Pontryagin forms on homogeneous spaces

ALLEN BACK*

I. Introduction

It has been well known for some time that the differential forms representing characteristic classes carry extra geometric information beyond the topological. (See e.g. [1].) In particular, there is the result of Chern and Simons [2] that the non-vanishing of the dual Pontryagin Form $p_k^\perp$ in dimension $4k$ is an obstruction to the isometric (or conformal) immersion of the manifold into Euclidean space with codimension $2k - 1$. On the other hand, effectively using this obstruction is quite difficult in all but the simplest cases.

This paper will study the obstruction to immersion in the case of a normal homogeneous space K/H where K and H are compact Lie groups. The main result is the existence of an effective algorithm for calculating the Pontryagin Forms in terms of geometrical properties of the roots of K and their projections into H.

When K/H is a symmetric space with K connected and containing the geodesic symmetries, then there are no odd dimensional invariant forms and consequently the vanishing of the Pontryagin forms is implied by their vanishing as cohomology classes. Examples in this category were studied by Donnelley [4] generalizing earlier work of Lawson and Heitsch [3]. Somewhat surprisingly, we find that even for simple non-symmetric homogeneous spaces with vanishing Pontryagin classes (such as Stiefel manifolds), the forms themselves in general do not vanish. As corollaries, we obtain minimal codimension results about isometric immersion of such spaces. (Thm. 9).

In all geometric conventions, we shall follow [5]. In using the Chevalley basis, the conventions of [6] will be followed.

II. Preliminaries and notation

A bi-invariant metric on K induces an orthogonal splitting $k = h + m$ where k and h are the Lie algebras of K and H respectively. Since the metric on K/H is

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induced from such a bi-invariant metric on K, the Riemannian connection is the k-invariant connection described by $\Lambda_m(X) = \frac{1}{2}\operatorname{ad} X$ and it is easy to verify (see [5]) that the curvature form is given by

$$\Omega_{u_0}(X, Y)Z = -\tfrac{1}{8}[\Omega_1(X, Y) + 2\Omega_2(X, Y) + \Omega_3(X, Y) - \Omega_3(Y, X)]Z \qquad (*)$$

where

$$\Omega_1(X, Y) = P_m \circ \operatorname{ad}[X, Y]$$

$$\Omega_2(X, Y) = \operatorname{ad} P_h[X, Y]$$

$$\Omega_3(X, y) = \operatorname{ad} X \circ P_h \circ \operatorname{ad} Y$$

Note Ω_3 is not a differential form although $\Omega_3(X, Y) - \Omega_3(Y, X)$ is.

Of course it is quite mechanical to plug this expression into an invariant polynomial and find an expression for the Pontryagin forms of the tangent or normal bundles in this metric. However carrying this out directly leads even in the calculation of p_1 to extremely messy manipulations with indices.

Let $T_1 \subset H$ be a fixed maximal torus with $T_1 \subset T$ where T is a maximal torus of K. Let $\mathbf{T}_1$ and $\mathbf{T}$ respectively be the Lie algebras. Then the structure of k is completely described by the roots of K with respect to T. The isotropy representation of H on m may be described as $\operatorname{Ad}_K|_H - \operatorname{Ad}_H$ and so is readily computable. If X_α is a root vector in $k \otimes c$ associated to a root $\alpha \in \Delta(K)$, then since $T_1 \subset T$, X_α is also a weight vector for $\operatorname{Ad}_K|_H$. The intrinsic metric on k determines a projection $\pi_1 : \mathbf{T} \to \mathbf{T}_1$ and so we may write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = \pi_1(\alpha)$ is the weight associated to the isotropy representation of H. Thus the Chevalley basis for k immediately gives a weight space decomposition for the representation $\operatorname{Ad}_K|_H$ and hence for the isotropy representation.

