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## Compactification of the space of vector bundles on a singular curve

C. J. REGO

Let  $X$  be a singular, integral, projective curve of genus greater than one over an algebraically closed field  $k$ . It has been verified by Narasimhan and Newstead [N] that the method of [S] extends to construct a projective moduli space for semi-stable torsion free  $\mathcal{O}_X$  modules of rank  $n$  and degree  $d$  which we denote by  $\bar{M}(n, d)$ . The points of  $\bar{M}(n, d)$  corresponding to vector bundles form an open, irreducible subset. The object of this article is to prove the

**THEOREM.** *If  $X$  is embeddable in a smooth surface  $Z$  then  $\bar{M}(n, d)$  is irreducible.*

To prove irreducibility it suffices (and is equivalent) to verify

(0.1) Given a torsion free  $\mathcal{O}_X$ -module  $N$  there is an  $\mathcal{O}_{X \times \text{Spec } k[[t]]}$ -module  $\mathcal{L}$  with  $\mathcal{L}/t \cdot \mathcal{L} \approx N$  and  $\mathcal{L} \otimes k((t))$  a vector bundle on  $X \times \text{Spec } k((t))$ .

If the singularities of  $X$  are not all planar then we have verified in [R] that there are rank one modules not deformable to a line bundle, hence  $\bar{M}$  cannot be irreducible in that case. The case  $n = 1$  was first established in [A] using Iarrobino's calculation of the dimension of the Punctual Hilbert Scheme of ideals in  $k[[x, y]]$  of colength  $m$ ,

(0.2)  $\dim \text{Hilb}_0^m(k[[x, y]]) \leq m - 1$ .

In [R] we gave a self contained proof of the irreducibility of  $\bar{M}(1, d)$  by induction on the multiplicity of the singular points and derived (0.2) as a consequence. The case of rank greater than one does not follow “module theoretically” from the rank one result except for very simple plane singularities for which modules split locally into a direct sum of rank one modules. By [B] this happens only when the multiplicity of each singular point is less than or equal to two.

An important ingredient in the arguments of [A] and [R] was the fact that every component of  $\text{Hilb}^m(X)$  is of dimension greater than or equal to  $m$ . This follows from the observation that  $\text{Hilb}^m(X)$  is (locally) the zero set of a section of

a rank  $m$  vector bundle on the  $2m$  dimensional space  $\text{Hilb}^m(Z)$ . For  $n > 1$  we work with  $\text{Quot}^m(n, Z)$ , the space of quotients of length  $m$  of a fixed free sheaf on  $Z$  of rank  $n$ . However  $\text{Quot}^m(n, Z)$  is singular and  $\text{Quot}^m(n, X) \hookrightarrow \text{Quot}^m(n, Z)$  does not have a simple description as a subscheme. There is thus no way of extending the ideas of [A] to the case of  $n > 1$ . In [R] we use the fact that  $\text{Hilb}^m(Z)$  is at least irreducible for  $Z$  a smooth connected surface. Again, we have no à priori proof that  $\text{Quot}^m(n, Z)$  is irreducible for  $n > 1$  and this result is deduced below as a corollary of the main theorem.

When  $n = 1$  the irreducibility of  $\text{Hilb}^m(Z) = \text{Quot}^m(1, Z)$  follows from the Hilbert–Schaps’ lemma “codim 2 + cohen-macaulay  $\Rightarrow$  smoothable,” where the matrices defining the presentation of the codimension 2 ideal are deformed. As the quotients in  $\text{Quot}^m(n, Z)$  are also defined by two term complexes it would be interesting to obtain a proof of the irreducibility of  $\text{Quot}^m(n, Z)$  along these lines. The main difficulty here is that for  $n > 1$  the matrices cannot be deformed “arbitrarily” as  $\text{Quot}^m(n, Z)$  is singular.

We are unable to prove that  $\bar{M}(n, d)$  is reduced for  $n > 1$ . It would suffice to know that  $\text{Quot}^m(n, X)$  is reduced. In the case when  $X$  has only ordinary double points Seshadri has recently proved that  $\bar{M}$  is reduced. He writes down the completion of the local rings of  $\bar{M}$  in determinantal form so that they can be described by available techniques. The general case is completely open.

No use is made here of the analogue of the scheme  $E$  introduced in [R] and we are able to avoid the somewhat precise (see (3.1.2.) to (3.1.7.) of [R]) dimension calculations used there. The analogue of (0.2) follows from the main theorem, as in the case of rank one, but as we have no applications details are omitted.

