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Integral means of derivatives of monotone slit mappings

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1. Introduction

Let Δ denote the unit disk $\{z : |z| < 1\}$ and S the class of functions f analytic and univalent in Δ with $f(0)=0, f'(0)=1$. In [1, p. 139], Baernstein proved that the Koebe function $k(z)=z(1-z)^{-2}$ is extremal for a large class of problems about integral means. We denote by K the class of all convex increasing functions $\Phi(x)$ defined on $(-\infty, \infty)$.

THEOREM A. *For $f \in S$, $\Phi \in K$, and $r \in (0, 1]$,*

$$\int_{-\pi}^{\pi} \Phi(\pm\log|f(re^{i\theta})|)\, d\theta \leq \int_{-\pi}^{\pi} \Phi(\pm\log|k(re^{i\theta})|)\, d\theta.$$

In particular, the Koebe function has the largest L^p means

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p\, d\theta, \quad 0<p<\infty,$$

among all the functions in S.

In the corresponding problem involving derivatives the Koebe function ceases to be extremal, at least for small values of p. Indeed,

$$k'(z)=\frac{1+z}{(1-z)^3}$$

belongs to the Hardy space H^p if $p<\frac{1}{3}$, whereas Lohwater, Piranian, and Rudin [15] have constructed a function in S whose derivative belongs to no H^p class.

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In this paper we show that if we restrict attention to *support points* of S then the Koebe function is restored to its extremal position, at least in terms of order of magnitude, in problems about integral means of derivatives. A function $f \in S$ is called a support point if there exists a continuous non-constant linear functional L on the space of functions analytic in Δ such that

$$\operatorname{Re} L(f) = \max_{g \in S} \operatorname{Re} L(g).$$

Here, and throughout the paper, the space of functions analytic in Δ is endowed with the topology of uniform convergence on compact subsets.

Denote by σ the set of support points of S. By a theorem of Pfluger [17] and Brickman and Wilken [4], each $f \in \sigma$ is a *monotone slit mapping*. That is, $\mathbf{C} - f(\Delta)$ is a Jordan arc Γ with one endpoint at ∞ and the other at a finite point w_0, called the tip, which intersects each circle $|w| = R$ at most once. Furthermore, Γ is an analytic arc which is asymptotic to a straight line at ∞ and at the tip, and Γ has the $\pi/4$-property: at each point on Γ the angle between the tangent vector and the radius vector from the origin is in absolute value smaller than $\pi/4$, except possibly at the tip, where it might be equal to $\pm\pi/4$.

Our results apply more generally to functions f for which $\mathbf{C} - f(\Delta)$ is a not necessarily analytic monotone slit having the property analogous to the $\pi/4$ property for any number λ strictly less than $\pi/2$. We define a subclass $\mathcal{M}(\lambda) \subseteq S$ as follows. $f \in \mathcal{M}(\lambda)$ if $\mathbf{C} - f(\Delta)$ is a Jordan arc with parametrization $w(t)$, $t_0 \le t \le \infty$, so that $w(\infty) = \infty$ and $0 < |w(t_0)| < |w(t_1)| < |w(t_2)| < \infty$ whenever $t_0 < t_1 < t_2 < \infty$, and such that

$$\overline{\lim_{t \to t_1^+}} \left| \arg \frac{w(t) - w(t_1)}{w(t_1)} \right| \le \lambda, \qquad \overline{\lim_{t \to t_1^-}} \left| \arg \frac{w(t_1) - w(t)}{w(t_1)} \right| \le \lambda \tag{1}$$

for every $t_1 \in (t_0, \infty)$. The first inequality should hold also for $t_1 = t_0$.

It is a simple exercise to show that a C^1 curve for which the angle between the tangent vector and the radius vector always has magnitude less than or equal to λ, $0 < \lambda < \pi/2$, satisfies (1). In particular, $\sigma \subset \mathcal{M}(\pi/4)$.

According to a theorem of Brickman [3], if f is an extreme point of S then $\mathbf{C} - f(\Delta)$ is a monotone slit. However, it is not known whether extreme points must be sufficiently monotone to belong to a class $\mathcal{M}(\lambda)$ for $\lambda < \pi/2$. On the other hand, if $f \in S$ is an extreme point of the *closed convex hull* of S, it is known ([5], [13]) that f must belong to the closure of σ. The classes $\mathcal{M}(\lambda)$ are closed sets.

(This point is discussed in §2.) Consequently, we have the inclusions

$$\mathscr{E}(\overline{\text{co}}\, S) \subset \bar{\sigma} \subset \mathscr{M}\left(\frac{\pi}{4}\right). \tag{2}$$

Here $\mathscr{E}(A)$ denotes the extreme points of a set A.

Now we can state our results. In the following $C(\lambda)$ denotes a positive constant depending only on λ, not necessarily the same in different occurrences.