If X_γ and $X_{\gamma'}$ are in weight spaces (for $\operatorname{Ad}_K|_H$) γ_1 and γ_1' respectively, then it is easy to verify that $[X_\gamma, X_{\gamma'}]$ is in weight space $\gamma_1 + \gamma_1'$. Consequently the 0-weight space j of $\operatorname{Ad}_K|_H$ forms a subalgebra which is the Lie algebra of the centralizer of T_1 in $k \otimes C$. Let T_2 be the torus whose Lie algebra is the orthogonal complement of T_1 in T. Then the orthogonal complement of $\mathbf{T}_1 \otimes C$ in j gives a subalgebra $j \otimes c$ with associated compact group $J \subset K$ (and maximal torus T_2) measuring complementary information from that given by $\operatorname{Ad}_K|_H$. The root vector X_α of K will become a weight vector of $\operatorname{Ad}_{K|J}$ with weight α_2.

Notice that the terms in $(*)$ have a simple interpretation with respect to this decomposition. Let $R(\alpha_1) = \{\beta \in \Delta(K) : \beta_1 = \alpha_1\}$ where α_1 is a root of H with associated root vector Y_{α_1}. Similarly, set $S(\alpha_1) = \operatorname{span}\{X_\beta : \beta \in R(\alpha_1)\}$, $S = \bigoplus_{\alpha_1 \in \Delta(H)} S(\alpha_1)$ and $S' = \operatorname{span}\{X_\beta : X_\beta \notin S\}$. Then $P_h X_\beta \neq 0$ implies $X_\beta \in S$ and on each $S(\alpha_1)$, the range of P_h is the one dimensional line in the direction Y_{α_1}.

Recall also that by means of Newton's formula for symmetric functions, the calculation of p_i is reduced to computing $\mathrm{Tr}\, \Omega^n$. (The p_i correspond to elementary symmetric functions; $\mathrm{Tr}\, \Omega^n$ to symmetric sums.)

III. Regular element case

Although the above point of view is helpful in all cases, it becomes especially easy to work with in the case that T_1 contains a regular element of K. Throughout this section, we will work under this assumption. Now the spaces $S(\alpha_1)$ are one-dimensional and the projection P_h is to easy to keep track of. For $\alpha \in \Delta(K)$, define

$$P_1(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha_2 \neq 0 \\ 1 & \text{if} \quad \alpha_2 = 0 \end{cases}$$

and $P_2(\alpha) = 1 - P_1(\alpha)$. Thus $P_h X_\alpha = P_1(\alpha) X_\alpha$ and $P_m(X_\alpha) = P_2(\alpha) X_\alpha$.

We're now ready to obtain a fairly neat formulation of how the complexification of $\Omega_i(X, Y)$ acts on root spaces. Since the Pontryagin forms are defined in terms of traces, we shall only be interested in special combinations; e.g. $p_k(X_{\alpha_1}, \ldots, X_{\alpha_{4k}}) = 0$ unless $\sum_{i=1}^{4k} \alpha_i = 0$. Using the Chevalley basis, we can establish the following. Here α, β, $\gamma \in \Delta(K) - \Delta(H)$ and $Z \in T_2$.

LEMMA 1. I. *If $\alpha \neq -\beta$ and*
(a) $\alpha + \beta \neq -\gamma$, *then* $\Omega_1(X_\alpha, X_\beta)X_\gamma = P_2(\alpha + \beta + \gamma)N_{\alpha,\beta}N_{\alpha+\beta,\gamma}X_{\alpha+\beta+\gamma}$
(b) $\alpha + \beta = -\gamma$, *then* $\Omega_1(X_\alpha, X_\beta)X_\gamma = -iN_{\alpha,\beta}P_m(H_\gamma)$.
   II. $\Omega_1(X_\alpha, X_{-\alpha})X_\gamma = -\gamma(H_\alpha)X_\gamma$

LEMMA 2. I. *If $\alpha \neq -\beta$ and*
(a) $\alpha + \beta \neq -\gamma$, *then* $\Omega_2(X_\alpha, X_\beta)X_\gamma = P_1(\alpha + \beta)N_{\alpha,\beta}N_{\alpha+\beta,\gamma}X_{\alpha+\beta+\gamma}$
(b) $\alpha + \beta = -\gamma$, *then* $\Omega_2(X_\alpha, X_\beta)X_\gamma = 0$
   II. $\Omega_2(X_\alpha, X_{-\alpha})X_\gamma = -\gamma_1(H_\alpha)X_\gamma$

