

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 57 (1982)

Artikel: On classification of immersions of n -manifolds in $(2n-1)$ -manifolds.
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DOI: <https://doi.org/10.5169/seals-43878>

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On classification of immersions of n -manifolds in $(2n-1)$ -manifolds

LI BANGHE

Abstract. Under the assumption of (f, M^n, N^{2n-1}) being trivial, the classification of immersions homotopic to $f: M^n \rightarrow N^{2n-1}$ is obtained in many cases. The triviality of (f, M^n, P^{2n-1}) is proved for any M^n and f .

Let M, N be differentiable manifolds of dimension n and $2n-1$ respectively. For a map $f: M \rightarrow N$, denote by $I[M, N]_f$ the set of regular homotopy classes of immersions homotopic to f . It has been proved in [1] that, when $n > 1$, $I[M, N]_f$ is nonempty for any f . In this paper we will determine the set $I[M, N]_f$ in some cases.

For example, if $N = P^{2n-1}$, or more generally, the lens spaces $S_m^{2n-1} = Z_m \backslash S^{2n-1}$, M is any orientable n -manifold or nonorientable but $n \equiv 0, 1, 3 \pmod{4}$, then, for any f , the $I[M, N]_f$ is determined completely.

When $N = R^{2n-1}$, the set $I[M, N]$ of regular homotopy classes of all immersions has been enumerated by James and Thomas in [2] and McClendon in [3] for $n > 3$. Applying our results to $N = R^{2n-1}$ we obtain their results again, except for the case $n \equiv 2 \pmod{4}$ and M nonorientable.

When $n = 3$, McClendon's results cannot be used. Our results include the cases $n = 3$, M orientable or not (for orientable M , $I[M, R^5]$ is known by Wu [4]).

§1. Preliminaries

Suppose M and N are differentiable manifolds with $\dim M < \dim N$. Let $\text{Hom}(T(M), T(N))$ be the bundle over $M \times N$ with fibre $\{\text{orthogonal homomorphisms } T_x(M) \rightarrow T_y(N)\}$ at (x, y) and structure group $O(\dim N) \times O(\dim M)$. For a map $f: M \rightarrow N$, define $F: M \rightarrow M \times N$ by $F(x) = (x, f(x))$, then F induces a bundle \mathcal{B}_f from $\text{Hom}(T(M), T(N))$.

Denote the space consisting of all maps $M \rightarrow N$ with the compact-open topology by N^M , and the fundamental group of the path-component of N^M containing f by $\pi_1(N^M, f)$. $\pi_1(N^M, f)$ operates on the set of homotopy classes of sections of \mathcal{B}_f as follows: for a section s_0 of \mathcal{B}_f i.e. a map $s_0: M \rightarrow \text{Hom}(T(M), T(N))$ covering F , and a homotopy $f_t: M \rightarrow N$ with $f_0 = f = f_1$, by the homotopy covering property of bundle, there exists a homotopy $s_t: M \rightarrow \text{Hom}(T(M), T(N))$ covering $F_t: M \rightarrow M \times N$, where $F_t(x) = (x, f_t(x))$, then s_1 defines a section of \mathcal{B}_f and the homotopy class $[s_1]$ is the image of $[f_t] \in \pi_1(N^M, f)$ operating on $[s_0]$. It is known that the orbits of this operation correspond one to one to the regular homotopy classes of immersions homotopic to f (see [5], [6]).

DEFINITION. We say that a triad (f, M, N) is trivial if $f: M \rightarrow N$ is a map and $\pi_1(N^M, f)$ operates on \mathcal{B}_f trivially.

LEMMA 1. Suppose $\dim N > \dim M + 1$, $\pi_1(N)$ is abelian, and $\pi_i(N) = 0$ for $2 \leq i \leq \dim M + 1$, then $\pi_1(N^M, f) \approx \pi_1(N)$ for any f .

Proof. As a set, $\pi_1(N^M, f)$ consists of the homotopy classes of maps $F: M \times [0, 1] \rightarrow N$ with $F(x, 0) = f(x) = F(x, 1)$ relative to $M \times \partial[0, 1]$. Under the assumptions of the lemma, by using obstruction theory, we know that $\pi_1(N^M, f)$ corresponds one to one with $H^1(M \times [0, 1], M \times \partial[0, 1]; \pi_1(N))$, and the latter corresponds one to one with $H^0(M, \pi_1(N)) \approx \pi_1(N)$. It is also easy to see that we can choose the bijections to be isomorphisms of groups, so $\pi_1(N^M, f) \approx \pi_1(N)$. The proof is complete.