THEOREM 1. *For $f \in \mathscr{M}(\lambda)$, $\Phi \in K$, $r \in (0, 1]$ and $\lambda \in (0, \pi/2)$,*

$$\int_{-\pi}^{\pi} \Phi(\pm\log|f'(re^{i\theta})|)\, d\theta \le \int_{-\pi}^{\pi} \Phi(\pm\log|C(\lambda)k'(re^{i\theta})|)\, d\theta \tag{3}$$

and

$$\int_{-\pi}^{\pi} \Phi\left(\pm\log\left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\pm\log\left|C(\lambda)\frac{rk'(re^{i\theta})}{k(re^{i\theta})}\right|\right) d\theta. \tag{4}$$

COROLLARY 1. *For $f \in \mathscr{M}(\lambda)$, $0 < \lambda < \pi/2$, $r \in (0, 1]$, and $p \in (-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p\, d\theta \le C(\lambda)^p \int_{-\pi}^{\pi} |k'(re^{i\theta})|^p\, d\theta \tag{5}$$

and

$$\int_{-\pi}^{\pi} \left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|^p d\theta \le C(\lambda)^p \int_{-\pi}^{\pi} \left|\frac{rk'(re^{i\theta})}{k(re^{i\theta})}\right|^p d\theta. \tag{6}$$

Note especially that for support points f these inequalities hold with absolute constants $C = C(\pi/4)$.

Since

$$k'(z) = \frac{1+z}{(1-z)^3}, \qquad \frac{zk'(z)}{k(z)} = \frac{1+z}{1-z},$$

Corollary 1 shows that, for $f \in \mathscr{M}(\lambda)$, f' belongs to H^p for $p < \frac{1}{3}$, while $1/f'$, $zf'(z)/f(z)$ and $f(z)/zf'(z)$, belong to H^p for $p < 1$. In each case the norm is bounded above by a constant depending only on p and λ, and for support points the constants depend only on p. Since support points are analytic in the closed disk except for a double pole at the point on $|z| = 1$ where $f(z) = \infty$, and $f'(z)$ has no zero except for a simple one at the point on $|z| = 1$ corresponding to the finite tip of the slit, the statement about membership in H^p classes for support points is clearly true. However, the statement about uniform boundedness of norms seems to be new.

Another case of particular interest is that of $p=2$ in (6). We state this as a separate corollary.

COROLLARY 2. *For* $f \in \mathcal{M}(\lambda)$, $\lambda \in (0, \pi/2)$ *and* $r \in (0, 1)$,

$$\int_{-\pi}^{\pi} \left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta \leq C(\lambda)\frac{1}{1-r}. \tag{7}$$

Inequality (8) is equivalent to the statement that the functions $\log[f(z)/z]$, $f \in \mathcal{M}(\lambda)$, belong to the mean smoothness class $\Lambda^2_{1/2}$ [8, p. 78], and, moreover, for fixed $\lambda \in (0, \pi/2)$, actually form a bounded set in this class. In particular, the support points form a bounded set in $\Lambda^2_{1/2}$. J. Cima and K. Petersen [7] noted that $\log[f(z)/z] \in \Lambda^2_{1/2}$ for $f \in \sigma$, but they did not state or prove the uniform boundedness assertion.

For a long time it was conjectured on the basis of the behavior of the Koebe function that $\log[f(z)/z]$ would belong to $\Lambda^2_{1/2}$ for every $f \in S$. However, W. K. Hayman [11] has recently constructed a function $f \in S$ for which

$$\int_{-\pi}^{\pi} \left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta \neq O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right), \qquad r \to 1.$$

Thus, the simple estimate [18, p. 130]

$$\int_{-\pi}^{\pi} \left|\frac{rf'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right)$$

is best possible in the full class S. Hayman has also constructed (unpublished) a monotone slit mapping in S for which the integral above is not

$$O\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right).$$

This shows that Corollary 2 and Theorem 1 become false in the limiting case $\lambda = \pi/2$. P. L. Duren and Y. J. Leung [10] showed that $\log f(z)/z$ belongs to $\Lambda^2_{1/2}$ if $f \in S$ has positive Hayman index, that is, grows sufficiently rapidly. It is not known whether $\log f(z)/z$ belongs to $\Lambda^2_{1/2}$ for extreme points of S, but our results and (2) show that this is indeed the case for $f \in \mathscr{E}(\overline{\text{co}}\, S)$.

Corollary 2 has an interesting reformulation. For $f \in S$ write

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

The "2" is inserted so that $\gamma_n = 1/n$ when $f = k$.

COROLLARY 3. *For* $f \in \mathcal{M}(\lambda)$, $\lambda \in (0, \pi/2)$, *and* $n = 1, 2, 3, \ldots,$

$$\sum_{j=1}^{n} j^2 |\gamma_j|^2 \leq C(\lambda)n.$$

A well-known argument (see, e.g., [10, p. 38]) shows that coefficient estimates of this form are equivalent to means estimates of the form (7). Hayman's examples show that $\sum_{j=1}^{n} j^2|\gamma_j|^2 = O(n)$ is false in the full class S, and even false for monotone slit mappings not belonging to a class $\mathcal{M}(\lambda)$ for some $\lambda < \pi/2$.

