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On classification of immersions of n-manifolds in $(2n-1)$-manifolds

LI BANGHE

Abstract. Under the assumption of (f, M^n, N^{2n-1}) being trivial, the classification of immersions homotopic to $f: M^n \to N^{2n-1}$ is obtained in many cases. The triviality of (f, M^n, P^{2n-1}) is proved for any M^n and f.

Let M, N be differentiable manifolds of dimension n and $2n-1$ respectively. For a map $f: M \to N$, denote by $I[M, N]_f$ the set of regular homotopy classes of immersions homotopic to f. It has been proved in [1] that, when $n>1$, $I[M, N]_f$ is nonempty for any f. In this paper we will determine the set $I[M, N]_f$ in some cases.

For example, if $N = P^{2n-1}$, or more generally, the lens spaces $S_m^{2n-1} = Z_m \backslash S^{2n-1}$, M is any orientable n-manifold or nonorientable but $n \equiv 0, 1, 3 \bmod 4$, then, for any f, the $I[M, N]_f$ is determined completely.

When $N = R^{2n-1}$, the set $I[M, N]$ of regular homotopy classes of all immersions has been enumerated by James and Thomas in [2] and McClendon in [3] for $n > 3$. Applying our results to $N = R^{2n-1}$ we obtain their results again, except for the case $n \equiv 2 \bmod 4$ and M nonorientable.

When $n = 3$, McClendon's results cannot be used. Our results include the cases $n = 3$, M orientable or not (for orientable M, $I[M, R^5]$ is known by Wu [4]).

§1. Preliminaries

Suppose M and N are differentiable manifolds with $\dim M < \dim N$. Let $\operatorname{Hom}(T(M), T(N))$ be the bundle over $M \times N$ with fibre {orthogonal homomorphisms: $T_x(M) \to T_y(N)$} at (x, y) and structure group $O(\dim N) \times O(\dim M)$. For a map $f: M \to N$, define $F: M \to M \times N$ by $F(x) = (x, f(x))$, then F induces a bundle $\mathcal{B}_f$ from $\operatorname{Hom}(T(M), T(N))$.

Denote the space consisting of all maps $M \to N$ with the compact-open topology by N^M, and the fundamental group of the path-component of N^M containing f by $\pi_1(N^M, f)$. $\pi_1(N^M, f)$ operates on the set of homotopy classes of sections of $\mathcal{B}_f$ as follows: for a section s_0 of $\mathcal{B}_f$ i.e. a map $s_0: M \to \operatorname{Hom}(T(M), T(N))$ covering F, and a homotopy $f_t: M \to N$ with $f_0 = f = f_1$, by the homotopy covering property of bundle, there exists a homotopy $s_t: M \to \operatorname{Hom}(T(M), T(N))$ covering $F_t: M \to M \times N$, where $F_t(x) = (x, f_t(x))$, then s_1 defines a section of $\mathcal{B}_f$ and the homotopy class $[s_1]$ is the image of $[f_t] \in \pi_1(N^M, f)$ operating on $[s_0]$. It is known that the orbits of this operation correspond one to one to the regular homotopy classes of immersions homotopic to f (see [5], [6]).

DEFINITION. We say that a triad (f, M, N) is trivial if $f: M \to N$ is a map and $\pi_1(N^M, f)$ operates on $\mathscr{B}_f$ trivially.

LEMMA 1. *Suppose* $\dim N > \dim M + 1$, $\pi_1(N)$ *is abelian, and* $\pi_i(N) = 0$ *for* $2 \leq i \leq \dim M + 1$, *then* $\pi_1(N^M, f) \approx \pi_1(N)$ *for any* f.

Proof. As a set, $\pi_1(N^M, f)$ consists of the homotopy classes of maps $F: M \times [0, 1] \to N$ with $F(x, 0) = f(x) = F(x, 1)$ relative to $M \times \partial[0, 1]$. Under the assumptions of the lemma, by using obstruction theory, we know that $\pi_1(N^M, f)$ corresponds one to one with $H^1(M \times [0, 1], M \times \partial[0, 1]; \pi_1(N))$, and the later corresponds one to one with $H^0(M, \pi_1(N)) \approx \pi_1(N)$. It is also easy to see that we can choose the bijections to be isomorphisms of groups, so $\pi_1(N^M, f) \approx \pi_1(N)$. The proof is complete.

