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On classification of immersions of n -manifolds in $(2n - 1)$ -manifolds

LI BANGHE

Abstract. Under the assumption of (f, M^n, N^{2n-1}) being trivial, the classification of immersions homotopic to $f: M^n \rightarrow N^{2n-1}$ is obtained in many cases. The triviality of (f, M^n, P^{2n-1}) is proved for any M^n and f .

Let M, N be differentiable manifolds of dimension n and $2n - 1$ respectively. For a map $f: M \rightarrow N$, denote by $I[M, N]_f$ the set of regular homotopy classes of immersions homotopic to f . It has been proved in [1] that, when $n > 1$, $I[M, N]_f$ is nonempty for any f . In this paper we will determine the set $I[M, N]_f$ in some cases.

For example, if $N = P^{2n-1}$, or more generally, the lens spaces $S_m^{2n-1} = Z_m \backslash S^{2n-1}$, M is any orientable n -manifold or nonorientable but $n \equiv 0, 1, 3 \pmod{4}$, then, for any f , the $I[M, N]_f$ is determined completely.

When $N = R^{2n-1}$, the set $I[M, N]$ of regular homotopy classes of all immersions has been enumerated by James and Thomas in [2] and McClendon in [3] for $n > 3$. Applying our results to $N = R^{2n-1}$ we obtain their results again, except for the case $n \equiv 2 \pmod{4}$ and M nonorientable.

When $n = 3$, McClendon's results cannot be used. Our results include the cases $n = 3$, M orientable or not (for orientable M , $I[M, R^5]$ is known by Wu [4]).

§1. Preliminaries

Suppose M and N are differentiable manifolds with $\dim M < \dim N$. Let $\text{Hom}(T(M), T(N))$ be the bundle over $M \times N$ with fibre $\{\text{orthogonal homomorphisms: } T_x(M) \rightarrow T_y(N)\}$ at (x, y) and structure group $O(\dim N) \times O(\dim M)$. For a map $f: M \rightarrow N$, define $F: M \rightarrow M \times N$ by $F(x) = (x, f(x))$, then F induces a bundle \mathcal{B}_f from $\text{Hom}(T(M), T(N))$.

Denote the space consisting of all maps $M \rightarrow N$ with the compact-open topology by N^M , and the fundamental group of the path-component of N^M containing f by $\pi_1(N^M, f)$. $\pi_1(N^M, f)$ operates on the set of homotopy classes of sections of \mathcal{B}_f as follows: for a section s_0 of \mathcal{B}_f i.e. a map $s_0: M \rightarrow \text{Hom}(T(M), T(N))$ covering F , and a homotopy $f_t: M \rightarrow N$ with $f_0 = f = f_1$, by the homotopy covering property of bundle, there exists a homotopy $s_t: M \rightarrow \text{Hom}(T(M), T(N))$ covering $F_t: M \rightarrow M \times N$, where $F_t(x) = (x, f_t(x))$, then s_1 defines a section of \mathcal{B}_f and the homotopy class $[s_1]$ is the image of $[f_t] \in \pi_1(N^M, f)$ operating on $[s_0]$. It is known that the orbits of this operation correspond one to one to the regular homotopy classes of immersions homotopic to f (see [5], [6]).

DEFINITION. We say that a triad (f, M, N) is trivial if $f : M \rightarrow N$ is a map and $\pi_1(N^M, f)$ operates on \mathcal{B}_f trivially.

LEMMA 1. *Suppose $\dim N > \dim M + 1$, $\pi_1(N)$ is abelian, and $\pi_i(N) = 0$ for $2 \leq i \leq \dim M + 1$, then $\pi_1(N^M, f) \approx \pi_1(N)$ for any f .*

Proof. As a set, $\pi_1(N^M, f)$ consists of the homotopy classes of maps $F : M \times [0, 1] \rightarrow N$ with $F(x, 0) = f(x) = F(x, 1)$ relative to $M \times \partial[0, 1]$. Under the assumptions of the lemma, by using obstruction theory, we know that $\pi_1(N^M, f)$ corresponds one to one with $H^1(M \times [0, 1], M \times \partial[0, 1]; \pi_1(N))$, and the later corresponds one to one with $H^0(M, \pi_1(N)) \approx \pi_1(N)$. It is also easy to see that we can choose the bijections to be isomorphisms of groups, so $\pi_1(N^M, f) \approx \pi_1(N)$. The proof is complete.