## §1. Initial definitions and propositions

Let  $Y$  be a scheme over  $k$  and  $V = \mathcal{O}_Y^n$ . The functor of  $\mathcal{O}_Y \otimes \mathcal{O}_T$  submodules of  $V \otimes \mathcal{O}_T$ ,  $N_T$ , satisfying “ $V \otimes \mathcal{O}_T / N_T$  is a locally free  $\mathcal{O}_T$  module of rank  $m$ ” is represented by a projective scheme denoted by  $\text{Quot}^m(n, Y)$ . In the sequel  $Y$  will usually be a smooth surface or a curve on a smooth surface. Note that if  $W$  is a subscheme of  $Y$  we have a closed immersion  $Q^m(n, W) \hookrightarrow Q^m(n, Y)$  where  $N \in Q^m(n, W)$  iff  $\mathcal{I}_W \cdot V \subset N$ , where  $\mathcal{I}_W$  is the defining ideal of  $W$ .

**PROPOSITION 1.1.** *Let  $Z$  be a smooth surface. Then  $Q^m(n, Z)$  is singular for  $n > 1, m > 1$ .*

*Proof.* The tangent space at a point corresponding to  $N \subset V$  is canonically

identified with  $\text{Hom}(N, V/N)$ . Suppose  $V/N$  is supported at  $m$  distinct points of  $Z$ . We claim  $V/N$  defines a smooth point of  $\text{Quot}$ . To see this first compute the tangent space. Since it is a local question it suffices to fix a local ring  $\mathcal{O}$  of  $Z$  with maximal ideal  $\mathfrak{M}$  and suppose  $N_0 = \mathfrak{M} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \subset \mathcal{O}^n = V$ . Then  $\text{Hom}(N_0, V/N_0)$  has dimension  $(n+1)$ . If  $V/N$  is supported at  $m$  distinct points its tangent space has rank  $m(n+1)$ . We now check that  $\text{Quot}^m$  has dimension  $m(n+1)$  at  $V/N$ . Again it suffices to prove that at  $N_0 = \mathfrak{M} \oplus \mathcal{O} \cdots \oplus \mathcal{O} \subset V$   $\text{Quot}^1(n, Z)$  has dimension  $(n+1)$ . Note that  $N_0$  defines a point of  $\mathbf{P}(V/\mathfrak{M} \cdot V) \hookrightarrow \text{Quot}^1$ . For each point of  $Z$  we thus obtain a  $\mathbf{P}^{n-1} \subset \text{Quot}^1$  of quotients supported at that point. As  $\dim Z = 2$  we find  $\dim \text{Quot}^1 \geq (n-1) + 2 = n+1$ . By the tangent space computation  $\dim \text{Quot}^1 = (n+1)$ .

We denote by  $U^m$  the smooth open subset of  $\text{Quot}^m$  defined by quotients supported at  $m$  distinct points. To see that  $\text{Quot}^m$  is singular for  $m \geq 2$ ,  $n \geq 2$  we pick a point in the closure of  $U^m$  which has a tangent space of rank greater than  $(n+1)m$ . One such point is defined by the module  $N \subset V$  of colength 1 at  $(m-2)$  points and of the type  $\mathfrak{M} \oplus \mathfrak{M} \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O} \subset V$  at one point. It is clear how to deform this quotient so that it has support at  $m$  points. If  $x, y$  are generators of  $\mathfrak{M}$  just take the  $k[[t]]$  deformation  $((x+t, y) \oplus \mathfrak{M} \cdots \oplus \mathcal{O}) \subset V \otimes k[[t]]$ . This shows that  $V/N$  is in the closure of  $U^m$ . However its tangent space has rank equal to  $(n+1)(m-2) + 2(n+2)$  which is greater than  $(n+1) \cdot m$ . This proves the proposition.

**PROPOSITION 1.2.** *Let  $X$  be a projective integral curve with singular points  $P_1 \cdots P_r$  and  $N$  a torsion free  $\mathcal{O}_X$ -module of rank  $n$ . Then  $N$  is deformable (over  $\text{Spec } k[[t]])$  to a vector bundle on  $X \times \text{Spec } k((t))$  if and only if  $N_{P_i}$  is deformable to a projective module over  $\mathcal{O}_{X, P_i} \otimes k[[t]] \forall i$ .*

*Proof.* One way is clear so suppose  $N_{P_i}$  is deformable to a projective module  $\forall i$  and let  $N_{P_i}[t]$  be the  $\mathcal{O}_{P_i} \otimes k[[t]]$  modules representing these deformations. Choose imbeddings  $q_i: N_{P_i}[t] \subset \mathcal{O}_{X, P_i}^n \otimes k[[t]]$  and observe that  $(\text{coker } q_i)$  is a finite  $k[[t]]$  module iff it is not supported at any height one maximal ideals. In any case there is an  $N'_{P_i}[t] \forall i$  with

$$N_{P_i}[t] \subset N'_{P_i}[t] \xrightarrow{q'_i} \mathcal{O}_{X, P_i}^n \otimes k[[t]]$$

with  $(\text{coker } q'_i)$  a finite free  $k[[t]]$  module and  $N'_{P_i}[t]$  specializes to  $N_{P_i}$ . Let  $U_i = X - (\bigcup_{j \neq i} P_j)$  and increasing the number of  $P_i$ 's if necessary we can assume  $N$  is trivial over  $U_i \cap U_j \forall i, j$ . Then the  $q'_i$ 's define sheaves  $\mathcal{N}_i$  on  $U_i \times \text{Spec } k[[t]]$  which are vector bundles outside  $(P_i) \times (0)$  and trivial on  $(U_i - (P_i)) \times \text{Spec } k[[t]]$ .



