)$.

We also note the exact homology sequence:

$$0 \rightarrow B \rightarrow B_0 \rightarrow H_1(\partial\tilde{X}) \rightarrow A \rightarrow A_0 \rightarrow H_0(\partial\tilde{X}) \rightarrow H_0(\tilde{X}) \rightarrow 0. \quad (8)$$

It is easy to see that $H_*(\partial\tilde{X})$ depends only on the linking number l and is given as follows:

$$H_0(\partial\tilde{X}) = \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1) \quad (9)$$

$$H_1(\partial\tilde{X}) = \begin{cases} 0 & l \neq 0 \\ \Lambda/(x-1) \oplus \Lambda/(y-1) & l = 0 \end{cases} \quad (10)$$

In (10), when $l = 0$, generators are given by the two longitudes, lifted into \tilde{X} .

§4

In the case $r=0$, it follows from (8) that $\text{rank } A_0 = \text{rank } B_0 = 0$ also. Thus $A^* = B^* = A_0^* = B_0^* = 0$. From (3) and (7) we conclude $B_0 \approx H^1(\tilde{H}; \Lambda) = 0$ and $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda) = 0$. From (4) and (5), we conclude $e^2(A) = 0 = e^3(A)$ and so A is one-dimensional (note $e^q = 0$ for $q > 3$, since Λ has homological dimension 3).

The Blanchfield pairing $\beta: \bar{A} \rightarrow \text{Hom}_\Lambda(A, S(\Lambda)) \approx e^1(A)$ can be written as the composition (according to (1)):

$$\bar{A} \rightarrow \bar{A}_0 \approx H^2(X; \Lambda) = J_{11} \rightarrow e^1(A).$$

If P denotes the null-space of β , and C the conull-space, we can deduce from (3) and (8) an exact sequence:

$$0 \rightarrow \overline{H_1(\partial\tilde{X})} \rightarrow P \rightarrow Z \rightarrow \bar{K} \rightarrow C \rightarrow 0 \quad (11)$$

where $K = \text{Kernel } \{H_0(\partial\tilde{X}) \rightarrow H_0(\tilde{X}) \approx Z\}$ – from (8).

In order to analyze the map $Z \rightarrow \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}$, we first recall that the edge homomorphism

$$\text{Ext}_\Lambda^q(H_0(C), N) = E_{0q}^2 \twoheadrightarrow E_{0q}^\infty = J_{0q} \subseteq J_{q0} = H^q(C; N)$$

is equivalent to the homomorphism induced by a chain map $C_* \rightarrow F_*$, where F_* is a free resolution of $H_0(C)$, which induces the identity map on $H_0(C) = H_0(F)$. In case $\Lambda = \mathbb{Z}\pi$ and $C_* = C_*(\tilde{X})$, where \tilde{X} is a regular π -covering of X , this coincides

with the homomorphism $\text{Ext}_\Lambda^q(Z, N) = H^q(\pi; N) \rightarrow H^q(\tilde{X}, N)$ induced by the classifying map $X \rightarrow B\pi$ of the covering $\tilde{X} \rightarrow X$. Now our map $Z \rightarrow \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}$ is the composition

$$Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\tilde{X}; \Lambda) \approx \overline{H_1(\tilde{X}, \partial\tilde{X})} \xrightarrow{\partial^*} \overline{H_0(\partial\tilde{X})},$$

where ε' is the edge homomorphism of the universal coefficient spectral sequence of $H^*(\tilde{X}; \Lambda)$, which coincides with the composition $Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\partial\tilde{X}; \Lambda) \approx \overline{H_0(\partial\tilde{X})}$, where ε' is the edge homomorphism of the universal coefficient spectral sequence of $H^*(\partial\tilde{X}; \Lambda)$. Now the map $\partial X \rightarrow BG$, which classifies the covering $\partial\tilde{X} \rightarrow \partial X$, is an l -fold covering on each component of ∂X (∂X is the disjoint union of two tori and BG a single torus). Therefore the induced map $H^2(G; \Lambda) \rightarrow H^2(\partial\tilde{X}; \Lambda) \approx \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1)$ maps a generator onto $(\phi_l(y), \phi_l(x))$, where $\phi_l(x) = \frac{x^l-1}{x-1}$. If $l \neq 0$, this is a monomorphism, and, since $H_1(\partial\tilde{X}) = 0$ (see (10)), we conclude $P = 0$. Furthermore we now see that $\text{Cok}\{Z \rightarrow \bar{K} \subseteq H_0(\partial\tilde{X})\}$ has a presentation $\{\alpha, \beta: (x-1)\alpha = 0 = (y-1)\beta, \phi_l(y)\alpha = \phi_l(x)\beta\}$, and it, therefore, follows from (11) that C corresponds to the submodule of elements $\lambda\alpha + \mu\beta$ ($\lambda, \mu \in \Lambda$) satisfying:

$$\lambda(1, 1) + \mu(1, 1) = 0.$$

It is not hard to see that C will, therefore, be generated by $\gamma = \alpha - \beta$, subject to the relations

$$(x-1)(y-1)\gamma = 0 = (\phi_l(y) + \phi_l(x) - l)\gamma.$$

To complete the proof of (B) it suffices to check that:

$$\phi_l(xy) \equiv \phi_l(x) + \phi_l(y) - l \pmod{(x-1)(y-1)}.$$

But this follows from the easy fact that, for any $f(x, y) \in \Lambda$:

$$f(x, y) \equiv f(x, 1) + f(1, y) - f(1, 1) \pmod{(x-1)(y-1)}.$$

§5

The longitudinal elements ξ_x, ξ_y of (C) are the generators of the image $H_1(\partial\tilde{X}) \rightarrow A$ in (8). According to (10) $(x-1)\xi_x = 0 = (y-1)\xi_y$. If $r = 0$, then $B_0 = 0$

and, from (10), we see that $x-1$ ($y-1$) generates the annihilator of ξ_x (ξ_y). Note that our computation of $Z \rightarrow \bar{K}$, in the preceding paragraph, shows that it is zero, when $l=0$, and, therefore, (11) contains the short exact sequence: $0 \rightarrow H_1(\partial\tilde{X}) \rightarrow P \rightarrow Z \rightarrow 0$.