Let S^n be a sphere of odd dimension. There is a natural action of Z_m on S^n , where m is any positive integer. Denote by S_m^n the quotient space $Z_m \backslash S^n$, then $S_1^n = S^n$ and $S_2^n = P^n$.

THEOREM 1. *Let $n > \dim M + 1$, $f : M \rightarrow S_m^n$ be any map, then (f, M, S_m^n) is trivial.*

Proof. There is a natural C^∞ -flow Φ_t on S_m^n such that, for any fixed $x \in S_m^n$, the closed orbit $\Phi_t(x), t \in [0, 1]$ represents the generator of $\pi_1(S_m^n) \approx Z_m$. For $q = 1, 2, \dots, m$, define $f_t^q : M \times [0, 1] \rightarrow S_m^n$ by $f_t^q(x) = \Phi_{qt}(f(x))$, then $f_0^q = f = f_1^q$. From Lemma 1 we know that $f_1^1, f_1^2, \dots, f_1^m$ represent all the elements of $\pi_1((S_m^n)^M, f) \approx Z_m$.

Let $\Phi_{t*} : T(S_m^n) \rightarrow T(S_m^n)$ be induced from the diffeomorphism Φ_t and $s : T(M) \rightarrow T(S_m^n)$ be any orthogonal homomorphism covering f (representing a section of \mathcal{B}_f), then $\Phi_{qt*} \circ s : T(M) \rightarrow T(S_m^n)$ is a homotopy of orthogonal homomorphisms covering f_t^q . Because $\Phi_q = \text{identity}$, so $\Phi_{q*} \circ s = s$. This shows that the operation of $\pi_1((S_m^n)^M, f)$ is trivial and the proof is complete.

§2. Boltyanski's theorem

In [4] using Postnikov formula, Wu has calculated $I[M, R^5]$ for orientable 3-manifolds M . But in the general case, in order to classify the sections of \mathcal{B}_f we have to use Boltyanski's generalization of the Postnikov formula (see [7]).

Let \mathcal{B} be a fiber bundle over a simplicial complex B with fiber F and structure

group G such that $\pi_0(F) = \pi_1(F) = \cdots = \pi_{r-1}(F) = 0$, $r \geq 2$, and let G be a connected Lie group operating on F effectively and transitively with path-connected isotropy group Γ . A section σ_0 given on B^{r+1} (the $(r+1)$ -skeleton of B) determines a Γ -principal bundle over B^{r+1} , whose characteristic class in $H^2(B^{r+1}, \pi_1(\Gamma))$ is denoted by $Y_{\sigma_0}^2$. Let σ and σ' be the sections of \mathcal{B} over B^{r+1} , if the first difference $D^r(\sigma, \sigma') \in H^r(B^{r+1}, \pi_r(F))$ is zero, i.e. $D^r(\sigma_0, \sigma) = D^r(\sigma_0, \sigma') = D^r$, then we can define a secondary difference $D^{r+1} = D^{r+1}(\sigma, \sigma') \in H^{r+1}(B^{r+1}, \pi_{r+1}(F))$. According to Boltyanski's theorem, σ and σ' are homotopic on B^{r+1} if and only if there exists a $\Lambda^{r-1} \in H^{r-1}(B^{r+1}, \pi_r(F))$ such that

$$D^{r+1} = \begin{cases} \Lambda^{r-1} \cup Y_{\sigma_0}^2 + sq^2 \Lambda^{r-1} & \text{if } r > 2, \\ \Lambda^1 \cup Y_{\sigma_0} + \Lambda^1 \cup D^2 + kv^2 \Lambda^1 & \text{if } r = 2. \end{cases}$$