Let S^n be a sphere of odd dimension. There is a natural action of Z_m on S^n, where m is any positive integer. Denote by S^n_m the quotient space $Z_m \backslash S^n$, then $S^n_1 = S^n$ and $S^n_2 = P^n$.

THEOREM 1. *Let* $n > \dim M + 1, f: M \to S^n_m$ be any map, then (f, M, S^n_m) *is trivial.*

Proof. There is a natural C^∞-flow Φ_t on S^n_m such that, for any fixed $x \in S^n_m$, the closed orbit $\Phi_t(x)$, $t \in [0, 1]$ represents the generator of $\pi_1(S^n_m) \approx Z_m$. For $q = 1, 2, \ldots, m$, define $f^q_t: M \times [0, 1] \to S^n_m$ by $f^q_t(x) = \Phi_{qt}(f(x))$, then $f^q_o = f = f^q_1$. From Lemma 1 we know that $f^1_t, f^2_t, \ldots, f^m_t$ represent all the elements of $\pi_1((S^n_m)^M, f) \approx Z_m$.

Let $\Phi_{t*}: T(S^n_m) \to T(S^n_m)$ be induced from the diffeomorphism Φ_t and $s: T(M) \to T(S^n_m)$ be any orthogonal homomorphism covering f (representing a section of $\mathscr{B}_t$), then $\Phi_{qt*} \circ s: T(M) \to T(S^n_m)$ is a homotopy of orthogonal homomorphisms covering f^q_t. Because $\Phi_q = \text{identity}$, so $\Phi_{q*} \circ s = s$. This shows that the operation of $\pi_1((S^n_m)^M, f)$ is trivial and the proof is complete.

§2. Boltyanski's theorem

In [4] using Postnikov formula, Wu has calculated $I[M, R^5]$ for orientable 3-manifolds M. But in the general case, in order to classify the sections of $\mathscr{B}_f$ we have to use Boltyanski's generalization of the Postnikov formula (see [7]).

Let $\mathscr{B}$ be a fiber bundle over a simplicial complex B with fiber F and structure group G such that $\pi_0(F)=\pi_1(F)=\cdots=\pi_{r-1}(F)=0$, $r\geq 2$, and let G be a connected Lie group operating on F effectively and transitively with path-connected isotropy group Γ. A section σ_0 given on B^{r+1} (the $(r+1)$-skeleton of B) determines a Γ-principal bundle over B^{r+1}, whose characteristic class in $H^2(B^{r+1}, \pi_1(\Gamma))$ is denoted by $Y^2_{\sigma_0}$. Let σ and σ' be the sections of $\mathcal{B}$ over B^{r+1}, if the first difference $D^r(\sigma,\sigma')\in H^r(B^{r+1},\pi_r(F))$ is zero, i.e. $D^r(\sigma_0,\sigma)=D^r(\sigma_0,\sigma')=D^r$, then we can define a secondary difference $D^{r+1}=D^{r+1}(\sigma,\sigma')\in H^{r+1}(B^{r+1},\pi_{r+1}(F))$. According to Boltyanski's theorem, σ and σ' are homotopic on B^{r+1} if and only if there exists a $\Lambda^{r-1}\in H^{r-1}(B^{r+1},\pi_r(F))$ such that

$$D^{r+1}=\begin{cases}\Lambda^{r-1}\cup Y^2_{\sigma_0}+sq^2\Lambda^{r-1} & \text{if } r>2,\\ \Lambda^1\cup Y_{\sigma_0}+\Lambda^1\cup D^2+kv^2\Lambda^1 & \text{if } r=2.\end{cases}$$