If we can show that $P \approx \Lambda/(x-1)(y-1)$ (with generator α), then it follows that we may choose $\xi_x = (y-1)\alpha$, $\xi_y = (x-1)\alpha$ as longitudinal elements, i.e. they are images, under $H_1(\partial\tilde{X}) \rightarrow N$, of generators of the respective summands (see (10)). Since we have already proved $C \approx \Lambda/(x-1)(y-1)$, the remaining assertions of (D) follows from the Hermitian property of the Blanchfield pairing together with:

LEMMA. *Let A be a one-dimensional torsion Λ -module equipped with a sesquilinear Hermitian pairing $\langle \ , \ \rangle$ with null-space K and conull-space C . Then $K \approx e^1(\bar{C})$ and, if $A' = A/K$, the induced pairing on A' is non-degenerate with conull-space $\approx e^2(\bar{C})$. If $e^3(C) = 0$, then A' is one-dimensional.*

Proof of Lemma:

Denote the adjoint of $\langle \ , \ \rangle$ by $\phi: A \rightarrow e^1(\bar{A})$; we have, by hypothesis an exact sequence: $0 \rightarrow K \rightarrow A \xrightarrow{\phi} e^1(\bar{A}) \rightarrow C \rightarrow 0$. The transpose of $\phi: A \rightarrow e^1 e^1 A \xrightarrow{e^2 \phi} e^1 \bar{A}$ coincides with ϕ (this is what Hermitian means), where $A \rightarrow e^1 e^1 A$ is a standard "double dual" map. Since A is one-dimensional this double dual map is an isomorphism. Now consider the diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \rightarrow & K & \rightarrow & A & \rightarrow & A' \rightarrow 0 \\
 & & & & \searrow \phi & & \downarrow \\
 & & & & & & e^1 \bar{A} \\
 & & & & & & \downarrow \\
 & & & & & & C \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

From this we derive the diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \rightarrow & e^1(C) & \rightarrow & e^1 e^1(\bar{A}) & \rightarrow & e^1(A') \rightarrow e^2(C) \rightarrow e^2 e^1(\bar{A}) = 0 \\
 & & & & \parallel & \searrow e^1 \phi & \downarrow \\
 & & & & \bar{A} & & e^1(A)
 \end{array}$$

as well as the isomorphism $e^i(A') \approx e^{i+1}(C)$, $i \geq 2$. We immediately see that $\bar{K} \approx e^1(C)$, the cokernel of the map $\bar{A}' \rightarrow e^1(A')$, induced by $e^1\phi = \bar{\phi}$, is $e^2(C)$, and that A' is one-dimensional if $e^3(C) = 0$.

§6

From now on we will assume $r = 1$, since all the statements for $r = 0$ have been proved. We first point out that $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda)$, by duality, and, by (7), we then conclude $B \approx A_0^*$, which is free – over a unique factorization domain, R^* is free for any module R of rank ≤ 1 .

We examine the longitudinal elements. We can define $\xi_x, \xi_y \in A$, when $l = 0$, by choosing translates of the components K_x, K_y of L into X which have 0 linking number with their associated components – since $l = 0$ these translates lift into \tilde{X} defining ξ_x, ξ_y up to multiplication by a unit of Λ . Clearly ξ_x, ξ_y generate $\text{Image}\{H_1(\partial\tilde{X}) \rightarrow H_1(\tilde{X})\}$, and we have $(x-1)\xi_x = 0 = (y-1)\xi_y$ (this distinguishes ξ_x from ξ_y). We now show the existence of $\lambda(x), \mu(y)$, as in (C).

Consider the infinite cyclic covering X_x of X defined by the homomorphism $\Pi \rightarrow G \rightarrow Z$, which sends $x \rightarrow 1$ and $y \rightarrow 0$. Thus \tilde{X} is an infinite cyclic covering of X_x , and in fact, $C_*(X_x) \approx C_*(\tilde{X})/(y-1)C_*(\tilde{X})$. We obtain, by tensoring $C_*(\tilde{X})$ with the short exact sequence:

$$0 \rightarrow \Lambda \xrightarrow{y-1} \Lambda \rightarrow \Lambda/(y-1) \rightarrow 0$$

the following exact homology sequence:

$$\begin{aligned} 0 \rightarrow H_2(\tilde{X}) \xrightarrow{y-1} H_2(\tilde{X}) \rightarrow H_2(X_x) \rightarrow H_1(\tilde{X}) \xrightarrow{y-1} H_1(\tilde{X}) \\ \rightarrow H_1(X_x) \rightarrow H_0(\tilde{X}) \xrightarrow{y-1} H_0(\tilde{X}) \end{aligned} \quad (12)$$

Now X_x is closely related to the infinite cyclic covering Y_x of the complement of K_x . In fact $\overline{Y_x - X_x}$ is the union of translates, by powers of x , of the solid torus formed by lifting a tubular neighborhood of K_y into Y_x . Thus $H_i(Y_x, X_x) \approx \Lambda/(y-1)$, if $i = 2, 3$, and zero otherwise. By considering the exact sequence of the pair (Y_x, X_x) and the facts that $H_i(Y_x) = 0$ if $i \geq 2$, we see easily that $H_2(X_x) \approx \Lambda/(y-1)$ and obtain an exact sequence:

$$0 \rightarrow \Lambda/(y-1) \rightarrow H_1(X_x) \rightarrow H_1(Y_x) \rightarrow 0. \quad (13)$$

The sequence (12) can now be put in the simpler form:

$$0 \rightarrow \Lambda/(y-1) \rightarrow \Lambda/(y-1) \rightarrow A \xrightarrow{y-1} A \rightarrow H_1(X_x) \rightarrow Z \rightarrow 0 \quad (12')$$

since $H_2(\tilde{X}) = B \approx \Lambda$. The image of a generator, under the injection $\Lambda/(y-1) \rightarrow \Lambda/(y-1)$ is represented by a non-zero polynomial $\lambda(x)$. Since a generator $\hat{\xi}_y$ of $H_2(X_x) \approx \Lambda/(y-1)$ is represented by the boundary torus of a tubular neighborhood of K_y (lifted into X_x), it is straightforward to check, from the definition of the boundary homomorphism $H_2(X_x) \rightarrow H_1(\tilde{X}) = A$, that $\hat{\xi}_y \rightarrow \xi_y \in A$. It follows immediately that $\lambda(x)$ and $y-1$ generate the annihilator ideal of ξ_y . A similar argument establishes the existence of $\mu(y)$.

Note from (12') that ξ_y generates the submodule of elements invariant under y . Thus $\lambda(x)$ is defined, purely algebraically, up to unit multiple, by the property of being a generator, together with $y-1$, of the annihilator ideal of this submodule – similarly for $\mu(y)$.

We now show $\lambda(x) \mid \Delta(x)$, where $\Delta(x)$ is the Alexander polynomial of K_x – this will imply $\lambda(1) = \pm 1$. Let T be the torsion sub-module of A . We first derive from (12') and (13) an exact sequence:

$$0 \rightarrow R \rightarrow T \xrightarrow{y-1} T \rightarrow S \rightarrow 0 \quad (14)$$

where $R = \Lambda/(\lambda(x), y-1)$, $S \subseteq H_1(Y_x)$ is the image of T under $A \rightarrow H_1(X_x) \rightarrow H_1(Y_x)$. The only point not immediately obvious is: $\text{Ker}\{T \rightarrow S\} \subseteq (y-1)T$. Suppose $\alpha \in T$ and $\alpha \rightarrow 0$ in S . If $\alpha \rightarrow 0$ in $H_1(X_x)$, then $\alpha = (y-1)\beta$ for some $\beta \in A$, by exactness of (12'). But then $\alpha \in T$ implies $\beta \in T$. To see $\alpha \rightarrow 0$ in $H_1(X_x)$ it suffices by (13) to show $f(x)\alpha \rightarrow 0$ for any non-zero $f(x)$. But, since $\alpha \in T$, $f(x, y)\alpha = 0$ for some non-zero $f(x, y)$. If we write $f(x, y) = f(x) + (y-1)g(x, y)$, then $0 = f(x)\alpha + (y-1)g(x, y)\alpha$. Since $(y-1)A \rightarrow 0$ in $H_1(X_x)$, so does $f(x)\alpha$. If $f(x) = 0$, then, by (12'), $\lambda(x)g(x, y)\alpha = 0$. But this would be impossible if we had chosen $f(x, y)$ with the smallest number of $y-1$ factors.