Assume that $T(M)$ and $f^*T(N)$ are orientable with $\dim M = n \geq 3$, $\dim N = 2n - 1$, and (f, M, N) is trivial, then $I[M, N]_f$ is in one to one correspondence with the set of homotopy classes of sections of \mathcal{B}_f . \mathcal{B}_f has fiber $F = V_{2n-1, n}$ and structure group $G = SO(2n-1) \times SO(n)$. Let $a = (a_{kh}) \in SO(n)$, $b = (b_{uv}) \in SO(n-1)$, define $i(a) = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix} \in SO(2n-1)$, $j(b) = \begin{pmatrix} I & 0 \\ 0 & b \end{pmatrix} \in SO(2n-1)$, then the isotropy group $\Gamma = \{(i(a)j(b), a) / a \in SO(n), b \in SO(n-1)\} \approx SO(n) \times SO(n-1)$.

Now we are going to determine the pairing of $\pi_1(\Gamma)$ and $\pi_{n-1}(V_{2n-1, n})$ into $\pi_n(V_{2n-1, n})$. Let E^2 and E^{n-1} be the cubes of dimension 2 and $n-1$ respectively and embed $S^{n-1} = \{(x_1, \dots, x_n) / \sum x_i^2 = 1\}$ in $V_{2n-1, n}$ by sending (x_1, \dots, x_n) to $(v_1, \dots, v_{n-1}, v_n) \in V_{2n-1, n}$ such that $v_1 = (1, 0, \dots, 0)$, $v_2 = (0, 1, 0, \dots, 0)$, \dots , $v_{n-1} = (0, \dots, 0, 1, 0, \dots, 0)$ and $v_n = (0, \dots, 0, x_1, \dots, x_n)$. Choose any map $\phi = E^{n-1} \rightarrow S^{n-1}$ carrying ∂E^{n-1} into $(1, 0, \dots, 0)$ and homeomorphic on $E^{n-1} - \partial E^{n-1}$, then as a map $E^{n-1} \rightarrow V_{2n-1, n}$, ϕ determines a generator of $\pi_{n-1}(V_{2n-1, n})$ denoted by $[\phi]$. Let α be a generator of $\pi_n(S^{n-1})$, set $\beta = \phi_*(\alpha) \in \pi_n(V_{2n-1, n})$, then the pairing of $\pi_{n-1}(V_{2n-1, n})$ and $\pi_{n-1}(V_{2n-1, n})$ into $\pi_n(V_{2n-1, n})$ is determined by

$$[\phi] \cdot [\phi] = \begin{cases} \beta, & \text{if } n > 3 \\ 2\beta, & \text{if } n = 3. \end{cases}$$

Define $a(\theta) = \begin{pmatrix} A(\theta) & 0 \\ 0 & I \end{pmatrix} \in SO(n)$, $b(\theta) = \begin{pmatrix} A(\theta) & 0 \\ 0 & I \end{pmatrix} \in SO(n-1)$, where

$A(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $0 \leq \theta \leq 2\pi$, then $(i(a(\theta)), a(\theta))$ and $(j(b(\theta)), I) \in \Gamma \subset SO(2n-1) \times SO(n)$ determine the two generators of $\pi_1(\Gamma) \approx \pi_1(SO(n)) + \pi_1(SO(n-1))$, denoted by ξ and η respectively. As in [8] (see p. 80–81) we see

that pairing $\eta \in \pi_1(\Gamma)$ and $[\phi] \in \pi_{n-1}(V_{2n-1,n})$ yields

$$\eta \cdot [\phi] = \pm\beta \in \pi_n(V_{2n-1,n}).$$

Because the operation of $(a, b) \in SO(2n-1) \times SO(n)$ on $V \in V_{2n-1,n}$ is defined by $(a, b) \cdot V = aVb^{-1}$, we have $(i(a(\theta)), a(\theta)) \cdot \phi(x) = \phi(x)$ for any $\theta \in [0, 2\pi]$ and $x \in E^{n-1}$. $\xi \cdot [\phi]$ is determined by

$$\psi(k, x) = \begin{cases} (i(a(\theta)), a(\theta)) \cdot \phi(x), & k = k(\theta) \in \partial E^2, x \in E^{n-1}, \\ \phi(\partial E^{n-1}), & k \in E^2, x \in \partial E^{n-1}. \end{cases}$$

Now ψ can be extended to a map $\tilde{\psi}: E^2 \times E^{n-1} \rightarrow V_{2n-1,n}$ by defining $\tilde{\psi}(k, x) = \phi(x)$, so $\xi \cdot [\phi] = 0$.