Let S^n be a sphere of odd dimension. There is a natural action of Z_m on S^n , where m is any positive integer. Denote by S_m^n the quotient space $Z_m \backslash S^n$, then $S_1^n = S^n$ and $S_2^n = P^n$.

THEOREM 1. Let $n > \dim M + 1$, $f: M \rightarrow S_m^n$ be any map, then (f, M, S_m^n) is trivial.

Proof. There is a natural C^∞ -flow Φ_t on S_m^n such that, for any fixed $x \in S_m^n$, the closed orbit $\Phi_t(x)$, $t \in [0, 1]$ represents the generator of $\pi_1(S_m^n) \approx Z_m$. For $q = 1, 2, \dots, m$, define $f_t^q: M \times [0, 1] \rightarrow S_m^n$ by $f_t^q(x) = \Phi_{qt}(f(x))$, then $f_0^q = f = f_1^q$. From Lemma 1 we know that $f_1^1, f_1^2, \dots, f_1^m$ represent all the elements of $\pi_1((S_m^n)^M, f) \approx Z_m$.

Let $\Phi_{t*}: T(S_m^n) \rightarrow T(S_m^n)$ be induced from the diffeomorphism Φ_t and $s: T(M) \rightarrow T(S_m^n)$ be any orthogonal homomorphism covering f (representing a section of \mathcal{B}_f), then $\Phi_{qt*} \circ s: T(M) \rightarrow T(S_m^n)$ is a homotopy of orthogonal homomorphisms covering f_t^q . Because $\Phi_q = \text{identity}$, so $\Phi_{q*} \circ s = s$. This shows that the operation of $\pi_1((S_m^n)^M, f)$ is trivial and the proof is complete.

§2. Boltyanski's theorem

In [4] using Postnikov formula, Wu has calculated $I[M, R^5]$ for orientable 3-manifolds M . But in the general case, in order to classify the sections of \mathcal{B}_f we have to use Boltyanski's generalization of the Postnikov formula (see [7]).

Let \mathcal{B} be a fiber bundle over a simplicial complex B with fiber F and structure

group G such that $\pi_0(F) = \pi_1(F) = \dots = \pi_{r-1}(F) = 0$, $r \geq 2$, and let G be a connected Lie group operating on F effectively and transitively with path-connected isotropy group Γ . A section σ_0 given on B^{r+1} (the $(r+1)$ -skeleton of B) determines a Γ -principal bundle over B^{r+1} , whose characteristic class in $H^2(B^{r+1}, \pi_1(\Gamma))$ is denoted by $Y_{\sigma_0}^2$. Let σ and σ' be the sections of \mathcal{B} over B^{r+1} , if the first difference $D^r(\sigma, \sigma') \in H^r(B^{r+1}, \pi_r(F))$ is zero, i.e. $D^r(\sigma_0, \sigma) = D^r(\sigma_0, \sigma') = D^r$, then we can define a secondary difference $D^{r+1} = D^{r+1}(\sigma, \sigma') \in H^{r+1}(B^{r+1}, \pi_{r+1}(F))$. According to Boltyanski's theorem, σ and σ' are homotopic on B^{r+1} if and only if there exists a $\Lambda^{r-1} \in H^{r-1}(B^{r+1}, \pi_r(F))$ such that

$$D^{r+1} = \begin{cases} \Lambda^{r-1} \cup Y_{\sigma_0}^2 + sq^2 \Lambda^{r-1} & \text{if } r > 2, \\ \Lambda^1 \cup Y_{\sigma_0} + \Lambda^1 \cup D^2 + kv^2 \Lambda^1 & \text{if } r = 2. \end{cases}$$

Assume that $T(M)$ and $f^*T(N)$ are orientable with $\dim M = n \geq 3$, $\dim N = 2n-1$, and (f, M, N) is trivial, then $I[M, N]_f$ is in one to one correspondence with the set of homotopy classes of sections of \mathcal{B}_f . \mathcal{B}_f has fiber $F = V_{2n-1, n}$ and structure group $G = SO(2n-1) \times SO(n)$. Let $a = (a_{kh}) \in SO(n)$, $b = (b_{uv}) \in SO(n-1)$, define $i(a) = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix} \in SO(2n-1)$, $j(b) = \begin{pmatrix} I & 0 \\ 0 & b \end{pmatrix} \in SO(2n-1)$, then the isotropy group $\Gamma = \{(i(a)j(b), a) / a \in SO(n), b \in SO(n-1)\} \approx SO(n) \times SO(n-1)$.