Assume that $T(M)$ and $f^*T(N)$ are orientable with $\dim M=n\geq 3$, $\dim N=2n-1$, and (f,M,N) is trivial, then $I[M,N]_f$ is in one to one correspondence with the set of homotopy classes of sections of $\mathcal{B}_f$. $\mathcal{B}_f$ has fiber $F=V_{2n-1,n}$ and structure group $G=SO(2n-1)\times SO(n)$. Let $a=(a_{kh})\in SO(n)$, $b=(b_{uv})\in SO(n-1)$, define $i(a)=\begin{pmatrix}a & 0\\ 0 & I\end{pmatrix}\in SO(2n-1)$, $j(b)=\begin{pmatrix}I & 0\\ 0 & b\end{pmatrix}\in SO(2n-1)$, then the isotropy group $\Gamma=\{(i(a)j(b),a)/a\in SO(n), b\in SO(n-1)\}\approx SO(n)\times SO(n-1)$.

Now we are going to determine the pairing of $\pi_1(\Gamma)$ and $\pi_{n-1}(V_{2n-1,n})$ into $\pi_n(V_{2n-1,n})$. Let E^2 and E^{n-1} be the cubes of dimension 2 and $n-1$ respectively and embed $S^{n-1}=\{(x_1,\ldots,x_n)/\sum x_i^2=1\}$ in $V_{2n-1,n}$ by sending $(x_1,\ldots,x_n)$ to $(v_1,\ldots,v_{n-1},v_n)\in V_{2n-1,n}$ such that $v_1=(1,0,\ldots,0)$, $v_2=(0,1,0,\ldots,0),\ldots$, $v_{n-1}=(0,\ldots,0,1,0,\ldots,0)$ and $v_n=(0,\ldots,0,x_1,\ldots,x_n)$. Choose any map $\phi=E^{n-1}\to S^{n-1}$ carrying ∂E^{n-1} into $(1,0,\ldots,0)$ and homeomorphic on $E^{n-1}-\partial E^{n-1}$, then as a map $E^{n-1}\to V_{2n-1,n}$, ϕ determines a generator of $\pi_{n-1}(V_{2n-1,n})$ denoted by $[\phi]$. Let α be a generator of $\pi_n(S^{n-1})$, set $\beta=\phi_*(\alpha)\in\pi_n(V_{2n-1,n})$, then the pairing of $\pi_{n-1}(V_{2n-1,n})$ and $\pi_{n-1}(V_{2n-1,n})$ into $\pi_n(V_{2n-1,n})$ is determined by

$$[\phi]\cdot[\phi]=\begin{cases}\beta, & \text{if } n>3\\ 2\beta, & \text{if } n=3.\end{cases}$$

Define $a(\theta)=\begin{pmatrix}A(\theta) & 0\\ 0 & I\end{pmatrix}\in SO(n)$, $b(\theta)=\begin{pmatrix}A(\theta) & 0\\ 0 & I\end{pmatrix}\in SO(n-1)$, where $A(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, $0\leq\theta\leq 2\pi$, then $(i(a(\theta)),a(\theta))$ and $(j(b(\theta)),I)\in\Gamma\subset SO(2n-1)\times SO(n)$ determine the two generators of $\pi_1(\Gamma)\approx\pi_1(SO(n))+\pi_1(SO(n-1))$, denoted by ξ and η respectively. As in [8] (see p. 80–81) we see that pairing $\eta\in\pi_1(\Gamma)$ and $[\phi]\in\pi_{n-1}(V_{2n-1,n})$ yields

$$\eta\cdot[\phi]=\pm\beta\in\pi_n(V_{2n-1,n}).$$

Because the operation of $(a,b)\in SO(2n-1)\times SO(n)$ on $V\in V_{2n-1,n}$ is defined by $(a,b)\cdot V=aVb^{-1}$, we have $(i(a(\theta)),a(\theta))\cdot\phi(x)=\phi(x)$ for any $\theta\in[0,2\pi]$ and $x\in E^{n-1}$. $\xi\cdot[\phi]$ is determined by

$$\psi(k,x)=\begin{cases}(i(a(\theta)),a(\theta))\cdot\phi(x), & k=k(\theta)\in\partial E^2, x\in E^{n-1},\\ \phi(\partial E^{n-1}), & k\in E^2, x\in\partial E^{n-1}.\end{cases}$$

Now ψ can be extended to a map $\tilde{\psi}:E^2\times E^{n-1}\to V_{2n-1,n}$ by defining $\tilde{\psi}(k,x)=\phi(x)$, so $\xi\cdot[\phi]=0$.