Let σ be a section of \mathfrak{B}_f , i.e. an orthogonal homomorphism $T(M) \rightarrow f^*T(N)$, then $f^*T(N)$ is the whitney sum $T(M) \oplus N_f$, where N_f is the normal bundle of σ if we regard σ as an immersion. Thus the principal bundle associated to $f^*T(N)$ has a principal subbundle with fibre $i(SO(n))j(SO(n-1)) \subset SO(2n-1)$, where

$$i(SO(n))j(SO(n-1)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} / a \in SO(n), b \in SO(n-1) \right\}.$$

Take any sections γ and δ of the principal bundles associated to $T(M)$ and N_f respectively, over M^1 , the 1-skeleton of M . For a 2-simplex T in M , regarding the restriction of the principal bundle of \mathfrak{B}_f on T as $T \times SO(2n-1) \times SO(n)$, we have a map $h: \partial T \rightarrow SO(2n-1) \times SO(n)$ defined by $h(x) = (i(\gamma(x))j(\delta(x)), \gamma(x))$. h defines a section of the Γ -principal bundle determined by σ , over M^1 . Let $Y_\sigma^2 \in H^2(M, \pi_1(\Gamma))$ and $W_0^2 \in H^2(M, \pi_1(SO(n-1)))$ be the characteristic classes of the Γ -principal bundle and the normal bundle of σ respectively, then from $\xi \cdot [\phi] = 0, \eta \cdot [\phi] = \pm\beta$ and the relation of h, γ and δ , we have $\Lambda^{n-2} \cup Y_\sigma^2 = \pm\Lambda^{n-1} \cup W_\sigma^2$ for any $\Lambda^{n-2} \in H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))$. Thus, by Boltyanski's theorem, if σ' is another section of \mathfrak{B}_f homotopic to σ on M^{n-1} , and $D^n \in H^n(M, \pi_n(V_{2n-1,n}))$ is a secondary difference, then σ and σ' are homotopic if and only if there exists a $\Lambda^{n-2} \in H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))$ such that

$$D^n = \begin{cases} \Lambda^{n-2} \cup W_\sigma^2 + sq^2\Lambda^{n-2}, & \text{if } n > 3 \\ \Lambda^{n-2} \cup W_\sigma^2 + kv^2\Lambda^{n-2}, & \text{if } n = 3. \end{cases}$$

§3. The orientable case

Following Wu [4], [9], first, we give a more precise description of $\beta = \phi_*(\alpha)$.

For $n > 3$, we have $\pi_n(V_{2n-1,n}) \approx \pi_n(V_{n+2,3}) \approx 0, Z_2 + Z_2, Z_2, Z_4$ as $n \equiv 0, 1, 2, 3 \pmod 4$, and $\pi_n(V_{n+1,2}) \approx Z_2 + Z, Z_2$ as $n \equiv 0, 1 \pmod 2$, respectively (see [10]). Let $[\psi] \in \pi_{n-1}(V_{n+1,2}) \approx \pi_{n-1}(V_{2n-1,n})$ be the standard generator, then