Now we are going to determine the pairing of $\pi_1(\Gamma)$ and $\pi_{n-1}(V_{2n-1, n})$ into $\pi_n(V_{2n-1, n})$. Let E^2 and E^{n-1} be the cubes of dimension 2 and $n-1$ respectively and embed $S^{n-1} = \{(x_1, \dots, x_n) / \sum x_i^2 = 1\}$ in $V_{2n-1, n}$ by sending (x_1, \dots, x_n) to $(v_1, \dots, v_{n-1}, v_n) \in V_{2n-1, n}$ such that $v_1 = (1, 0, \dots, 0)$, $v_2 = (0, 1, 0, \dots, 0)$, \dots , $v_{n-1} = (0, \dots, 0, 1, 0, \dots, 0)$ and $v_n = (0, \dots, 0, x_1, \dots, x_n)$. Choose any map $\phi = E^{n-1} \rightarrow S^{n-1}$ carrying ∂E^{n-1} into $(1, 0, \dots, 0)$ and homeomorphic on $E^{n-1} - \partial E^{n-1}$, then as a map $E^{n-1} \rightarrow V_{2n-1, n}$, ϕ determines a generator of $\pi_{n-1}(V_{2n-1, n})$ denoted by $[\phi]$. Let α be a generator of $\pi_n(S^{n-1})$, set $\beta = \phi_*(\alpha) \in \pi_n(V_{2n-1, n})$, then the pairing of $\pi_{n-1}(V_{2n-1, n})$ and $\pi_{n-1}(V_{2n-1, n})$ into $\pi_n(V_{2n-1, n})$ is determined by

$$[\phi] \cdot [\phi] = \begin{cases} \beta, & \text{if } n > 3 \\ 2\beta, & \text{if } n = 3. \end{cases}$$

Define $a(\theta) = \begin{pmatrix} A(\theta) & 0 \\ 0 & I \end{pmatrix} \in SO(n)$, $b(\theta) = \begin{pmatrix} A(\theta) & 0 \\ 0 & I \end{pmatrix} \in SO(n-1)$, where

$A(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $0 \leq \theta \leq 2\pi$, then $(i(a(\theta)), a(\theta))$ and $(j(b(\theta)), I) \in \Gamma \subset SO(2n-1) \times SO(n)$ determine the two generators of $\pi_1(\Gamma) \approx \pi_1(SO(n)) + \pi_1(SO(n-1))$, denoted by ξ and η respectively. As in [8] (see p. 80–81) we see

that pairing $\eta \in \pi_1(\Gamma)$ and $[\phi] \in \pi_{n-1}(V_{2n-1,n})$ yields

$$\eta \cdot [\phi] = \pm \beta \in \pi_n(V_{2n-1,n}).$$

Because the operation of $(a, b) \in SO(2n-1) \times SO(n)$ on $V \in V_{2n-1,n}$ is defined by $(a, b) \cdot V = aVb^{-1}$, we have $(i(a(\theta)), a(\theta)) \cdot \phi(x) = \phi(x)$ for any $\theta \in [0, 2\pi]$ and $x \in E^{n-1}$. $\xi \cdot [\phi]$ is determined by

$$\psi(k, x) = \begin{cases} (i(a(\theta)), a(\theta)) \cdot \phi(x), & k = k(\theta) \in \partial E^2, x \in E^{n-1}, \\ \phi(\partial E^{n-1}), & k \in E^2, x \in \partial E^{n-1}. \end{cases}$$

Now ψ can be extended to a map $\tilde{\psi}: E^2 \times E^{n-1} \rightarrow V_{2n-1,n}$ by defining $\tilde{\psi}(k, x) = \phi(x)$, so $\xi \cdot [\phi] = 0$.