Now recall that $\Delta(x) = \Delta(H_1(Y_x))$, where $\Delta(A)$, for any Λ_x -module A ($\Lambda_x = z[x, x^{-1}] \approx \Lambda/(y-1)$) is the greatest common divisor of the *order ideal* of A (see [L]). We also recall the following property of $\Delta(A)$: if $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is a short exact sequence of Λ_x -modules, then $\Delta(A) = \Delta(A') \Delta(A'')$ (see [L] for a proof). Thus, for example, $\Delta(S) \mid \Delta(x)$ and, so, it suffices to prove that $\lambda(x)$ ($= \Delta(R)$, R considered as a Λ_x -module) divides $\Delta(S)$. Define

$$T_i = \frac{\text{Ker } \phi^{i+1}}{\text{Ker } \phi^i} \quad \text{and} \quad T^i = \frac{\phi^i T}{\phi^{i+1} T},$$

where $\phi: T \rightarrow T$ is multiplication by $y-1$. These are Λ_x -modules and we have a family of short exact sequences: $0 \rightarrow T_{i+1} \rightarrow T_i \rightarrow T^i \rightarrow T^{i+1} \rightarrow 0$, for $i \geq 0$ (see [L2]). From (14) we see that $T_0 \approx R$ and $T^0 \approx S$. From the above-mentioned multiplicative property of Δ we have $\Delta(T_{i+1})\Delta(T^i) = \Delta(T_i)\Delta(T^{i+1})$ for $i \geq 0$. Therefore, we see that $\Delta(T_{i+1}) \mid \Delta(T^{i+1})$ would imply $\Delta(T_i) \mid \Delta(T^i)$ —note that these are all non-zero, since $\Delta(T_{i+1}) \mid \Delta(T_i)$, $\Delta(T^{i+1}) \mid \Delta(T^i)$ and $\Delta(T_0), \Delta(T^0)$ are non-zero. Thus it suffices to show $\Delta(T_i) \mid \Delta(T^i)$ for some value of i . But $T_i = 0$, for large enough i , since $\{\text{Ker } \phi_i\}$ is an increasing sequence of submodules in a finitely-generated module over a Noetherian ring. This completes the proof.

Of course, by a similar argument, we can show $\mu(y) \mid \Delta(y)$.

We can now show that P_0 , the submodule of A generated by ξ_x and ξ_y , is the submodule of elements annihilated by $(x-1)(y-1)$. Suppose $(x-1)(y-1)\alpha = 0$; then $(y-1)\alpha = f\xi_x$ for some $f \in \Lambda$. So $\mu(y)(y-1)\alpha = 0$ which means $\mu(y)\alpha = g\xi_y$. Since $\mu(1) = 1$, we may write $\mu(y) = 1 + (y-1)\mu'(y)$ and so $\alpha + (y-1)\mu'(y)\alpha = g\xi_y$ or $\alpha + \mu'(y)f\xi_x = g\xi_y$. Thus $\alpha \in P_0$.

§7

We now examine fA and prove $fA \otimes_{\Lambda} Z$ is infinite cyclic. (over Z) We already know $A \otimes_{\Lambda} Z$ is infinite cyclic, which implies $fA \otimes_{\Lambda} Z$ is cyclic. If $fA \otimes_{\Lambda} Z$ were finite of order $k > 0$, then $fA \otimes_{\Lambda} Z/p = 0$, for any p relatively prime to k . If so, by Nakayama's lemma, $\Delta \cdot fA = 0$ for some $\Delta \notin M_p$, where $M_p = \ker \{\Lambda \rightarrow Z/p\}$. But fA is torsion-free. If we define I to be the ideal of Λ with greatest common divisor 1 which is isomorphic to fA , then $I + M = \Lambda$. To see this choose $\lambda \in I$ which generates $I/MI \approx I \otimes Z \approx Z$ —we will show $\lambda(1, 1) = \pm 1$. $M(I/(\lambda)) = I/(\lambda)$, which implies, by Nakayama's lemma, that $\Delta \cdot I/(\lambda) = 0$, i.e. $\Delta I \subseteq (\lambda)$, for some $\Delta \equiv 1 \pmod{M}$ —i.e. $\Delta(1, 1) = \pm 1$. Since I has greatest common divisor one, $\Delta \in (\lambda)$ and so $\lambda(1, 1) = \pm 1$. To see that $tA \otimes_{\Lambda} Z = 0$ (when $r = 1$) consider the short exact sequence $0 \rightarrow tA \rightarrow A \rightarrow fA \rightarrow 0$ and apply $\otimes_{\Lambda} Z$ to obtain $\text{Tor}^1(fA, Z) \rightarrow tA \otimes Z \rightarrow A \otimes Z \rightarrow fA \otimes Z \rightarrow 0$. Since $A \otimes Z \approx Z \approx fA \otimes Z$, it suffices to show $\text{Tor}^1(fA, Z) = \text{Tor}^1(I, Z) = 0$. Now $\text{Tor}^1(I, Z) = \text{Tor}^2(\Lambda/I, Z)$ which can be considered to be the submodule of *invariant* elements of Λ/I —i.e. of elements α satisfying $x\alpha = y\alpha = \alpha$. But $\lambda\alpha = 0$, where $\lambda \in I$ satisfying $\lambda(1, 1) = 1$ has been found in the preceding paragraph, and so $0 = (1 + (x-1)\lambda' + (y-1)\lambda'')\alpha = \alpha + \lambda'(x-1)\alpha + \lambda''(y-1)\alpha = \alpha$.