Now  $N$  can be defined by matrices in  $Gl_n(\mathcal{O}_{U_i \cap U_j})$ . Lifting these matrices to elements of  $Gl_n(\mathcal{O}_{U_i \cap U_j} \otimes k[[t]])$  defines an  $\mathcal{O}_{X \times \text{Spec} k[[t]]}$  module  $\mathcal{N}$  which is generically a vector bundle and specializes to  $N$ . This proves the proposition.

**PROPOSITION (1.2.0).** *Let  $X$  be a smooth irreducible curve; then  $\text{Quot}^m(n, X)$  is irreducible. In particular, for any irreducible curve, the open subset of  $\text{Quot}$  supported at smooth points is irreducible.*

*Proof.* Write  $Q = \text{Quot}^m(n, X)$  and recall we have an exact sequence

$$(1.1.1) \quad 0 \rightarrow \mathcal{N} \hookrightarrow \mathcal{O}_{X \times Q}^n \rightarrow H \rightarrow 0$$

where  $\mathcal{N}$  is a rank  $n$  vector bundle on  $X \times Q$  and  $p_{2*}H$  is a rank  $m$  vector bundle on  $Q$ . The determinant defines a map  $d: \bigwedge^n \mathcal{N} \rightarrow \mathcal{O}_{X \times Q}$  with cokernel finite of rank  $m$  over  $Q$ . Hence we get a morphism  $p$  from  $Q$  to  $\text{Hilb}^m(X) = \text{Quot}^m(1, X)$  with fibres representing quotients which are supported at the “determinantal cycle.” Consider the subset  $U^m(n, X) \subset \text{Quot}^m(n, X)$  of quotients supported at  $m$  distinct points. As in the proof of Proposition (1.1) we see that the fibres of  $p$  are  $m$ -fold products of  $\mathbf{P}^{n-1}$ 's. Since  $\text{Hilb}^m(X)$  is irreducible and the image of  $U^m(n, X)$  is dense open we find  $U^m(n, X)$  is irreducible. It remains to prove that  $U^m(n, X)$  is dense. But any  $N \subset \mathcal{O}_X^n$  is locally free as  $X$  is smooth so the arguments used in the proof of Proposition (1.1) show that  $N$  can be deformed over  $k[[t]]$  so that the quotient is supported at  $m$  distinct points. This proves the proposition.

**Remark (1.2.1).** Since any finite set of points on a smooth irreducible projective variety  $Z$  can be joined by a smooth irreducible curve  $X$ , given two points on  $U^m(n, Z)$  we can find an  $X$  with  $U^m(n, X)$  containing them. Hence  $U^m(n, Z)$  is irreducible.

**PROPOSITION (1.3) [D'Souza].** *Let  $N$  be a torsion free module of rank  $n$  over a one dimensional Gorenstein ring  $\mathcal{O}$ . Let  $A' \rightarrow A$  be a surjective map of complete local  $k$ -algebras (with residue field  $k$ ). Given an embedding  $N_A \hookrightarrow \mathcal{O}^n \otimes_k A$  with  $\mathcal{O}^n \otimes A / N_A$  a flat  $A$  module and a flat deformation  $N_{A'}$  of  $N$  over  $A'$ , lifting  $N_A$ , there is an embedding  $N_{A'} \hookrightarrow \mathcal{O}^n \otimes A'$  so that the diagram*

$$\begin{array}{ccc} N_{A'} & \hookrightarrow & \mathcal{O}^n \otimes A' \\ \downarrow & & \downarrow \\ N_{A'} \otimes_{A'} A \approx N_A & \hookrightarrow & \mathcal{O}^n \otimes A \end{array} \text{ is commutative.}$$

*Proof.* [O-S, Appendix].

(1.4) From now on fix a singular local ring  $\mathcal{O}$  of an integral Gorenstein curve  $X$  and write  $F = \mathcal{O}^n$ ,  $\bar{\mathcal{O}}$  the normalization of  $\mathcal{O}$ ,  $K$  the quotient field of  $\mathcal{O}$ ,  $\bar{F} = \bar{\mathcal{O}}^n$ ,  $\delta = \text{length}(\bar{\mathcal{O}}/\mathcal{O})$ ,  $C \subset \mathcal{O}$  the conductor of  $\mathcal{O}$  in  $\bar{\mathcal{O}}$ . Let  $N$  be a torsion free  $\mathcal{O}$  module of rank  $n$ . Write  $\bar{N} = N \cdot \bar{\mathcal{O}} = N \otimes \bar{\mathcal{O}} / \text{Torsion}$  and as  $\bar{N}$  is torsion free over a P.I.D. it is free. Choose  $n$  elements in  $N$  which generate  $\bar{N}$  over  $\bar{\mathcal{O}}$ . These define an imbedding  $F \hookrightarrow N$  so that  $F \cdot \bar{\mathcal{O}} = N \cdot \bar{\mathcal{O}} = \bar{F}$ . Thus every isomorphism class of  $\mathcal{O}$  modules is represented by one between  $F$  and  $\bar{F}$ .