Let σ be a section of \mathcal{B}_f , i.e. an orthogonal homomorphism $T(M) \rightarrow f^*T(N)$, then $f^*T(N)$ is the Whitney sum $T(M) \oplus N_f$, where N_f is the normal bundle of σ if we regard σ as an immersion. Thus the principal bundle associated to $f^*T(N)$ has a principal subbundle with fibre $i(SO(n))j(SO(n-1)) \subset SO(2n-1)$, where

$$i(SO(n))j(SO(n-1)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle/ a \in SO(n), b \in SO(n-1) \right\}.$$

Take any sections γ and δ of the principal bundles associated to $T(M)$ and N_f respectively, over M^1 , the 1-skeleton of M . For a 2-simplex T in M , regarding the restriction of the principal bundle of \mathcal{B}_f on T as $T \times SO(2n-1) \times SO(n)$, we have a map $h: \partial T \rightarrow SO(2n-1) \times SO(n)$ defined by $h(x) = (i(\gamma(x))j(\delta(x)), \gamma(x))$. h defines a section of the Γ -principal bundle determined by σ , over M^1 . Let $Y_\sigma^2 \in H^2(M, \pi_1(\Gamma))$ and $W_\sigma^2 \in H^2(M, \pi_1(SO(n-1)))$ be the characteristic classes of the Γ -principal bundle and the normal bundle of σ respectively, then from $\xi \cdot [\phi] = 0$, $\eta \cdot [\phi] = \pm \beta$ and the relation of h, γ and δ , we have $\Lambda^{n-2} \cup Y_\sigma^2 = \pm \Lambda^{n-1} \cup W_\sigma^2$ for any $\Lambda^{n-2} \in H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))$. Thus, by Boltyanski's theorem, if σ' is another section of \mathcal{B}_f homotopic to σ on M^{n-1} , and $D^n \in H^n(M, \pi_n(V_{2n-1,n}))$ is a secondary difference, then σ and σ' are homotopic if and only if there exists a $\Lambda^{n-2} \in H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))$ such that

$$D^n = \begin{cases} \Lambda^{n-2} \cup W_\sigma^2 + sq^2 \Lambda^{n-2}, & \text{if } n > 3 \\ \Lambda^{n-2} \cup W_\sigma^2 + kv^2 \Lambda^{n-2}, & \text{if } n = 3. \end{cases}$$

§3. The orientable case

Following Wu [4], [9], first, we give a more precise description of $\beta = \phi_*(\alpha)$.

For $n > 3$, we have $\pi_n(V_{2n-1,n}) \approx \pi_n(V_{n+2,3}) \approx 0$, $Z_2 + Z_2, Z_2, Z_4$ as $n \equiv 0, 1, 2, 3 \pmod{4}$, and $\pi_n(V_{n+1,2}) \approx Z_2 + Z, Z_2$ as $n \equiv 0, 1 \pmod{2}$, respectively (see [10]). Let $[\psi] \in \pi_{n-1}(V_{n+1,2}) \approx \pi_{n-1}(V_{2n-1,n})$ be the standard generator, then

$\psi_*(\alpha) \in \pi_n(V_{n+1,2})$. By the natural fibration $S^{n-1} \subset V_{n+1,2} \rightarrow S^n$, and the exact sequence

$$\pi_n(S^{n-1}) \xrightarrow{\psi_*} \pi_n(V_{n+1,2}) \longrightarrow \pi_n(S^n),$$

we see that ψ_* is an isomorphism of $\pi_n(S^{n-1})$ and the subgroup Z_2 of $\pi_n(V_{n+1,2})$. Let $c: V_{n+1,2} \rightarrow V_{n+2,3}$ be the natural inclusion, then $\beta = c_*\psi_*(\alpha)$, by the natural isomorphism of $\pi_n(V_{n+2,3})$ and $\pi_n(V_{2n-1,n})$. From the following exact sequence

$$\pi_{n+1}(S^{n+1}) \longrightarrow \pi_n(V_{n+1,2}) \xrightarrow{c_*} \pi_n(V_{n+2,3}) \longrightarrow \pi_n(S^{n+1}),$$

we know that, except for $n \equiv 0 \pmod{4}$, $\beta = c_*\psi_*(\alpha)$ is a non-zero element of order 2 in $\pi_n(V_{n+2,3}) \approx \pi_n(V_{2n-1,n})$.

Let $\rho: Z$ (or Z_2) $\rightarrow Z_2$ be the mod 2 homomorphism and $s: Z_2 \rightarrow \pi_n(V_{2n-1,n})$ the homomorphism defined by $s(1 \pmod{2}) = \beta$. Notice that, in case $n > 3$, $W_\sigma^2 \in H^2(M, Z_2)$ is independent of σ , depending only on f . In reality, it is the stable normal class of f , we denote it by W_f^2 .

