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Autor: Levine, J.
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The module of a 2-component link

J. LEVINE

The most prominent algebraic invariant of a link L in 3-space is the fundamental group Π of the complement. One might try to extract "abelian" invariants from Π. The most obvious candidate: Π/Π', where Π' is the commutator subgroup of Π, is not very useful since, by Alexander duality, this is just the free abelian group with rank the multiplicity (i.e. number of components) of L. A reasonable next candidate is $A(L)=\Pi'/\Pi''$, considered as a module over Π/Π'. If L is oriented, a canonical basis of Π/Π' is defined by the meridians of L. Thus $A(L)$ has a well-defined structure as modulue over $\Lambda_\mu = Z[t_1, t_1^{-1}, \ldots, t_\mu, t_\mu^{-1}]$ (μ = multiplicity of L). We refer to this as the *module of* L. An alternative description can be given by considering the universal abelian covering $\tilde{X}$ of the complement X of L. The group of covering translations of $\tilde{X}$ is canonically identified with Π/Π' and then $H_1(\tilde{X})\approx A(L)$, as a Π/Π'-module.

A closely related invariant of L is what is sometimes called the *Alexander module of* L, $\tilde{A}(L)$. This is classically defined as the Λ_μ-module presented by the Jacobian matrix of any presentation of Π. Equivalently $\tilde{A}(L)\approx H_1(\tilde{X}, \tilde{*})$, where $\tilde{*}$ is the inverse image of a base-point $*$ of X. Thus we have an exact sequence: $0\to A(L)\to \tilde{A}(L)\to M\to 0$, where M is the "augmentation ideal" of Λ_μ generated by $t_1-1, \ldots, t_\mu-1$.

A classical collection of invariants considered by Fox [F] is the sequence of elementary ideals, or Fitting invariants, $\tilde{E}_i(L)$, $i\geq 0$. $\tilde{E}_i(L)$ is defined to be the ideal of Λ_μ generated by the $(n-i)$-order minors of a presentation matrix of $\tilde{A}(L)$ obtained from n generators. One also considers the greatest common divisior $\tilde{\Delta}_i(L)$ of $\tilde{E}_i(L)$ – note that $\tilde{E}_{i+1}(L)\supseteq \tilde{E}_i(L)$, and so $\tilde{\Delta}_{i+1}(L)\mid \tilde{\Delta}_i(L)$. Furthermore $\tilde{E}_0(L)=0=\tilde{\Delta}_0(L)$: $\tilde{\Delta}_1(L)$ is the *Alexander polynomial* of L. One can define $E_i(L)$ and $\Delta_i(L)$ from $A(L)$ in the same way; then $\Delta_i(L)=\tilde{\Delta}_{i+1}(L)$, but $E_i(L)\neq \tilde{E}_{i+1}(L)$, in general. If $\mu=1$, then $E_i(L)=\tilde{E}_{i+1}(L)$, in fact, $\tilde{A}(L)=A(L)\oplus \Lambda_1$, and $E_0(L)$ is principal and non-zero.

See [C], [F], [H], [H1], [L], [M] for details and more information.

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pairing $\langle\ ,\ \rangle$ with values in $S(\Lambda)=Q(\Lambda)/\Lambda$ ($Q(\Lambda)$ is the quotient field of Λ), referred to as the *Blanchfield pairing* (see [B], [L1]). If $\beta:\bar{A}\to \mathrm{Hom}_\Lambda(A, S(\Lambda))$ is the adjoint of $\langle\ ,\ \rangle$, ($\bar{A}$ is the conjugate of A, defined by changing the action of Λ on A via the anti-automorphism $f(x, y)\to f(x^{-1}, y^{-1})$) then Kernel β is referred to as the *null-space* of $\langle\ ,\ \rangle$ and cokernel β as the *conull-space*. If $\mu=1$, the pairing is non-singular. See [B], [H] for more information.

The problem of giving a purely algebraic characterization of $A(L)$, with the Blanchfield pairing, has been solved in the case $\mu=1$ (see [L1]). Bailey [By] has given a characterization of $A(L)$ in terms of the presentation matrix, when $\mu=2$. The present paper is devoted to a further examination of $A(L)$ when $\mu=2$; in paricular the identification of some of its algebraic properties and a characterization of certain natural "parts" of $A(L)$.

We write $\Lambda=\Lambda_2=Z\ [x, x^{-1}, y, y^{-1}]$, and use the notation $G=\pi/\pi'$, $A=A(L)$, $B=H_2(\tilde{X})$ – note that $H_i(\tilde{X})=0$, for $i>2$. We begin by presenting the main results.

A. $r=\operatorname{rank} A=\operatorname{rank} B\leq 1$. B is a free Λ-module. If l is the linking number of the link components, then $r=1$ implies $l=0$. $A\otimes Z=Z/l$.

B. If $l\neq 0$, then A has projective dimension one, (we will say A is *one-dimensional*), the Blanchfield pairing is non-degenerate (i.e. null-space$\,=0$) and the conull-space $\approx\Lambda/I_l$, where I_l is the ideal generated by

$$(x-1)(y-1) \quad\text{and}\quad \frac{(xy)^l-1}{xy-1}.$$

C. If $l=0$, we define *longitudinal elements* $\xi_x, \xi_y\in A$ by lifting into $\tilde{X}$ "longitudinal" circles parallel to the x and y components of L which link neither component (ξ_x, ξ_y are, therefore, determined up to multiplication by elements of Π/Π'). ξ_x (resp ξ_y) generates the submodule of elements invariant under x (resp. y). The annihilator ideal of ξ_x (resp. ξ_y) is generated by $x-1$ (resp. $y-1$) and one more element $\mu(y)$ (resp. $\lambda(x)$). Thus $\mu(y)$ (resp. $\lambda(x)$) is well-defined up to unit multiple in $Z\ [y, y^{-1}]$ (resp. $Z[x, x^{-1}]$); $\lambda(x), \mu(y)$ will be called the *longitudinal orders* of L and depend only on A.

D. If $l=0$ and $r=0$, then $\lambda(x)=0=\mu(y)$ and A is one-dimensional and contains an element α such that $(y-1)\ \alpha=\xi_x$ and $(x-1)\ \alpha=\xi_y$. Thus the annihilator ideal of α is generated by $(x-1)(y-1)$. The null-space of $\langle\ ,\ \rangle$ is generated by α, while the conull-space$\,\approx\Lambda/(x-1)(y-1)$. In fact, $A/(\alpha)$ is one-dimensional and the pairing on $A/(\alpha)$ induced by the Blanchfield pairing is non-singular.

E. If $r=1$, then, we may choose $\lambda(1)=1=\mu(1)$ and, in fact, $\lambda(x)\mid\Delta(x)$ and $\mu(y)\mid\Delta(y)$, where $\Delta(x), \Delta(y)$ are the Alexander polynomials of the individual components of L considered as knots.

Furthermore, $tA \otimes Z = 0$ and $fA = A/tA$ is isomorphic to an ideal I of Λ. I may be uniquely specified by demanding that its greatest common divisor be 1; in that case, $I + M = \Lambda$. Another ideal $J \subseteq I$ can be defined from L; J is generated by $(x-1)(y-1)$ I and an element $\sigma(x, y) \in I$, which is well-defined modulo $(x-1)(y-1)$ I. Then $\sigma(x, y) \equiv \lambda(x^{-1}) + \mu(y^{-1}) - 1 \bmod (x-1)(y-1)$ and so $\sigma(x, y)$ defines a slightly sharper invariant of L than the pair $(\lambda(x), \mu(y))$, since $I/(x-1)(y-1)$ $I \to \Lambda/(x-1)(y-1)$ has kernel

$$\frac{I \cap (x-1)(y-1)\Lambda}{(x-1)(y-1)I}.$$

F. If $r = 1$, the null-space of $\langle\ ,\ \rangle$ is the "pseudo-null" submodule $P(\bar{A})$ of $\bar{A}$ (i.e. the set of all elements whose annihilator ideal has greatest common divisor 1 see [Bo]. $P(A)$ contains the submodule P_0 generated by ξ_x, ξ_y which coincides with the submodule generated by $\xi = \xi_x + \xi_y$, whose annihilator ideal is generated by $\sigma(x, y)$ and $(x-1)(y-1)$. P_0 is the submodule of elements annihilated by $(x-1)(y-1)$. $P(\bar{A})/\bar{P}_0 \approx e^1(I)$ – we use the notation $e^i(R) = \mathrm{Ext}^i_\Lambda(R, \Lambda)$ for any Λ-module R. In fact, $P(\bar{A}) \approx e^1(J)$. The conull-space C is isomorphic to the kernel of a homomorphism $e^2(I) \to \Lambda/\bar{J}$, whose cokernel is isomorphic to $e^2(tA)$. A and tA have projective dimension ≤ 2.