Corollary 3 may have some bearing on the coefficient problem in the class S. Writing $C = C(\pi/4)$, we have

COROLLARY 4. *For* $f \in \sigma$ *and* $n = 1, 2, 3, \ldots$

$$\sum_{j=1}^{n} j^2 |\gamma_j|^2 \leq Cn. \tag{8}$$

I. M. Milin proved [16], [18, §3.5] that in the full class S,

$$\sum_{j=1}^{n} j |\gamma_j|^2 \leq \sum_{j=1}^{n} \frac{1}{j} + \delta,$$

where $0 < \delta < 0.312$. It is known that one cannot take $\delta = 0$ here, so that the Koebe function is not extremal for this inequality. However, Milin has conjectured that perhaps the Koebe function is sharp for a smoothed-out version.

MILIN'S CONJECTURE. For $f \in S$,

$$\sum_{k=1}^{n} \sum_{j=1}^{k} j |\gamma_j|^2 \leq \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{j}.$$

An inequality of Milin and Lebedev (see [9, p. 897]) shows that Milin's conjecture implies Bieberbach's conjecture. In fact, to prove Bieberbach's conjecture for the full class S it would be sufficient to prove Milin's conjecture for support points. Our inequality (8), which holds for support points but not for the whole class S is encouraging since it indicates that maybe the γ_n are better behaved for support points than they are in general.

Concerning integral means inequalities like those of Theorem 1, we mention that Leung [14] has proved the sharp inequalities

$$\int_{-\pi}^{\pi} \Phi(\log |f'(re^{i\theta})|)\, d\theta \le \int_{-\pi}^{\pi} \Phi(\log |k'(re^{i\theta})|)\, d\theta \tag{9}$$

for f close-to-convex. Brown [6] has solved the corresponding problems for close-to-convex functions of order β. In the full class S, standard results show that

$$\int_{-\pi}^{\pi} |f'(re^{i\theta})|^p\, d\theta \le C(p) \int_{-\pi}^{\pi} |k'(re^{i\theta})|^p\, d\theta$$

holds for $\frac{1}{2} < p < \infty$. Such an estimate still holds for $\frac{2}{5} < p < \infty$, ([18, p. 130], [21, p. 210]), and possibly also for $\frac{1}{3} < p \le \frac{2}{5}$, but this is not known. The best constant $C(p)$ is not known either, except that $C(\infty) = 1$, by the distortion theorem. It would be very interesting to prove or disprove that $C(p) = 1$ for various other values of p. For example, if the smallest $C(2)$ is larger than one, then Bieberbach's conjecture is false.

Sharp order of magnitude estimates for L^p means of the logarithmic derivative zf'/f do not seem to be known in the full class S except when $p = \infty$ (the distortion theorem) and $p = 2$ (discussed above). An estimate for the case $p = 1$ appears in [18, p. 129].

Our proof of Theorem 1 is based on a representation theorem for the logarithmic derivative of functions in $\mathcal{M}(\lambda)$. This is obtained in §2. In §3 we use this representation to prove that if $f \in \mathcal{M}(\lambda)$ then zf'/f belongs to the space "weak H^1". From this and some considerations involving *-functions it is easy to conclude the proof of Theorem in §4.

2. A representation theorem

Let us denote by $A(\lambda), 0 < \lambda < \pi/2$, the subclass of $\mathcal{M}(\lambda)$ consisting of functions for which the omitted arcs are analytic except at infinity and their finite tips, where they have well-defined linear asymptotic directions. A function $f \in A(\lambda)$ has a meromorphic extension to the closed unit disk $\bar{\Delta}$. Its only pole there is a double one at the point on $|z| = 1$ where $f(z) = \infty$. The only zero of f' in $\bar{\Delta}$ is a simple one at the point on $|z| = 1$ corresponding to the finite tip. At each point on the omitted arc it follows from (1) that the angle between the tangent vector and the radius vector is at most λ in absolute value.

W. E. Kirwan and R. Pell [13] proved that the closure of $A(\pi/4)$ is contained in $\mathcal{M}(\pi/4)$. Conversely, they also proved that certain functions in $\mathcal{M}(\lambda)$ are in the closure of $A(\lambda), 0<\lambda<\pi/2$. The Kirwan–Pell arguments, together with a few embellishments which we leave to the reader, show that

The closure (in the topology of local uniform convergence) of $A(\lambda)$ *is* $\mathcal{M}(\lambda)$, $0<\lambda<\pi/2$.

To state our representation theorem we introduce the class $\mathcal{G}(\lambda)$ of functions g which are analytic and zero free in Δ, and satisfy

$$|\arg g(z)| \leq \lambda, \qquad \forall z \in \Delta$$

$$|g(0)| = 1.$$

It is also convenient to introduce a bit more special notation. Since $f \in \mathcal{M}(\lambda)$ is a monotone slit mapping there are unique real numbers $\varphi_1 < \varphi_2 < \varphi_1 + 2\pi$ such that

$$f(e^{i\varphi_1}) = \infty, \qquad f(e^{i\varphi_2}) = w_0,$$

where w_0 is the finite tip. Define

$$S_f(z) = e^{-i(\varphi_2-\varphi_1)/2} \frac{e^{i\varphi_2}-z}{e^{i\varphi_1}-z}. \tag{10}$$