LEMMA 3. I. *If $\beta \neq -\gamma$ and*
(a) $\alpha \neq -(\beta + \gamma)$, *then* $\Omega_3(X_\alpha, X_\beta)X_\gamma = P_1(\beta + \gamma)N_{\beta,\gamma}N_{\alpha,\beta+\gamma}X_{\alpha+\beta+\gamma}$
(b) $\alpha = -(\beta + \gamma)$, *then* $\Omega_3(X_\alpha, X_\beta)X_\gamma = 0$.
   II. $\Omega_3(X_\alpha, X_\beta)X_{-\beta} = \alpha_1(H_\beta)X_\alpha$

LEMMA 4. I. *If $\alpha \neq -\beta$, then* $\Omega_1(X_\alpha, X_\beta)Z = -iP_2(\alpha + \beta)((\alpha + \beta)(Z))$
$N_{\alpha,\beta}X_{\alpha+\beta}$
   II. $\Omega_1(X_\alpha, X_{-\alpha})Z = 0$

LEMMA 5. $\Omega_2(X_\alpha, X_\beta)Z = 0$

LEMMA 6. $\Omega_3(X_\alpha, X_\beta)Z = 0$

These lemmas may be translated into statements about when $\Omega_1(X_\alpha, X_\beta)X_\gamma$ and $\Omega_1(X_\alpha, X_\beta)Z$ can be nonzero.

PROPOSITION 7. I. For $\Omega_1(X_\alpha, X_\beta)X_\gamma$ to be nonzero, it is necessary that
(a) $\alpha + \beta \in \Delta(K) \cup \{0\}$ and
(b) Either
     (1) $\alpha \neq -\beta$, $\alpha + \beta + \gamma \in \Delta(K)$ and $(\alpha + \beta + \gamma)_2 \neq 0$
or   (2) $\gamma = -(\alpha + \beta)$ and $\gamma_2 \neq 0$
or   (3) $\alpha = -\beta$ and $\gamma(H_\alpha) \neq 0$.
II. For $\Omega_2(X_\alpha, X_\beta)X_\gamma$ to be nonzero, it is necessary that
(a) $\alpha + \beta \in \Delta(H) \cup \{0\}$
(b) $\alpha + \beta + \gamma \in \Delta(K)$
(c) If $\alpha = -\beta$, then $\gamma_1(H_\alpha) \neq 0$.
III. For $\Omega_3(X_\alpha, X_\beta)X_\gamma$ to be nonzero, it is necessary that
(a) $\beta + \gamma \in \Delta(H) \cup \{0\}$
(b) $\alpha + \beta + \gamma \in \Delta(K)$
(c) If $\beta = -\gamma$, then $\alpha_1(H_\beta) \neq 0$.

PROPOSITION 8. For $\Omega_1(X_\alpha, X_\beta)Z \neq 0$, it is necessary that
(a) $\alpha + \beta \in \Delta(K)$ and $(\alpha + \beta)_2 \neq 0$
(b) $(\alpha + \beta)(Z) \neq 0$

With somewhat more cumbersome notation, similar formulas can be written down in the non-regular case. However the reduction of all trace computations to one dimensional spaces does not apply; now the spaces $S(\alpha_1)$ are the basic components.

IV. Stiefel manifolds

In this section, we will illustrate the above technique to prove the following theorem.

THEOREM 9. The normal homogeneous space $SU(n)/SU(k)$ does not immerse isometrically in codimension less than $2M$ where $M = \min([k/2], [(n-k)/2])$.

The above does not represent anywhere near the limit of this approach. It is simply the outcome of calculating Pontryagin forms under assumptions designed to minimize the intricacy of applying Propositions 7 and 8. It is natural to conjecture for this example (and other homogeneous spaces without symmetric factors) that nonzero obstructions to isometric immersion are obtained from all nonzero Pontryagin forms which are not zero for dimension reasons.