Let σ be a section of $\mathcal{B}_f$, i.e. an orthogonal homomorphism $T(M)\to f^*T(N)$, then $f^*T(N)$ is the whitney sum $T(M)\oplus N_f$, where N_f is the normal bundle of σ if we regard σ as an immersion. Thus the principal bundle associated to $f^*T(N)$ has a principal subbundle with fibre $i(SO(n))j(SO(n-1))\subset SO(2n-1)$, where

$$i(SO(n))j(SO(n-1))=\left\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\Big/ a\in SO(n), b\in SO(n-1)\right\}.$$

Take any sections γ and δ of the principal bundles associated to $T(M)$ and N_f respectively, over M^1, the 1-skeleton of M. For a 2-simplex T in M, regarding the restriction of the principal bundle of $\mathcal{B}_f$ on T as $T\times SO(2n-1)\times SO(n)$, we have a map $h:\partial T\to SO(2n-1)\times SO(n)$ defined by $h(x)=(i(\gamma(x))j(\delta(x)),\gamma(x))$. h defines a section of the Γ-principal bundle determined by σ, over M^1. Let $Y_\sigma^2\in H^2(M,\pi_1(\Gamma))$ and $W_0^2\in H^2(M,\pi_1(SO(n-1))$ be the characteristic classes of the Γ-principal bundle and the normal bundle of σ respectively, then from $\xi\cdot[\phi]=0, \eta\cdot[\phi]=\pm\beta$ and the relation of h,γ and δ, we have $\Lambda^{n-2}\cup Y_\sigma^2=\pm\Lambda^{n-1}\cup W_\sigma^2$, for any $\Lambda^{n-2}\in H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))$. Thus, by Boltyanski's theorem, if σ' is another section of $\mathcal{B}_f$ homotopic to σ on M^{n-1}, and $D^n\in H^n(M,\pi_n(V_{2n-1,n}))$ is a secondary difference, then σ and σ' are homotopic if and only if there exists a $\Lambda^{n-2}\in H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))$ such that

$$D^n=\begin{cases}\Lambda^{n-2}\cup W_\sigma^2+sq^2\Lambda^{n-2}, & \text{if } n>3\\ \Lambda^{n-2}\cup W_\sigma^2+kv^2\Lambda^{n-2}, & \text{if } n=3.\end{cases}$$

§3. The orientable case

Following Wu [4], [9], first, we give a more precise description of $\beta=\phi_*(\alpha)$.

For $n>3$, we have $\pi_n(V_{2n-1,n})\approx\pi_n(V_{n+2,3})\approx 0, Z_2+Z_2, Z_2, Z_4$ as $n\equiv 0,1,2,3 \bmod 4$, and $\pi_n(V_{n+1,2})\approx Z_2+Z, Z_2$ as $n\equiv 0, 1 \bmod 2$, respectively (see [10]). Let $[\psi]\in\pi_{n-1}(V_{n+1,2})\approx\pi_{n-1}(V_{2n-1,n})$ be the standard generator, then

$\psi_*(\alpha)\in\pi_n(V_{n+1,2})$. By the natural fibration $S^{n-1}\subset V_{n+1,2}\to S^n$, and the exact sequence

$$\pi_n(S^{n-1})\xrightarrow{\psi_*}\pi_n(V_{n+1,2})\longrightarrow\pi_n(S^n),$$

we see that ψ_* is an isomorphism of $\pi_n(S^{n-1})$ and the subgroup Z_2 of $\pi_n(V_{n+1,2})$. Let $c: V_{n+1,2}\to V_{n+2,3}$ be the natural inclusion, then $\beta=c_*\psi_*(\alpha)$, by the natural isomorphism of $\pi_n(V_{n+2,3})$ and $\pi_n(V_{2n-1,n})$. From the following exact sequence

$$\pi_{n+1}(S^{n+1})\longrightarrow\pi_n(V_{n+1,2})\xrightarrow{c_*}\pi_n(V_{n+2,3})\longrightarrow\pi_n(S^{n+1}),$$

we know that, except for $n\equiv 0 \bmod 4$, $\beta=c_*\psi_*(\alpha)$ is a non-zero element of order 2 in $\pi_n(V_{n+2,3})\approx\pi_n(V_{2n-1,n})$.