Then S_f is a conformal mapping of Δ onto the upper half plane. S_f is negative real on the arc $\{e^{i\varphi} : \varphi_1 < \varphi < \varphi_2\}$ on which $|f(e^{i\varphi})|$ is decreasing and is positive real on the arc $\{e^{i\varphi} : \varphi_2 < \varphi < \varphi_1 + 2\pi\}$ on which $|f(e^{i\varphi})|$ is increasing

PROPOSITION 1. *For* $f \in \mathcal{M}(\lambda)$, $0<\lambda<\pi/2$, *we can write*

$$\frac{izf'(z)}{f(z)} = S_f(z)g(z), \quad z \in \mathcal{M}(\lambda),$$

where S_f *is defined by* (10) *and* $g \in \mathcal{G}(\lambda)$. *Moreover*

$$\pi - 2\lambda \leq \varphi_2 - \varphi_1 \leq \pi + 2\lambda. \tag{11}$$

For $\lambda = \pi/4$ and $f \in A(\pi/4)$ this is a result of W. Hengartner and G. Schober [12, p. 211]. Inequality (11) asserts that the points on $\partial\Delta$ corresponding to ∞ and the finite tip are separated by at least a certain distance which depends only on λ.

Proof of Proposition 1. Suppose first that $f \in A(\lambda)$. If $\theta \neq \varphi_1, \varphi_2$ then

$$\frac{izf'(z)}{f(z)} = \frac{1}{f(z)} \frac{\partial f}{\partial \theta}(z)$$

exists at $z = e^{i\theta}$. The argument of this number represents the angle between the properly sensed tangent vector to the omitted arc and the radius vector. Hence, the number lies in the sector

$$K(\lambda) = \{z : |\arg z| \leq \lambda\}$$

if $\varphi_2 < \theta < \varphi_1 + 2\pi$ and in $-K(\lambda)$ if $\varphi_1 < \theta < \varphi_2$.

Define

$$g(z) = \frac{1}{S_f(z)} \frac{izf'(z)}{f(z)}.$$

Then g is analytic and zero free in $\bar{\Delta}$ and maps $\partial\Delta$ into $K(\lambda)$. By the Poisson representation, g maps Δ into $K(\lambda)$ as well. Since $|g(0)| = 1$ it follows that $g \in \mathscr{G}(\lambda)$. Since

$$i = g(0)S_f(0) = g(0)e^{i(\varphi_2 - \varphi_1)/2}$$

and $|\arg g(0)| \leq \lambda$, (11) holds.

Next suppose that f is any function in $\mathscr{M}(\lambda)$. Since $\mathscr{M}(\lambda)$ is the closure of $A(\lambda)$ there is a sequence $\{f_n\}$ in $A(\lambda)$ which converges to f uniformly on compact subsets of Δ. Write

$$\frac{izf_n'(z)}{f_n(z)} = S_{f_n}(z)g_n(z), \qquad g_n \in \mathscr{G}(\lambda).$$

The left-hand side approaches $izf'(z)/f(z)$ as $n \to \infty$, $z \in \Delta$. Since $\mathscr{G}(\lambda)$ is a normal family, a subsequence of $\{g_n\}$ converges locally uniformly to a function g, which also belongs to $\mathscr{G}(\lambda)$. The S_{f_n} for this subsequence converge locally uniformly to a function $S(z)$ which is either a conformal mapping of the disk onto the upper half plane or is constant. We need to verify that $S(z) = S_f(z)$. One way to do this is to observe that the assumption $\lambda<\pi/2$ forces the sequence of continua $\mathbf{C}-(1/f_n)(\Delta)$ to be uniformly locally connected [18, p. 283]. It follows that the normally converging sequence $\{f_n\}$ in fact converges uniformly on $\bar{\Delta}$ in the spherical metric. Monotonicity considerations now imply that the zero $e^{i\varphi_{2,n}}$ of S_{f_n} converges to the zero $e^{i\varphi_2}$ of S_f and similarily for the poles, so that $S=S_f$, as required. This argument shows also that (11) holds for f, since it holds for each f_n.

We close this section by considering an example. Suppose $f(z)=z(1-z)^{-1-e^{i\alpha}}$, $-\pi<\alpha<\pi$. Then f maps Δ onto the complement of a logarithmic spiral for which the angle between the tangent vector and the radius vector is constantly $\alpha/2$. The corresponding representation is

$$\frac{izf'(z)}{f(z)}=e^{i\alpha/2}e^{-(\pi-\alpha)/2}\frac{e^{i(\pi-\alpha)}-z}{1-z}.$$

Thus $\varphi_2-\varphi_1=\pi-\alpha$ and $g(z)$ is the constant $e^{i\alpha/2}$.

3. A weak-type inequality

The symmetrization theorem [1, p. 143] implies that L^p means of functions $g\in\mathcal{G}(\lambda)$ are majorized by those of the conformal mapping $[(1+z)/(1-z)]^{2\lambda/\pi}$. In particular, if $g\in\mathcal{G}(\lambda)$ then $g\in H^p$ for $p<\pi/2\lambda$.