**DEFINITION-PROPOSITION (1.5).** *The functor of  $\mathcal{O}$ -submodules of  $\bar{F}$  with colength  $d$  is denoted by  $E(d)$ . It is represented by a closed subset of a Grassmanian.*

(1.6) By the above  $\bigcup_{d \leq n\delta} E(d)$  'contains' every isomorphism class of  $\mathcal{O}$  modules. We claim  $E(n\delta)$  contains an open subset of free  $\mathcal{O}$  modules which has dimension  $\delta \cdot n^2$ . Openness is immediate. Now let  $F_1, F_2 \in E(n\delta)$ ,  $F_1 \cong F_2 \approx F$ . Then  $\varphi$  yields an element of  $\text{Aut}(K^n)$  which preserves  $\bar{F}$  i.e. an element of  $GL_n(\bar{\mathcal{O}})$ . Thus  $GL_n(\bar{\mathcal{O}})$  acts transitively on this open subset of  $E(n\delta)$  so to obtain its dimension we just calculate the isotropy at any one point, say,  $F$ . This is clearly  $GL_n(\mathcal{O})$  and the coset space  $GL_n(\bar{\mathcal{O}})/GL_n(\mathcal{O})$  has dimension equal to  $n^2 \cdot \text{length}(\bar{\mathcal{O}}/\mathcal{O}) = \delta n^2$ . For  $X$  rational with one singular point this open subset defines all vector bundles trivial on  $\tilde{X}$ . However the space of stable vector bundles should be  $\delta \cdot n^2 - (n^2 - 1)$ . This is accounted for by the fact that  $PGL_n(k)$  operates freely at the generic module in  $E(n\delta)$  and the moduli is got generically by taking a quotient.

(1.7) Take  $N, F \subset N \subset \bar{F}$ . Then

$$(1.7.1) \quad N^* = \left\{ (n_i^*) \in K^N \mid \sum n_i^* \cdot n_i \in \mathcal{O} \forall (n_i) \in N \right\}$$

is canonically identified with  $\text{Hom}(N, \mathcal{O})$ . Note that for  $N$  as above  $N^* \subset F$  and  $C \cdot F \subset N^*$ . By reflexivity

$$(1.7.2) \quad \text{length}(F/N^*) = \text{length}(N/F).$$

*(Remark.* It is a standard fact that rank one torsion free modules over  $\mathcal{O}$  are reflexive. For higher rank just use induction on the rank and the vanishing of  $\text{Ext}^1(N, \mathcal{O})$  for  $N$  torsion free.)

**PROPOSITION (1.7.2).** (a). *Every module  $N$  can be represented by  $C \cdot F \subset N \subset F$ .*

(b) *If  $N \subset F$ ,  $C \cdot F \not\subset N$  then there is an  $N' \approx N$  with  $C \cdot F \subset N' \subset F$  satisfying*

$$(1.7.4) \quad \text{length}(F/N') <_* \text{length}(F/N)$$

*Proof.* Writing  $N = P^*$ ,  $F \subset P \subset \bar{F}$ , (a) is clear by reflexivity.

To prove (b) use (a) to get  $N'$  with  $C \cdot F \subset N' \subset F$  and extend the isomorphism  $\varphi : N' \approx N$  to an isomorphism  $N' \otimes K \approx N \otimes K = K^n$  so  $\varphi \in Gl_n(K)$ . As  $\varphi(C \cdot F) \subset N \subset F$  all the entries of  $\varphi$  are in  $\bar{\mathcal{O}}$  so  $\varphi \in M_n(\bar{\mathcal{O}})$ . It is easy to verify

$$(1.7.5) \quad \text{length}(\bar{F}/\varphi(\bar{F})) = \text{length}(\bar{\mathcal{O}}/\det \varphi).$$

It follows that  $\text{length}(F/N') = \text{length}(F/N) - \text{length}(\bar{\mathcal{O}}/\det(\varphi))$ . Suppose  $\det(\varphi)$  is unit in  $\bar{\mathcal{O}}$  so  $\varphi \in Gl_n(\bar{\mathcal{O}})$ . Then as  $C \cdot F \subset N'$ ,  $C \cdot F \subset \varphi(C \cdot F) \subset \varphi(N') = N$  which contradicts our assumption. So  $\varphi$  is not in  $Gl_n(\bar{\mathcal{O}})$  and hence  $\text{length}(\bar{\mathcal{O}}/\det(\varphi)) > 0$ . This proves the proposition.

## §2.

In this section the curve  $X$  will be assumed to be embedded in a smooth surface  $Z$ . We first prove.