By the results of §7, we may break (8) up into two shorter exact sequences (for

$r = 1$):

$$0 \rightarrow B \rightarrow B_0 \rightarrow \Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow 0 \quad (15a)$$

$$0 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow A \rightarrow A_0 \rightarrow \Lambda/(x-1)(y-1) \rightarrow 0 \quad (15b)$$

where $\rho = \lambda(x) + \mu(y) - 1$ (choosing $\lambda(1) = 1 = \mu(1)$). Note that the quotient of $\Lambda/(x-1) \oplus \Lambda/(y-1)$ by the submodule generated by $(\mu(y), 0)$ and $(0, \lambda(x))$ is isomorphic to $\Lambda/(\rho, (x-1)(y-1))$, using the generator $(1, 1)$, and the kernel of the epimorphism $\Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow Z$ is isomorphic to $\Lambda/(x-1)(y-1)$, using the generator $(1, -1)$. Applying $\text{Hom}(_, \Lambda)$ to (15a) yields an exact sequence:

$$0 \rightarrow B_0^* \rightarrow B^* \rightarrow \Lambda/(x-1) \oplus \Lambda/(y-1) \rightarrow e^1(B_0) \rightarrow e^1(B).$$

Now $e^1(B) = 0$, since B is free. From (3), we conclude that $\bar{B}_0 \approx H^1(\tilde{X}; \Lambda)$ is free or isomorphic to M (the ideal in Λ generated by $(x-1, y-1)$), since A^* is free. But (15a) is possible only if $\bar{B}_0 \approx M$. Thus $e^1(B_0) \approx Z$. We, therefore, have the exact sequence:

$$0 \rightarrow B_0^* \rightarrow B^* \rightarrow \Lambda/(x-1)(y-1) \rightarrow 0 \quad (16)$$

We now apply $\text{Hom}(_, \Lambda)$ to (15b) and obtain exact sequences:

$$0 \rightarrow A_0^* \rightarrow A^* \rightarrow \Lambda/(x-1)(y-1) \rightarrow e^1(A_0) \rightarrow e^1(A) \rightarrow 0 \quad (17a)$$

$$0 \rightarrow e^2(A_0) \rightarrow e^2(A) \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow e^3(A_0) \rightarrow 0. \quad (17b)$$

Note that

$$e^1(\Lambda/(\rho, (x-1)(y-1))) = 0$$

$$e^2(\Lambda/(\rho, (x-1)(y-1))) \approx \Lambda/(\rho, (x-1)(y-1))$$

and $e^3(A) = 0$ (by (5), since B is free).

We now examine the homomorphisms $tA \rightarrow tA_0$ and $fA \rightarrow fA_0$, using (15b). Denoting the kernel and cokernel, respectively, by K_1, K_2 and C_1, C_2 , we can apply the snake lemma, using (15b) to obtain an exact sequence: $0 \rightarrow K_1 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow K_2 \rightarrow C_1 \rightarrow \Lambda/(x-1)(y-1) \rightarrow C_2 \rightarrow 0$. Since K_2 is torsion-free and C_1 torsion, we see that $K_2 = 0$. For some ideal $S \supseteq (x-1)(y-1)$, we have

$$C_1 \approx S/(x-1)(y-1), \quad C_2 \approx \Lambda/S$$

and we have:

$$0 \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow tA \rightarrow tA_0 \rightarrow S/(x-1)(y-1) \rightarrow 0 \quad (18a)$$

$$0 \rightarrow fA \rightarrow fA_0 \rightarrow \Lambda/S \rightarrow 0. \quad (18b)$$

§8

We now deduce some facts from (3)–(6) and duality:

$$\overline{tA} = e^1(A_0). \quad (19)$$

This follows from (6), since $e^1(A_0)$ is torsion and B_0^* is free.

$$0 \rightarrow \bar{B}_0 \rightarrow A^* \rightarrow Z \rightarrow \overline{tA_0} \rightarrow e^1(A) \rightarrow 0. \quad (20)$$

This is just (3), since $J_{11} \simeq \overline{tA_0}$ from (4).

$$0 \rightarrow \overline{fA_0} \rightarrow B^* \rightarrow e^2(A) \rightarrow 0. \quad (21)$$

This follows from (4).

$$0 \rightarrow \overline{fA} \rightarrow B_0^* \rightarrow e^2(A_0) \rightarrow 0, \quad e^3(A_0) \approx Z/k, \quad (\text{some } k > 0). \quad (22)$$

This follows from (6), since $H^3(\tilde{X}, \partial\tilde{X}) \approx Z$ and $e^3(A_0)$ cannot be isomorphic to Z , since $e^2(Z) \neq 0$ but $e^2e^3 = 0$ over Λ (see [Ba]).

We use the map $\tilde{X} \rightarrow (\tilde{X}, \partial\tilde{X})$ to map (22) \rightarrow (21). Using (18b), (16) and (17b), we obtain a commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \overline{fA} & \rightarrow & B_0^* & \rightarrow & e^2(A_0) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \rightarrow & \overline{fA_0} & \rightarrow & B^* & \rightarrow & e^2(A) & \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \Lambda/S & & \Lambda/(x-1)(y-1) & & \Lambda/(\rho, (x-1)(y-1)) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & e^3(A_0) & \\
 & & & & & \downarrow & \\
 & & & & & 0 &
 \end{array} \quad (23)$$

From the snake lemma we deduce an exact sequence:

$$0 \rightarrow \Lambda/S \rightarrow \Lambda/(x-1)(y-1) \rightarrow \Lambda/(\rho, (x-1)(y-1)) \rightarrow e^3(A_0) \rightarrow 0. \quad (24)$$

From this sequence we may deduce: $e^3(A_0) = 0$ and $S = (x-1)(y-1)$. The first of these follows from the fact that any epimorphism $\Lambda \rightarrow Z/k$ is of the form $f(x, y) \rightarrow af(1, 1)$, where $a \in Z$ is relatively prime to k , and $\rho(1, 1) = 1$. To see the second, let $\alpha \in \Lambda$ represent the image of 1 under $\Lambda/S \rightarrow \Lambda/(x-1)(y-1)$. Then $(\alpha, (x-1)(y-1)) = (\rho, (x-1)(y-1))$ and so $\alpha(1, 1) = \pm 1$. But $\beta \in S$ if and only if $(x-1)(y-1) \mid \alpha\beta$ and so $S \subseteq (x-1)(y-1)$. Since we already know $S \supseteq (x-1)(y-1)$, we have $S = (x-1)(y-1)$.

Now define J to be the ideal of Λ , with greatest common divisor 1, isomorphic to fA_0 . We can rewrite (23) as follows:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & I & \xrightarrow{\tau'} & \Lambda & \rightarrow & e^2(\bar{A}_0) & \rightarrow 0 \\ & \downarrow \tau & & \downarrow & & \downarrow & \\ 0 \rightarrow & J & \xrightarrow{\tau''} & \Lambda & \rightarrow & e^2(\bar{A}) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \Lambda/(x-1)(y-1) & \rightarrow & \Lambda/(x-1)(y-1) & \rightarrow & \Lambda/(\bar{\rho}, (x-1)(y-1)) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \quad (25)$$

The maps indicated by τ, τ', τ'' and τ_0 are all multiplication by elements of $Q(\Lambda)$ – we also use $\tau, \tau', \tau'', \tau_0$ to denote these elements. Obviously $\tau_0 \in \Lambda$ and is a unit multiple of $(x-1)(y-1)$ and, since I and J have greatest common divisor one, τ' and τ'' are also in Λ . Now $e^2(\bar{A}_0)$ and $e^2(\bar{A})$ are pseudo-null since they are $\text{grade} \geq 2$ (see [Ba]) and $\text{grade} \geq 2$ means pseudo-null (see [R]). Therefore τ' and τ'' must be units of Λ ; so $\tau \in \Lambda$ and is a unit multiple of τ_0 or $(x-1)(y-1)$.