Consider now an analytic function F in Δ of the form

$$F(z)=e^{i\alpha}\frac{e^{i\varphi_2}-z}{e^{i\varphi_1}-z}\,g(z) \tag{12}$$

where $\alpha, \varphi_1, \varphi_2\in\mathbf{R}$ and $g\in\mathcal{G}(\lambda)$. Hölder's inequality implies that this function belongs to H^p for $p<(1+2\lambda/\pi)^{-1}$, and the example

$$F(z)=\left(\frac{1+z}{1-z}\right)^{1+(2\lambda/\pi)}$$

shows that nothing better is true in general. If, however, we impose the additional condition

$$|F(z)|\leq\frac{2}{1-|z|} \tag{13}$$

then it will turn out that F belongs to H^p for every $p<1$, and even more. In the case of interest to us F will be the logarithmic derivative of a function in $\mathcal{M}(\lambda)$ and the distortion theorem will imply that (13) holds.

We define a Hardy-type space called weak H^1 as follows. $F \in$ weak H^1 if

(i) $F \in N^+$ and

(ii) $|\{e^{i\theta} \in \partial\Delta : |F(e^{i\theta})| > t\}| < bt^{-1}$, $\forall t > 0$.

In (i), N^+ denotes the uniform Nevanlinna class [8, p. 25]. In (ii), $|E|$ denotes the Lebesgue measure of a set E on $\partial\Delta$ and b is a constant depending on F. Define the weak H^1 "norm" $b(F)$ of F by

$$b(F) = \inf\{b: \text{(ii) is valid}\}. \tag{14}$$

Clearly, $H^1 \subset$ weak H^1. We will prove in the next section that weak $H^1 \subset H^p$ for every $p < 1$. The function $(1-z)^{-1}$ is typical of those which belong to weak H^1 but not to H^1.

PROPOSITION 2. *Suppose F has the form* (12) *where* $\lambda < \pi/2$, *and that* (13) *holds. Then* $F \in$ weak H^1 *and* $|b(F)| \leq C(\lambda)$.

The gist of the proposition is that (13) forces the F of (12) to grow no faster on average than its linear fractional part. However, as $\lambda \uparrow \pi/2$ the perturbation term $g(z)$ can become more influential, and for $\lambda = \pi/2$ the proposition is almost surely false, although we have not constructed any counter examples. The "2" in (13) is for convenience. Any bound A would do, and then the constant $C(\lambda)$ bounding $|b(F)|$ would have to depend on A as well.

Proof of Proposition 2. We noted above that functions of the form (12) belong to H^p for small p and hence to N^+. Thus the radial limit function $F(e^{i\theta})$ exists a.e.

Without loss of generality we may assume that $\alpha = \varphi_1 = 0$, so that F has the form

$$F(z) = \frac{e^{i\varphi} - z}{1 - z}\, g(z) \tag{15}$$

where $0 < \varphi < 2\pi$ and $g \in \mathcal{M}(\lambda)$ for some $\lambda \in (0, \pi/2)$.

Fix $t > 0$ and define

$$E(t) = \{e^{i\theta} : 0 < \theta < \varphi \text{ and } |F(e^{i\theta})| > t\}.$$

We will prove that

$$|E(t)| \leq C(\lambda) t^{-1}. \tag{16}$$

A similar analysis proves the same estimate for the arc $\{e^{i\theta}:\varphi<\theta<2\pi\}$, so that the proof will be finished once (16) is proved.

For $n=0,1,2,\ldots$ define

$$I_n=\{e^{i\theta}:2^{-n-1}\varphi<\theta<2^{-n}\varphi\},\qquad E_n=E(t)\cap I_n.$$

One verifies easily that for $e^{i\theta}\in I_n$

$$\left|\frac{e^{i\varphi}-e^{i\theta}}{1-e^{i\theta}}\right|<A2^n.$$

The letter A will stand for an absolute constant whose value can change from line to line. From (15) it follows that

$$E_n\subset\{e^{i\theta}\in I_n:|g(e^{i\theta})|>A2^nt\}. \tag{17}$$

Let $a_n\in\Delta$ denote the point such that $1-|a_n|=(1/2\pi)\,|I_n|$ and $a_n/|a_n|$ is the center if I_n. Also, let μ_n denote harmonic measure relative to Δ for a_n. A simple estimate for the Poisson kernel (see, e.g., [2, p. 8]) shows that for every measurable set $B\subset I_n$ we have

$$|B|\le A\,|I_n|\,\mu_n(B).$$

From (17) it follows that

$$|E_n|\le A\,|I_n|\,\mu_n(\{e^{i\theta}:|g(e^{i\theta})|>A2^nt\}). \tag{18}$$

To estimate the right-hand side we need a lemma. For $a\in\Delta$ let μ_a denote the harmonic measure at a relative to Δ.