$\psi_*(\alpha) \in \pi_n(V_{n+1,2})$. By the natural fibration $S^{n-1} \subset V_{n+1,2} \rightarrow S^n$, and the exact sequence

$$\pi_n(S^{n-1}) \xrightarrow{\psi_*} \pi_n(V_{n+1,2}) \longrightarrow \pi_n(S^n),$$

we see that ψ_* is an isomorphism of $\pi_n(S^{n-1})$ and the subgroup Z_2 of $\pi_n(V_{n+1,2})$. Let $c: V_{n+1,2} \rightarrow V_{n+2,3}$ be the natural inclusion, then $\beta = c_*\psi_*(\alpha)$, by the natural isomorphism of $\pi_n(V_{n+2,3})$ and $\pi_n(V_{2n-1,n})$. From the following exact sequence

$$\pi_{n+1}(S^{n+1}) \longrightarrow \pi_n(V_{n+1,2}) \xrightarrow{c_*} \pi_n(V_{n+2,3}) \longrightarrow \pi_n(S^{n+1}),$$

we know that, except for $n \equiv 0 \pmod{4}$, $\beta = c_*\psi_*(\alpha)$ is a non-zero element of order 2 in $\pi_n(V_{n+2,3}) \approx \pi_n(V_{2n-1,n})$.

Let $\rho: Z$ (or Z_2) $\rightarrow Z_2$ be the mod 2 homomorphism and $s: Z_2 \rightarrow \pi_n(V_{2n-1,n})$ the homomorphism defined by $s(1 \pmod{2}) = \beta$. Notice that, in case $n > 3$, $W_\sigma^2 \in H^2(M, Z_2)$ is independent of σ , depending only on f . In reality, it is the stable normal class of f , we denote it by W_f^2 .

THEOREM 2. *Let M and N be connected manifolds with $\dim M = n > 3$, $\dim N = 2n - 1$, and $f: M \rightarrow N$ a map such that $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \pi_{n-1}(V_{2n-1,n})) \\ \times [H^n(M, \pi_n(V_{2n-1,n})) / s_*(\rho_* H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup f^*(W_2))]$$

where $W_2 \in H^2(N, Z_2)$ is the Stiefel–Whitney class, s_* and f_* are the induced homomorphism of s and f respectively, and \cup stands for the ordinary cup product.

Proof. The primary differences of sections of \mathcal{B}_f fill $H^{n-1}(M, \pi_{n-1}(V_{2n-1,n}))$, and the secondary differences of those sections homotopic on M^{n-1} fill $H^n(M, \pi_n(V_{2n-1,n}))$. Now the set of secondary differences of sections homotopic on M is just

$$s_*[\rho_* H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup W_f^2 + sq^2 \rho_* H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))],$$

by what we have already known from the above.

Because $f^*T(N) = T(M) \oplus N_f$, so $f^*(W_2) = W_f^2 + W_2(M)$, here $W_2(M)$ denotes the Stiefel–Whitney class of M . By the Wu formula,

$$\rho_* H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup W_2(M) + sq^2 \rho_* H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) = 0.$$

From this, the conclusion of the theorem follows immediately.

COROLLARY 1. *Under the same assumptions as Theorem 1, if $n \equiv 0 \pmod 4$, $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$. Furthermore, suppose M is closed, then, if $n \equiv 2 \pmod 4$, we have $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2) \times Z_2$ if $f^*(W_2) = 0$ and $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$ if $f^*(W_2) \neq 0$; if $n \equiv 1 \pmod 2$, we have $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_4$ as $f^*(W_2) \in \rho_*T^2(M, Z)$, and $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_2$ as $f^*(W_2) \in \rho_*T^2(M, Z)$, where $T^2(M, Z)$ is the torsion subgroup of $H^2(M, Z)$.*

Proof. In the case $n \equiv 0 \pmod 4$, it is due to $\pi_n(V_{2n-1,n}) = 0$. In the case $n \equiv 2 \pmod 4$, when $f^*(W_2) \neq 0$, by the Poincaré duality theorem, we have $\rho_*H^{n-2}(M, Z_2) \cup f^*(W_2) = H^n(M, Z_2) \approx Z_2$. For $n = 1 \pmod 2$, by the result of Massey and Peterson [13], we know that

$$\rho_*H^{n-2}(M, Z) \cup f^*(W_2) = H^n(M, Z_2)$$

if and only if $\rho_*(W_2) \in f_*T^2(M, Z)$. Thus, noticing that $s(1 \pmod 2) = \beta$ is a non-zero element in the cases $n = 1, 2, 3 \pmod 4$, the corollary becomes clear.