LEMMA 2.0.  $\bar{M}$  is irreducible  $\Leftrightarrow \text{Quot}^m(n, X)$  is irreducible for every  $m$ .

*Proof.* Let  $N$  be an  $\mathcal{O}_X$  module of rank  $n$ ,  $N \subset \mathcal{O}_X^n$  with finite cokernel of length  $m$ . Suppose  $\text{Quot}^m(n, X)$  is irreducible so  $N$  can be deformed to  $N(t) \subset \mathcal{O}_X^n \otimes k[[t]]$  with the quotient supported at  $m$  distinct  $k[[t]]$  rational primes and none of them singular. Then clearly  $N(t) \otimes k((t))$  is locally free on  $X \times \text{Spec } k((t))$ .

Conversely, let  $\bar{M}$  be irreducible and suppose  $\text{Quot}^m(n, X)$  is irreducible for  $m \leq m_0 - 1$ . Let  $N \subset \mathcal{O}_X^n$  define a point in  $\text{Quot}^{m_0}$ . Recall that as  $X$  is irreducible the quotients  $\mathcal{O}_X^n/N$  supported at  $m_0$  distinct smooth points form an irreducible open subset  $U = U^{m_0}(n, X)$ . Also if  $\mathcal{O}_X^n/N$  is supported at smooth points of  $X$  it lies in the closure of  $U$  as may be verified by treating  $\mathcal{O}_X^n/N$  as a sheaf on the normalization of  $X$ . Suppose  $\mathcal{O}_X^n/N$  is supported at  $y_1, y_2, \dots, y_s$ ,  $s > 1$ . Then for every  $i$  we have  $s_i = \text{length}(\mathcal{O}_{X, y_i}^n/N_{y_i}) < m_0$ . Since each quotient  $\mathcal{O}_{X, y_i}^n/N_{y_i}$  defines a point in  $\text{Quot}^{s_i}(n, X)$  which is in the closure of  $U^{s_i}(n, X)$  there exists a deformation of  $\mathcal{O}_X^n/N$  generically having support at  $m_0 = \sum s_i$  distinct smooth points. We may therefore assume that  $\mathcal{O}_X^n/N$  and all its small deformations are supported at one point  $x \in X$ . This means that the Punctual Quot scheme  $\text{Quot}_x^{m_0}(n, X)$  contains a component  $W$  of  $\text{Quot}^{m_0}(n, X)$  and the map  $\text{Quot}_x^{m_0}(n, X) \hookrightarrow \text{Quot}^{m_0}(n, X)$  is bijective in a neighbourhood of  $\mathcal{O}_X^n/N$ . Since  $\bar{M}$  is irreducible  $N$  can be deformed to a locally free  $\mathcal{O}_X$  module. Let  $N[t]$  be an  $\mathcal{O}_X \otimes k[[t]]$  module

defining this deformation. If we localize around  $x$  then we can use (1.3) to lift the given imbedding  $N_x \hookrightarrow \mathcal{O}_{X,x}^n$  to  $N_x[t] \hookrightarrow \mathcal{O}_{X,x}^n \otimes k[[t]]$  with cokernel a free  $k[[t]]$  module of rank  $m_0$ . Now the imbedding  $N_x[t] \hookrightarrow \mathcal{O}_{X,x}^n \otimes k[[t]]$  is the restriction of an inclusion  $N'[t] \hookrightarrow \mathcal{O}_X^n \otimes k[[t]]$  with the same cokernel and  $N'[t]$  is generically a vector bundle specializing to  $N$  on  $X$ . As  $\text{Quot}_x^{m_0}(n, X)$  is bijective with  $\text{Quot}^{m_0}(n, X)$  in a neighbourhood of  $\mathcal{O}_X^n/N$ ,  $\mathcal{O}_X^n \otimes k[[t]]/N'[t]$  is supported at  $(x) \times \text{Spec } k[[t]]$  and so there are points of  $W$  in every neighbourhood of  $\mathcal{O}_X^n/N$  defined by vector bundles. We will derive a contradiction.

Let  $h_1, \dots, h_n \in \mathcal{O}_{X,x}^n$  define a free  $\mathcal{O}_{X,x}^n$  module  $P$  with  $\mathcal{O}_{X,x}^n/P$  having length  $m_0$ . Then the deformation  $(h_i + t)$  is a flat deformation that is not supported at  $x$ . But then  $\text{Quot}^{m_0}(n, X)$  cannot be bijective with  $\text{Quot}_x^{m_0}(n, X)$  in a neighbourhood of  $\mathcal{O}_X^n/P$  and the proposition is proved.

*Remark 2.1.* The proof of the above proposition yields the fact that if  $N \subset \mathcal{O}_X^n$  is deformable over  $k[[t]]$  to a vector bundle then the given injection lifts to a (possibly different) deformation that is supported generically at  $m$  distinct points where  $m = \text{length}(\mathcal{O}_X^n/N)$ . We will use this remark later.