LEMMA. *Suppose that $a\in\Delta$ and that $g\in\mathcal{M}(\lambda)$, $0<\lambda<\infty$. Then*

$$\mu_a(\{e^{i\theta}:|g(e^{i\theta})|>s\})<A(|g(a)|\,s^{-1})^{\pi/2\lambda},\qquad s>0. \tag{19}$$

Proof. Suppose first that $\lambda=\pi/2$ and that $a=0$. Then $\operatorname{Re} g\ge0$, so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\operatorname{Re} g(Re^{i\theta})\,d\theta=\operatorname{Re} g(0)\le g(0)|=1,$$

for $0<R<1$. An application of Kolmogorov's weak $1-1$ inequality [20, p. 134] to the function $\operatorname{Re} g(Re^{i\theta})$ gives

$$|\{e^{i\theta}: |\operatorname{Im} g(Re^{i\theta}) - \operatorname{Im} g(0)| > s\}| < As^{-1}.$$

Since also $|\operatorname{Im} g(0)| \leq 1$ and

$$|\{e^{i\theta}: \operatorname{Re} g(Re^{i\theta}) > s\}| < 2\pi s^{-1}$$

it follows easily that

$$\mu_0(\{e^{i\theta}: |g(Re^{i\theta})| > s\}) = \frac{1}{2\pi} |\{e^{i\theta}: |g(Re^{i\theta})| > s\}| < As^{-1}.$$

A passage to the limit $R = 1$ shows that (19) holds when $\lambda = \pi/2$ and $a = 0$.

For the general case, let $g_1(z) = g((z+a)/(1+\bar{a}z))$. By the conformal invariance of harmonic measure

$$\mu_a(\{e^{i\theta}: |g(e^{i\theta})| > s\} = \frac{1}{2\pi} |\{e^{i\theta}: |g_1(e^{i\theta})| > s\}|.$$

The function $g_2(z) = [g_1(z)]^{\pi/2\lambda} |g(a)|^{-\pi/2\lambda}$ belongs to $\mathcal{G}(\pi/2)$, and (19) follows from the special case proved above.

Continuing now with the proof of Proposition 2, we see from (19) and (18) that

$$|E_n| \leq C(\lambda)(|g(a_n)|\, 2^n t^{-1})^{\pi/2\lambda}\, |I_n|.$$

Now we use hypothesis (13). It implies that

$$|g(z)| \leq \left|\frac{1-z}{e^{i\varphi}-z}\right| \frac{2}{1-|z|}.$$

It is easy to see that $|1-a_n| \leq A(1-|a_n|)$ and that $|e^{i\varphi} - a_n| > A\varphi$. Hence

$$|g(a_n)| \leq A\varphi^{-1}.$$

Since $\lambda < \pi/2$ we can write $\pi/2\lambda = 1 + \varepsilon$ where $\varepsilon > 0$. Also $|I_n| = 2^{-n-1}\varphi$. So, it follows from (20) and (21) that

$$|E_n| \leq C(\lambda)\varphi^{-\varepsilon} 2^{n\varepsilon} t^{-1-\varepsilon}, \qquad n \geq 0. \tag{22}$$

If $t\varphi<1$ then, since $|E(t)|\leq\varphi$, estimate (16) is true as long as $C(\lambda)\geq 1$. Suppose $t\varphi\geq 1$. Let ν be the positive integer satisfying

$$t\varphi\leq 2^{\nu}<2t\varphi.$$

Then, from (22),

$$|E(t)|=\sum_{n=0}^{\infty}|E_n|\leq C(\lambda)\varphi^{-\varepsilon}t^{-1-\varepsilon}\sum_{n=0}^{\nu-1}2^{n\varepsilon}+\sum_{n=\nu}^{\infty}|I_n|.$$

The first term on the right equals

$$C(\lambda)\varphi^{-\varepsilon}t^{-1-\varepsilon}\frac{2^{\nu\varepsilon}-1}{2^{\varepsilon}-1}\leq C(\lambda)\varphi^{-\varepsilon}t^{-1-\varepsilon}(2t\varphi)^{\varepsilon}$$
$$=C(\lambda)t^{-1},$$

while the second term on the right equals

$$\sum_{n=\nu}^{\infty}2^{-n}\varphi=2^{1-\nu}\varphi\leq 2t^{-1}.$$

Thus $|E(t)|\leq C(\lambda)t^{-1}$, as asserted by (16).

4. Completion of the proof

We first show that a function in weak H^1 can behave no worse on the average than the function $1/(1-z)$. Because of the way Theorem 1 is stated it is more convenient to use

$$\frac{1+z}{1-z}=\frac{zk'(z)}{k(z)}$$

as comparison function.

PROPOSITION 3. *Suppose* $F\in$ weak H^1. *Then*

$$\int_{-\pi}^{\pi}\varphi(\log|F(re^{i\theta})|)\,d\theta\leq\int_{-\pi}^{\pi}\varphi\left(\log\left|b(F)\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right)d\theta$$

for every $\varphi\in K$ *and* $r\in(0,1]$.

Here $b(F)$ is the weak H^1 norm defined by (14).