G. *Realization:* Let $\lambda(x)$, $\mu(y)$ be polynomials and I an ideal of Λ satisfying: (i) $\lambda(1) = 1 = \mu(1)$; (ii) greatest common divisor of I is 1 and (iii) $\lambda(x^{-1}) + \mu(y^{-1}) - 1 \in I$. Then there exists a 2-component link whose module A has longitudinal orders $\lambda(x)$, $\mu(y)$ and $fA \approx I$. Note (i), (ii) and (iii) are necessary conditions (see (C) and (E)).

We refer the reader to work of Hillman [H], [H1], [H2] and Sato [S] for related and overlapping results.

§1

We begin by considering the Cartan–LeRay spectral sequence of the covering $\tilde{X} \to X$. $E^2_{pq} = H_p(G; H_q(\tilde{X})) = 0$ for $p > 2$ or $q > 2$ and so $E^3_{pq} = E^\infty_{pq}$. Straightforward examination obtains an exact sequence: $H_2(X) \xrightarrow{\phi} H_2(G) \to A \otimes Z \to 0$ where ϕ is induced by the map $X \to K(G, 1)$ corresponding to the covering $\tilde{X}$. Now $H_2(X) = H_2(G) = Z$ and ϕ = multiplication by l; thus $A \otimes Z$ is infinite cyclic, if $l = 0$, and cyclic of order l, if $l \neq 0$. Now a standard Nakayama lemma argument allows us to construct $\Delta \in \Lambda$ such that $\Delta A = 0$ and $\Delta(1, 1) = l^k$, for some integer $k > 0$: if $\{\alpha_i\}$ generate A, then we may write $l\alpha_i = \Sigma\, \lambda_{ij}\alpha_j$, where $\lambda_{ij} \in M$, and, thus, $\Delta = \det(l\delta_{ij} - \lambda_{ij})$ annihilates A. This shows that A is a torsion module if $l \neq 0$.

That rank A = rank B follows from consideration of the Euler characteristic:

rank B − rank $A = \chi_\Lambda(\tilde{X}) = \chi(X) = 0$. ($\chi_\Lambda$ is the Euler characteristic using rank as a Λ-module.) To see that rank $B \leqslant 1$, choose a finite 2-dimensional cellular structure on X (actually a compact-deformation retract of X) and let C_*, $\tilde{C}_*$ denote the corresponding chain complexes of X and $\tilde{X}$. If D_{ij} and d_{ij} are matrix representatives, with respect to the cell basis, of the boundary maps $C_2(\tilde{X}) \to C_1(\tilde{X})$ and $C_2(X) \to C_1(X)$, then $d_{ij} = D_{ij}(1, 1)$. Now rank $B = \mathrm{null}_\Lambda\,(D_{ij}) \leq \mathrm{null}_Z\,(D_{ij}(1, 1)) = \mathrm{rank}\, H_2(X) = 1$. Note that this argument shows rank $H_2(\tilde{X}) \leq \mu - 1$ for a μ-component link.

§2

We now define the Blanchfield pairing $\langle\ ,\ \rangle$ on tA with values in $S(\Lambda)$.

Let K be a triangulation of X and K' the dual triangulation – let $\tilde{K}$ and $\tilde{K}'$ be the induced triangulations of $\tilde{X}$. If $\alpha, \beta \in tA$, choose representative cycles z of α in $\tilde{K}$ and w of β in $\tilde{K}'$. If $\lambda\alpha = 0$, $\lambda \in \Lambda$, choose a chain c in $\tilde{K}$ such that $\partial c = \lambda z$. Now define $\langle \alpha, \beta \rangle = \dfrac{c \cdot w}{\lambda} \bmod \Lambda$. Standard arguments (see [L1]) show this is well-defined. Furthermore $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$, using the usual symmetry properties of intersection. An alternative definition of the adjoint β of $\langle\ ,\ \rangle$ is obtained by composing the maps:

$$\begin{aligned} &\overline{tH_1(\tilde{X})} \subseteq \overline{H_1(\tilde{X})} \xrightarrow{j^*} \overline{H_1(\tilde{X}, \partial\tilde{X})} \overset{D}{\approx} H^2(\tilde{X}; \Lambda) \\ &\qquad \xrightarrow{\rho} e^1(H_1(\tilde{X})) \longrightarrow e^1(tH_1(X)) \approx \mathrm{Hom}_\Lambda\,(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \tag{1}$$

D is the Reidemeister–Milnor duality isomorphism ([M]) and ρ is a "universal coefficient" homomorphism defined on $Dj_*\overline{tH_1}(\tilde{X})$ which will be explained below. We are now taking X to be a compact manifold, the complement of an open tubular neighborhood of L.

It is not hard to equate this definition with the following reformulation;

$$\begin{aligned} &\overline{H_1}(\tilde{X}) \xleftarrow{\partial_*} \overline{H_2(\tilde{X}; S(\Lambda))} \longrightarrow \overline{H_2(\tilde{X}, \partial\tilde{X}; S(\Lambda))} \overset{D}{\approx} H^1(\tilde{X}; S(\Lambda)) \\ &\qquad \xrightarrow{\rho} \mathrm{Hom}_\Lambda\,(H_1(\tilde{X}), S(\Lambda)) \longleftarrow \mathrm{Hom}_\Lambda\,(tH_1(\tilde{X}), S(\Lambda)) \end{aligned} \tag{2}$$

where $\bar{\rho}$ is the standard Kronecker map on cohomology, and ∂_* is the Bockstein from the coefficient sequence $0 \to \Lambda \to Q(\Lambda) \to S(\Lambda) \to 0$ Note that Image $\partial_* = \overline{tH_1}(\tilde{X})$ and so any element α of $tH_1(\tilde{X})$, can be pulled back to $\alpha' \in \overline{H_2(X; S(\Lambda))}$. Any two pull-backs α', α'' differ by the image of an element of $\overline{H_2(\tilde{X}\colon Q(\Lambda))}$.

Using naturality of the maps of (2) with respect to the homomorphism $Q(\Lambda)\to S(\Lambda)$, we see that $\alpha'-\alpha''$ passes to an element of $\mathrm{Hom}_\Lambda\,(tH_1(\tilde{X}), S(\Lambda))$ which comes from $\mathrm{Hom}_\Lambda\,(tH_1(\tilde{X}), Q(\Lambda))=0$. Thus the composition defined by (2) is well-defined on $tH_1(\tilde{X})$. This reformulation is seen to be equivalent to our first definition using the definition of D via the intersection pairing.

§3

To understand the maps $\rho, \bar{\rho}$ used in our definitions of the Blanchfield pairing we need a "universal coefficient" consideration of the relation between homology and cohomology. Recall the universal coefficient spectral sequence (see [Mc]): Given a free left chain complex C_* over a ring Λ and a left module N, there exists a spectral sequence "converging" to $H^*(C; N)$, with E^2-terms given by $E^2_{pq}=\mathrm{Ext}^q_\Lambda\,(H_p(C), N)$, and differential d_r in E^r of degree $(1-r, r)$. There is a filtration

$$H^m(C; N)=J_{m0}\supseteq J_{m-1,1}\supseteq\cdots\supseteq J_{1,m-1}\supseteq J_{0,m}$$

where $J_{pq}/J_{p-1,q+1}\approx E^\infty_{pq}$. To define $\bar{\rho}$, we simply consider $H^m(C; N)=J_{m,0}\twoheadrightarrow E^\infty_{m0}\subseteq E^2_{m0}=\mathrm{Hom}_\Lambda\,(H_m(C), N)$. To define ρ (on $\mathrm{Ker}\,\bar{\rho}$), we take $\mathrm{Ker}\,\bar{\rho}=J_{m-1,1}\twoheadrightarrow E^\infty_{m-1,1}\subseteq E^2_{m-1,1}=\mathrm{Ext}^1_\Lambda(H_{m-1}(C), N)$. Looking back at (1), we see that ρ is well-defined on elements coming from $\overline{tH_1}(\tilde{X})$, since $\bar{\rho}$ is obviously zero on any torsion element when $N=\Lambda$ (and Λ is a domain).

We will consider the universal coefficient spectral sequences for $C=C^*(\tilde{X})$ and $C=C^*(\tilde{X}, \partial\tilde{X})$, with $N=\Lambda$. In each case the spectral sequence can be reduced to one or more exact sequences. This reduction is straightforward and we omit the details. The exact sequences obtained are the following:

$$0\to H^1(\tilde{X};\Lambda)\xrightarrow{\bar{\rho}}A^*\to Z\to J_{11}\xrightarrow{\rho}e^1(A)\to 0 \tag{3}$$

$$0\to J_{11}\to H^2(\tilde{X};\Lambda)\xrightarrow{\bar{\rho}}B^*\to e^2(A)\to 0 \tag{4}$$

$$e^3(A)\approx e^1(B) \tag{5}$$

$$0\to e^1(A_0)\to H^2(\tilde{X},\partial\tilde{X};\Lambda)\to B_0^*\to e^2(A_0)\to H^3(\tilde{X},\partial\tilde{X})$$

$$\to e^1(B_0)\to e^3(A_0)\to 0 \tag{6}$$

$$A_0^*\approx H^1(\tilde{X},\partial\tilde{X};\Lambda) \tag{7}$$

where we use the notation $A=H_1(\tilde{X})$, $B=H_2(\tilde{X})$, (as before) $A_0=H_1(\tilde{X},\partial\tilde{X})$, $B_0=H_2(\tilde{X},\partial\tilde{X})$, $e^i=\mathrm{Ext}^i_\Lambda(\ ,\Lambda)$ and $*=e^0=\mathrm{Hom}_\Lambda\,(\ ,\Lambda)$.

