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Hölder continuity of conformal mappings and non-quasiconformal Jordan curves

JOCHEN BECKER and CHRISTIAN POMMERENKE

Let $\Gamma \subset \mathbf{C}$ be a quasiconformal curve (quasicircle), that is the image of the unit circle under a quasiconformal mapping of the plane. Let f and f^* denote conformal mappings of the unit disk \mathbf{D} onto $\text{int } \Gamma$ and $\text{ext } \Gamma$ respectively.

It is well-known that f and f^* have quasiconformal extensions onto the plane [3], p. 98. Because of the Hölder continuity of quasiconformal mappings ([3], p. 71), f and f^* as well as their inverse mappings f^{-1} and f^{*-1} satisfy Hölder conditions:

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq K |z_2 - z_1|^\alpha & (z_1, z_2 \in \mathbf{D}), \\ |f^{-1}(w_1) - f^{-1}(w_2)| &\leq L |w_1 - w_2|^\beta & (w_1, w_2 \in \text{int } \Gamma) \end{aligned} \quad (1)$$

and similarly for f^* where one has to use the spherical metric in $\text{ext } \Gamma$. The Hölder exponents α ($0 < \alpha \leq 1$) and β ($1 \leq \beta < 2$) depend only on Γ [4], [5], [8], p. 287, 289, 347.

By means of a simple geometrical characterization of (1) (Theorem 1), we construct a non-quasiconformal Jordan curve (Theorem 2) such that nevertheless f , f^* , and also f^{-1} , f^{*-1} remain Hölder continuous. This shows that Hölder continuity of all these conformal mappings is only a necessary, but not a sufficient condition for quasicircles.

Let $G \subsetneq \mathbf{C}$ be a simply connected domain and let $h(w_1, w_2)$ denote the hyperbolic distance of the points $w_1, w_2 \in G$ defined by

$$h(w_1, w_2) = \log \frac{|1 - \bar{z}_1 z_2| + |z_2 - z_1|}{|1 - \bar{z}_1 z_2| - |z_2 - z_1|} \quad (w_i = f(z_i), i = 1, 2) \quad (2)$$

where f is a conformal mapping of \mathbf{D} onto G . Let $\delta(w) = \text{dist}(w, \partial G)$ denote the (Euclidean) boundary distance.

THEOREM 1. *If G is a bounded simply connected domain, $w_0 \in G$, and f a conformal mapping of \mathbf{D} onto G , then (1) is satisfied if and only if*

$$\limsup_{\delta(w) \rightarrow 0} \left[h(w_0, w) + \frac{1}{\alpha} \log \delta(w) \right] < +\infty. \quad (3)$$

Proof. It is well-known that (1) is equivalent to

$$|f'(z)| \leq M(1 - |z|^2)^{\alpha-1} \quad (|z| < 1) \quad (4)$$

(see for instance [2], p. 361–363). Introducing the non-Euclidean length element

$$\rho(w) |dw| = \frac{2 |dz|}{1 - |z|^2} \quad (w = f(z))$$

corresponding to (2) we see that (4) means

$$h(f(0), f(z)) \leq \frac{1}{\alpha} \log \left[\frac{M}{2} (1 + |z|)^{2\alpha} \rho(f(z)) \right].$$

Since G is bounded we have $\delta(f(z)) \rightarrow 0$ if and only if $|z| \rightarrow 1$. Thus we obtain as an equivalent condition

$$\limsup_{\delta(w) \rightarrow 0} \left[h(f(0), w) - \frac{1}{\alpha} \log \rho(w) \right] < +\infty. \quad (5)$$

Because of

$$\frac{1}{2} \leq \rho(w) \delta(w) \leq 2$$

[8], p. 22 it follows by the triangle inequality for $h(w_1, w_2)$ that (5) is the same as (3) which proves Theorem 1.

The following corollary gives a convenient sufficient condition for Hölder continuity.

COROLLARY. *Let f be a bounded univalent function defined in \mathbf{D} , $G = f(\mathbf{D})$, then f satisfies (1) if there are positive numbers M, δ_0, δ_1 with the property that, for every $w \in G$ with $\delta(w) < \delta_0$, there exists $w_1 \in G$ with $\delta(w_1) \geq \delta_1$ and a connecting arc $C \subset G$, such that*

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq M + \frac{1}{2\alpha} \log \frac{1}{\delta(w)}. \quad (6)$$

Proof. We choose $w_0 = f(0)$. Then we have

$$\begin{aligned} h(w_0, w) &\leq h(w_0, w_1) + \int_C \rho(\omega) |d\omega| \\ &\leq \log \frac{1 + |f^{-1}(w_1)|}{1 - |f^{-1}(w_1)|} + 2 \int_C \frac{|d\omega|}{\delta(\omega)}. \end{aligned}$$

From (6) and $\delta(w_1) \geq \delta_1 > 0$ it follows that (3) is satisfied which implies (1) by Theorem 1.

We are now ready to construct a Jordan curve with the desired properties.