Now choose an element $\sigma \in J$ which maps onto a generator of $\Lambda/(x-1)(y-1)$. Therefore $J = (x-1)(y-1)I + (\sigma)$. From the left-most vertical row of (25) we see that $f\sigma \in (x-1)(y-1)I$ if and only if $(x-1)(y-1) \mid f$. If $f = (x-1)(y-1)$, this says $\sigma \in I$, and so $J \subseteq I$. If σ' is another element such that $J = (x-1)(y-1)I + (\sigma')$, then a straight-forward computation shows $\sigma' = \alpha\sigma \bmod (x-1)(y-1)I$, where $\alpha = u \bmod (x-1)(y-1)$ for some unit u of Λ . Since $\sigma \in I$, we have $\sigma' \equiv u\sigma \bmod (x-1)(y-1)I$ and so σ is well-defined up to unit multiple, $\bmod (x-1)(y-1)I$. Finally, it follows from (25) that $(\sigma, (x-1)(y-1)) = (\bar{\rho}, (x-1)(y-1))$, which implies $\sigma \equiv u'\bar{\rho} \bmod (x-1)(y-1)$ for some unit u' of Λ .

§9

We now determine the null-space N and co-null space C of the Blanchfield pairing. Its adjoint $\overline{tA} \rightarrow e^1(tA)$, whose kernel and cokernel are N and C , can be described as the composition:

$$\overline{tA} \rightarrow e^1(A_0) \rightarrow e^1(tA_0) \rightarrow e^1(tA) \quad (26)$$

where the first homomorphism is the isomorphism of (19) and the others are induced by inclusion $tA_0 \subseteq A_0$ and $A \rightarrow A_0$. By its Hermitian property this coincides with the composition;

$$\overline{tA} \rightarrow \overline{tA_0} \rightarrow e^1(A) \rightarrow e^1(tA). \quad (27)$$

The middle map comes from (20).

In (26), the last map is also an isomorphism – this follows from (18a), since $S = (x-1)(y-1)$ and $e^0(\Lambda/(\rho, (x-1)(y-1))) = e^1(\Lambda/(\rho, (x-1)(y-1))) = 0$. Thus N and C are isomorphic to the kernel and cokernel, respectively, of $e^1(A_0) \rightarrow e^1(tA_0)$. From the short exact sequence $0 \rightarrow tA_0 \rightarrow A_0 \rightarrow fA_0 \rightarrow 0$, we conclude $N \approx e^1(fA_0) = e^1(J)$. Since $e^1(tA) \approx \text{Hom}_\Lambda(tA, S(\Lambda))$ is pseudo-null free, and $e^1(J) \approx e^2(\Lambda/J)$ is pseudo-null, it follows that N is the pseudo-null submodule of \overline{tA} .

We show that the map $\overline{tA_0} \rightarrow e^1(A)$ in (27) is an isomorphism. Referring to (20) we have already seen that $A^* \rightarrow Z$ is non-trivial, since $B_0 \approx M$ and A^* is free. It remains to show that tA_0 cannot contain a submodule isomorphic to Z/k , unless $k = 0$ or 1 . But we have seen $e^3(A_0) = 0$ and an inclusion $Z/k \rightarrow A_0$ would induce an epimorphism $e^3(A_0) \rightarrow e^3(Z/k) \approx Z/k$ (if $k > 0$).

From the short exact sequence $0 \rightarrow tA \rightarrow A \rightarrow fA \rightarrow 0$ we deduce an exact sequence.

$$0 \rightarrow e^1(I) \rightarrow e^1(A) \rightarrow e^1(tA) \rightarrow e^2(I) \rightarrow e^2(A) \rightarrow e^2(tA) \rightarrow 0 \quad (28)$$

since $fA \approx I$ and $e^3(I) \approx e^4(\Lambda/I) = 0$. From (28) and (18a), we can deduce exact sequences:

$$0 \rightarrow \Lambda/(\bar{\rho}, (x-1)(y-1)) \rightarrow N \rightarrow e^1(I) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow C \rightarrow e^2(I) \rightarrow e^2(A) \rightarrow e^2(tA) \rightarrow 0.$$

Recall $S = (x-1)(y-1)$. Since $e^2(A) \approx \Lambda/\bar{J}$ from (25), we have completed the proof of (F).

§10

To prove (G) we will use the following special case of a result of Bailey [By]

THEOREM. *Let (λ_{ij}) be an $(n \times n)$ -matrix over Λ satisfying (i) $\lambda_{11} = 0$; (ii) $\lambda_{ij} = \bar{\lambda}_{ji}$ if $n \geq i, j > 1$ (iii) $\lambda_{1j} = (x^{-1} - 1)(y^{-1} - 1)\bar{\lambda}_{j1}$ for $1 \leq j \leq n$ (iv) $\lambda_{ij}(1, 1) = \pm \delta_{ij}$ if $n \geq i, j > 1$. Then there exists a link, with $l = 0$, whose module A has presentation*

$$\left\{ \alpha_1, \dots, \alpha_n : \sum_{j=1}^n \lambda_{ij} \alpha_j = 0, i = 1, \dots, n \right\}.$$

Since a proof this theorem has not appeared in a journal, we present one in the Appendix. We point out that our proof is very different from Bailey's. Also see [N]. ((λ_{ij}) is referred to as a *presentation matrix* of A .) We prove two lemmas.

LEMMA 1. *Let A be a link module with a presentation matrix (λ_{ij}) satisfying (i)–(iv). Suppose (σ_i) is an $(n \times 1)$ -row vector, whose entries are relatively prime, such that $\sum_{i=1}^n \sigma_i \lambda_{ij} = 0$ for $j = 1, \dots, n$. Then $\sigma_1(s, 1)$ and $\sigma_1(1, y)$ are the longitudinal orders of A .*

The next lemma deals with a more general situation.