THEOREM 2. *Let M and N be connected manifolds with $\dim M = n > 3$, $\dim N = 2n-1$, and $f: M \rightarrow N$ a map such that $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \pi_{n-1}(V_{2n-1,n})) \times [H^n(M, \pi_n(V_{2n-1,n}))/s_*(\rho_*H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup f^*(W_2))]$$

where $W_2 \in H^2(N, Z_2)$ is the Stiefel–Whitney class, s_* and f_* are the induced homomorphism of s and f respectively, and \cup stands for the ordinary cup product.

Proof. The primary differences of sections of \mathcal{B}_f fill $H^{n-1}(M, \pi_{n-1}(V_{2n-1,n}))$, and the secondary differences of those sections homotopic on M^{n-1} fill $H^n(M, \pi_n(V_{2n-1,n}))$. Now the set of secondary differences of sections homotopic on M is just

$$s_*[\rho_*H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup W_f^2 + sq^2\rho_*H^{n-2}(M, \pi_{n-1}(V_{2n-1,n}))],$$

by what we have already known from the above.

Because $f^*T(N) = T(M) \oplus N_f$, so $f^*(W_2) = W_f^2 + W_2(M)$, here $W_2(M)$ denotes the Stiefel–Whitney class of M . By the Wu formula,

$$\rho_*H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) \cup W_2(M) + sq^2\rho_*H^{n-2}(M, \pi_{n-1}(V_{2n-1,n})) = 0.$$

From this, the conclusion of the theorem follows immediately.

COROLLARY 1. *Under the same assumptions as Theorem 1, if $n \equiv 0 \pmod{4}$, $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$. Furthermore, suppose M is closed, then, if $n \equiv 2 \pmod{4}$, we have $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2) \times Z_2$ if $f^*(W_2) = 0$ and $I[M, N]_f \leftrightarrow H^{n-1}(M, Z_2)$ if $f^*(W_2) \neq 0$; if $n \equiv 1 \pmod{2}$, we have $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_4$ as $f^*(W_2) \in \rho_* T^2(M, Z)$, and $I[M, N]_f \leftrightarrow H^{n-1}(M, Z) \times Z_2$ as $f^*(W_2) \notin \rho_* T^2(M, Z)$, where $T^2(M, Z)$ is the torsion subgroup of $H^2(M, Z)$.*

Proof. In the case $n \equiv 0 \pmod{4}$, it is due to $\pi_n(V_{2n-1,n}) = 0$. In the case $n \equiv 2 \pmod{4}$, when $f^*(W_2) \neq 0$, by the Poincaré duality theorem, we have $\rho_* H^{n-2}(M, Z_2) \cup f^*(W_2) = H^n(M, Z_2) \approx Z_2$. For $n \equiv 1 \pmod{2}$, by the result of Massey and Peterson [13], we know that

$$\rho_* H^{n-2}(M, Z) \cup f^*(W_2) = H^n(M, Z_2)$$

if and only if $\rho_*(W_2) \notin f_* T^2(M, Z)$. Thus, noticing that $s(1 \pmod{2}) = \beta$ is a non-zero element in the cases $n = 1, 2, 3 \pmod{4}$, the corollary becomes clear.

EXAMPLE 1. Let $n = 4m + 2, m > 1$, and M orientable and closed, then $I[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2) \times H^{n-1}(M, Z_2) \times Z_2$, because $W_2(P^{2n-1}) = 0$ and $[M, P^{2n-1}] \leftrightarrow H^1(M, Z_2)$.

EXAMPLE 2. Let $n \geq 5$ be odd, then $I[P^n, P^{2n-1}]$ consists of 16 elements. Because $H^{n-2}(P^n, Z) = 0$, thus by Theorem 2,

$$I[P^n, P^{2n-1}] \leftrightarrow H^1(P^n, Z_2) \times H^{n-1}(P^n, Z) \times H^n(P^n, Z_4) \leftrightarrow Z_2 \times Z_2 \times Z_4.$$