The roots of $SU(n)$ are given by $\Delta(SU(n)) = \{\pm(\theta_i - \theta_j) : 1 \le i < j \le n\}$ and $\Delta(SU(k)) = \{\pm(\theta_i - \theta_j) : 1 \le i < j \le k\}$. Our Killing form is normalized so that $\theta_i \cdot \theta_j = \delta_{ij} - 1/n$. For $1 \le i \le n$, let α_i be the root $\theta_i - \theta_{n+1-i}$. The roots α_i are mutually orthogonal; sums and differences of distinct α_i are never roots. Theorem 9 follows immediately from:

PROPOSITION 10. *For* $r \le M$, $p_r^\perp(X_{\alpha_1}, X_{-\alpha_1}, X_{\alpha_2}, X_{-\alpha_2}, \ldots, X_{\alpha_{2r}}, X_{-\alpha_{2r}}) \ne 0$ *for* $SU(n)/SU(k)$.

To prove Proposition 10, we shall need the following straightforward consequences of Proposition 7 for $SU(n)/SU(k)$.

LEMMA 11. (1) $\Omega_1(X_{\alpha_i}, X_{-\alpha_i})X_\epsilon \ne 0 \Rightarrow \epsilon = \pm(\theta_j - \theta_{n+1-i})$ *or* $\epsilon = \pm(\theta_p - \theta_i)$ *with* $p \ge k + 1$.
 (2) $\Omega_2(X_{\alpha_i}, X_{-\alpha_i})X_\epsilon \ne 0 \Rightarrow \epsilon = \pm(\theta_p - \theta_i)$ *with* $p \ge k + 1$.
 (3) $\Omega_3(X_{\alpha_i}, X_{-\alpha_i})X_\epsilon \ne 0 \Rightarrow \epsilon = \theta_j - \theta_{n+1-i}$ *with* $1 \le j \le k$.
 (4) $\Omega_3(X_{-\alpha_i}, X_{\alpha_i})X_\epsilon \ne 0 \Rightarrow \epsilon = \theta_{n+1-i} - \theta_j$ *with* $1 \le j \le k$.

LEMMA 12. (1) *For* $p = 1$ *or* 2 *and* $i \ne j$, $\Omega_p(X_{\pm\alpha_i}, X_{\alpha_j}) = 0$.
 (2) *For* $i \ne j$, $\Omega_3(X_{\alpha_i}, X_{-\alpha_j}) = \Omega_3(X_{-\alpha_i}, X_{\alpha_j}) = 0$.
 (3) *For* $i \ne j$, $\Omega_3(X_{\pm\alpha_i}, X_{\pm\alpha_j})X_\epsilon \ne 0 \Rightarrow \epsilon = \pm(\theta_{n-j+1} - \theta_i)$
and the range of Ω_3 *is a multiple of* $X_{\pm(\theta_i - \theta_{n-i+1})}$.

Let $A = \{X_{\pm\alpha_i}; 1 \le i \le 2M\}$. The above show it is not easy for $\Omega_p(v_1, v_2)X_\epsilon \ne 0$ when $v_1, v_2 \in A$. If $\epsilon = \theta_a - \theta_b$, let $\bar{a} = \min(a, n+1-a)$ and $\bar{b} = \min(b, n+1-b)$. Then v_1 and v_2 must be distinct root vectors associated to roots in the set $Y(\epsilon) = \{\pm\alpha_{\bar{a}}, \pm\alpha_{\bar{b}}\}$. But $Y(\epsilon)$ has only four elements and in all cases, $\Omega_p(v_1, v_2)X_\epsilon$ will be a multiple of $X_{\epsilon'}$ with $Y(\epsilon') = Y(\epsilon)$. So we have proven the following very helpful corollary.

PROPOSITION 13. *For* v_p *distinct elements of* A, $\Omega_{i_1}(v_1, v_2)\Omega_{i_2}(v_3, v_4)\Omega_{i_3}(v_5, v_6) = 0$.