EXAMPLE 1. Let $n = 4m + 2, m > 1$, and M orientable and closed, then $I[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2) \times H^{n-1}(M, Z_2) \times Z_2$, because $W_2(P^{2n-1}) = 0$ and $[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2)$.

EXAMPLE 2. Let $n \geq 5$ be odd, then $I[P^n, P^{2n-1}]$ consists of 16 elements. Because $H^{n-2}(P^n, Z) = 0$, thus by Theorem 2,

$$I[P^n, P^{2n-1}] \leftrightarrow H^1(P^n, Z_2) \times H^{n-1}(P^n, Z) \times H^n(P^n, Z_4) \leftrightarrow Z_2 \times Z_2 \times Z_4.$$

Now we consider the case $n = 3$. Just as Wu did in [4], we see that $\beta = \phi_*(\alpha)$ is two times a generator of $\pi_3(V_{5,3}) \approx Z$. Since $\pi_2(V_{5,3}) \approx Z$ is free, according to the definition of kv^2 (see [7]), we know that $kv^2H^1(M, \pi_2(V_{5,3})) = 0$. If σ and σ' are two sections of \mathfrak{B}_f homotopic on M^2 , i.e. $D^2(\sigma_0, \sigma) = D^2(\sigma_0, \sigma') = D \in H^2(M, \pi_2(V_{5,3})) \approx H^2(M, Z)$, for a fixed section σ_0 , then $W_\sigma^2 = W_{\sigma'}^2 \in H^2(M, SO(2)) \approx H^2(M, Z)$. So we can denote them by an element W_D^2 in $H^2(M, Z)$. Thus we have

THEOREM 3. Let M and N be connected manifolds with $\dim M = 3$ and $\dim N = 5$. Suppose $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then

$$I[M, N]_f \leftrightarrow \bigcup_{D \in H^2(M, Z)} [H^3(M, Z)/2W_D^2 \cup H^1(M, Z)],$$

where \cup stands for the ordinary cup product.

REMARK. All normal classes W_D^2 form a coset of $2H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{Z})$, which is just the inverse image of $f^*(W_2(N))$ under the mod 2 homomorphism $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2)$. All D with the same W_D^2 form a coset of the subgroup of $H^2(M, \mathbb{Z})$ consisting of all elements of order 2.

EXAMPLE 3. $I[P^3, P^5] \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$.

§4. Nonorientable case

If at least one of $T(M)$ and $f^*T(N)$ is nonorientable, Boltyanski's theorem does not work. So we have to appeal to other methods. The case $n = 4s$ is trivial, because $\pi_n(V_{2n-1,n}) = 0$, and only primary obstructions are concerned. In order to deal with the case n odd, we embed \mathcal{B}_f in a larger bundle, in which the original problem to determine secondary obstructions turns out to be one to determine primary obstructions. To do this, we need some lemmas first.

LEMMA 2. Let $n \geq 3$ be odd, $i: V_{2n-1,n} \rightarrow V_{2n,n}$ the natural embedding, then $i_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n})$ is surjective.

Proof. Let $j: V_{2n-1,n} \rightarrow V_{2n,n+1}$ be the natural embedding and $p: V_{2n,n+1} \rightarrow V_{2n,n}$ the natural projection, then we have $i = pj$. From the fibrations $V_{2n-1,n} \subset V_{2n,n+1} \rightarrow S^{2n-1}$ and $S^{n-1} \subset V_{2n,n+1} \rightarrow V_{2n,n}$, we know that $j_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n+1})$ is an isomorphism, and $p_*: \pi_n(V_{2n,n+1}) \rightarrow \pi_n(V_{2n,n})$ is surjective. Thus $i_* = p_*j_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n})$ is surjective. This proves the lemma.

Let λ be the map $V_{n,k} \rightarrow V_{n,k}$ which changes the sign of every column, we have.