LEMMA 2. *Let (λ_{ij}) be an $(n \times n)$ matrix over a domain Λ , a presentation matrix of a module A of rank one. Let M be the $(n-1) \times (n-1)$ -matrix $(\lambda_{ij}), 2 \leq i, j \leq n$ and suppose $\Delta = \det M \neq 0$. Let $(\mu_{ij}) = \Delta \cdot M^{-1} (2 \leq i, j \leq n)$, the cofactor matrix of M and set $\rho_i = \sum_{j=2}^n \mu_{ij} \lambda_{j1}$. Then fM is isomorphic to the ideal of Λ generated by $(\Delta, \rho_2, \dots, \rho_n)$.*

Proof of Lemma 1. Let $0 \rightarrow W \rightarrow F_0 \xrightarrow{d} F_1 \rightarrow A \rightarrow 0$ be the resolution defined by (λ_{ij}) , i.e. F_0 and F_1 are free modules of rank n with bases $\{\alpha_i\}, \{\beta_i\}$ with $d(\beta_i) = \sum_j \lambda_{ij} \alpha_j$. Since $\text{rank } A = 1$ and projective dimension $A \leq 2$, W is free of rank one. If a generator of $W \subset F_0$ is $\sum_i \sigma'_i \beta_i$, then $\sigma'_i = u \sigma_i$, for some unit u in Λ . Let $\Lambda_x = \Lambda/(y-1)$: then $\text{Tor}_1^\Lambda(A, \Lambda_x)$ is the submodule of elements of A annihilated by $y-1$ (using exact sequence $0 \rightarrow \Lambda \xrightarrow{y-1} \Lambda \rightarrow \Lambda_x \rightarrow 0$) which, by (C), is isomorphic to $\Lambda_x/(\lambda(x))$. Using the resolution of A given above $\text{Tor}_1^\Lambda(A, \Lambda_x)$ is the homology of the chain complex:

$$W \otimes_{\Lambda} \Lambda_x \xrightarrow{d'} F_0 \otimes_{\Lambda} \Lambda_x \xrightarrow{d''} F_1 \otimes_{\Lambda} \Lambda_x$$

where the modules are free over Λ_x and d', d'' are represented by the matrices: $(\sigma_i(x, 1))$ and $(\lambda_{ij}(x, 1))$, respectively. Since $\lambda_{1j}(x, 1) = 0$, for all j , and $\lambda_{ij}(1, 1) = \pm \delta_{ij}$ for $i, j \geq 2$, it follows easily that kernel d'' is the free submodule of $F_0 \otimes_{\Lambda} \Lambda_x$ generated by $\beta_1 \otimes 1$. Thus, since $\text{Image } d' \subseteq \text{kernel } d''$, $\sigma_i(x, 1) = 0$ for $i > 1$, and $\text{Tor}_1^{\Lambda}(A, \Lambda_x) \approx \Lambda/(\sigma_1(x, 1))$.

A similar argument for $\mu(y)$ completes the proof.

Proof of Lemma 2. Suppose (ρ_{ij}) , $1 \leq i, j \leq n$, is any matrix over Λ ; consider the module A' presented by the product matrix $(\rho_{ij})(\lambda_{ij})$ —i.e. $A' = \{\beta_1, \dots, \beta_n : \sum_j \rho_{ij} \lambda_{sj} \beta_j = 0, i = 1, \dots, n\}$. If $\{\alpha_i\}$ are the generators of A , subject to relations $\sum_j \lambda_{ij} \alpha_j = 0$ ($i = 1, \dots, n$), then $\beta_i \rightarrow \alpha_i$ defines an epimorphism $\phi: A' \rightarrow A$. The kernel of ϕ is generated by $\{\gamma_i\}$, where $\gamma_i = \sum_j \lambda_{ij} \beta_j$ ($i = 1, \dots, n$), and the $\{\gamma_i\}$ are subject to relations $\sum_j \rho_{ij} \gamma_j = 0$ ($i = 1, \dots, n$). We apply these observations to the matrix (ρ_{ij}) given by

$$\rho_{ij} = \begin{cases} \mu_{ij} & i, j \geq 2 \\ \delta_{ij} & i = 1 \text{ or } j = 1. \end{cases}$$

The matrix $(\sigma_{ij}) = (\rho_{ij})(\lambda_{ij})$ is given by

$$\sigma_{ij} = \begin{cases} \lambda_{ij} & i = 1 \\ \rho_i & j = 1, \quad i > 1 \\ \Delta \delta_{ij} & i, j \geq 2. \end{cases}$$

Now $\det(\rho_{ij}) = \Delta \neq 0$, which implies, since (ρ_{ij}) is a relation matrix for $\text{Ker } \phi$, that $\text{Ker } \phi$ is a torsion module. Thus ϕ induces an isomorphism $fA \approx fA'$. To compute fA' , we define a homomorphism $\psi: A' \rightarrow \Lambda$ by $\psi(\beta_1) = -\Delta$, $\psi(\beta_i) = \rho_i$ for $i \geq 2$. This is well-defined since it preserves the relations given by all the rows of (σ_{ij}) , except perhaps the first—but, since $\text{rank } A' = 1$, the rows of (σ_{ij}) are linearly dependent and, therefore, the relations given by the first row must also be preserved (note that rows 2 through n are linearly independent). Since $\text{rank } A' = 1$, ψ induces an isomorphism $fA' \approx \text{Image } \psi = (\Delta, \rho_2, \dots, \rho_n)$. This completes the proof of lemma 2.

§11

We can now prove the realization theorem (G). Let $\sigma(x, y) = \lambda(x^{-1}) + \mu(y^{-1}) - 1$ and choose elements $\tau_1, \dots, \tau_k \in I$ so that $(\sigma, \tau_1, \dots, \tau_k) = I$.

Define the $(n \times n)$ -matrix (λ_{ij}) , where $n = 2k + 1$ as follows:

$$\lambda_{ij} = \begin{cases} 0 & i = 1 = j \\ \bar{\sigma}\tau_{i-1} & 2 \leq i \leq k+1, j = 1 \\ \tau_{i-k-1} & k+1 < i \leq n, j = 1 \\ (x^{-1} - 1)(y^{-1} - 1)\sigma\bar{\tau}_{j-1} & i = 1, 2 \leq j \leq k+1 \\ (x^{-1} - 1)(y^{-1} - 1)\bar{\tau}_{j-k-1} & i = 1, k+1 < j \leq n \\ -\delta_{ij}\sigma\bar{\sigma} & k+1 \geq i \geq 2 \\ \delta_{ij} & n \geq i \geq k+1. \end{cases}$$

This matrix satisfies the conditions of Bailey's theorem and is, therefore, the presentation matrix of a 2-link module A .

We can define a row-vector (σ_i) satisfying the hypothesis of lemma 1 by setting:

$$\sigma_i = \begin{cases} \bar{\sigma} & i = 1 \\ (x^{-1} - 1)(y^{-1} - 1)\bar{\tau}_{i-1} & 2 \leq i \leq k+1 \\ -(x^{-1} - 1)(y^{-1} - 1)\bar{\sigma}\bar{\tau}_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Since $\sigma, \tau_1, \dots, \tau_k$ are relatively prime, and $\sigma(1, 1) = 1$, the $\{\sigma_i\}$ are relatively prime. Clearly $\sigma_1(x, 1) = \lambda(x)$, $\sigma_1(1, y) = \mu(y)$ and so, by lemma 1, these are the longitudinal orders of A . To show $fA \approx I$, we apply lemma 2. For our matrix (λ_{ij}) , $\Delta = (-\sigma\bar{\sigma})^k$ and (μ_{ij}) is given by:

$$\mu_{ij} = \begin{cases} (-\sigma\bar{\sigma})^{k-1}\delta_{ij} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\delta_{ij} & k+1 < i \leq n. \end{cases}$$

Then

$$\rho_i = \sum_j \mu_{ij}\lambda_{j1} = \begin{cases} (-\sigma)^{k-1}\bar{\sigma}^k\tau_{i-1} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\tau_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Thus $fA \approx$ ideal generated by $\{(-\sigma\bar{\sigma})^k, (-\sigma)^{k-1}\bar{\sigma}^k\tau_i (1 \leq i \leq k), (-\sigma\bar{\sigma})^k\tau_i (1 \leq i \leq k)\}$.