We also note the exact homology sequence:

$$0 \to B \to B_0 \to H_1(\partial\tilde{X}) \to A \to A_0 \to H_0(\partial\tilde{X}) \to H_0(\tilde{X}) \to 0. \tag{8}$$

It is easy to see that $H_*(\partial\tilde{X})$ depends only on the linking number l and is given as follows:

$$H_0(\partial\tilde{X}) = \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1) \tag{9}$$

$$H_1(\partial\tilde{X}) = \begin{cases} 0 & l \neq 0 \\ \Lambda/(x-1) \oplus \Lambda/(y-1) & l = 0 \end{cases} \tag{10}$$

In (10), when $l=0$, generators are given by the two longitudes, lifted into $\tilde{X}$.

§4

In the case $r=0$, it follows from (8) that rank $A_0 =$ rank $B_0 = 0$ also. Thus $A^* = B^* = A_0^* = B_0^* = 0$. From (3) and (7) we conclude $B_0 \approx H^1(\tilde{H}{:}\ \Lambda) = 0$ and $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda) = 0$. From (4) and (5), we conclude $e^2(A) = 0 = e^3(A)$ and so A is one-dimensional (note $e^q = 0$ for $q > 3$, since Λ has homological dimension 3).

The Blanchfield pairing $\beta: \bar{A} \to \mathrm{Hom}_\Lambda(A, S(\Lambda)) \approx e^1(A)$ can be written as the composition (according to (1)):

$$\bar{A} \to \bar{A}_0 \approx H^2(X; \Lambda) = J_{11} \to e^1(A).$$

If P denotes the null-space of β, and C the conull-space, we can deduce from (3) and (8) an exact sequence:

$$0 \to \overline{H_1(\partial\tilde{X})} \to P \to Z \to \bar{K} \to C \to 0 \tag{11}$$

where $K = \mathrm{Kernel}\{H_0(\partial\tilde{X}) \to H_0(\tilde{X}) \approx Z\}$ – from (8).

In order to analyze the map $Z \to \bar{K} \subseteq \overline{H_0}(\partial\tilde{X})$, we first recall that the edge homomorphism

$$\mathrm{Ext}^q_\Lambda(H_0(C), N) = E^2_{0q} \twoheadrightarrow E^\infty_{0q} = J_{0q} \subseteq J_{q0} = H^q(C; N)$$

is equivalent to the homomorphism induced by a chain map $C_* \to F_*$, where F_* is a free resolution of $H_0(C)$, which induces the identity map on $H_0(C) = H_0(F)$. In case $\Lambda = Z\pi$ and $C_* = C_*(\tilde{X})$, where $\tilde{X}$ is a regular π-covering of X, this coincides with the homomorphism $\mathrm{Ext}^q_\Lambda(Z, N) = H^q(\pi; N) \to H^q(\tilde{X}, N)$ induced by the classifying map $X \to B\pi$ of the covering $\tilde{X} \to X$. Now our map $Z \to \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}$ is the composition

$$Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\tilde{X}; \Lambda) \approx \overline{H_1(\tilde{X}, \partial\tilde{X})} \xrightarrow{\partial^*} \overline{H_0(\partial\tilde{X})},$$

where ε' is the edge homomorphism of the universal coefficient spectral sequence of $H^*(\tilde{X}; \Lambda)$, which coincides with the composition $Z = e^2(Z) \xrightarrow{\varepsilon'} H^2(\partial\tilde{X}: \Lambda) \approx \overline{H_0(\partial\tilde{X})}$, where ε' is the edge homomorphism of the universal coefficient spectral sequence of $H^*(\partial\tilde{X}; \Lambda)$. Now the map $\partial X \to BG$, which classifies the covering $\partial\tilde{X} \to \partial X$, is an l-fold covering on each component of ∂X (∂X is the disjoint union of two tori and BG a single torus). Therefore the induced map $H^2(G; \Lambda) \to H^2(\partial\tilde{X}; \Lambda) \approx \Lambda/(x-1, y^l-1) \oplus \Lambda/(y-1, x^l-1)$ maps a generator onto $(\phi_l(y), \phi_l(x))$, where $\phi_l(x) = \dfrac{x^l-1}{x-1}$. If $l \neq 0$, this is a monomorphism, and, since $H_1(\partial\tilde{X}) = 0$ (see (10)), we conclude $P = 0$. Furthermore we now see that $\mathrm{Cok}\{Z \to \bar{K} \subseteq \overline{H_0(\partial\tilde{X})}\}$ has a presentation $\{\alpha, \beta : (x-1)\alpha = 0 = (y-1)\beta, \phi_l(y)\alpha = \phi_l(x)\beta\}$, and it, therefore, follows from (11) that C corresponds to the submodule of elements $\lambda\alpha + \mu\beta$ ($\lambda, \mu \in \Lambda$) satisfying:

$$\lambda(1,1) + \mu(1,1) = 0.$$

It is not hard to see that C will, therefore, be generated by $\gamma = \alpha - \beta$, subject to the relations

$$(x-1)(y-1)\gamma = 0 = (\phi_l(y) + \phi_l(x) - l)\gamma.$$

To complete the protf of (B) it suffices to check that:

$$\phi_l(xy) \equiv \phi_l(x) + \phi_l(y) - l \bmod (x-1)(y-1).$$

But this follows from the easy fact that, for any $f(x, y) \in \Lambda$:

$$f(x, y) \equiv f(x, 1) + f(1, y) - f(1, 1) \bmod (x-1)(y-1).$$

§5

The longitudinal elements ξ_x, ξ_y of (C) are the generators of the image $H_1(\partial\tilde{X}) \to A$ in (8). According to (10) $(x-1)\xi_x = 0 = (y-1)\xi_y$. If $r = 0$, then $B_0 = 0$

and, from (10), we see that $x-1$ $(y-1)$ generates the annihilator of $\xi_x(\xi_y)$. Note that our computation of $Z \to \bar{K}$, in the preceding paragraph, shows that it is zero, when $l=0$, and, therefore, (11) contains the short exact sequence: $0 \to H_1(\partial\tilde{X}) \to P \to Z \to 0$.

If we can show that $P \approx \Lambda/(x-1)(y-1)$ (with generator α), then it follows that we may choose $\xi_x = (y-1)\alpha$, $\xi_y = (x-1)\alpha$ as longitudinal elements, i.e. they are images, under $H_1(\partial\tilde{X}) \to N$, of generators of the respective summands (see (10)). Since we have already proved $C \approx \Lambda/(x-1)(y-1)$, the remaining assertions of (D) follows from the Hermitian property of the Blanchfield pairing together with:

LEMMA. *Let A be a one-dimensional torsion Λ-module equipped with a sesquilinear Hermitian pairing $\langle\ ,\ \rangle$ with null-space K and conull-space C. Then $K \approx e^1(\bar{C})$ and, if $A' = A/K$, the induced pairing on A' is non-degenerate with conull-space $\approx e^2(\bar{C})$. If $e^3(C)=0$, then A' is one-dimensional.*

Proof of Lemma:

Denote the adjoint of $\langle\ ,\ \rangle$ by $\phi: A \to e^1(\bar{A})$; we have, by hypothesis an exact sequence: $0 \to K \to A \xrightarrow{\varphi} e^1(\bar{A}) \to C \to 0$. The transpose of $\phi: A \to e^1e^1A \xrightarrow{\overline{e^2\phi}} e^1\bar{A}$ coincides with ϕ (this is what Hermitian means), where $A \to e^1e^1A$ is a standard "double dual" map. Since A is one-dimensional this double dual map is an isomorphism. Now consider the diagram of exact sequences:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] & A \arrow[r] \arrow[dr, "\phi"'] & A' \arrow[r] \arrow[d] & 0 \\
 & & & e^1\bar{A} \arrow[d] & \\
 & & & C \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

From this we derive the diagram of exact sequences:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & e^1(C) \arrow[r] & e^1e^1(\bar{A}) \arrow[r] \arrow[d, phantom, "\wr"] \arrow[dr, "e^1\phi"] & e^1(A') \arrow[r] \arrow[d] & e^2(C) \arrow[r] & e^2e^1(\bar{A}) = 0 \\
 & & \bar{A} & e^1(A) & &
\end{tikzcd}$$

as well as the isomorphism $e^i(A') \approx e^{i+1}(C)$, $i \geq 2$. We immediately see that $\bar{K} \approx e^1(C)$, the cokernel of the map $\bar{A}' \to e^1(A')$, induced by $e^1\phi = \bar{\phi}$, is $e^2(C)$, and that A' is one-dimensional if $e^3(C) = 0$.