This means that all $\mathrm{Tr}\,\Omega^r(r \ge 3)$ evaluated on elements of A will be zero. Thus to determine $p_r^\perp$ on elements of A, we really only need to know $\mathrm{Tr}\,\Omega^2$.

LEMMA 14. (1) *For* $v_p \in A$, $\operatorname{Tr} \Omega \wedge \Omega(v_1, v_2, v_3, v_4) = 0$ *unless* $\{v_1, v_2, v_3, v_4\} = \{X_{\alpha_i}, X_{-\alpha_i}, X_{\alpha_j}, X_{-\alpha_j}\}$.

(2) *For* $i \neq j$, $\operatorname{Tr} \Omega \wedge \Omega(X_{\alpha_i}, X_{-\alpha_i}, X_{\alpha_j}, X_{-\alpha_j}) \neq 0$ *and is independent of i and j.*

Proof. Statement (1) follows since $\Omega \wedge \Omega(X_a, X_b, X_c, X_d) = 0$ if $a + b + c + d \neq 0$. The proof of (2) is not difficult once we have the reductions indicated in Lemmas 11 and 12. For the remainder of this proof, we shall let $\alpha = \alpha_i$ and $\beta = \alpha_j$ with $i \neq j$. The table below summaries the computation. The "important roots" are the roots ϵ with $\operatorname{Tr}(\Omega \wedge \Omega)$ getting a potentially nonzero contribution from X_ϵ.

In obtaining the last column of Table 1, we must use appropriate Chevalley basis identities. In (3) and (5), $N_{-\beta,\epsilon}N_{\beta,\epsilon-\beta} = (|\beta - \epsilon|^2/|\epsilon|^2)N_{\beta-\epsilon,-\beta}N_{\beta,\epsilon-\beta} = -|N_{\beta,\epsilon-\beta}|^2 = -1$ by noting that $\epsilon - \beta$ is at the bottom of a β-root string. Similarly in (7), $(N_{-\alpha,\epsilon}N_{\alpha,\epsilon-\alpha})(N_{\beta,\epsilon-\alpha-\beta}N_{-\beta,\epsilon-\alpha}) = |N_{\alpha,\epsilon-\alpha}|^2 |N_{\beta,\epsilon-\beta}|^2 = 1$.

To fully utilize the results in Table 1, one should rember that the $\Omega_i(X_a, X_b)$ are skew symmetric matrices for $i = 1, 2$ and $\Omega_3'(X_a, X_b) = \Omega_3(X_b, X_a)$. And, of course, if A is skew symmetric $\operatorname{Tr} AB = -\operatorname{Tr} B^t A = -\operatorname{Tr} AB^t = \operatorname{Tr} BA$.

Thus

$$192 \operatorname{Tr} \Omega \wedge \Omega(X_\alpha, X_{-\alpha}, X_\beta, X_{-\beta})$$

$$= 64[\operatorname{Tr} \Omega(X_\alpha, X_{-\alpha})\Omega(X_\beta, X_{-\beta})$$

$$-\operatorname{Tr} \Omega(X_\alpha, X_{-\beta})\Omega(X_\beta, X_{-\alpha}) + \operatorname{Tr} \Omega(X_\alpha, X_\beta)\Omega(X_{-\beta}, X_{-\alpha})]$$

$$= \operatorname{Tr} \Omega_1(X_\alpha, X_{-\alpha})\Omega_1(X_\beta, X_{-\beta}) + 4 \operatorname{Tr} \Omega_1(X_\alpha, X_{-\alpha})\Omega_2(X_\beta, X_{-\beta})$$

$$+ 4 \operatorname{Tr} \Omega_1(X_\alpha, X_{-\alpha})\Omega_3(X_\beta, X_{-\beta}) + 8 \operatorname{Tr} \Omega_2(X_\alpha, X_{-\alpha})\Omega_3(X_\beta, X_{-\beta})$$

$$+ 2 \operatorname{Tr} \Omega_3(X_\alpha, X_\beta)\Omega_3(X_{-\beta}, X_{-\alpha}) - 2 \operatorname{Tr} \Omega_3(X_\alpha, X_\beta)\Omega_3'(X_{-\beta}, X_{-\alpha}) = 24 \neq 0.$$

We are now ready to prove Proposition 10.