LEMMA 3. If $n = 4s + 3, s \geq 0$ then $\lambda_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n-1,n})$ is the identity.

Proof. By a theorem of James (see [12], chapter 13), $\lambda_* = 1 - i_* \circ \Sigma \circ \Delta$, where $\Delta: \pi_n(V_{2n-1,n}) \rightarrow \pi_{n-1}(S^{n-2})$ denotes the boundary operator in the homotopy sequence for the fibration $S^{n-2} \subset V_{2n-1,n+1} \rightarrow V_{2n-1,n}$, $\Sigma: \pi_{n-1}(S^{n-2}) \rightarrow \pi_n(S^{n-1})$ the suspension homomorphism, and $i_*: \pi_n(S^{n-1}) \rightarrow \pi_n(V_{2n-1,n})$ the induced homomorphism of the fiber inclusion for the fibration $S^{n-1} \subset V_{2n-1,n} \rightarrow V_{2n-1,n-1}$.

For $n = 3$, because $\pi_2(S^1) = 0$, thus $\Delta = 0$, and $\lambda_* = 1$.

For $n > 3$, the exact sequence

$$\begin{array}{ccccccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n+1}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n}) \\ \cong & & \cong & & \cong & & \cong \\ Z_4 & & Z_2 & & Z_2 & & Z \end{array}$$

tells us $\Delta = 0$, thus $\lambda_* = 1$. The proof is complete.

LEMMA 4. *If $n = 4s + 1$, $s > 1$, then λ_* is an automorphism of $\pi_n(V_{2n-1,n}) \approx Z_2 + Z_2$ which fixes two elements and exchanges other two elements.*

Proof. From the exact sequence

$$\begin{array}{ccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) \longrightarrow \pi_{n-1}(V_{2n-1,n+1}) \\ \cong & & \cong \\ Z_2 + Z_2 & & Z_2 \quad 0 \end{array}$$

we know that Δ is surjective. Now from the following exact sequence

$$\begin{array}{ccccccc} \pi_n(S^{n-1}) & \xrightarrow{i_*} & \pi_n(V_{2n-1,n}) & \longrightarrow & \pi_n(V_{2n-1,n-1}) & \longrightarrow & \pi_n(S^n) \\ \cong & & \cong & & \cong & & \cong \\ Z_2 & & Z_2 + Z_2 & & Z_2 & & Z \end{array}$$

it follows that i_* must be an injection. Thus the image of $i_* \circ \Sigma \circ \Delta$ has two elements, and the conclusion of the lemma follows from $\lambda_* = 1 - i_* \circ \Sigma \circ \Delta$. The proof is complete.

THEOREM 4. *Let $n = 4s + 1$, $s > 0$, M and N be connected differentiable manifolds with $\dim M = n$ and $\dim N = 2n - 1$. Furthermore, assume that (f, M, N) is trivial, M closed and $f^*W_1(N) \neq W_1(M)$, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \mathfrak{B}(\pi_{n-1})) \times Z_2,$$

where $W_1(N)$ and $W_1(M)$ are the Stiefel–Whitney classes of N and M respectively, and $\mathfrak{B}(\pi_{n-1})$ is the coefficient bundle associated to \mathfrak{B}_f with fiber $\pi_{n-1}(V_{2n-1,n}) \approx Z$.

Proof. For a fixed section σ_0 of \mathfrak{B}_f the primary differences fill $H^{n-1}(M, \mathfrak{B}(\pi_{n-1}))$. Let $\mathfrak{B}(\pi_n(V_{2n-1,n}))$ (abbreviated to $\mathfrak{B}(\pi_n)$) be the coefficient bundle associated to \mathfrak{B}_f . Given a $D^{n-1} \in H^{n-1}(M, \mathfrak{B}(\pi_{n-1}))$, the secondary obstructions $D^n(\sigma, \sigma')$ of all sections σ and σ' with $D^{n-1}(\sigma_0, \sigma) = D^{n-1}(\sigma_0, \sigma') = D^{n-1}$ fill $H^n(M, \mathfrak{B}(\pi_n))$.