Let $\rho: Z$ (or Z_2) $\to Z_2$ be the mod 2 homomorphism and $s: Z_2\to\pi_n(V_{2n-1,n})$ the homomorphism defined by $s(1 \bmod 2)=\beta$. Notice that, in case $n>3$, $W_\sigma^2\in H^2(M, Z_2)$ is independent of σ, depending only on f. In reality, it is the stable normal class of f, we denote it by W_f^2.

THEOREM 2. *Let M and N be connected manifolds with* $\dim M=n>3$, $\dim N=2n-1$, *and $f: M\to N$ a map such that $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then*

$$I[M,N]_f\leftrightarrow H^{n-1}(M,\pi_{n-1}(V_{2n-1,n}))$$
$$\times[H^n(M,\pi_n(V_{2n-1,n}))/s_*(\rho_*H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))\cup f^*(W_2))$$

where $W_2\in H^2(N, Z_2)$ is the Stiefel–Whitney class, s_* and f_* are the induced homomorphism of s and f respectively, and $\cup$ stands for the ordinary cup product.

Proof. The primary differences of sections of $\mathscr{B}_f$ fill $H^{n-1}(M, \pi_{n-1}(V_{2n-1,n}))$, and the secondary differences of those sections homotopic on M^{n-1} fill $H^n(M, \pi_n(V_{2n-1,n}))$. Now the set of secondary differences of sections homotopic on M is just

$$s_*[\rho_*H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))\cup W_f^2+sq^2\rho_*H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))],$$

by what we have already known from the above.

Because $f^*T(N)=T(M)\oplus N_f$, so $f^*(W_2)=W_f^2+W_2(M)$, here $W_2(M)$ denotes the Stiefel–Whitney class of M. By the Wu formula,

$$\rho_*H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))\cup W_2(M)+sq^2\rho_*H^{n-2}(M,\pi_{n-1}(V_{2n-1,n}))=0.$$

From this, the conclusion of the theorem follows immediately.

COROLLARY 1. *Under the same assumptions as Theorem* 1, *if* $n \equiv 0 \bmod 4$, $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$. *Furthermore, suppose M is closed, then, if* $n \equiv 2 \bmod 4$, *we have* $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2) \times Z_2$ *if* $f^*(W_2) = 0$ *and* $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$ *if* $f^*(W_2) \neq 0$; *if* $n \equiv 1 \bmod 2$, *we have* $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_4$ *as* $f^*(W_2) \in \rho_* T^2(M, Z)$, *and* $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_2$ *as* $f^*(W_2) \bar{\in} \rho_* T^2(M, Z)$, *where* $T^2(M, Z)$ *is the torsion subgroup of* $H^2(M, Z)$.

Proof. In the case $n \equiv 0 \bmod 4$, it is due to $\pi_n(V_{2n-1,n}) = 0$. In the case $n \equiv 2 \bmod 4$, when $f^*(W_2) \neq 0$, by the Poincaré duality theorem, we have $\rho_* H^{n-2}(M, Z_2) \cup f^*(W_2) = H^n(M, Z_2) \approx Z_2$. For $n = 1 \bmod 2$, by the result of Massey and Peterson [13], we know that

$$\rho_* H^{n-2}(M, Z) \cup f^*(W_2) = H^n(M, Z_2)$$

if and only if $\rho_*(W_2) \bar{\in} f_* T^2(M, Z)$. Thus, noticing that $s(1 \bmod 2) = \beta$ is a non-zero element in the cases $n = 1, 2, 3 \bmod 4$, the corollary becomes clear.