**COROLLARY 2.2.** *If  $\bar{M}$  is irreducible then*

$$(2.2.0) \quad \dim \text{Quot}_x^m(n, X) \leq n \cdot m - 1$$

for all  $x \in X$  and  $m \geq 1$ .

*Proof.* Since  $\text{Quot}_x^m$  is a proper closed subset of  $\text{Quot}^m$  and  $\dim U^m(n, X) = n \cdot m$  the result follows.

From now on  $X$  is an irreducible and reduced curve on a smooth surface  $Z$  and  $C$  is the conductor of  $\mathcal{O}_X$  in its normalization.

**LEMMA 2.3.** *If  $\nu$  is the multiplicity of  $\mathcal{O}_{X,x}$  and  $\mathfrak{M}$  the maximal ideal then*

$$(2.3.1) \quad C \subset \mathfrak{M}^{\nu-1}, \quad C \not\subset \mathfrak{M}^\nu$$

*Proof.* The conductor is defined by the set of curves  $g=0$ ,  $g \in \mathcal{O}_Z$  with multiplicity greater than or equal to  $(\text{mult } \mathcal{O}_{X,x} - 1)$  at  $x$  as well as at all infinitely near points. Hence  $C \subset \mathfrak{M}^{\nu-1}$ . Recall that if  $\mathcal{O}'$  is the blow up of  $\mathcal{O} = \mathcal{O}_{X,x}$  then  $\mathfrak{M}^{\nu-1}$  is the conductor of  $\mathcal{O}$  in  $\mathcal{O}'$  and  $C = C_1 \cdot \mathfrak{M}^{\nu-1}$  where  $C_1$  is the conductor of  $\mathcal{O}'$  in  $\mathcal{O}$ . Also by the definition of blowing up there is a  $z$  in  $\mathfrak{M}$  satisfying  $z \cdot \mathcal{O}' = \mathfrak{M} \cdot \mathcal{O}'$  so that  $\mathfrak{M}^{\nu-1} \cdot \mathcal{O}' = z^{\nu-1} \cdot \mathcal{O}'$ . Assume that  $C \subset \mathfrak{M}^\nu$ ; we will derive a

contradiction. We have

$$\begin{aligned}
 C &\subset \mathcal{M}^v \\
 \Rightarrow C_1 \cdot \mathcal{M}^{v-1} &\subset \mathcal{M}^v \\
 \Rightarrow C_1 &\subset \text{Hom}(\mathcal{M}^{v-1}, \mathcal{M}^v) = \text{Hom}(\mathcal{M}^{v-1}\mathcal{O}', \mathcal{M}^v\mathcal{O}') \\
 &= \text{Hom}(z^{v-1} \cdot \mathcal{O}', z^{v-1} \cdot z \cdot \mathcal{O}') \\
 &= \text{Hom}(z^{-1} \cdot \mathcal{O}', \mathcal{O}') = z \cdot \mathcal{O}'.
 \end{aligned}$$

This says that  $z^{-1} \cdot C_1 \subset \mathcal{O}'$ ,  $z$  a non unit in  $\bar{\mathcal{O}}$  and contradicts the definition of  $C_1$  as the largest  $\bar{\mathcal{O}}$  ideal in  $\mathcal{O}'$ . The lemma is thereby proved.

*Remark 2.4.* In characteristic zero a polar of the equation of  $X$  in  $Z$  at  $x$  gives an element of  $C$  not in  $\mathcal{M}^v$  since a derivative has lower order than that of the equation. (see Coolidge – A Treatise on Algebraic Plane Curves). In general we refer to any  $g \in C - \mathcal{M}^v$  as a polar.

**THEOREM (2.5).**  $\text{Quot}^m(n, X)$  is irreducible for all  $m$ .

*Proof.* Since the problem is local around the singular points we use induction on the multiplicity of one singular point,  $x \in X$ . Assume the result true for a curve with multiplicity less than  $\nu = \text{mult}(\mathcal{O}_{X,x})$ . By Lemma 2.3 there is a  $g \in \mathcal{O}_Z$  with  $g$  of order  $(\nu - 1)$  and  $g$  defining an element of  $C$ . We have

$$(2.5.1) \quad \text{Quot}^m(n, \mathcal{O}_X/C) \hookrightarrow \text{Quot}_x^m(n, \mathcal{O}_Z/(g)).$$

By adding a general element of  $\mathcal{O}_Z$  of high order to  $g$  we can assume that  $g=0$  defines (locally) a reduced curve in  $Z$  irreducible in a neighbourhood of  $x$ . Now induction and Cor. 2.2 gives

$$(2.5.2) \quad \dim \text{Quot}^m(n, \mathcal{O}_X/C) \leq \dim \text{Quot}_x^m(n, \mathcal{O}_Z/(g)) \leq n \cdot m - 1.$$

If  $x_1, \dots, x_r$  are the singular points of  $X$  and  $N$  a torsion free rank  $n$   $\mathcal{O}_X$  module then to show that  $N$  is deformable to a vector bundle it suffices to know this for  $N_{x_i}$ ,  $\forall i$ . We may therefore assume that  $X$  has one singular point,  $x$ .