If we divide out $\pm\sigma^{k-1}\bar{\sigma}^k$ from these elements, we find $fA \approx$ ideal generated by $\{\sigma, \tau_i, \sigma\tau_i\} = \{\sigma, \tau_i\} = I$.

This completes the proof of (G).

Appendix

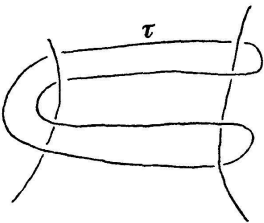
We outline a proof of Bailey's Theorem as stated in §11. The construction of the desired link proceeds, in the spirit of [L], by surgery on the complement of the "unlink", i.e. the link formed by the boundary of two disjoint 2-disks in 3-space.

Let X_0 be the complement of the unlink – then $H_1(\tilde{X}_0)$ is free of rank one. Choose a generator e of $H_1(\tilde{X}_0)$ and let $\{\sigma_i\}$ ($2 \leq i \leq n$) be disjoint imbedded circles in X_0 which lift to imbedded circles $\{\tilde{\sigma}_i\}$ in \tilde{X}_0 such that $\tilde{\sigma}_i$ represents $\lambda_{i1}e$. We would also like $\{\sigma_i\}$, considered as a link in 3-space, to be the $(n-1)$ -component unlink. If we give each σ_i the normal framing which winds once around and do surgery on S^3 , using these framed imbedded circles, the result Σ , as in [L], is again diffeomorphic to S^3 . The desired link L will be the original unlink regarded, now, as a link in Σ .

Let Y be the complement of the $\{\sigma_i\}$ in X_0 and X be the complement of L in Σ . \tilde{Y} and \tilde{X} will be the coverings of Y and X inherited from \tilde{X}_0 ; \tilde{X} is the universal abelian covering of X . To compute $H_1(\tilde{Y})$ we examine the homology sequence of (\tilde{X}_0, \tilde{Y}) . From this we conclude that $H_1(\tilde{Y})$ is generated by elements $\{e', \varepsilon_2, \dots, \varepsilon_n\}$ where $e' \rightarrow e$ under the inclusion $\tilde{Y} \rightarrow \tilde{X}_0$, and ε_i is represented by a small circle which links $\tilde{\sigma}_i$ simply. There is a single relation $\sum_{i=2}^n \alpha_i \varepsilon_i = 0$, where $\alpha_i = E \cdot \tilde{\sigma}_i$, the intersection in Λ of a generator E of $H_2(\tilde{X}_0)$ with $\tilde{\sigma}_i$. Since $\tilde{\sigma}_i$ represents $\lambda_{i1}e$, we have

$$\alpha_i = \bar{\lambda}_{i1}(E \cdot e).$$

Finally, one may calculate $E \cdot e = (x-1)(y-1)$ by a direct computation: E is represented by a 2-sphere separating the components of the unlink and e is represented by the loop τ as follows:



So the relation is $(x-1)(y-1) \sum_{i=2}^n \bar{\lambda}_{i1} \varepsilon_i = 0$.

To compute $H_1(\tilde{X})$ we now examine the homology sequence of (\tilde{X}, \tilde{Y}) . From this we conclude that $H_1(\tilde{X})$ has generators $e'', \varepsilon'_2, \dots, \varepsilon'_n$, the images of $e', \varepsilon_2, \dots, \varepsilon_n$ under the inclusion $\tilde{Y} \rightarrow \tilde{X}$, with the relation:

$$(x-1)(y-1) \sum_{i=2}^n \bar{\lambda}_{i1} \varepsilon'_i = 0$$

and, in addition, new relations

$$(*) \quad \lambda_{i1}e'' + \sum_j \lambda'_{ij}\varepsilon'_j = 0, \quad \text{for some } \{\lambda'_{ij}\}.$$

$\lambda_{i1}e' + \sum_j \lambda'_{ij}\varepsilon_j \in H_1(\tilde{Y})$ is the class represented by the circle $\tilde{\sigma}'_i$ obtained by translating $\tilde{\sigma}_i$ along one of the vector fields of the normal framing of $\tilde{\sigma}_i$ used in the surgery. That the coefficient of e' is λ_{i1} follows from the fact that $\tilde{\sigma}_i$ represents $\lambda_{i1}e$ in $H_1(\tilde{X}_0)$. We show that the correct original choice of e' results in the following properties:

$$(i) \quad \lambda'_{ij} = \bar{\lambda}'_{ji}$$

$$(ii) \quad \phi(\lambda'_{ij}) = \delta_{ij}$$

where $\phi: \Lambda \rightarrow Z$ is the usual augmentation $f(x, y) \rightarrow f(1, 1)$.

LEMMA. Suppose X is a compact oriented 3-manifold, $\tilde{X} \rightarrow X$ a regular covering with τ as the group of covering transformations. Let T_1, \dots, T_n be tori components of ∂X which lift to $\tilde{T}_i \subseteq \tilde{X}$ trivially covering T_i , for each i . Let α_i, β_i be the canonical generators of $H_1(\tilde{T}_i)$ represented by meridian and longitude circles. Satisfying $\alpha_i\alpha_j = 0 = \beta_i\beta_j$ and $\alpha_i \cdot \beta_j = \delta_{ij}$. If $\sum_j \lambda_{ij}i_*(\alpha_j) + \sum_j \mu_{ij}u_*(\beta_j) = 0$, $i = 1, \dots, m$, is any set of relations in $H_1(\tilde{X})$, $i: \tilde{T}_i \subseteq \tilde{X}$, then, for any i, j

$$\sum_s \lambda_{is}\bar{\mu}_{js} = \sum_s \mu_{is}\bar{\lambda}_{js},$$

where $\mu \rightarrow \bar{\mu}$ is the usual conjugation in $Z\pi$.