Proof of Proposition 10. Let s_m denote the differential form $\operatorname{Tr} \Omega^{2m}$. By Proposition 13, $s_m = 0$ for $m > 1$. By Lemma 14, it is easy to see that $(s_1)^r(X_{\alpha_1}, X_{-\alpha_1}, \ldots, X_{\alpha_2r}, X_{-\alpha_{2r}}) \neq 0$ for $r \leq M$ since all nonzero terms in the skew symmetrization have the same sign.

When all symmetric sums above s_1 are zero, the Waring formula (or an easy direct argument from the Newton formulas) shows that the k'th elementary symmetric function σ_k is $1/k!$ times $(s_1)^k$. Thus for the Pontryagin classes of the tangent bundle of K/H, $p_r = (-1)^r[(s_1)^r/r!]$ when $r \leq m$ – these have generating function e^{-ts_1}. The dual Pontryagin forms $p_r^\perp$ are defined by $(1 + p_1 + p_2 + \ldots)$

Table 1

Term	Important Roots ϵ	Contribution	Numerical Value
(1) $\mathrm{Tr}\,\Omega_1(X_\alpha, X_{-\alpha})\Omega_1(X_\beta, X_{-\beta})$	$\pm(\theta_i - \theta_{n+l-j})$ $\pm(\theta_{n+l-i} - \theta_{n+1-j})$ $\pm(\theta_{n+l-i} - \theta_j)$	$\sum_\epsilon \epsilon(H_\alpha)\epsilon(H_\beta)$	2
(2) $\mathrm{Tr}\,\Omega_1(X_\alpha, X_{-\alpha})\Omega_2(X_\beta, X_{-\beta})$	$\pm(\theta_j - \theta_{n+1-i})$	$\sum_\epsilon \epsilon(H_\alpha)\epsilon_1(H_\beta)$	2
(3) $\mathrm{Tr}\,\Omega_1(X_\alpha, X_{-\alpha})\Omega_3(X_\beta, X_{-\beta})$	$\theta_i - \theta_{n+l-j}$	$-\epsilon(H_\alpha)N_{-\beta,\epsilon}N_{\beta,\epsilon-\beta}$	1
(4) $\mathrm{Tr}\,\Omega_2(X_\alpha, X_{-\alpha})\Omega_2(X_\beta, X_{-\beta})$	None	0	0
(5) $\mathrm{Tr}\,\Omega_2(X_\alpha, X_{-\alpha})\Omega_3(X_\beta, X_{-\beta})$	$\theta_i - \theta_{n+l-j}$	$-\epsilon_1(H_\alpha)N_{-\beta,\epsilon}N_{\beta,\epsilon-\beta}$	1
(6) $\mathrm{Tr}\,\Omega_3(X_\alpha, X_\beta)\Omega_3(X_{-\beta}, X_{-\alpha})$	$\theta_j - \theta_{n+l-i}$	$N_{\beta,\epsilon-\alpha-\beta}N_{\alpha,\epsilon-\alpha}N_{-\alpha,\epsilon}N_{-\beta,\epsilon-\alpha}$	1
(7) $\mathrm{Tr}\,\Omega_3(X_\alpha, X_\beta)\Omega_3(X_{-\alpha}, X_{-\beta})$	None	0	0

$(1 + p_1^\perp + p_2^\perp + \ldots) = 1$. So for $r \leq M$, the generating function for $p_r^\perp$ is e^{ts_1} and $p_r^\perp = (s_1)^r/r! \neq 0$. This proves Proposition 10.

It is interesting to note that the algorithm described herein could easily be carried out on a computer. Experiments in this direction might suggest stronger non-vanishing theorems as well as other situations in which general calculations can easily be made.

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