Assume all  $\mathcal{O}_X$  modules  $N$  which have an embedding  $N \hookrightarrow \mathcal{O}_X^n$  with  $\text{length}(\mathcal{O}_X^n/N) < m_0$  can be deformed to locally free modules. By Remark (2.1) this is equivalent to the assumption that  $U^m(n, X)$  is dense in  $\text{Quot}^m(n, X)$  for  $m < m_0$  and in particular  $\text{Quot}^m(n, X)$  is irreducible for  $m < m_0$ . We want to show that  $\text{Quot}^{m_0}(n, X)$  is irreducible. By Lemma 2.0 and induction on  $m_0$  this would yield the theorem.

Suppose  $\mathcal{O}_X^n/N$  is supported at  $y_1, y_2, \dots, y_s$ ,  $s > 1$ . Then for every  $i$  we have  $s_i = \text{length}(\mathcal{O}_{X,y_i}^n/N_{y_i}) < m_0$ .

Since each quotient  $\mathcal{O}_{X,y_i}^n/N_{y_i}$  defines a point in  $\text{Quot}^s(n, X)$  which is the closure of  $U^s(n, X)$  there exists a deformation of  $\mathcal{O}_X^n/N$  generically having support at  $m_0$  distinct smooth points. Hence  $\mathcal{O}_X^n/N$  lies in the closure of  $U^{m_0}(n, X)$  so that  $N$  is deformable to a vector bundle. Therefore suppose  $\mathcal{O}_X^n/N$  is supported at one point  $y \in X$ . If  $y \neq x$ , i.e. if  $y$  is a smooth point then  $N$  is actually locally free since  $N_x = \mathcal{O}_{X,x}^n$  and  $N_y$  is free when  $\mathcal{O}_{X,y}$  is a discrete valuation ring.

We can thus restrict ourselves to quotients in  $\text{Quot}_x^{m_0}(n, X)$ . The above discussion says that the open set  $U = \text{Quot}^{m_0}(n, X) - \text{Quot}_x^{m_0}(n, X)$  has  $U^{m_0}(n, X)$  as a dense (irreducible) subset. If  $\text{Quot}^{m_0}(n, X)$  is reducible then  $\text{Quot}_x^{m_0}(n, X)$  must contain a component  $W$  of  $\text{Quot}^{m_0}(n, X)$ . We will prove the theorem by deriving a contradiction.

Let  $Q \in \text{Quot}_x^{m_0}(n, X) \subset \text{Quot}^{m_0}(X)$  be a general point of  $W$  so that if  $\mathcal{O}_X^n/N$  represents  $Q$  every small deformation of  $\mathcal{O}_X^n/N$  is supported at  $x$ . By Remark (2.1),  $N$  cannot be deformed to a locally free module. If  $C \cdot \mathcal{O}_X^n \not\subset N$  then by Proposition (1.7.3) there is an  $N'$  with  $N'_x \approx N_x$  and with  $C \cdot \mathcal{O}_X^n \subset N' \subset \mathcal{O}_X^n$ . Further

$$(2.5.3) \quad \text{length}(\mathcal{O}_X^n/N') \not\leq \text{length}(\mathcal{O}_X^n/N) = m_0$$

and so by assumption  $N'$  is deformable to a vector bundle. But for  $y \neq x$   $N'_y = \mathcal{O}_{X,y}^n$  and  $N'_x \approx N_x$  so that Proposition (1.2) implies that  $N$  is also deformable to a vector bundle. This contradicts Remark (2.1) and hence we may assume  $C \cdot \mathcal{O}_X^n \subset N$ . As  $\mathcal{O}_X^n/N$  is a general point of  $W$  the natural map  $\text{Quot}^{m_0}(n, \mathcal{O}_X/C) \rightarrow \text{Quot}^{m_0}(n, X)$  is a bijection in a neighbourhood of  $P$ . Of course, as  $W$  is a component,  $\text{Quot}^{m_0}(n, X)$  is bijective with  $\text{Quot}_x^{m_0}(n, X)$  in a neighbourhood of  $P$ .