By Lemma 4, we can choose a basis of $\pi_n(V_{2n-1,n}) \approx Z_2 + Z_2$ in which λ_* has the form: $\lambda_*(1, 0) = (0, 1)$, $\lambda_*(0, 1) = (1, 0)$ and $\lambda_*(1, 1) = (1, 1)$. Let Δ_0 be an n -simplex in M , then any n -cocycle with coefficient in $\mathcal{B}(\pi_n)$ is cohomologous to an n -cocycle T_γ such that $\langle T_\gamma, \Delta_0 \rangle = \gamma \in \pi_n(V_{2n-1,n})$, $\langle T_\gamma, \Delta \rangle = 0$ for other n -simplices Δ in a triangulation of M including Δ_0 . Now $f^*W_1(N) \neq W_1(M)$, so there must exist a loop in M which preserves the orientation of $T(M)$ but reverses the orientation of $f^*T(N)$, or reverses the former but preverses the later. By the existence of such a loop, we see that $T_{\gamma+\lambda,\gamma} = T_\gamma + T_{\lambda,\gamma}$ must be cohomologous to 0 for any $\gamma \in \pi_n(V_{2n-1,n})$. Thus $T_{(1,0)} \sim T_{(0,1)}$, and $T_{(1,1)} = T_{(1,0)} - T_{(0,1)} \sim 0$. So $H^n(M, \mathcal{B}(\pi_n))$ has at most two elements.

Now let $N' = N \times R$, define $f' : M \rightarrow N'$ by $f'(x) = (f(x), 0)$, then \mathcal{B}_f may be regarded as a subbundle of $\mathcal{B}_{f'}$. By Lemma 2, there exists a $\gamma_0 \in \pi_n(V_{2n-1,n})$ such that $i_*\gamma_0 \neq 0$. For any $D^{n-1} \in H^{n-1}(M, \mathcal{B}(\pi_{n-1}))$, we can choose two sections σ and σ' of \mathcal{B}_f such that $D^{n-1}(\sigma_0, \sigma) = D^{n-1}(\sigma_0, \sigma') = D^{n-1}$, and one of their secondary differences includes T_{γ_0} . Thus as sections of $\mathcal{B}_{f'}$, σ and σ' have a non-zero primary difference. It is obvious that σ and σ' are not homotopic in \mathcal{B}_f (see [5]). From this, we conclude that $H^n(M, \mathcal{B}(\pi_n)) \approx Z_2$ and two sections with non-zero secondary difference are not homotopic. The proof is complete.

THEOREM 5. *Let $n = 4s + 3$, $s \geq 0$, M and N be connected differentiable manifolds with $\dim M = n$ and $\dim N = 2n - 1$. Furthermore, assume that M is closed and nonorientable, and (f, M, N) is trivial, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \mathcal{B}(\pi_{n-1})) \times Z_2,$$

where $\mathcal{B}(\pi_{n-1})$ is the coefficient bundle associated to \mathcal{B}_f with fibre $\pi_{n-1}(V_{2n-1,n}) \approx Z$.

Proof. In virtue of Lemma 3, $\mathcal{B}(\pi_n)$ is trivial. Thus, by the assumption that M is closed and nonorientable, we have $H^n(M, \mathcal{B}(\pi_n)) \approx H^n(M, Z_4) \approx Z_2$ for $n > 3$, and $H^n(M, \mathcal{B}(\pi_n)) \approx H^n(M, Z) \approx Z_2$ for $n = 3$. Now, using the similar argument as that for theorem 4, the conclusion of this theorem follows.

From Theorems 1, 4, 5, we have

COROLLARY 2. *Let M be a nonorientable closed manifold with $\dim M = 2n + 1$, $n \geq 1$, then*

$$I[M, S_m^{4n+1}] \leftrightarrow H^1(M, Z_m) \times H^{2n}(M, [Z]_\tau) \times Z_2,$$

where $[Z]_\tau$ is the coefficient bundle of integers determined by $T(M)$.

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