Now we consider the case $n = 3$. Just as Wu did in [4], we see that $\beta = \phi_*(\alpha)$ is two times a generator of $\pi_3(V_{5,3}) \approx Z$. Since $\pi_2(V_{5,3}) \approx Z$ is free, according to the definition of kv^2 (see [7]), we know that $kv^2 H^1(M, \pi_2(V_{5,3})) = 0$. If σ and σ' are two sections of \mathfrak{B}_f homotopic on M^2 , i.e. $D^2(\sigma_0, \sigma) = D^2(\sigma_0, \sigma') = D \in H^2(M, \pi_2(V_{5,3})) \approx H^2(M, Z)$, for a fixed section σ_0 , then $W_\sigma^2 = W_{\sigma'}^2 \in H^2(M, SO(2)) \approx H^2(M, Z)$. So we can denote them by an element W_D^2 in $H^2(M, Z)$. Thus we have

THEOREM 3. Let M and N be connected manifolds with $\dim M = 3$ and $\dim N = 5$. Suppose $T(M)$ and $f^*T(N)$ are orientable, and (f, M, N) is trivial, then

$$I[M, N]_f \leftrightarrow \bigcup_{D \in H^2(M, Z)} [H^3(M, Z)/2W_D^2 \cup H^1(M, Z)],$$

where \cup stands for the ordinary cup product.

REMARK. All normal classes W_D^2 form a coset of $2H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{Z})$, which is just the inverse image of $f^*(W_2(N))$ under the mod 2 homomorphism $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2)$. All D with the same W_D^2 form a coset of the subgroup of $H^2(M, \mathbb{Z})$ consisting of all elements of order 2.

EXAMPLE 3. $I[P^3, P^5] \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$.

§4. Nonorientable case

If at least one of $T(M)$ and $f^*T(N)$ is nonorientable, Boltyanski's theorem does not work. So we have to appeal to other methods. The case $n = 4s$ is trivial, because $\pi_n(V_{2n-1,n}) = 0$, and only primary obstructions are concerned. In order to deal with the case n odd, we embed \mathcal{B}_f in a larger bundle, in which the original problem to determine secondary obstructions turns out to be one to determine primary obstructions. To do this, we need some lemmas first.

LEMMA 2. *Let $n \geq 3$ be odd, $i: V_{2n-1,n} \rightarrow V_{2n,n}$ the natural embedding, then $i_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n})$ is surjective.*

Proof. Let $j: V_{2n-1,n} \rightarrow V_{2n,n+1}$ be the natural embedding and $p: V_{2n,n+1} \rightarrow V_{2n,n}$ the natural projection, then we have $i = pj$. From the fibrations $V_{2n-1,n} \subset V_{2n,n+1} \rightarrow S^{2n-1}$ and $S^{n-1} \subset V_{2n,n+1} \rightarrow V_{2n,n}$, we know that $j_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n+1})$ is an isomorphism, and $p_*: \pi_n(V_{2n,n+1}) \rightarrow \pi_n(V_{2n,n})$ is surjective. Thus $i_* = p_*j_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n,n})$ is surjective. This proves the lemma.

Let λ be the map $V_{n,k} \rightarrow V_{n,k}$ which changes the sign of every column, we have.

LEMMA 3. *If $n = 4s + 3$, $s \geq 0$ then $\lambda_*: \pi_n(V_{2n-1,n}) \rightarrow \pi_n(V_{2n-1,n})$ is the identity.*

Proof. By a theorem of James (see [12], chapter 13), $\lambda_* = 1 - i_* \circ \Sigma \circ \Delta$, where $\Delta: \pi_n(V_{2n-1,n}) \rightarrow \pi_{n-1}(S^{n-2})$ denotes the boundary operator in the homotopy sequence for the fibration $S^{n-2} \subset V_{2n-1,n+1} \rightarrow V_{2n-1,n}$, $\Sigma: \pi_{n-1}(S^{n-2}) \rightarrow \pi_n(S^{n-1})$ the suspension homomorphism, and $i_*: \pi_n(S^{n-1}) \rightarrow \pi_n(V_{2n-1,n})$ the induced homomorphism of the fiber inclusion for the fibration $S^{n-1} \subset V_{2n-1,n} \rightarrow V_{2n-1,n-1}$.

For $n = 3$, because $\pi_2(S^1) = 0$, thus $\Delta = 0$, and $\lambda_* = 1$.

For $n > 3$, the exact sequence

$$\begin{array}{ccccccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n+1}) & \longrightarrow & \pi_{n-1}(V_{2n-1,n}) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ Z_4 & & Z_2 & & Z_2 & & Z \end{array}$$

tells us $\Delta = 0$, thus $\lambda_* = 1$. The proof is complete.