§6

From now on we will assume $r = 1$, since all the statements for $r = 0$ have been proved. We first point out that $B \approx H^1(\tilde{X}, \partial\tilde{X}; \Lambda)$, by duality, and, by (7), we then conclude $B \approx A_0^*$, which is free – over a unique factorization domain, R^* is free for any module R of rank ≤ 1.

We examine the longitudinal elements. We can define $\xi_x, \xi_y \in A$, when $l = 0$, by choosing translates of the components K_x, K_y of L into X which have 0 linking number with their associated components – since $l = 0$ these translates lift into $\tilde{X}$ defining ξ_x, ξ_y up to multiplication by a unit of Λ. Clearly ξ_x, ξ_y generate Image $\{H_1(\partial\tilde{X}) \to H_1(\tilde{X})\}$, and we have $(x-1)\xi_x = 0 = (y-1)\xi_y$ (this distinguishes ξ_x from ξ_y). We now show the existence of $\lambda(x)$, $\mu(y)$, as in (C).

Consider the infinite cyclic covering X_x of X defined by the homorphism $\Pi \to G \to Z$, which sends $x \to 1$ and $y \to 0$. Thus $\tilde{X}$ is an infinite cyclic covering of X_x, and in fact, $C_*(X_x) \approx C_*(\tilde{X})/(y-1)C_*(\tilde{X})$. We obtain, by tensoring $C_*(\tilde{X})$ with the short exact sequence:

$$0 \to \Lambda \xrightarrow{y-1} \Lambda \to \Lambda/(y-1) \to 0$$

the following exact homology sequence:

$$\begin{aligned} 0 \to H_2(\tilde{X}) \xrightarrow{y-1} H_2(\tilde{X}) \to H_2(X_x) \to H_1(\tilde{X}) \xrightarrow{y-1} H_1(\tilde{X}) \\ \to H_1(X_x) \to H_0(\tilde{X}) \xrightarrow{y-1} H_0(\tilde{X}) \end{aligned} \tag{12}$$

Now X_x is closely related to the infinite cyclic covering Y_x of the complement of K_x. In fact $\overline{Y_x - X_x}$ is the union of translates, by powers of x, of the solid torus formed by lifting a tubular neighborhood of K_y into Y_x. Thus $H_i(Y_x, X_x) \approx \Lambda/(y-1)$, if $i = 2, 3$, and zero otherwise. By considering the exact sequence of the pair (Y_x, Y_x) and the facts that $H_i(Y_x) = 0$ if $i \geq 2$, we see easily that $H_2(X_x) \approx \Lambda/(y-1)$ and obtain an exact sequence:

$$0 \to \Lambda/(y-1) \to H_1(X_x) \to H_1(Y_x) \to 0. \tag{13}$$

The sequence (12) can now be put in the simpler form:

$$0 \to \Lambda/(y-1) \to \Lambda/(y-1) \to A \xrightarrow{y-1} A \to H_1(X_x) \to Z \to 0 \tag{12'}$$

since $H_2(\tilde{X}) = B \approx \Lambda$. The image of a generator, under the injection $\Lambda/(y-1) \to \Lambda/(y-1)$ is represented by a non-zero polynomial $\lambda(x)$. Since a generator $\hat{\xi}_y$ of $H_2(X_x) \approx \Lambda/(y-1)$ is represented by the boundary torus of a tubular neighborhood of K_y (lifted into X_x), it is straightforward to check, from the definition of the boundary homomorphism $H_2(X_x) \to H_1(\tilde{X}) = A$, that $\hat{\xi}_y \to \xi_y \in A$. It follows immediately that $\lambda(x)$ and $y-1$ generate the annihilator ideal of ξ_y. A similar argument establishes the existence of $\mu(y)$.

Note from (12′) that ξ_y generates the submodule of elements invariant under y. Thus $\lambda(x)$ is defined, purely algebraically, up to unit multiple, by the property of being a generator, together with $y-1$, of the annihilator ideal of this submodule – similarly for $\mu(y)$.

We now show $\lambda(x) \mid \Delta(x)$, where $\Delta(x)$ is the Alexander polynomial of K_x – this will imply $\lambda(1) = \pm 1$. Let T be the torsion sub-module of A. We first derive from (12′) and (13) an exact sequence:

$$0 \to R \to T \xrightarrow{y-1} T \to S \to 0 \tag{14}$$

where $R = \Lambda/(\lambda(x), y-1)$, $S \subseteq H_1(Y_x)$ is the image of T under $A \to H_1(X_x) \to H_1(Y_x)$. The only point not immediately obvious is: $\operatorname{Ker}\{T \to S\} \subseteq (y-1)T$. Suppose $\alpha \in T$ and $\alpha \to 0$ in S. If $\alpha \to 0$ in $H_1(X_x)$, then $\alpha = (y-1)\beta$ for some $\beta \in A$, by exactness of (12′). But then $\alpha \in T$ implies $\beta \in T$. To see $\alpha \to 0$ in $H_1(X_x)$ it suffices by (13) to show $f(x)\alpha \to 0$ for any non-zero $f(x)$. But, since $\alpha \in T$, $f(x, y)\ \alpha = 0$ for some non-zero $f(x, y)$. If we write $f(x, y) = f(x) + (y-1)\ g(x, y)$, then $0 = f(x)\ \alpha + (y-1)\ g(x, y)\alpha$. Since $(y-1)\ A \to 0$ in $H_1(X_x)$, so does $f(x)\alpha$. If $f(x) = 0$, then, by (12′), $\lambda(x)\ g(x, y)\ \alpha = 0$. But this would be impossible if we had chosen $f(x, y)$ with the smallest number of $y-1$ factors.

Now recall that $\Delta(x) = \Delta(H_1(Y_x))$, where $\Delta(A)$, for any Λ_x-module A ($\Lambda_x = z[x, x^{-1}] \approx \Lambda/(y-1)$) is the greatest common divisor of the *order ideal* of A (see [L]). We also recall the following property of $\Delta(A)$: if $0 \to A' \to A \to A'' \to 0$ is a short exact sequence of Λ_x-modules, then $\Delta(A) = \Delta(A')\ \Delta(A'')$ (see [L] for a proof). Thus, for example, $\Delta(S) \mid \Delta(x)$ and, so, it suffices to prove that $\lambda(x)$ ($= \Delta(R)$, R considered as a Λ_x-module) divides $\Delta(S)$. Define

$$T_i = \frac{\operatorname{Ker} \phi^{i+1}}{\operatorname{Ker} \phi^i} \quad \text{and} \quad T^i = \frac{\phi^i T}{\phi^{i+1} T},$$

where $\phi: T \to T$ is multiplication by $y-1$. These are Λ_x-modules and we have a family of short exact sequences: $0 \to T_{i+1} \to T_i \to T^i \to T^{i+1} \to 0$, for $i \geq 0$ (see [L2]). From (14) we see that $T_0 \approx R$ and $T^\circ \approx S$. From the above-mentioned multiplicative property of Δ we have $\Delta(T_{i+1})\Delta(T^i) = \Lambda(T_i)\,\Delta(T^{i+1})$ for $i \geq 0$. Therefore, we see that $\Delta(T_{i+1}) \mid \Delta(T^{i+1})$ would imply $\Delta(T_i) \mid \Delta(T^i)$ – note that these are all non-zero, since $\Delta(T_{i+1}) \mid \Delta(T_i)$, $\Delta(T^{i+1}) \mid \Delta(T^i)$ and $\Delta(T_0)$, $\Delta(T^0)$ are non-zero. Thus it suffices to show $\Delta(T_i) \mid \Delta(T^i)$ for some value of i. But $T_i = 0$, for large enough i, since $\{\operatorname{Ker} \phi_i\}$ is an increasing sequence of submodules in a finitely-generated module over a Noetherian ring. This completes the proof.

Of course, by a similar argument, we can show $\mu(y) \mid \Delta(y)$.

We can now show that P_0, the submodule of A generated by ξ_x and ξ_y, is the submodule of elements annihilated by $(x-1)(y-1)$. Suppose $(x-1)(y-1)\ \alpha = 0$; then $(y-1)\alpha = f\xi_x$ for some $f \in \Lambda$. So $\mu(y)(y-1)\alpha = 0$ which means $\mu(y)\ \alpha = g\xi_y$. Since $\mu(1) = 1$, we may write $\mu(y) = 1 + (y-1)\mu'(y)$ and so $\alpha + (y-1)\mu'(y)\ \alpha = g\xi_y$ or $\alpha + \mu'(y)\ f\xi_x = g\xi_y$. Thus $\alpha \in P_0$.