THEOREM 2. *There is a non-quasiconformal Jordan curve Γ such that the conformal mappings f and f^* of \mathbf{D} onto $\text{int } \Gamma$ and $\text{ext } \Gamma$ respectively, and also their inverse mappings are Hölder continuous.*

Proof. We set $Q = \{w = u + iv : |u| < 1, |v| < 1\}$ and $R_n = \{w = u + iv : 0 \leq u - 1 < a_n, |v - v_n| < \varepsilon_n\}$ where we choose $v_n = 1/n$, $\varepsilon_n = 2^{-n}$, $a_n = -\varepsilon_n \log \varepsilon_n$, $n = 2, 3, \dots$. We consider the domain $G = Q \cup \bigcup_{n=2}^{\infty} R_n$. Since $\Gamma = \partial G$ is a locally connected continuum without cut points it is a Jordan curve [8], p. 281. But it is not a quasicircle, because the Ahlfors criterion [1] is not satisfied. For we have

$$\frac{\text{diam } R_n}{2\varepsilon_n} \geq \frac{a_n}{2\varepsilon_n} \rightarrow +\infty (n \rightarrow \infty).$$

On the other hand, we shall show below that, for every $w \in G$, $\delta(w) < \frac{1}{2}$, there exists $w_1 \in G$, $\delta(w_1) \geq \frac{1}{2}$, and a connecting arc C such that

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq 2 \log \frac{1}{\delta(w)}. \quad (7)$$

Thus (6) is satisfied with $\alpha = \frac{1}{4}$ which shows that f is α -Hölder-continuous for some $\alpha \geq \frac{1}{4}$.

The Hölder continuity of f^{-1} , f^* , f^{*-1} follows easily from results by Näkki and Palka [6], [7]:

Let $d(w_1, w_2) = \inf_C \int_C |dw|$ denote the inner distance of $w_1, w_2 \in G$ where the infimum is taken over all connecting arcs $C \subset G$. Then, obviously, there exists a constant A such that

$$d(w_1, w_2) \leq A |w_2 - w_1| \quad (w_1, w_2 \in G).$$

It follows from [6] that f^* and f^{-1} are Hölder continuous.

Considering the inner distance $d^*(w_1, w_2)$ with regard to $G^* = \text{ext } \Gamma$ one easily sees that, for every β ($0 < \beta < 1$), and $R > 0$, there exists a constant B such that

$$d^*(w_1, w_2) \leq B |w_2 - w_1|^\beta \quad (w_i \in G^*, |w_i| \leq R, i = 1, 2)$$

This implies, by another result of Näkki and Palka [6], [7], that f^{*-1} is also Hölder continuous.

Hence it remains to prove (7). We have to consider the following cases.

(a) $w = u + iv \in Q$, and, if $u \geq v \geq 0$, $|v - v_n| \geq \varepsilon_n$, $n = 2, 3, \dots$. Hence $\delta(w) = \min(1 - |u|, 1 - |v|)$. Choose $w_1 = 0$ and $C = [w_1, w_2]$ (segment with end points w_1, w_2). Then

$$\int_C \frac{|d\omega|}{\delta(\omega)} \leq \int_{\delta(w)}^1 \frac{\sqrt{2} dt}{t} = \sqrt{2} \log \frac{1}{\delta(w)}.$$

(b) $w = u + iv \in Q$, $u \geq v \geq 0$, $|v - v_n| < \varepsilon_n$ for some n (hence $\delta(w) > 1 - u$). Choose, for $\delta(w) < \frac{1}{2}$, $w_1 = \frac{1}{2} + iv$, $C = [w_1, w]$. Because of

$$\delta(w) = \sqrt{((1 - x)^2 + (\varepsilon_n - |v - v_n|)^2)}, \quad \omega = x + iv \in C,$$

we obtain

$$\begin{aligned} \int_C \frac{|d\omega|}{\delta(\omega)} &= \log \frac{\frac{1}{2} + \sqrt{(\frac{1}{4} + (\varepsilon_n - |v - v_n|)^2)}}{1 - u + \sqrt{((1 - u)^2 + (\varepsilon_n - |v - v_n|)^2)}} \\ &\leq \log \frac{1 + \sqrt{(1 + 4\varepsilon_n^2)}}{2\delta(w)} < 2 \log \frac{1}{\delta(w)}. \end{aligned}$$

(c) $w = u + iv \in R_n$ for some n , $0 \leq u - 1 \leq a_n - \varepsilon_n$ hence $\delta(w) = \varepsilon_n - |v - v_n|$. Choose $w_1 = \frac{1}{2} + iv_n$ and $C = [w_1, u + iv_n] \cup [u + iv_n, w]$. Then

$$\begin{aligned} \int_C \frac{|d\omega|}{\delta(\omega)} &= \int_{1/2}^1 \frac{dx}{\sqrt{((1 - x)^2 + \varepsilon_n^2)}} + \int_1^u \frac{dx}{\varepsilon_n} + \int_0^{|v - v_n|} \frac{dy}{\varepsilon_n - y} \\ &< \frac{a_n}{\varepsilon_n} - \log(\varepsilon_n - |v - v_n|) \leq 2 \log \frac{1}{\delta(w)}. \end{aligned}$$

(d) $w = u + iv \in R_n$ for some n , $a_n - \varepsilon_n < u - 1$ hence $\delta(w) = \min(\varepsilon_n - |v - v_n|, 1 + a_n - u)$. Choose $w_1 = \frac{1}{2} + iv_n$, $C = [w_1, 1 + a_n - \varepsilon_n + iv_n] \cup [1 + a_n - \varepsilon_n + iv_n, w]$.

Then

$$\begin{aligned} \int_C \frac{|d\omega|}{\delta(\omega)} &\leq \int_{1/2}^1 \frac{dx}{\sqrt{(1-x)^2 + \varepsilon_n^2}} + \int_0^{a_n - \varepsilon_n} \frac{dx}{\varepsilon_n} + \int_{\delta(w)}^{\varepsilon_n} \frac{\sqrt{2} dt}{t} \\ &< -\log \varepsilon_n + \frac{a_n}{\varepsilon_n} + \sqrt{2} \log \frac{\varepsilon_n}{\delta(w)} \leq 2 \log \frac{1}{\delta(w)}. \end{aligned}$$

Thus we have proved that (7) holds in each case which completes the proof of Theorem 2.

Remark. The Hölder continuity of f^* could have also been shown by this method.

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