LEMMA 4. *If $n = 4s + 1$, $s > 1$, then λ_* is an automorphism of $\pi_n(V_{2n-1,n}) \approx Z_2 + Z_2$ which fixes two elements and exchanges other two elements.*

Proof. From the exact sequence

$$\begin{array}{ccc} \pi_n(V_{2n-1,n}) & \xrightarrow{\Delta} & \pi_{n-1}(S^{n-2}) \longrightarrow \pi_{n-1}(V_{2n-1,n+1}) \\ \Downarrow & & \Downarrow \\ Z_2 + Z_2 & & Z_2 \end{array}$$

we know that Δ is surjective. Now from the following exact sequence

$$\begin{array}{ccccccc} \pi_n(S^{n-1}) & \xrightarrow{i_*} & \pi_n(V_{2n-1,n}) & \longrightarrow & \pi_n(V_{2n-1,n-1}) & \longrightarrow & \pi_n(S^n) \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ Z_2 & & Z_2 + Z_2 & & Z_2 & & Z \end{array}$$

it follows that i_* must be an injection. Thus the image of $i_* \circ \Sigma \circ \Delta$ has two elements, and the conclusion of the lemma follows from $\lambda_* = 1 - i_* \circ \Sigma \circ \Delta$. The proof is complete.

THEOREM 4. *Let $n = 4s + 1$, $s > 0$, M and N be connected differentiable manifolds with $\dim M = n$ and $\dim N = 2n - 1$. Furthermore, assume that (f, M, N) is trivial, M closed and $f^*W_1(N) \neq W_1(M)$, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \mathcal{B}(\pi_{n-1})) \times Z_2,$$

where $W_1(N)$ and $W_1(M)$ are the Stiefel–Whitney classes of N and M respectively, and $\mathcal{B}(\pi_{n-1})$ is the coefficient bundle associated to \mathcal{B}_f with fiber $\pi_{n-1}(V_{2n-1,n}) \approx Z$.

Proof. For a fixed section σ_0 of \mathcal{B}_f the primary differences fill $H^{n-1}(M, \mathcal{B}(\pi_{n-1}))$. Let $\mathcal{B}(\pi_n(V_{2n-1,n}))$ (abbreviated to $\mathcal{B}(\pi_n)$) be the coefficient bundle associated to \mathcal{B}_f . Given a $D^{n-1} \in H^{n-1}(M, \mathcal{B}(\pi_{n-1}))$, the secondary obstructions $D^n(\sigma, \sigma')$ of all sections σ and σ' with $D^{n-1}(\sigma_0, \sigma) = D^{n-1}(\sigma_0, \sigma') = D^{n-1}$ fill $H^n(M, \mathcal{B}(\pi_n))$.

By Lemma 4, we can choose a basis of $\pi_n(V_{2n-1,n}) \approx Z_2 + Z_2$ in which λ_* has the form: $\lambda_*(1, 0) = (0, 1)$, $\lambda_*(0, 1) = (1, 0)$ and $\lambda_*(1, 1) = (1, 1)$. Let Δ_0 be an n -simplex in M , then any n -cocycle with coefficient in $\mathcal{B}(\pi_n)$ is cohomologous to an n -cocycle T_γ such that $\langle T_\gamma, \Delta_0 \rangle = \gamma \in \pi_n(V_{2n-1,n})$, $\langle T_\gamma, \Delta \rangle = 0$ for other n -simplices Δ in a triangulation of M including Δ_0 . Now $f^*W_1(N) \neq W_1(M)$, so there must exist a loop in M which preserves the orientation of $T(M)$ but reverses the orientation of $f^*T(N)$, or reverses the former but preserves the later. By the existence of such a loop, we see that $T_{\gamma+\lambda,\gamma} = T_\gamma + T_{\lambda,\gamma}$ must be cohomologous to 0 for any $\gamma \in \pi_n(V_{2n-1,n})$. Thus $T_{(1,0)} \sim T_{(0,1)}$, and $T_{(1,1)} = T_{(1,0)} - T_{(0,1)} \sim 0$. So $H^n(M, \mathcal{B}(\pi_n))$ has at most two elements.