§7

We now examine fA and prove $fA \otimes_\Lambda Z$ is infinite cyclic. (over Z) We already know $A \otimes_\Lambda Z$ is infinite cyclic, which implies $fA \otimes_\Lambda Z$ is cyclic. If $fA \otimes_\Lambda Z$ were finite of order $k > 0$, then $fA \otimes_\Lambda Z/p = 0$, for any p relatively prime to k. If so, by Nakayama's lemma, $\Delta \cdot fA = 0$ for some $\Delta \notin M_p$, where $M_p = \ker\{\Lambda \to Z/p\}$. But fA is torsion-free. If we define I to be the ideal of Λ with greatest common divisor 1 which is isomorphic to fA, then $I + M = \Lambda$. To see this choose $\lambda \in I$ which generates $I/MI \approx I \otimes Z \approx Z$ – we will show $\lambda(1,1) = \pm 1$. $M(I/(\lambda)) = I/(\lambda)$, which implies, by Nakayma's lemma, that $\Delta \cdot I/(\lambda) = 0$, i.e. $\Delta I \subseteq (\lambda)$, for some $\Delta \equiv 1 \bmod M$ – i.e. $\Delta(1,1) = \pm 1$. Since I has greatest common divisor one, $\Delta \in (\lambda)$ and so $\lambda(1,1) = \pm 1$. To see that $tA \otimes_\Lambda Z = 0$ (when $r = 1$) consider the short exact sequence $0 \to tA \to A \to fA \to 0$ and apply $\otimes_\Lambda Z$ to obtain $\operatorname{Tor}^1(fA, Z) \to tA \otimes Z \to A \otimes Z \to fA \otimes Z \to 0$. Since $A \otimes Z \approx Z \approx fA \otimes Z$, it suffices to show $\operatorname{Tor}^1(fA, Z)\ \operatorname{Tor}^1(I, Z) = 0$. Now $\operatorname{Tor}^1(I, Z) = \operatorname{Tor}^2(\Lambda/I, Z)$ which can be considered to be the submodule of *invariant* elements of Λ/I – i.e. of elements α satisfying $x\alpha = y\alpha = \alpha$. But $\lambda\alpha = 0$, where $\lambda \in I$ satisfying $\lambda(1,1) = 1$ has been found in the preceding paragraph, and so $0 = (1 + (x-1)\lambda' + (y-1)\lambda'')\alpha = \alpha + \lambda'(x-1)\alpha + \lambda''(y-1)\ \alpha = \alpha$.

By the results of §7, we may break (8) up into two shorter exact sequences (for

$r = 1$):

$$0 \to B \to B_0 \to \Lambda/(x-1) \oplus \Lambda/(y-1) \to 0 \tag{15a}$$

$$0 \to \Lambda/(\rho, (x-1)(y-1)) \to A \to A_0 \to \Lambda/(x-1)(y-1) \to 0 \tag{15b}$$

where $\rho = \lambda(x) + \mu(y) - 1$ (choosing $\lambda(1) = 1 = \mu(1)$). Note that the quotient of $\Lambda/(x-1) \oplus \Lambda/(y-1)$ by the submodule generated by $(\mu(y), 0)$ and $(0, \lambda(x))$ is isomorphic to $\Lambda/(\rho, (x-1)(y-1))$, using the generator $(1, 1)$, and the kernel of the epimorphism $\Lambda/(x-1) \oplus \Lambda/(y-1) \to Z$ is isomorphic to $\Lambda/(x-1)(y-1)$, using the generator $(1, -1)$. Applying Hom (, Λ) to (15a) yields an exact sequence:

$$0 \to B_0^* \to B^* \to \Lambda/(x-1) \oplus \Lambda/(y-1) \to e^1(B_0) \to e^1(B).$$

Now $e^1(B) = 0$, since B is free. From (3), we conclude that $\bar{B}_0 \approx H^1(\tilde{X}; \Lambda)$ is free or isomorphic to M (the ideal in Λ generated by $(x-1, y-1)$), since A^* is free. But (15a) is possible only if $\bar{B}_0 \approx M$. Thus $e^1(B_0) \approx Z$. We, therefore, have the exact sequence:

$$0 \to B_0^* \to B^* \to \Lambda/(x-1)(y-1) \to 0 \tag{16}$$

We now apply Hom (, Λ) to (15b) and obtain exact sequences:

$$0 \to A_0^* \to A^* \to \Lambda/(x-1)(y-1) \to e^1(A_0) \to e^1(A) \to 0 \tag{17a}$$

$$0 \to e^2(A_0) \to e^2(A) \to \Lambda/(\rho, (x-1)(y-1)) \to e^3(A_0) \to 0. \tag{17b}$$

Note that

$$e^1(\Lambda(\rho, (x-1)(y-1)) = 0$$

$$e^2(\Lambda/(\rho, (x-1)(y-1)) \approx \Lambda/(\rho, (x-1)(y-1))$$

and $e^3(A) = 0$ (by (5), since B is free).

We now examine the homomorphisms $tA \to tA_0$ and $fA \to fA_0$, using (15b). Denoting the kernel and cokernel, respectively, by K_1, K_2 and C_1, C_2, we can apply the snake lemma, using (15b) to obtain an exact sequence: $0 \to K_1 \to \Lambda/(\rho, (x-1)(y-1)) \to K_2 \to C_1 \to \Lambda/(x-1)(y-1) \to C_2 \to 0$. Since K_2 is torsion-free and C_1 torsion, we see that $K_2 = 0$. For some ideal $S \supseteq (x-1)(y-1)$, we have

$$C_1 \approx S/(x-1)(y-1), \qquad C_2 \approx \Lambda/S$$

and we have:

$$0 \to \Lambda/(\rho, (x-1)(y-1)) \to tA \to tA_0 \to S/(x-1)(y-1) \to 0 \tag{18a}$$

$$0 \to fA \to fA_0 \to \Lambda/S \to 0. \tag{18b}$$

§8

We now deduce some facts from (3)–(6) and duality:

$$\overline{tA} = e^1(A_0). \tag{19}$$

This follows from (6), since $e^1(A_0)$ is torsion and B_0^* is free.

$$0 \to \bar{B}_0 \to A^* \to Z \to \overline{tA_0} \to e^1(A) \to 0. \tag{20}$$

This is just (3), since $J_{11} \approx \overline{tA_0}$ from (4).

$$0 \to \overline{fA_0} \to B^* \to e^2(A) \to 0. \tag{21}$$

This follows from (4).

$$0 \to \overline{fA} \to B_0^* \to e^2(A_0) \to 0, \; e^3(A_0) \approx Z/k, \qquad \text{(some } k > 0). \tag{22}$$

This follows from (6), since $H^3(\tilde{X}, \partial\tilde{X}) \approx Z$ and $e^3(A_0)$ cannot be isomorphic to Z, since $e^2(Z) \neq 0$ but $e^2 e^3 = 0$ over Λ (see [Ba]).

We use the map $\tilde{X} \to (\tilde{X}, \partial\tilde{X})$ to map (22) $\to$ (21). Using (18b), (16) and (17b), we obtain a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{fA} & \to & B_0^* & \to & e^2(A_0) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{fA_0} & \to & B^* & \to & e^2(A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Lambda/S & & \Lambda/(x-1)(y-1) & & \Lambda/(\rho, (x-1)(y-1)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & e^3(A_0) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{23}$$

From the snake lemma we deduce an exact sequence:

$$0 \to \Lambda/S \to \Lambda/(x-1)(y-1) \to \Lambda/(\rho, (x-1)(y-1)) \to e^3(A_0) \to 0. \tag{24}$$

From this sequence we may deduce: $e^3(A_0)=0$ and $S=(x-1)(y-1)$. The first of these follows from the fact that any epimorphism $\Lambda \to Z/k$ is of the form $f(x, y) \to af(1, 1)$, where $a \in Z$ is relatively prime to k, and $\rho(1, 1)=1$. To see the second, let $\alpha \in \Lambda$ represent the image of 1 under $\Lambda/S \to \Lambda/(x-1)(y-1)$. Then $(\alpha, (x-1)(y-1))=(\rho, (x-1)(y-1))$ and so $\alpha(1, 1)=\pm 1$. But $\beta \in S$ if and only if $(x-1)(y-1) \mid \alpha\beta$ and so $S \subseteq (x-1)(y-1)$. Since we already know $S \supseteq (x-1)(y-1)$, we have $S=(x-1)(y-1)$.

Now define J to be the ideal of Λ, with greatest common divisor 1, isomorphic to fA_0. We can rewrite (23) as follows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r, "\tau'"] \arrow[d, "\tau"] & \Lambda \arrow[r] \arrow[d] & e^2(\bar{A}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r, "\tau''"'] \arrow[d] & \Lambda \arrow[r] \arrow[d] & e^2(\bar{A}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda/(x-1)(y-1) \arrow[r] \arrow[d] & \Lambda/(x-1)(y-1) \arrow[r] \arrow[d] & \Lambda/(\bar{\rho}, (x-1)(y-1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{25}$$

The maps indicated by τ, τ', τ'' and τ_0 are all multiplication by elements of $Q(\Lambda)$ – we also use $\tau, \tau', \tau'', \tau_0$ to denote these elements. Obviously $\tau_0 \in \Lambda$ and is a unit multiple of $(x-1)(y-1)$ and, since I and J have greatest common divisor one, τ' and τ'' are also in Λ. Now $e^2(\bar{A}_0)$ and $e^2(\bar{A})$ are pseudo-null since they are grade ≥ 2 (see [Ba]) and grade ≥ 2 means pseudo-null (see [R]). Therefore τ' and τ'' must be units of Λ; so $\tau \in \Lambda$ and is a unit multiple of τ_0 or $(x-1)(y-1)$.