Proof. Since $F \in N^+$, $|F(z)|$ is majorized by the outer factor in its canonical factorization, while $|F(e^{i\theta})|$ equals its outer factor a.e. Hence, we may assume F is an outer function. Also, we may assume that $b(F)=1$. This can be verified by a homogeneity argument and the observation that for each constant a $\Phi(x+a)$ belongs to K whenever $\Phi(x)$ does. Thus, we are assuming that

$$|\{e^{i\theta} : |F(e^{i\theta})| > t\}| \le t^{-1}, \qquad 0 < t < \infty. \tag{24}$$

Write

$$u(z) = \log|F(z)|, \qquad v(z) = \log\left|\frac{1+z}{1-z}\right|.$$

Let Δ^+ denote the upper half of the closed unit disk with the origin deleted,

$$\Delta^+ = \{z : |z| \le 1, \operatorname{Im} z \ge 0, z \ne 0\}.$$

For $z = re^{i\theta} \in \Delta^+$ define

$$u^*(z) = \sup_{|E|=2\theta} \int_E u(re^{i\theta})\, d\varphi$$

where the supremum is taken over all measurable subsets of $\partial\Delta$ whose measure is exactly 2θ. The function v^* is defined analogously. By [1, p. 150] the desired conclusion (23) is equivalent to

$$u^*(z) \le v^*(z), \qquad \forall z \in \Delta^+. \tag{25}$$

Now u^* is subharmonic in the interior of Δ^+ [1, p. 141]. Also, u^* is continuous on Δ^+. This is easy to deduce using the fact that u is the Poisson integral of its boundary values. Since $v(re^{i\theta})$ is a symmetric decreasing function of θ for each r it follows as in [1, p. 153] that v^* is harmonic in the interior of Δ^+, and continuous on Δ^+. Both u^* and v^* are bounded in Δ^+, so, applying the extended maximum principle [19, p. 77] to the subharmonic function $u^* - v^*$ in the upper half of the unit disk with exceptional boundary set $\{0\}$, we see that it suffices to establish (25) for $z \in \partial\Delta^+$.

We have $u^*(r) = v^*(r) = 0$ for $0 < r < 1$. Since u and v are harmonic in Δ (recall F is outer), the mean value property and continuity give, for $0 < r < 1$,

$$2\pi u(0) = u^*(re^{i\pi}) = u^*(e^{i\pi}), \qquad v^*(re^{i\pi}) = v^*(e^{i\pi}).$$

Hence, it suffices to establish (25) for $z = e^{i\theta}$, $0 \le \theta \le \pi$.

Let $\tilde{u}(e^{i\theta})$ denote the symmetric non-increasing rearrangement of $u(e^{i\theta})$ [1, p. 149]. From (24) we have, for $t>0$,

$$|\{e^{i\theta}: u(e^{i\theta})>t\}| = |\{e^{i\theta}: \tilde{u}(e^{i\theta})>t\}| < e^{-t}.$$

Take $0\in(0, \pi]$. Let $y=\tilde{u}(e^{i\theta})$. Then

$$2\theta \le |\{e^{i\theta}: \tilde{u}(e^{i\varphi}) \ge y\}| \le e^{-y},$$

so that

$$\tilde{u}(e^{i\theta}) = y \le \log\frac{1}{2\theta}, \qquad 0\le\theta\le\pi.$$

Since $\cot x > 1/4x$ for $x\in(0, \pi/4)$ it follows from the last inequality that

$$\tilde{u}(e^{i\theta}) \le \log\cot\frac{\theta}{2} = v(e^{i\theta}), \qquad 0<\theta<\frac{\pi}{2}.$$

Hence, for $0<\theta<\pi/2$,

$$u^*(e^{i\theta}) = \int_{-\theta}^{\theta} \tilde{u}(e^{i\varphi})\, d\varphi \le \int_{-\theta}^{\theta} v(e^{i\varphi})\, d\varphi = v^*(e^{i\theta}).$$

For $\pi/2\le\theta\le\pi$, it follows from (26) that

$$u^*(e^{i\theta}) \le \int_{-\theta}^{\theta} \log\frac{1}{2\,|\varphi|}\, d\varphi = 2\theta\left(\log\frac{e}{2}+\log\frac{1}{\theta}\right) \le 0,$$

while

$$v^*(e^{i\theta}) \ge v^*(e^{i\pi}) = \int_{-\pi}^{\pi} \log\left|\frac{1+e^{i\varphi}}{1-e^{i\varphi}}\right| d\varphi = 0.$$

We have thus shown that $u^*(e^{i\theta}) \le v^*(e^{i\theta})$ for $\theta\in[0, \pi]$, and the proof of Proposition 3 is complete.

Proof of Theorem 1. Suppose $f\in\mathcal{M}(\lambda)$, $0<\lambda<\pi/2$. Let

$$F(z) = \frac{izf'(z)}{f(z)}.$$

By Proposition 1, F has a representation of the form (12) and by the distortion theorem [18, p. 21] F satisfies (13). Hence, by Proposition 2, $F\in$ weak H^1 with norm $|b(F)|$ less than or equal to $C(\lambda)$. Proposition 3 now asserts that conclusion (4) of theorem holds for the positive sign. To see that it holds for the negative sign as well, apply the above reasoning to $F(z)=f(z)/izf'(z)$ and note that

$$\frac{1+z}{1-z}=\frac{zk'(z)}{k(z)}=-\frac{k(-z)}{zk'(-z)},$$

so that $\log|zk'(z)/k(z)|$ and $-\log|zk'(z)/k(z)|$ have the same integral means.