EXAMPLE 1. Let $n = 4m + 2, m > 1$, and M orientable and closed, then $I[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2) \times H^{n-1}(M, Z_2) \times Z_2$, because $W_2(P^{2n-1}) = 0$ and $[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2)$.

EXAMPLE 2. Let $n \geq 5$ be odd, then $I[P^n, P^{2n-1}]$ consists of 16 elements. Because $H^{n-2}(P^n, Z) = 0$, thus by Theorem 2,

$$I[P^n, P^{2n-1}] \leftrightarrow H^1(P^n, Z_2) \times H^{n-1}(P^n, Z) \times H^n(P^n, Z_4) \leftrightarrow Z_2 \times Z_2 \times Z_4.$$

Now we consider the case $n = 3$. Just as Wu did in [4], we see that $\beta = \phi_*(\alpha)$ is two times a generator of $\pi_3(V_{5,3}) \approx Z$. Since $\pi_2(V_{5,3}) \approx Z$ is free, according to the definition of kv^2 (see [7]), we know that $kv^2 H^1(M, \pi_2(V_{5,3}) = 0$. If σ and σ' are two sections of $\mathcal{B}_f$ homotopic on M^2, i.e. $D^2(\sigma_0, \sigma) = D^2(\sigma_0, \sigma') = D \in H^2(M, \pi_2(V_{5,3})) \approx H^2(M, Z)$, for a fixed section σ_0, then $W_\sigma^2 = W_{\sigma'}^2 \in H^2(M, SO(2)) \approx H^2(M, Z)$. So we can denote them by an element W_D^2 in $H^2(M, Z)$. Thus we have

THEOREM 3. Let M and N be connected manifolds with $\dim M = 3$ and $\dim N = 5$. Suppose $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then

$$I[M, N]_f \leftrightarrow \bigcup_{D \in H^2(M,Z)} [H^3(M, Z)/2W_D^2 \cup H^1(M, Z)],$$

where $\cup$ stands for the ordinary cup product.

REMARK. All normal classes W_D^2 form a coset of $2H^2(M, Z)$ in $H^2(M, Z)$, which is just the inverse image of $f^*(W_2(N))$ under the mod 2 homomorphism $H^2(M, Z) \to H^2(M, Z_2)$. All D with the same W_D^2 form a coset of the subgroup of $H^2(M, Z)$ consisting of all elements of order 2.

EXAMPLE 3. $I[P^3, P^5] \to Z_2 \times Z_2 \times Z$.

§4. Nonorientable case

If at least one of $T(M)$ and $f^*T(N)$ is nonorientable, Boltyanski's theorem does not work. So we have to appeal to other methods. The case $n = 4s$ is trivial, because $\pi_n(V_{2n-1,n}) = 0$, and only primary obstructions are concerned. In order to deal with the case n odd, we embed $\mathscr{B}_f$ in a larger bundle, in which the original problem to determine secondary obstructions turns out to be one to determine primary obstructions. To do this, we need some lemmas first.

LEMMA 2. *Let $n \geq 3$ be odd, $i: V_{2n-1,n} \to V_{2n,n}$ the natural embedding, then $i_*: \pi_n(V_{2n-1,n}) \to \pi_n(V_{2n,n})$ is surjective.*

Proof. Let $j: V_{2n-1,n} \to V_{2n,n+1}$ be the natural embedding and $p: V_{2n,n+1} \to V_{2n,n}$ the natural projection, then we have $i = pj$. From the fibrations $V_{2n-1,n} \subset V_{2n,n+1} \to S^{2n-1}$ and $S^{n-1} \subset V_{2n,n+1} \to V_{2n,n}$, we know that $j_*: \pi_n(V_{2n-1,n}) \to \pi_n(V_{2n,n+1})$ is an isomorphism, and $p_*: \pi_n(V_{2n,n+1}) \to \pi_n(V_{2n,n})$ is surjective. Thus $i_* = p_* j_*: \pi_n(V_{2n-1,n}) \to \pi_n(V_{2n,n})$ is surjective. This proves the lemma.