Now choose an element $\sigma \in J$ which maps onto a generator of $\Lambda/(x-1)(y-1)$. Therefore $J=(x-1)(y-1)I+(\sigma)$. From the left-most vertical row of (25) we see that $f\sigma \in (x-1)(y-1)I$ if and only if $(x-1)(y-1) \mid f$. If $f=(x-1)(y-1)$, this says $\sigma \in I$, and so $J \subseteq I$. If σ' is another element such that $J=(x-1)(y-1)I+(\sigma')$, then a straight-forward computation shows $\sigma'=\alpha\sigma \bmod (x-1)(y-1)\ I$, where $\alpha = u \bmod (x-1)(y-1)$ for some unit u of Λ. Since $\sigma \in I$, we have $\sigma' \equiv u\sigma \bmod (x-1)(y-1)\ I$ and so σ is well-defined up to unit multiple, mod $(x-1)(y-1)\ I$. Finally, it follows from (25) that $(\sigma, (x-1)(y-1))=(\bar{\rho}, (x-1)(y-1))$, which implies $\sigma \equiv u'\bar{\rho} \bmod (x-1)(y-1)$ for some unit u' of Λ.

§9

We now determine the null-space N and co-null space C of the Blanchfield pairing. Its adjoint $\overline{tA} \to e^1(tA)$, whose kernel and cokernel are N and C, can be described as the composition:

$$\overline{tA} \to e^1(A_0) \to e^1(tA_0) \to e^1(tA) \tag{26}$$

where the first homomorphism is the isomorphism of (19) and the others are induced by inclusion $tA_0 \subseteq A_0$ and $A \to A_0$. By its Hermitian property this coincides with the composition;

$$\overline{tA} \to \overline{tA_0} \to e^1(A) \to e^1(tA). \tag{27}$$

The middle map comes from (20).

In (26), the last map is also an isomorphism – this follows from (18a), since $S = (x-1)(y-1)$ and $e^0(\Lambda/(\rho, (x-1)(y-1)) = e^1(\Lambda/(\rho, (x-1)(y-1)) = 0$. Thus N and C are isomorphic to the kernel and cokernel, respectively, of $e^1(A_0) \to e^1(tA_0)$. From the short exact sequence $0 \to tA_0 \to A_0 \to fA_0 \to 0$, we conclude $N \approx e^1(fA_0) = e^1(J)$. Since $e^1(tA) \approx \mathrm{Hom}_\Lambda(tA, S(\Lambda))$ is pseudo-null free, and $e^1(J) \approx e^2(\Lambda/J)$ is pseudo-null, it follows that N is the pseudo-null submodule of $\overline{tA}$.

We show that the map $\overline{tA_0} \to e^1(A)$ in (27) is an isomorphism. Referring to (20) we have already seen that $A^* \to Z$ is non-trivial, since $B_0 \approx M$ and A^* is free. It remains to show that tA_0 cannot contain a submodule isomorphic to Z/k, unless $k = 0$ or 1. But we have seen $e^3(A_0) = 0$ and an inclusion $Z/k \to A_0$ would induce an epimorphism $e^3(A_0) \to e^3(Z/k) \approx Z/k$ (if $k > 0$).

From the short exact sequence $0 \to tA \to A \to fA \to 0$ we deduce an exact sequence.

$$0 \to e^1(I) \to e^1(A) \to e^1(tA) \to e^2(I) \to e^2(A) \to e^2(tA) \to 0 \tag{28}$$

since $fA \approx I$ and $e^3(I) \approx e^4(\Lambda/I) = 0$. From (28) and (18a), we can deduce exact sequences:

$$0 \to \Lambda/(\bar{\rho}, (x-1)(y-1)) \to N \to e^1(I) \to 0 \quad \text{and}$$

$$0 \to C \to e^2(I) \to e^2(A) \to e^2(tA) \to 0.$$

Recall $S = (x-1)(y-1)$. Since $e^2(A) \approx \Lambda/\bar{J}$ from (25), we have completed the proof of (F).

§10

To prove (G) we will use the following special case of a result of Bailey [By]

THEOREM. *Let* (λ_{ij}) *be an* $(n\times n)$*-matrix over* Λ *satisfying* (i) $\lambda_{11}=0$; (ii) $\lambda_{ij}=\bar{\lambda}_{ji}$ *if* $n\geq i, j>1$ (iii) $\lambda_{1j}=(x^{-1}-1)\,(y^{-1}-1)\,\bar{\lambda}_{j1}$ *for* $1\leq j\leq n$ (iv) $\lambda_{ij}(1,1)=\pm\delta_{ij}$ *if* $n\geq i, j>1$. *Then there exists a link, with* $l=0$, *whose module A has presentation*

$$\left\{\alpha_1,\dots,\alpha_n:\sum_{j=1}^{n}\lambda_{ij}\alpha_j=0,\ i=1,\dots,n\right\}.$$

Since a proof this theorem has not appeared in a journal, we present one in the Appendix. We point out that our proof is very different from Bailey's. Also see [N]. ((λ_{ij}) is referred to as a *presentation* matrix of A.) We prove two lemmas.

LEMMA 1. *Let A be a link module with a presentation matrix* (λ_{ij}) *satisfying* (i)–(iv). *Suppose* (σ_i) *is an* $(n\times 1)$*-row vector, whose entries are relatively prime, such that* $\sum_{i=1}^{n}\sigma_i\lambda_{ij}=0$ *for* $j=1,\dots,n$. *Then* $\sigma_1(s,1)$ *and* $\sigma_1(1,y)$ *are the longitudinal orders of A.*

The next lemma deals with a more general situation.

LEMMA 2. *Let* (λ_{ij}) *be an* $(n\times n)$ *matrix over a domain* Λ, *a presentation matrix of a module A of rank one. Let M be the* $(n-1)\times(n-1)$*-matrix* (λ_{ij}), $2\leqslant i,j\leqslant n$ *and suppose* $\Delta=\det M\neq 0$. *Let* $(\mu_{ij})=\Delta\cdot M^{-1}(2\leq i,j\leq n)$, *the cofactor matrix of M and set* $\rho_i=\sum_{j=2}^{n}\mu_{ij}\lambda_{j1}$. *Then fM is isomorphic to the ideal of* Λ *generated by* $(\Delta,\rho_2,\dots,\rho_n)$.

Proof of Lemma 1. Let $0\to W\to F_0\xrightarrow{d}F_1\to A\ 0$ be the resolution defined by (λ_{ij}), i.e. F_0 and F_1 are free modules of rank n with bases $\{\alpha_i\}\{\beta_i\}$ with $d(\beta_i)=\sum_j\lambda_{ij}\alpha_j$. Since rank $A=1$ and projective dimension $A\leq 2$, W is free of rank one. If a generator of $W\subset F_0$ is $\sum_i\sigma_i'\beta_i$, then $\sigma_i'=u\sigma_i$, for some unit u in Λ. Let $\Lambda_x=\Lambda/(y-1)$: then $\mathrm{Tor}_1^{\Lambda}(A,\Lambda_x)$ is the submodule of elements of A annihilated by $y-1$ (using exact sequence $0\to\Lambda\xrightarrow{y-1}\Lambda\to\Lambda_x\to 0$) which, by (C), is isomorphic to $\Lambda_x/(\lambda(x))$. Using the resolution of A given above $\mathrm{Tor}_1^{\Lambda}(A,\Lambda_x)$ is the homology of the chain complex:

$$W\otimes_{\Lambda}\Lambda_x\xrightarrow{d'}F_0\otimes_{\Lambda}\Lambda_x\xrightarrow{d''}F_1\otimes_{\Lambda}\Lambda_x$$

where the modules are free over Λ_x and d', d'' are represented by the matrices: $(\sigma_i(x, 1))$ and $(\lambda_{ij}(x, 1))$, respectively. Since $\lambda_{1j}(x, 1) = 0$, for all j, and $\lambda_{ij}(1, 1) = \pm\delta_{ij}$ for $i, j \geq 2$, it follows easily that kernel d'' is the free submodule of $F_0 \otimes_\Lambda \Lambda_x$ generated by $\beta_1 \otimes 1$. Thus, since Image $d' \subseteq$ kernel d'', $\sigma_i\ (x, 1) = 0$ for $i > 1$, and $\mathrm{Tor}_1^\Lambda(A, \Lambda_x) \approx \Lambda/(\sigma_1(x, 1))$.

A similar argument for $\mu(y)$ completes the proof.