If  $X'$  is an affine open subset of  $X$  then  $\text{Quot}^m(n, X')$  is an open subset of  $\text{Quot}^m(n, X)$  and if  $x \in X'$  then  $\text{Quot}_x^m(n, X') = \text{Quot}_x^m(n, X)$ . As we will encounter only quotients supported at  $x$  and their small deformations we can restrict ourselves to an open neighbourhood of  $x$ . Let  $Z' \subset Z$  be an affine open set with  $X' = X \cap Z'$  satisfying  $x \in X'$  and  $X'$  defined by one equation,  $\{f=0\}$ ,  $f \in \mathcal{O}_Z$ . Let  $g \in \mathcal{O}_Z$  define a polar of  $X'$  at  $x$ . If  $g$  is chosen sufficiently general and  $Z'$  small enough we can arrange so that  $\{f=0\} \cap \{g=0\} = z \in Z$  and  $\{f+tg=0\} \subset Z' \times \text{Spec } k[[t]]$  is smooth outside  $(z) \times \text{Spec } k[[t]]$ , where of course,  $\mathcal{O}_{Z,z}/(f) = \mathcal{O}_{X,x}$ . Write  $S = \text{Spec } k[[t]]$  and  $\{f+tg=0\} = X'_S$  and  $\varphi': X'_S \rightarrow S$  the restriction of the projection map. The family  $\varphi$  is smooth outside  $(z) \times S$  and the singular point of the generic fibre has multiplicity  $\nu - 1 = (\text{order } g \text{ at } z)$ , by the definition of a polar. As in the proper case we have a corresponding family  $\rho': \text{Quot}^{m_0}(n, X'_S | S) \rightarrow S$  where the fibres of  $\rho'$  are open subsets of the Quot schemes of the fibres of  $\varphi'$ . Since  $f+tg \in C \subset N$ ,  $\mathcal{O}_X^n/N$  or rather  $\mathcal{O}_{X \times S}^n/N \otimes \mathcal{O}_S$ , defines a section  $\sigma$  of  $\rho'$ . By

induction on the multiplicity of the singular point the generic fibre of  $\rho'$  is irreducible of dimension  $n \cdot m_0$ . As the section  $\sigma(S)$  passes through  $P \in W' = W \cap \text{Quot}^{m_0}(n, X')$ , semicontinuity of dimension gives  $\dim_P \text{Quot}^{m_0}(n, X) \geq n \cdot m_0$ . But we have seen that

$$\text{Quot}^{m_0}(n, \mathcal{O}_X/C) \rightarrow \text{Quot}_x^{m_0}(n, X) \rightarrow \text{Quot}^{m_0}(n, X)$$

are bijections around  $P$  so we have

$$(2.5.4) \quad \dim_P \text{Quot}_x^{m_0}(n, \mathcal{O}_X/C) \geq n \cdot m_0.$$

However  $g$  defines an element of  $C$  and hence  $\mathcal{O}_X/C$  is a quotient of  $\mathcal{O}_{Z'}/(g)$ . This means  $\text{Quot}^{m_0}(n, \mathcal{O}_X/C) \subset \text{Quot}_y^{m_0}(n \cdot \mathcal{O}_{Z'}/(g))$  where  $y \in \text{Spec } \mathcal{O}_{Z'}/(g)$  maps to  $z \in Z'$ . As  $\text{Quot}^{m_0}(n, \mathcal{O}_{Z'}/(g))$  is irreducible of dimension  $n \cdot m_0$  and  $\text{Quot}_y^{m_0}(n, \mathcal{O}_{Z'}/(g))$  is a proper closed subset of  $\text{Quot}^{m_0}(n, \mathcal{O}_{Z'}/(g))$  we have

$$(2.5.5) \quad \dim_P \text{Quot}_x^{m_0}(n, \mathcal{O}_X/C) < n \cdot m_0.$$

This contradicts (2.5.4) and proves the theorem.

**COROLLARY 2.6.**  $\text{Quot}^m(n, Z)$  is irreducible  $\forall m$ .

*Proof.* Note that for  $n=1$   $\text{Quot}^m$  is just  $\text{Hilb}^m(Z)$  which is smooth and connected of dimension  $2m$ . For  $n>1$  we need to use the irreducibility of  $\text{Quot}^m(n, X) \subset \text{Quot}^m(n, Z)$  for a suitable irreducible curve  $X$  in  $Z$ . Now it is clear that if any two points of a scheme can be joined by an irreducible subscheme the scheme is irreducible. We claim that given two quotients  $\mathcal{O}_Z^n/M$  and  $\mathcal{O}_Z^n/N$  of length  $m$  there is an irreducible curve  $X$  with  $\text{Quot}^m(n, X)$  containing both the given quotients.

Let  $Q_1, Q_2, \dots, Q_s$ , and  $P_1, P_2, \dots, P_t$  be the supports of  $\mathcal{O}_Z^n/M$  and  $\mathcal{O}_Z^n/N$  respectively. The annihilator of  $\mathcal{O}_Z^n/M \oplus \mathcal{O}_Z^n/N$  is an ideal  $I \subset \mathcal{O}_Z$  with  $\mathcal{O}_Z/I$  supported at  $(P_1, P_2, \dots, Q_1, Q_2, \dots)$ . Let  $B$  be the semi local ring of the  $(P_i, Q_j)$  which exists as  $Z$  is projective. Then all we need to define a suitable  $X$  is to find a height one prime  $\mathfrak{P} \subset I$ . As  $B$  is a U.F.D. any irreducible element in  $I$  defines a  $\mathfrak{P}$  and this proves the corollary.

*Remark (2.7).* We do not know if  $\text{Quot}^m(n, Z)$  is reduced for  $n > 1$ .

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