Let λ be the map $V_{n,k} \to V_{n,k}$ which changes the sign of every column, we have.

LEMMA 3. *If $n = 4s+3$, $s \geq 0$ then $\lambda_*: \pi_n(V_{2n-1,n}) \to \pi_n(V_{2n-1,n})$ is the identity.*

Proof. By a theorem of James (see [12], chapter 13), $\lambda_* = 1 - i_* \circ \Sigma \circ \Delta$, where $\Delta: \pi_n(V_{2n-1,n}) \to \pi_{n-1}(S^{n-2})$ denotes the boundary operator in the homotopy sequence for the fibration $S^{n-2} \subset V_{2n-1,n+1} \to V_{2n-1,n}$, $\Sigma: \pi_{n-1}(S^{n-2}) \to \pi_n(S^{n-1})$ the suspension homomorphism, and $i_*: \pi_n(S^{n-1}) \to \pi_n(V_{2n-1,n})$ the induced homomorphism of the fiber inclusion for the fibration $S^{n-1} \subset V_{2n-1,n} \to V_{2n-1,n-1}$.

For $n = 3$, because $\pi_2(S^1) = 0$, thus $\Delta = 0$, and $\lambda_* = 1$.

For $n>3$, the exact sequence

$$\begin{array}{ccccccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n+1}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n}) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ Z_4 & & Z_2 & & Z_2 & & Z \end{array}$$

tells us $\Delta=0$, thus $\lambda_*=1$. The proof is complete.

LEMMA 4. *If $n=4s+1$, $s>1$, then λ_* is an automorphism of $\pi_n(V_{2n-1,n})\approx Z_2+Z_2$ which fixes two elements and exchanges other two elements.*

Proof. From the exact sequence

$$\begin{array}{ccccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n+1}) \\ \wr\wr & & \wr\wr & & \wr\wr \\ Z_2+Z_2 & & Z_2 & & 0 \end{array}$$

we know that Δ is surjective. Now from the following exact sequence

$$\begin{array}{ccccccc} \pi_n(S^{n-1}) & \xrightarrow{i_*} & \pi_n(V_{2n-1,n}) & \longrightarrow & \pi_n(V_{2n-1,n-1}) & \longrightarrow & \pi_n(S^n) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ Z_2 & & Z_2+Z_2 & & Z_2 & & Z \end{array}$$

it follows that i_* must be an injection. Thus the image of $i_*\circ\Sigma\circ\Delta$ has two elements, and the conclusion of the lemma follows from $\lambda_*=1-i_*\circ\Sigma\circ\Delta$. The proof is complete.

THEOREM 4. *Let $n=4s+1$, $s>0$, M and N be connected differentiable manifolds with $\dim M=n$ and $\dim N=2n-1$. Furthermore, assume that (f, M, N) is trivial, M closed and $f^*W_1(N)\neq W_1(M)$, then*

$$I[M,N]_f \leftrightarrow H^{n-1}(M,\mathscr{B}(\pi_{n-1}))\times Z_2,$$

where $W_1(N)$ and $W_1(M)$ are the Stiefel–Whitney classes of N and M respectively, and $\mathscr{B}(\pi_{n-1})$ is the coefficient bundle associated to $\mathscr{B}_f$ with fiber $\pi_{n-1}(V_{2n-1,n})\approx Z$.

Proof. For a fixed section σ_0 of $\mathscr{B}_f$ the primary differences fill $H^{n-1}(M,\mathscr{B}(\pi_{n-1}))$. Let $\mathscr{B}(\pi_n(V_{2n-1,n}))$ (abbreviated to $\mathscr{B}(\pi_n)$) be the coefficient bundle associated to $\mathscr{B}_f$. Given a $D^{n-1}\in H^{n-1}(M,\mathscr{B}(\pi_{n-1}))$, the secondary obstructions $D^n(\sigma,\sigma')$ of all sections σ and σ' with $D^{n-1}(\sigma_0,\sigma)=D^{n-1}(\sigma_o,\sigma')=D^{n-1}$ fill $H^n(M,\mathscr{B}(\pi_n))$.