Proof. Write $\sum_j (\lambda_{ij}\alpha_j + \mu_{ij}\beta_j) = \partial_*\theta_i$ for some $\theta_i \in H_2(\tilde{X}, \tilde{T})$ where $\partial_*: H_2(\tilde{X}, \tilde{T}) \rightarrow H_1(\tilde{T})$ is the boundary homomorphism. Then, using the property: If $\alpha \in H_1(\tilde{T})$, $\theta \in H_2(\tilde{X}, \tilde{T})$, then $\partial_*\theta \cdot \alpha = \theta \cdot i_*\alpha$ we conclude that $\theta_i \cdot i_*(\alpha_j) = -\mu_{ij}$; $\theta_i \cdot i_*(\beta_j) = \lambda_{ij}$. Now

$$\begin{aligned} 0 &= \theta_i \cdot (i_*\partial_*\theta_j) = \theta_i \cdot \sum_k (\lambda_{jk}i_*(\alpha_k) + \mu_{jk}i_*(\beta_k)) \\ &= \sum_k (\bar{\lambda}_{jk}\theta_i \cdot i_*(\alpha_k) + \bar{\mu}_{jk}\theta_i \cdot i_*(\beta_k)) \\ &= \sum_k (-\bar{\lambda}_{jk}\mu_{ik} + \bar{\mu}_{jk}\lambda_{ik}). \end{aligned}$$

We have the equality $\tilde{\sigma}'_i = \lambda_{i1}e' + \sum_j \lambda'_{ij}\varepsilon_j$ in $H_1(\tilde{Y})$. If we remove a tubular neighborhood of the loop τ , representing e' , from Y to obtain a new manifold W , we obtain new equations: $\tilde{\sigma}'_{0i} = \lambda_{i1}e'_0 + \sum \lambda'_{ij}\varepsilon_{0j} + \mu_i C$ in $H_1(\tilde{W})$ where C is represented by a meridian of the newly removed tube, e'_0 is represented by a

translate $\tilde{\tau}'$ of $\tilde{\tau}$ into \tilde{W} , and $\varepsilon_{0j} \rightarrow \varepsilon_j$, $\sigma'_{0i} \rightarrow \sigma'_i$. We apply the lemma to these relations and conclude:

$$\lambda'_{ij} - \lambda_{i1}\mu_j = \bar{\lambda}'_{ji} - \bar{\mu}_i\bar{\lambda}_{j1}$$

assuming that $\{\varepsilon_j\}$ and C are oriented correctly. We now replace our original choice of e' by $e' + \sum_j \mu_j \varepsilon_j$ and check that λ'_{ij} is replaced by $\lambda'_{ij} - \lambda_{i1}\mu_j$. Now property (i) is satisfied.

To verify property (ii), we need to add to the above argument the constraint that τ' be chosen to have linking number 0 with τ in S^3 . If we now project everything to $W \subseteq S^3$, the above equations imply:

$$(a) \quad \phi(\mu_i) = l(\tau, -\sigma_i + \phi(\lambda_{i1})\tau')$$

$$(b) \quad \phi(\lambda'_{ij}) = l(\sigma_j, \sigma'_i - \phi(\lambda_{i1})\tau)$$

where l denotes linking number in S^3 . Since $l(\tau, \tau') = 0$ by choice, and $l(\sigma_i, \sigma'_j) = \delta_{ij}$ by definition of σ'_j , (a) and (b) imply:

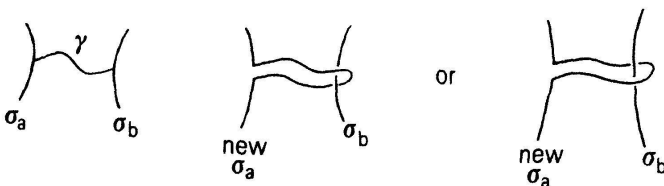
$$\phi(\lambda'_{ij}) = \delta_{ij} + \phi(\lambda_{i1})\phi(\mu_j)$$

or $\phi(\lambda'_{ij} - \lambda_{i1}\mu_j) = \delta_{ij}$, as desired.

We finally propose to alter the $\{\sigma_i\}$ in order to change the $\{\lambda'_{ij}\}$ to the prescribed $\{\lambda_{ij}\}$ for $2 \leq i, j \leq n$. As a preliminary consideration we show how to make certain elementary changes in the $\{\lambda'_{ij}\}$. Choose $g \in G$, and $2 \leq a, b \leq n$; we will change σ_a to effect the change:

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm g & i = a, j = b, a \neq b \\ \lambda'_{ij} \pm g^{-1} & i = b, j = a, a \neq b \\ \lambda'_{ij} \pm (g + g^{-1}) & i = j = a = b \\ \lambda'_{ij} & (i, j) \neq (a, b) \text{ or } (b, a). \end{cases}$$

Choose an arc $\tilde{\gamma}$ in \tilde{X}_0 from $\tilde{\sigma}_a$ to $g\tilde{\sigma}_b$ avoiding all lifts of σ_i , τ , and use γ to form a connected sum of σ_a with a small circle linking σ_b , as in the following picture:

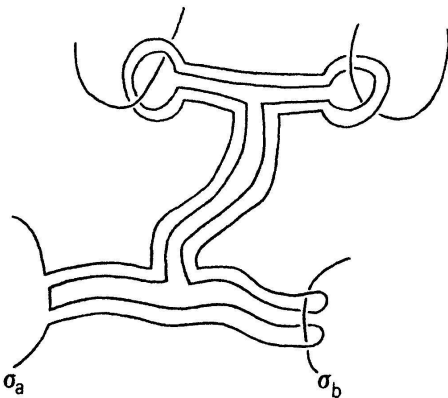


To see that the $\{\lambda'_{ij}\}$ are changed as claimed, we use the following characterization: given chains θ_i in \tilde{X}_0 such that $\tilde{\sigma}_i - \lambda_{i1}\tilde{\tau} = \partial\theta_i$, then $\lambda'_{ij} = \theta_i \cdot \tilde{\sigma}'_j$. If we now make the obvious change in θ_a to accompany our change of σ_a , it is straight forward to verify the new values of $\{\lambda'_{ij}\}$. The ambiguity in sign is achieved by the ambiguity in the connected sum, as in the picture.

Note that this construction will destroy the property that $\{\sigma_i\}$ should form a trivial link in S^3 , as well as property (ii) of $\{\lambda'_{ij}\}$. The elementary changes in $\{\lambda'_{ij}\}$ which would generate an arbitrary change preserving properties (i), (ii) are of the following type: give $g \in G$ and $2 \leq a, b \leq n$:

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm (g-1) & i = a, j = b, a \neq b \\ \lambda'_{ij} \pm (g^{-1}-1) & i = b, j = a, a \neq b. \\ \lambda'_{ij} \pm (g + g^{-1} - 2) & i = j = a = b \\ \lambda'_{ij} & (i, j) \neq (a, b) \text{ or } (b, a) \end{cases}$$

But this change is realized by a pair of changes of the original type and, therefore, we will be done if such a pair can be effected without changing the link type of $\{\sigma_i\}$ in S^3 . To see this it is merely necessary to choose the two arcs from σ_a to σ_b so that, in $S^3 - \{\sigma_i\}$, they will be isotopic rel boundary, as suggested by the following picture.



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