To obtain (3) from (4) we use an idea of Leung's [14].

LEMMA. *Suppose that u_1 and u_2 are real-valued functions defined in $\bar{\Delta}$ and integrable on circles $|z|=r$. Then, for $z\in\Delta^+$,*

$$(u_1+u_2)^*(z)\le u_1^*(z)+u_2^*(z),$$

and equality holds if u_1 and u_2 are either both symmetric increasing or both symmetric decreasing functions of θ for each r.

The proof follows easily from the definitions of the *-function.

Write

$$\log|f'(z)|=\log\left|\frac{zf'(z)}{f(z)}\right|+\log\left|\frac{f(z)}{z}\right|.$$

We have

$$\left(\log\left|\frac{zf'(z)}{f(z)}\right|\right)^*\le\left(\log\left|C(\lambda)\frac{zk'(z)}{k(z)}\right|\right)^*$$

by (4) and the equivalence between *-function inequalities and φ mean inequalities [1, p. 150], while from Theorem A it follows that

$$\left(\log\left|\frac{f(z)}{z}\right|\right)^*\le\left(\log\left|\frac{k(z)}{z}\right|\right)^*.$$

The lemma implies that

$$(\log|f'|)^*(z) \leq (\log|C(\lambda)k'|)^*(z), \qquad z \in \Delta^+,$$

and this is equivalent to (3) with the positive sign. The same reasoning, with negative signs inserted, shows that (3) also holds with the negative sign. Here one uses the fact that

$$-\log\left|\frac{zk'(z)}{k(z)}\right| = \log\left|\frac{1-z}{1+z}\right| \quad \text{and} \quad -\log\left|\frac{k(z)}{z}\right| = \log|1-z|^2$$

are both symmetric increasing functions.

REFERENCES

[1] A. BAERNSTEIN, *Integral means, univalent functions and circular symmetrzation*, Acta. Math. 133 (1974), 139–169.

[2] A. BAERNSTEIN, Analytic functions of bounded mean oscillation, *Aspects of Contemporary Complex Analysis*, pp. 3–36, edited by D. Brannan and J. Clunie, Academic Press, London, 1980.

[3] L. BRICKMAN, *Extreme points of the set of univalent functions*, Bull. Amer. Math. Soc. 76 (1970), 372–374. MR41 #448.

[4] L. BRICKMAN and D. WILKEN, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc. 43 (1974), 523–528. MR 48 #6399.

[5] L. BRICKMAN and D. WILKEN, *Subordination and insuperable elements*, Michigan Math. J. 23 (1976), 225–233.

[6] J. E. BROWN, *Derivatives of close-to-convex functions, integral means and bounded mean oscillation*, Math. Z. 178 (1981), 353–359.

[7] J. CIMA and K. PETERSEN, *Some analytic functions whose boundary values have bounded mean oscillation*, Math. Z. 151 (1976), 295–300.

[8] P. L. DUREN, *Theory of H^p Spaces*, Academic Press, New York, 1970. MR 42 #3552.

[9] P. L. DUREN, *Coefficients of univalent functions*, Bull. Amer. Math. Soc. 83 (1977), 891–911.

[10] P. L. DUREN and Y. J. LEUNG, *Logarithmic coefficients of univalent functions*, J. d'Analyse Math. 36 (1979), 36–43.

[11] W. K. HAYMAN, *The logarithmic derivative of multivalent functions*, Michigan Math. J. 27 (1980), 149–179.

[12] W. HENGARTNER and G. SCHOBER, *Some new properties of support points for compact families of univalent functions in the unit disk*, Michigan Math. J. 23 (1976), 207–216.

[13] W. E. KIRWAN and R. PELL, *Extremal properties of a class of slit conformal mappings*, Michigan Math. J. 25 (1978), 223–232.

[14] Y. J. LEUNG, *Integral means of the derivatives of some univalent functions*, Bull. London Math. Soc. 11 (1979), 289–294.

[15] A. J. LOHWATER, G. PIRANIAN and W. RUDIN, *The derivative of a schlicht function*, Math. Scand. 3 (1955), 103–106. MR 17, 249.

[16] I. M. MILIN, *On the coefficients of univalent functions*, Dokl. Akad. Nauk SSSR 176 (1967), 1015–1018-Soviet Math. Dokl. 8 (1967), 1255–1258. MR 36 #5328.
[17] A. PFLUGER, *Lineare Extremalprobleme bei schlichten Funktionen*, Ann. Acad. Sci. Fenn. Ser. A I No. 489 (1971), 32 pp. MR 45 #5337.
[18] CH. POMMERENKE, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1974.
[19] M. TSUJI, *Potential theory in modern function theory*, Maruzen Company, Tokyo, 1959.
[20] A. ZYGMUND, *Trigonometric Series*, 2nd Ed., Cambridge University Press, Cambridge, 1968.
[21] J. FENG and T. MACGREGOR, "Estimates on integral means of the derivatives of univalent functions", J. Analyse Math. 29 (1976), 203–231.

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