Proof of Lemma 2. Suppose (ρ_{ij}), $1 \leq i, j \leq n$, is any matrix over Λ; consider the module A' presented by the product matrix $(\rho_{ij})(\lambda_{ij})$ – i.e. $A' = \{\beta_1, \ldots, \beta_n : \sum_{j,s} \rho_{is}\lambda_{sj}\ \beta_j = 0,\ i = 1, \ldots, n\}$. If $\{\alpha_i\}$ are the generators of A, subject to relations $\sum_j \lambda_{ij}\alpha_j = 0$ $(i = 1, \ldots, n)$, then $\beta_i \to \alpha_i$ defines an epimorphism $\phi: A' \to A$. The kernel of ϕ is generated by $\{\gamma_i\}$, where $\gamma_i = \sum_j \lambda_{ij}\beta_j$ $(i = 1, \ldots, n)$, and the $\{\gamma_i\}$ are subject to relations $\sum_j \rho_{ij}\gamma_j = 0$ $(i = 1, \ldots, n)$. We apply these observations to the matrix (ρ_{ij}) given by

$$\rho_{ij} = \begin{cases} \mu_{ij} & i, j \geq 2 \\ \delta ij & i = 1 \quad \text{or} \quad j = 1. \end{cases}$$

The matrix $(\sigma_{ij}) = (\rho_{ij})(\lambda_{ij})$ is given by

$$\sigma_{ij} = \begin{cases} \lambda_{ij} & i = 1 \\ \rho_i & j = 1, \qquad i > 1 \\ \Delta\delta_{ij} & i, j \geq 2. \end{cases}$$

Now $\det(\rho_{ij}) = \Delta \neq 0$, which implies, since (ρ_{ij}) is a relation matrix for Ker ϕ, that Ker ϕ is a torsion module. Thus ϕ induces an isomorphism $fA \approx fA'$. To compute fA', we define a homomorphism $\psi: A' \to \Lambda$ by $\psi(\beta_1) = -\Delta$, $\psi(\beta_i) = \rho_i$ for $i \geq 2$. This is well-defined since it preserves the relations given by all the rows of (σ_{ij}), except perhaps the first – but, since rank $A' = 1$, the rows of (σ_{ij}) are linearly dependent and, therefore, the relations given by the first row must also be preserved (note that rows 2 through n are linearly independent). Since rank $A' = 1$, ψ induces an isomorphism $fA' \approx$ Image $\psi = (\Delta, \rho_2, \ldots, \rho_n)$. This completes the proof of lemma 2.

§11

We can now prove the realization theorem (G). Let $\sigma(x, y) = \lambda(x^{-1}) + \mu(y^{-1}) - 1$ and choose elements $\tau_1, \ldots, \tau_k \in I$ so that $(\sigma, \tau_1, \ldots, \tau_k) = I$.

Define the $(n \times n)$-matrix (λ_{ij}), where $n = 2k+1$ as follows:

$$\lambda_{ij} = \begin{cases} 0 & i = 1 = j \\ \bar{\sigma}\tau_{i-1} & 2 \leq i \leq k+1, j = 1 \\ \tau_{i-k-1} & k+1 < i \leq n, j = 1 \\ (x^{-1}-1)(y^{-1}-1)\sigma\bar{\tau}_{j-1} & i = 1, 2 \leq j \leq k+1 \\ (x^{-1}-1)(y^{-1}-1)\bar{\tau}_{j-k-1} & i = 1, k+1 < j \leq n \\ -\delta_{ij}\sigma\bar{\sigma} & k+1 \geq i \geq 2 \\ \delta_{ij} & n \geq i \geq k+1. \end{cases}$$

This matrix satisfies the conditions of Bailey's theorem and is, therefore, the presentation matrix of a 2-link module A.

We can define a row-vector (σ_i) satisfying the hypothesis of lemma 1 by setting:

$$\sigma_i = \begin{cases} \bar{\sigma} & i = 1 \\ (x^{-1}-1)(y^{-1}-1)\bar{\tau}_{i-1} & 2 \leq i \leq k+1 \\ -(x^{-1}-1)(y^{-1}-1)\bar{\sigma}\bar{\tau}_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Since $\sigma, \tau_1, \ldots \tau_k$ are relatively prime, and $\sigma(1,1) = 1$, the $\{\sigma_i\}$ are relatively prime. Clearly $\sigma_1(x, 1) = \lambda(x)$, $\sigma_1(1, y) = \mu(y)$ and so, by lemma 1, these are the longitudinal orders of A. To show $fA \approx I$, we apply lemma 2. For our matrix (λ_{ij}), $\Delta = (-\sigma\bar{\sigma})^k$ and (μ_{ij}) is given by:

$$\mu_{ij} = \begin{cases} (-\sigma\bar{\sigma})^{k-1}\delta_{ij} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\delta_{ij} & k+1 < i \leq n. \end{cases}$$

Then

$$\rho_i = \sum_j \mu_{ij}\lambda_{j1} = \begin{cases} (-\sigma)^{k-1}\bar{\sigma}^k\tau_{i-1} & 2 \leq i \leq k+1 \\ (-\sigma\bar{\sigma})^k\tau_{i-k-1} & k+1 < i \leq n. \end{cases}$$

Thus $fA \approx$ ideal generated by $\{(-\sigma\bar{\sigma})^k, (-\sigma)^{k-1}\bar{\sigma}^k\tau_i (1 \leq i \leq k), (-\sigma\bar{\sigma})^k\tau_i (1 \leq i \leq k)\}$.

If we divide out $\pm\sigma^{k-1}\bar{\sigma}^k$ from these elements, we find $fA \approx$ ideal generated by $\{\sigma, \tau_i, \sigma\tau_i\} = \{\sigma, \tau_i\} = I$.

This completes the proof of (G).

Appendix

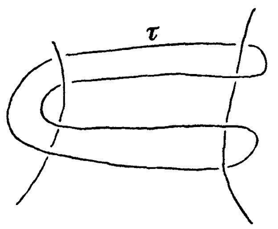
We outline a proof of Bailey's Theorem as stated in §11. The construction of the desired link proceeds, in the spirit of [L], by surgery on the complement of the "unlink", i.e. the link formed by the boundary of two disjoint 2-disks in 3-space.

Let X_0 be the complement of the unlink – then $H_1(\tilde{X}_0)$ is free of rank one. Choose a generator e of $H_1(\tilde{X}_0)$ and let $\{\sigma_i\}$ $(2 \le i \le n)$ be disjoint imbedded circles in X_0 which lift to imbedded circles $\{\tilde{\sigma}_i\}$ in $\tilde{X}_0$ such that $\tilde{\sigma}_i$ represents $\lambda_{i1}e$. We would also like $\{\sigma_i\}$, considered as a link in 3-space, to be the $(n-1)$-component unlink. If we give each σ_i the normal framing which winds once around and do surgery on S^3, using these framed imbedded circles, the result Σ, as in [L], is again diffeomorphic to S^3. The desired link L will be the original unlink regarded, now, as a link in Σ.

Let Y be the complement of the $\{\sigma_i\}$ in X_0 and X be the complement of L in Σ. $\tilde{Y}$ and $\tilde{X}$ will be the coverings of Y and X inherited from $\tilde{X}_0$; $\tilde{X}$ is the universal abelian covering of X. To compute $H_1(\tilde{Y})$ we examine the homology sequence of $(\tilde{X}_0, \tilde{Y})$. From this we conclude that $H_1(\tilde{Y})$ is generated by elements $\{e', \varepsilon_2, \ldots, \varepsilon_n\}$ where $e' \to e$ under the inclusion $\tilde{Y} \to \tilde{X}_0$, and ε_i is represented by a small circle which links $\tilde{\sigma}_i$ simply. There is a single relation $\sum_{i=2}^n \alpha_i\varepsilon_i = 0$, where $\alpha_i = E \cdot \tilde{\sigma}_i$, the intersection in Λ of a generator E of $H_2(\tilde{X}_0)$ with $\tilde{\sigma}_i$. Since $\tilde{\sigma}_i$ represents $\lambda_{i1}e$, we have

$$\alpha_i = \bar{\lambda}_{i1}(E \cdot e).$$

Finally, one may calculate $E \cdot e = (x-1)(y-1)$ by a direct computation: E is represented by a 2-sphere separating the components of the unlink and e is represented by the loop τ as follows:



So the relation is $(x-1)(y-1)\sum_{i=2}^n \bar{\lambda}_{i1}\varepsilon_i = 0$.

To compute $H_1(\tilde{X})$ we now examine the homology sequence of $(\tilde{X}, \tilde{Y})$. From this we conclude that $H_1(\tilde{X})$ has generators $e'', \varepsilon_2', \ldots, \varepsilon_n'$, the images of $e', \varepsilon_2, \ldots, \varepsilon_n$ under the inclusion $\tilde{Y} \to \tilde{X}$, with the relation:

$$(x-1)(y-1)\sum_{i=2}^n \bar{\lambda}_{i1}\varepsilon_i' = 0$$

and, in addition, new relations

$$(*)\quad \lambda_{i1}e''+\sum_j \lambda'_{ij}\varepsilon'_j=0, \quad \text{for some } \{\lambda'_{ij}\}.$$

$\lambda_{i1}e'+\sum_j \lambda'_{ij}\varepsilon_j\in H_1(\tilde{Y})$ is the class represented by the circle $\tilde{\sigma}'_i$ obtained by translating $\tilde{\sigma}_i$ along one of the vector fields of the normal framing of $\tilde{\sigma}_i$ used in the surgery. That the coefficient of e' is λ_{i1} follows from the fact that $\tilde{\sigma}_i$ represents $\lambda_{i1}e$ in $H_1(\tilde{X}_0)$. We show that the correct original choice of e' results in the following properties:

(i) $\lambda'_{ij}=\bar{\lambda}'_{ji}$

(ii) $\phi(\lambda'_{ij})=\delta_{ij}$

where $\phi: \Lambda\to Z$ is the usual augmentation $f(x, y)\to f(1, 1)$.