Now let $N' = N \times R$, define $f': M \rightarrow N'$ by $f'(x) = (f(x), 0)$, then \mathcal{B}_f may be regarded as a subbundle of $\mathcal{B}_{f'}$. By Lemma 2, there exists a $\gamma_0 \in \pi_n(V_{2n-1,n})$ such that $i_*\gamma_0 \neq 0$. For any $D^{n-1} \in H^{n-1}(M, \mathcal{B}(\pi_{n-1}))$, we can choose two sections σ and σ' of \mathcal{B}_f such that $D^{n-1}(\sigma_0, \sigma) = D^{n-1}(\sigma_0, \sigma') = D^{n-1}$, and one of their secondary differences includes T_{γ_0} . Thus as sections of $\mathcal{B}_{f'}$, σ and σ' have a non-zero primary difference. It is obvious that σ and σ' are not homotopic in \mathcal{B}_f (see [5]). From this, we conclude that $H^n(M, \mathcal{B}(\pi_n)) \approx Z_2$ and two sections with non-zero secondary difference are not homotopic. The proof is complete.

THEOREM 5. *Let $n = 4s + 3$, $s \geq 0$, M and N be connected differentiable manifolds with $\dim M = n$ and $\dim N = 2n - 1$. Furthermore, assume that M is closed and nonorientable, and (f, M, N) is trivial, then*

$$I[M, N]_f \leftrightarrow H^{n-1}(M, \mathcal{B}(\pi_{n-1})) \times Z_2,$$

where $\mathcal{B}(\pi_{n-1})$ is the coefficient bundle associated to \mathcal{B}_f with fibre $\pi_{n-1}(V_{2n-1,n}) \approx Z$.

Proof. In virtue of Lemma 3, $\mathcal{B}(\pi_n)$ is trivial. Thus, by the assumption that M is closed and nonorientable, we have $H^n(M, \mathcal{B}(\pi_n)) \approx H^n(M, Z_4) \approx Z_2$ for $n > 3$, and $H^n(M, \mathcal{B}(\pi_n)) \approx H^n(M, Z) \approx Z_2$ for $n = 3$. Now, using the similar argument as that for theorem 4, the conclusion of this theorem follows.

From Theorems 1, 4, 5, we have

COROLLARY 2. *Let M be a nonorientable closed manifold with $\dim M = 2n + 1$, $n \geq 1$, then*

$$I[M, S_m^{4n+1}] \leftrightarrow H^1(M, Z_m) \times H^{2n}(M, [Z]_\tau) \times Z_2,$$

where $[Z]_\tau$ is the coefficient bundle of integers determined by $T(M)$.

REFERENCES

- [1] B. H. LI and F. P. PETERSON. *On immersions of k -manifolds in $(2k - 1)$ -manifolds*, proc. A.M.S., 83 (1981), No. 1, 159–162.
- [2] I. JAMES and EMERY THOMAS. *On the enumeration of cross-section*, Topology, **5** (1966), 95–114.
- [3] J. F. MCCLENDON. *Obstruction theory in fiber spaces*, Math. Z., 120 (1971), 1–17.
- [4] W. T. WU. *On the immersion of C^∞ -3-manifolds in a Euclidean space*, Scientia Sinica, XIII (1964), 335–336.
- [5] B. H. LI. *On immersions of manifolds in manifolds*, Scientia Sinica, XXV (1982), No. 3, 255–263.
- [6] B. H. LI. *On immersions of m -manifolds in $(m + 1)$ -manifolds*, preprint of I.H.E.S.
- [7] V. G. BOLTYANSKI. *The homotopy classification of cross-sections*, A.M.S. Translations, ser. 2 Vol. 21, 87–117.
- [8] V. G. BOLTYANSKI. *Second obstructions for cross-sections*, A.M.S. Translations, ser. 2, Vol. 21, 51–86.
- [9] W. T. WU. *A theorem on immersion*, Scientia Sinica, XIII (1964), p. 160.
- [10] G. PAECHTER. *The groups $\pi_r(V_{n,m})I$* , Quart. J. Math., 7 (1956), 249–268.
- [11] W. T. WU. *Classes caractéristiques et i -carrés d'une variété*, C.R. Acad. Sci., Paris, 230 (1950), 508–511.
- [12] I. JAMES, *The Topology of Stiefel Manifolds*, Cambridge Univ. Press, 1976.
- [13] W. S. MASSEY and F. P. PETERSON. *On the dual Stiefel–Whitney classes of a manifold*, Bol. Soc. Mat. Mexicana, 8 (1963), 1–13.

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Received Jan. 29, 1982