By Lemma 4, we can choose a basis of $\pi_n(V_{2n-1,n}) \approx Z_2 + Z_2$ in which λ_* has the form: $\lambda_*(1,0)=(0,1)$, $\lambda_*(0,1)=(1,0)$ and $\lambda_*(1,1)=(1,1)$. Let Δ_0 be an n-simplex in M, then any n-cocycle with coefficient in $\mathscr{B}(\pi_n)$ is cohomologous to an n-cocycle T_γ such that $\langle T_\gamma, \Delta_0\rangle = \gamma \in \pi_n(V_{2n-1,n})$, $\langle T_\gamma, \Delta\rangle = 0$ for other n-simplices Δ in a triangulation of M including Δ_0. Now $f^*W_1(N) \neq W_1(M)$, so there must exist a loop in M which preserves the orientation of $T(M)$ but reverses the orientation of $f^*T(N)$, or reverses the former but preverses the later. By the existence of such a loop, we see that $T_{\gamma+\lambda_*\gamma} = T_\gamma + T_{\lambda_*\gamma}$ must be cohomologous to 0 for any $\gamma \in \pi_n(V_{2n-1,n})$. Thus $T_{(1,0)} \sim T_{(0,1)}$, and $T_{(1,1)} = T_{(1,0)} - T_{(0,1)} \sim 0$. So $H^n(M, \mathscr{B}(\pi_n))$ has at most two elements.

Now let $N' = N \times R$, define $f': M \to N'$ by $f(x) = (f(x), 0)$, then $\mathscr{B}_f$ may be regarded as a subbundle of $\mathscr{B}_{f'}$. By Lemma 2, there exists a $\gamma_0 \in \pi_n(V_{2n-1,n})$ such that $i_*\gamma_0 \neq 0$. For any $D^{n-1} \in H^{n-1}(M, \mathscr{B}(\pi_{n-1}))$, we can choose two sections σ and σ' of $\mathscr{B}_f$ such that $D^{n-1}(\sigma_0, \sigma) = D^{n-1}(\sigma_0, \sigma') = D^{n-1}$, and one of their secondary differences includes T_{γ_0}. Thus as sections of $\mathscr{B}_{f'}$, σ and σ' have a non-zero primary difference. It is obvious that σ and σ' are not homotopic in $\mathscr{B}_f$ (see [5]). From this, we concluse that $H^n(M, \mathscr{B}(\pi_n)) \approx Z_2$ and two sections with non-zero secondary difference are not homotopic. The proof is complete.

THEOREM 5. *Let $n = 4s+3$, $s \geq 0$, M and N be connected differentiable manifolds with* $\dim M = n$ *and* $\dim N = 2n-1$. *Furthermore, assume that M is closed and nonorientable, and (f, M, N) is trivial, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \mathscr{B}(\pi_{n-1})) \times Z_2,$$

where $\mathscr{B}(\pi_{n-1})$ is the coefficient bundle associated to $\mathscr{B}_f$ with fibre $\pi_{n-1}(V_{2n-1,n}) \approx Z$.

Proof. In virtue of Lemma 3, $\mathscr{B}(\pi_n)$ is trivial. Thus, by the assumption that M is closed and nonorientable, we have $H^n(M, \mathscr{B}(\pi_n)) \approx H^n(M, Z_4) \approx Z_2$ for $n > 3$, and $H^n(M, \mathscr{B}(\pi_n)) \approx H^n(M, Z) \approx Z_2$ for $n = 3$. Now, using the similar argument as that for theorem 4, the conclusion of this theorem follows.

From Theorems 1, 4, 5, we have

COROLLARY 2. *Let M be a nonorientable closed manifold with* $\dim M = 2n+1$, $n \geq 1$, *then*

$$I[M, S_m^{4n+1}] \leftrightarrow H^1(M, Z_m) \times H^{2n}(M, [Z]_\tau) \times Z_2,$$

where $[Z]_\tau$ is the coefficient bundle of integers determined by $T(M)$.

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Institute of Mathematics
University of Geneva
CH-1200 Geneva
and Institute of Systems Science
Academia Sinica

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