LEMMA. *Suppose X is a compact oriented 3-manifold, $\tilde{X}\to X$ a regular covering with τ as the group of covering transformations. Let $T_1, \ldots, T_n$ be tori components of ∂X which lift to $\tilde{T}_i\subseteq\tilde{X}$ trivially covering T_i, for each i. Let α_i, β_i be the canonical generators of $H_1(\tilde{T}_i)$ represented by meridian and longitude circles. Satisfying $\alpha_i\alpha_j=0=\beta_i\beta_j$ and $\alpha_i\cdot\beta_j=\delta_{ij}$. If $\sum_j \lambda_{ij}i_*(\alpha_j)+\sum_j \mu_{ij}u_*(\beta_j)=0$, $i=1,\ldots, m$, is any set of relations in $H_1(\tilde{X})$, $i: \tilde{T}_i\subseteq\tilde{X}$, then, for any i, j*

$$\sum_s \lambda_{is}\bar{\mu}_{js}=\sum_s \mu_{is}\bar{\lambda}_{js},$$

where $\mu\to\bar{\mu}$ is the usual conjugation in $Z\pi$.

Proof. Write $\sum_j(\lambda_{ij}\alpha_j+\mu_{ij}\beta_j)=\partial_*\theta_i$ for some $\theta_i\in H_2(\tilde{X}, \tilde{T})$ where $\partial_*: H_2(\tilde{X}, \tilde{T})\to H_1(\tilde{T})$ is the boundary homomorphism. Then, using the property: If $\alpha\in H_1(\tilde{T})$, $\theta\in H_2(\tilde{X}, \tilde{T})$, then $\partial_*\theta\cdot\alpha=\theta\cdot i_*\alpha$ we conclude that $\theta_i\cdot i_*(\alpha_j)=-\mu_{ij}$; $\theta_i\cdot i_*(\beta_j)=\lambda_{ij}$. Now

$$\begin{aligned}0=\theta_i\cdot(i_*\partial_*\theta_j)&=\theta_i\cdot\sum_k(\lambda_{jk}i_*(\alpha_k)+\mu_{jk}i_*(\beta_k))\\&=\sum_k(\bar{\lambda}_{jk}\theta_i\cdot i_*(\alpha_k)+\bar{\mu}_{jk}\theta_i\cdot i_*(\beta_k))\\&=\sum_k(-\bar{\lambda}_{jk}\mu_{ik}+\bar{\mu}_{jk}\lambda_{ik}).\end{aligned}$$

We have the equality $\tilde{\sigma}'_i=\lambda_{i1}e'+\sum_j\lambda'_{ij}\varepsilon_j$ in $H_1(\tilde{Y})$. If we remove a tubular neighborhod of the loop τ, representing e', from Y to obtain a new manifold W, we obtain new equations: $\tilde{\sigma}'_{0i}=\lambda_{i1}e'_0+\sum\lambda'_{ij}\varepsilon_{0j}+\mu_iC$ in $H_1(\tilde{W})$ where C is represented by a meridian of the newly removed tube, e'_0 is represented by a translate $\tilde{\tau}'$ of $\tilde{\tau}$ into $\tilde{W}$, and $\varepsilon_{0j} \to \varepsilon_j$, $\sigma'_{0i} \to \sigma'_i$. We apply the lemma to these relations and conclude:

$$\lambda'_{ij} - \lambda_{i1}\mu_j = \bar{\lambda}'_{ji} - \bar{\mu}_i\bar{\lambda}_{j1}$$

assuming that $\{\varepsilon_j\}$ and C are oriented correctly. We now replace our original choice of e' by $e' + \sum_j \mu_j\varepsilon_j$ and check that λ'_{ij} is replaced by $\lambda'_{ij} - \lambda_{i1}\mu_j$. Now property (i) is satisfied.

To verify property (ii), we need to add to the above argument the constraint that τ' be chosen to have linking number 0 with τ in S^3. If we now project everything to $W \subseteq S^3$, the above equations imply:

(a) $\phi(\mu_i) = l(\tau, -\sigma_i + \phi(\lambda_{i1})\tau')$

(b) $\phi(\lambda'_{ij}) = l(\sigma_j, \sigma'_i - \phi(\lambda_{i1})\tau)$

where l denotes linking number in S^3. Since $l(\tau, \tau') = 0$ by choice, and $l(\sigma_i, \sigma'_j) = \delta_{ij}$ by definition of σ'_j, (a) and (b) imply:

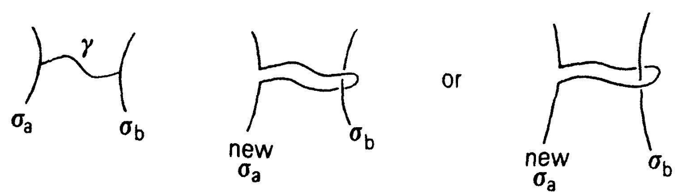
$$\phi(\lambda'_{ij}) = \delta_{ij} + \phi(\lambda_{i1})\phi(\mu_j)$$

or $\phi(\lambda'_{ij} - \lambda_{i1}\mu_j) = \delta_{ij}$, as desired.

We finally propose to alter the $\{\sigma_i\}$ in order to change the $\{\lambda'_{ij}\}$ to the prescribed $\{\lambda_{ij}\}$ for $2 \le i, j \le n$. As a preliminary consideration we show how to make certain elementary changes in the $\{\lambda'_{ij}\}$. Choose $g \in G$, and $2 \le a, b \le n$; we will change σ_a to effect the change:

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm g & i = a, j = b, a \ne b \\ \lambda'_{ij} \pm g^{-1} & i = b, j = a, a \ne b \\ \lambda'_{ij} \pm (g + g^{-1}) & i = j = a = b \\ \lambda'_{ij} & (i, j) \ne (a, b) \text{ or } (b, a). \end{cases}$$

Choose an arc $\tilde{\gamma}$ in $\tilde{X}_0$ from $\tilde{\sigma}_a$ to $g\tilde{\sigma}_b$ avoiding all lifts of σ_i, τ, and use γ to form a connected sum of σ_a with a small circle linking σ_b, as in the following picture:

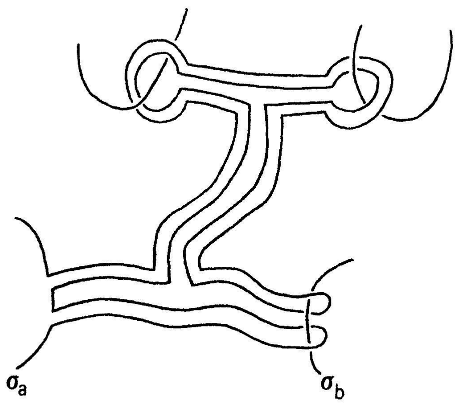


To see that the $\{\lambda'_{ij}\}$ are changed as claimed, we use the following characterization: given chains θ_i in $\tilde{X}_0$ such that $\tilde{\sigma}_i - \lambda_{i1}\tilde{\tau} = \partial\theta_i$, then $\lambda'_{ij} = \theta_i \cdot \tilde{\sigma}'_j$. If we now make the obvious change in θ_a to accompany our change of σ_a, it is straight forward to verify the new values of $\{\lambda'_{ij}\}$. The ambiguity in sign is achieved by the ambiguity in the connected sum, as in the picture.

Note that this construction will destroy the property that $\{\sigma_i\}$ should form a trivial link in S^3, as well as property (ii) of $\{\lambda'_{ij}\}$. The elementary changes in $\{\lambda'_{ij}\}$ which would generate an arbitrary change preserving properties (i), (ii) are of the following type: give $g \in G$ and $2 \le a, b \le n$:

$$\lambda'_{ij} \mapsto \begin{cases} \lambda'_{ij} \pm (g-1) & i=a, j=b, a \neq b \\ \lambda'_{ij} \pm (g^{-1}-1) & i=b, j=a, a\neq b. \\ \lambda'_{ij} \pm (g+g^{-1}-2) & i=j=a=b \\ \lambda'_{ij} & (i,j) \neq (a,b) \text{ or } (b,a) \end{cases}$$

But this change is realized by a pair of changes of the original type and, therefore, we will be done if such a pair can be effected without changing the link type of $\{\sigma_i\}$ in S^3. To see this it is merely necessary to choose the two arcs from σ_a to σ_b so that, in $S^3 - \{\sigma_i\}$, they will be isotopic rel boundary, as suggested by the following picture.



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Brandeis University
Waltham, Mass. 02153

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